On Hopf bifurcations of piecewise planar Hamiltonian systems

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A B S T R A C T

In this paper we study the number of limit cycles appearing in Hopf bifurcations of piecewise planar Hamiltonian systems. For the case that the Hamiltonian function is a piecewise polynomials of a general form we obtain lower and upper bounds of the number of limit cycles near the origin respectively. For some systems of special form we obtain the Hopf cyclicity.

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1. Introduction and main result

Consider a planar system of the form

\[\begin{align*}
\dot{x} &= f(x, y), \\
\dot{y} &= g(x, y), \quad x \neq 0,
\end{align*}\]

where

\[
(f(x, y), g(x, y)) = \begin{cases}
(f^+(x, y), g^+(x, y)), & x > 0, \\
(f^-(x, y), g^-(x, y)), & x < 0,
\end{cases}
\]

and \(f^\pm(x, y), g^\pm(x, y)\) are supposed to be \(C^\omega\) functions. Then the functions define two \(C^\omega\) systems below

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\[ \dot{x} = f^+(x, y), \quad \dot{y} = g^+(x, y) \quad (1.2) \]

and

\[ \dot{x} = f^-(x, y), \quad \dot{y} = g^-(x, y) \quad (1.3) \]

which are called the right subsystem and the left subsystem of (1.1) respectively.

The flow \( \varphi(t, A) \) of system (1.1) can be defined by using the flows \( \varphi^\pm(t, A) \) of (1.2) and (1.3) (see Bernardo, Budd, Champneys and Kowalczyk [1], Filippov [3], Kunze [6] and Han and Zhang [5]). Recently, [5] gave the following definitions.

**Definition 1.1.** Let \( A \) be a point on the \( y \)-axis. We call \( A \) a generalized singular point of (1.1) if \( f^+(A)f^-(A) \leq 0 \). In this case, we define

\[ \varphi(t, A) = A \quad \text{for all } t \in \mathbb{R}. \]

**Definition 1.2.** Let \( A_0 = (0, y_0) \) be an isolated generalized singular point of (1.1). Suppose for some \( \varepsilon_0 > 0 \), the solution \( \varphi(t, A) = (x(t, a), y(t, a)) \) of (1.1) starting at \( A = (0, a) \) satisfies for \( 0 < |a - y_0| < \varepsilon_0 \)

(i) \( \lim_{t \to \infty} \varphi(t, A) = A_0 \), \( 0 \leq t \leq T(a) \).

Define

\[ P(a) = \begin{cases} 
    y(T(a), a), & 0 < |a - y_0| < \varepsilon_0, \\
    y_0, & a = y_0.
\end{cases} \]

We call \( P : (y_0 - \varepsilon_0, y_0 + \varepsilon_0) \to \mathbb{R} \) a Poincaré return map of (1.1) near \( A_0 \). Further, we call \( A_0 \) a center of (1.1) if \( P(a) = a \) for \( 0 < a - y_0 < \varepsilon_0 \). We call \( A_0 \) a focus of (1.1) if \( P(a) \neq a \) for \( 0 < a - y_0 < \varepsilon_0 \). We call \( A_0 \) a stable (respectively, unstable) focus of (1.1) if \( P(a) < a \) (respectively, \( P(a) > a \)) for \( 0 < a - y_0 < \varepsilon_0 \).

We call \( A_0 \) a center-focus of (1.1) if it is neither a center nor a focus.

By (ii) in Definition 1.2, \( P \) is continuous at \( a = y_0 \).

In the following we suppose system (1.1) has a generalized singular point at the origin and the orbits near it are oriented clockwise. By [5], for \( 0 < a \ll 1 \), there exist \( T_1(a) \in (0, T(a)) \) and \( h^+(a) < 0 \) such that for \( A = (0, a) \)

\[ \varphi(T_1(a), A) = \varphi^+(T_1(a), A) = (0, h^+(a)) \equiv A_1. \]

For \( 0 < -a' \ll 1 \), there exist \( T_2(a') \in (0, T(a')) \) and \( h^-(a') > 0 \) such that for \( A' = (0, a') \)

\[ \varphi(T_2(a'), A') = \varphi^-(T_2(a'), A') = (0, h^-(a')). \]

Then the Poincaré return map near the origin can be denoted by the following way (see Fig. 1.1)

\[ P(a) = \begin{cases} 
    h^-(h^+(a)), & 0 < a < \varepsilon_0, \\
    0, & a = 0, \\
    h^+(h^-(a)), & 0 < -a < \varepsilon_0.
\end{cases} \]

Let
\[ d(a) = P(a) - a = \begin{cases} 
\ h^{-}(h^{+}(a)) - a, & a > 0, \\
0, & a = 0, \\
\ h^{+}(h^{-}(a)) - a, & a < 0. 
\end{cases} \] (1.4)

The function \( d(a) \) is called a successor function or a displacement function of system (1.1).

