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Indecomposable triple systems exist for all lambda

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Abstract

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A triple system $(v, 3, \lambda)$ is *indecomposable* if it is not the union of two triple systems $(v, 3, \lambda_1)$, $(v, 3, \lambda_2)$ with $\lambda = \lambda_1 + \lambda_2$. A triple system is *simple* if it has no repeated blocks. In this paper we show the existence of simple indecomposable triple systems for all v and λ satisfying the necessary conditions with v large. Specifically, for each λ we show that there is a $v_0(\lambda)$ (where $v_0(\lambda) = O(\lambda^4)$) such that there exists a simple indecomposable triple system $(v, 3, \lambda)$ for each $v \geqslant v_0(\lambda)$ with $\lambda v(v-1) \equiv 0 \pmod 6$ and $\lambda(v-1) \equiv 0 \pmod 2$. We then concentrate on the case of $\lambda = 5$ and show that $v_0(5) \leqslant 25$.

1. Introduction

We begin with some standard definitions from design theory. A (v, k, λ) *balanced incomplete block design* ((v, k, λ)-BIBD or simply a (v, k, λ) design) is a pair $(V, \mathscr{B})$ where V is a v-element set of *points* and $\mathscr{B}$ is a collection of k-element subsets of V, called *blocks*, such that each pair of points appears together in exactly λ blocks. When $k = 3$ the design is a *triple system*. If in addition $\lambda = 1$, then it is a *Steiner triple system*. A design is *simple* if it contains no repeated blocks.

A common way to form a block design with larger λ is to union the sets of blocks of smaller designs sharing a common point-set. In particular, the union of a (v, k, λ_1) design with a (v, k, λ_2) design is a $(v, k, \lambda_1 + \lambda_2)$ design. Conversely, suppose that we can partition the blocks of a (v, k, λ) design so that each part induces a triple system with a strictly smaller λ. Then we say that the design is

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decomposable; the partition is a *decomposition*. A design with no decompositions is *indecomposable*. Indecomposable block designs form the building blocks for general block designs under block unions.

The main problem. Construct indecomposable simple triple systems for all possible v and λ.

The well-known necessary conditions for the existence of a simple $(v, 3, \lambda)$-BIBD are:

$$\lambda v(v-1) \equiv 0 \pmod 6, \qquad \lambda(v-1) \equiv 0 \pmod 2, \quad \text{and} \quad \lambda \leqslant v-2.$$

In 1847, Kirkman [9] proved that these conditions are sufficient for Steiner triple systems, $(v, 3, 1)$ designs. Building on the work of others, Dehon [5] proved in 1983 that for every v and λ satisfying the above necessary conditions there exists a simple $(v, 3, \lambda)$-BIBD (see also Sarvate [14]). Indecomposable simple designs were introduced in 1974 by Kramer [11] who showed that an indecomposable simple $(v, 3, 2)$ design exists for all $v \equiv 0, 1 \pmod 3$, $v \geqslant 4$, except $v = 7$. He also showed that indecomposable simple $(v, 3, 3)$ designs exist for all $v \equiv 1 \pmod 2$, $v \geqslant 5$. Colbourn and Rosa [4] proved that indecomposable simple $(v, 3, 4)$ designs exist for all $v \equiv 0, 1 \pmod 3$, $v \geqslant 10$. In 1989, Dinitz [6] and independently Milici [12] showed that indecomposable simple $(v, 3, 6)$ designs exist for $v = 8, 14$ and all $v \geqslant 17$. Thus for $\lambda = 1, 2, 3, 4$ and 6 the necessary conditions for the existence of an indecomposable simple $(v, 3, \lambda)$ are sufficient, except for possibly a few small values of v.

For the general case of $\lambda > 6$, Colbourn and Colbourn [3] constructed a single indecomposable simple $(v, 3, \lambda)$-BIBD for each odd λ. As they noted in their paper, their technique does not extend to even λ. Shen [15] used the Colbourn and Colbourn result and some recursive constructions to prove that the necessary conditions are asymptotically sufficient. Specifically, if λ is odd, then there exists a constant v_0 depending on λ with an indecomposable simple $(v, 3, \lambda)$ for all $v \geqslant v_0$ satisfying the necessary conditions. This result was proved using Wilson's Theorem and so the value of v_0 was not specified.

In this paper we show that for each λ the necessary conditions are sufficient except for finitely many small values of v. In fact, in Section 2 we give a specific upper bound for $v_0(\lambda)$ such that if $v > v_0(\lambda)$ and v satisfies the necessary conditions, then there exists a simple indecomposable triple system $(v, 3, \lambda)$. In Section 3 we will consider the remaining small case $\lambda = 5$. We show that that there exists an indecomposable simple $(v, 3, 5)$ for all $v \geqslant 25$, $v \equiv 1, 3 \pmod 6$.

2. Main results

A $(v, 3, \lambda)$ *partial triple system* (denoted $(v, 3, \lambda)$-PTS) is a pair $(V, \mathscr{B})$, where $\mathscr{B}$ is a collection of triples from a point set V such that each pair occurs at most λ

times. A subset of the blocks in a triple system forms a partial triple system, but it is not the case that every $(v, 3, \lambda)$-PTS completes to a triple system $(v, 3, \lambda)$.

A partial triple system $(V, \mathscr{B})$ is *λ-indecomposable* if there is no partitioning of the triples $\mathscr{B}$ into sets $\mathscr{B}_1$, $\mathscr{B}_2$ such that for $i = 1, 2$, $(V, \mathscr{B}_i)$ is a $(v, 3, \lambda_i)$-PTS with $\lambda_i > 0$ and $\lambda_1 + \lambda_2 = \lambda$. Note that indecomposability depends on the value of λ for partial triple systems. For suppose that $(V, \mathscr{B})$ is a partial triple system in which no pair occurs more than k times. Then $(V, \mathscr{B})$ is also a partial triple system for any value of $\lambda' \geqslant k$. But for $\lambda' \geqslant k + 1$ it always decomposes into a partial triple system with $\lambda = k$ (containing all of the blocks) and a partial triple system with $\lambda = \lambda' - k$ (containing no blocks).

If a partial triple system P is contained in a triple system T, then a decomposition of T induces a decomposition of P. The contrapositive is stated in the following lemma.

Lemma 2.1. *A triple system $(v, 3, \lambda)$ which contains a λ-indecomposable $[w, 3, \lambda]$-PTS is itself indecomposable.*

We next construct a single λ-indecomposable partial triple system for each λ.

Theorem 2.2. *For each λ there exists a simple λ-indecomposable partial triple system $(v, 3, \lambda)$-PTS with $v = \lambda(\lambda + 2)$.*

Proof. We begin with a chain of triples shown as triangles in Fig. 1. Specifically, our point set is $V = \{1, 2, \ldots, \lambda\} \cup \{a, b\}$. The triples are $\{i, a, b\}$ for $1 \leqslant i \leqslant \lambda$, $\{i, i+1, b\}$ for odd i's between 1 and $\lambda - 1$, and $\{i, i+1, a\}$ for even i's in this range. The triples with both a and b are called *white*, the remaining triples are called *black*.

We modify this partial configuration by 'blowing up' each point λ times. Our new point set is $V \times I_\lambda$, where $I_\lambda = \{1, 2, \ldots, \lambda\}$ and we will say that a point $(v, i) \in V \times I_\lambda$ lies *above* the point $v \in V$. Similarly, the triangles in our derived partial triple system lie *above* triangles in the base PTS. Specifically, above each black triple $\{i, i+1, x\}$ (where $x \in \{a, b\}$) we put every triple $\{(i, k_1), (i+1, k_2), (x, k_3)\}$, except the one with $k_1 = k_2 = k_3 = 1$. Above each white triple we put in a single triple $\{(i, 1), (a, 1), (b, 1)\}$. These new triples are colored black and white, depending on the color of the corresponding triples in the base design. Let P denote the resulting partial triple system.

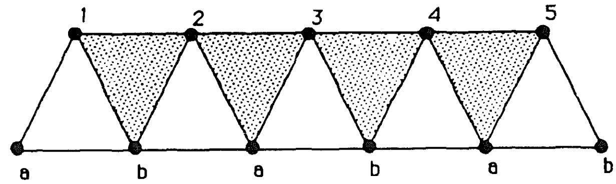

Fig. 1.

It is straightforward to show that P is simple, that no pair occurs more than λ times, and that λ is the smallest such value. We need only show that P is λ-indecomposable.

Let t_i denote the ith white triple, $\{(i, q), (a, 1), (b, 1)\}$. Color the edges lying above $(1, b)$ *red*, and those above $(2, b)$ *blue*. Note that every black triple above $(1, 2, b)$ contains one red and one blue edge. Moreover t_1 is the only triple with a red but not a blue edge, and t_2 is the only triple with a blue but not a red edge.

Suppose that P were decomposable into partial triple systems P_1, P_2 with multiplicities λ_1, λ_2, where $\lambda_1 + \lambda_2 = \lambda$. Each red edge (and similarly each blue edge) appears in exactly λ triples in P. Hence it appears in exactly λ_i triples in P_i. It follows that the total number of red edges appearing in triples of P_i is the same as the total number of blue edges in triples of P_i, namely $\lambda_i\lambda^2$. Each black triple of P either adds 1 to both sums or adds 0 to both sums. Suppose without loss of generality that $t_1 \in P_1$. Then there is one more red edge than blue edge in P_1. For the sums to equate there must be a triple with a blue but not a red edge. The only such triple is t_2. Summarizing, in any λ-decomposition the part with the triple t_1 must contain the triple t_2.

A similar argument shows that the part with triple t_2 also contains t_3. Inductively, we can conclude that one part contains each t_i, $1 \leq i \leq \lambda$. But these white triples contain the edge $\{(a, 1), (b, 1)\}$ a total of λ times, so $\lambda_1 = \lambda$. It follows that the decomposition is trivial, and that P is λ-indecomposable. □

We next complete this λ-indecomposable partial triple system to a triple system using a theorem of Rodger [13]. Our systems are simple, so here $\alpha = 0$.

Theorem 2.4 (Rodger). *A partial triple system $(n, 3, \lambda)$-PTS with α repeated triples can be embedded in a $(v, 3, \lambda)$ triple system with α repeated triples, for some $v \leq 3(2\lfloor\lambda/2\rfloor + 1)((3\lambda + 2)n + 1)$.*

Combining Lemma 2.1, Theorem 2.2 and Theorem 2.4 we get the following.

Theorem 2.5. *For each $\lambda \geq 1$ there exists an indecomposable simple $(v, 3, \lambda)$ triple system. Furthermore $v \leq 3(2\lfloor\lambda/2\rfloor + 1)((3\lambda + 2)\lambda(\lambda + 2) + 1)$.*

To prove our main theorem we will now embed this indecomposable simple triple system of order v into a simple triple system of order w for every $w \geq 2v + 1$. This will produce an indecomposable simple triple system $(w, 3, \lambda)$ for all w larger than some specified value (roughly $18\lambda^4$). We need only apply a new and powerful result of Shen [16] which is an analogue of the Doyen–Wilson theorem [7] for higher λ.

Theorem 2.6 (Shen). *A simple triple system $(v, 3, \lambda)$ can be embedded in a simple triple system $(w, 3, \lambda)$ for every $w \geq 2v + 1$ satisfying the necessary conditions.*

Using Theorems 2.5, 2.6 and Lemma 2.1 we have our main theorem.

Theorem 2.7. *An indecomposable simple triple system* $(v, 3, \lambda)$ *exists for all* $v \geq v_0(\lambda)$ *satisfying the necessary conditions where* $v_0(\lambda) \leq 6(2\lfloor\lambda/2\rfloor + 1)((3\lambda + 2)\lambda(\lambda + 2) + 1) + 1$.

The bound in Theorem 2.7 can be improved for odd λ by roughly a factor of $9\lambda^2$ if the triple systems constructed are not necessarily simple. Konig [10] gives examples of λ-regular graphs on $\lambda(\lambda + 2) + 1$ vertices without any regular factors (Hoffman, Rodger, and Rosa [8] show that these graphs are the smallest possible). Colbourn [2] shows that these graphs are the neighborhood of a point in a (not necessarily simple) triple system on $\lambda(\lambda + 2) + 2$ points. We now use the theorem of Stern [17] to embed this triple system in one of order v for all $v \geq 2\lambda(\lambda + 2) + 5$ which satisfies the necessary conditions.

3. Indecomposable triple systems with $\lambda = 5$

In this section we consider the case $\lambda = 5$. The necessary conditions for the existence of a $(v, 3, 5)$ triple system are $v \equiv 1, 3 \pmod 6$. We use the computer to construct several small examples of indecomposable simple $(v, 3, 5)$ designs and then again appeal to Shen's Theorem (Theorem 2.6) to complete the spectrum.

In [18], an extremely effective hill-climbing algorithm for finding Steiner triple system is discussed. It is straightforward to modify that algorithm to find simple triple system with higher λ. It is also an easy modification to the algorithm to fix a set of blocks that must occur in the final triple system. This is done by beginning with this set of blocks and then hill-climbing, never allowing any block from the initial set to be deleted.

Using these two modifications of Stinson's algorithm we found triple systems T with $\lambda = 5$ which contain the simple 5-indecomposable partial triple system P described in Theorem 2.2, so by Lemma 2.1 we have constructed a simple indecomposable triple system $(v, 3, 5)$. Note that P contains 35 points, so that necessarily T must contain at least 35 points. In fact, we were surprised to find a $(39, 3, 5)$ simple indecomposable triple system which contained P. Using this modified algorithm, we also found $(v, 3, 5)$ simple indecomposable triple system for all $v \equiv 1, 3 \pmod 6$, $39 \leq v \leq 49$. These triple systems are given in [1].

Using a smaller simple 5-indecomposable partial triple system and the aforementioned modification to Stinson's algorithm, Colbourn (personal communication) has constructed simple indecomposable triple systems $(v, 3, 5)$ for $v = 25$, 27, 31, 33 and 37.

We have the following theorem for $\lambda = 5$.

Theorem 3.1. *A simple indecomposable triple system* $(v, 3, 5)$ *exists for all* $v \equiv 1, 3 \pmod 6$, $v \geq 25$.

Proof. Assume $v \equiv 1, 3 \pmod 6$. If $25 \leqslant v \leqslant 49$, then a simple indecomposable triple system $(v, 3, 5)$ exists by the above computer constructions. If $v \geqslant 51$, then a simple indecomposable triple system $(v, 3, 5)$ exists by Theorem 2.6 and the existence of a simple indecomposable $(25, 3, 5)$ triple system. □

The values of v for which the existence of a simple indecomposible $(v, 3, 5)$ design remains open are $v = 13$, 15, 19, and 21.

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