**Definition 1.3.** (See [5].) Let \( A_0 = (0, y_0) \) be an isolated generalized singular point of (1.1). We call it elementary if one of the conditions is satisfied:

(i) \( A_0 \) is elementary as a singular point of both (1.2) and (1.3);
(ii) \( A_0 \) is elementary as a singular point of (1.2), and
\[ f^{-}(A_0) = 0, \quad f_{y}^{-}(A_0)g^{-}(A_0) \neq 0; \]
(iii) \( A_0 \) is elementary as a singular point of (1.3), and
\[ f^{+}(A_0) = 0, \quad f_{y}^{+}(A_0)g^{+}(A_0) \neq 0; \]
(iv) \( f^{\pm}(A_0) = 0, \ f_{y}^{\pm}(A_0)g^{\pm}(A_0) \neq 0. \)

Let \( A_0 \) be an elementary focus of (1.1). Then from [2,3] the functions \( P(a) \) and \( d(a) \) are analytic for \( 0 \leq a \ll 1 \). Thus
\[ d(a) = \sum_{k \geq 1} V_k a^k, \quad 0 \leq a \ll 1. \]

**Definition 1.4.** [2] The coefficient \( V_k \) is called the \( k \)th Lyapunov constant. The origin is called a focus of order \( k \) if \( V_1 = \cdots = V_{k-1} = 0, \ V_k \neq 0. \)

The system (1.1) has a limit cycle near the origin if and only if the function \( d(a) \) has a positive zero near \( a = 0. \) For \( 0 < |a| \ll 1, \) Liu and Han [7] gave a further property of \( d(a) \) as follows.
Lemma 1.1. Let the origin be an elementary focus. Then

(i) the function \( d(a) \) has a positive zero near \( a = 0 \) if and only if it has a negative zero near \( a = 0 \);

(ii) if

\[
d(a) = V_k a^k + O(a^{k+1}), \quad V_k \neq 0
\]

for \( 0 \leq a \ll 1 \), then

\[
d(a) = \frac{V_k}{[(h^+)'(0)]^{k-1}} a^k + O(a^{k+1})
\]

for \( 0 \leq -a \ll 1 \), where \((h^+)'(0) < 0\).

By [2], system (1.1) has four possible types of foci denoted by \( FF \), \( FP \), \( PF \) and \( PP \), where \( F \) means “focus” and \( P \) means “parabolic”. We only need to consider the three cases \( FF \), \( FP \) and \( PP \) since the case of \( PF \) type is similar to the one of \( FP \) type.

In this paper we consider a piecewise Hamiltonian system of the form

\[
\begin{align*}
\dot{x} &= H_y, \quad \dot{y} = -H_x, \quad x \neq 0, \\
H(x, y) &= \begin{cases} 
H^+(x, y), & x > 0, \\
H^-(x, y), & x < 0,
\end{cases}
\end{align*}
\]

where

\[
H^\pm(x, y) \in C^\omega \text{ with } H^\pm(0, 0) = 0.
\]

Then we have for \((x, y)\) near the origin

\[
H^\pm(x, y) = \sum_{i+j \geq 1} h_{ij}^\pm x^i y^j.
\]

Then the right and left subsystems are

\[
\begin{align*}
\dot{x} &= H^+_y = h_{01}^+ + h_{11}^+ x + 2h_{02}^+ y + O(|x, y|^2), \\
\dot{y} &= -H^+_x = -h_{10}^+ - 2h_{20}^+ x - h_{11}^+ y + O(|x, y|^2)
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= H^-_y = h_{01}^- + h_{11}^- x + 2h_{02}^- y + O(|x, y|^2), \\
\dot{y} &= -H^-_x = -h_{10}^- - 2h_{20}^- x - h_{11}^- y + O(|x, y|^2).
\end{align*}
\]

By Definitions 1.2 and 1.3 and [2] we easily have the following lemma.

Lemma 1.2. Suppose the origin is an elementary focus or center of system (1.7) and the orbits near it have a clockwise orientation. Then

(i) the origin is of \( FF \) type if and only if

\[
h_{01}^\pm = 0, \quad h_{10}^\pm = 0, \quad h_{11}^\pm - 4h_{20}^\pm h_{02}^\pm < 0, \quad h_{20}^\pm > 0;
\]
(ii) the origin is of FP type if and only if

\begin{align*}
    h_{01}^+ &= 0, \quad h_{10}^- < 0, \quad h_{02}^- > 0, \\
    h_{01}^- &= 0, \quad h_{10}^+ = 0, \quad h_{11}^+ - 4h_{20}^+ h_{02}^+ < 0, \quad h_{20}^+ > 0;
\end{align*}

(iii) the origin is of PP type if and only if

\begin{align*}
    h_{01}^- &= 0, \quad h_{10}^- < 0, \quad h_{10}^+ > 0, \quad h_{02}^+ > 0.
\end{align*}

Therefore, in any case of the above we have

\[ H^\pm(0, y) = h_{02}^\pm y^2 + O(y^3), \quad h_{02}^\pm > 0. \]

Let \( H_0^+(a) = H^+(0, y) \) and consider the equations

\[ H_0^+(a) = H_0^+(a_1), \quad H_0^-(a) = H_0^-(a_2), \quad aa_1 < 0, \quad aa_2 < 0 \quad (1.10) \]

for \( 0 < |a| \ll 1 \). By Lemma 1.2 and the implicit function theorem the equations above define two \( C^\omega \) functions

\[ a_1 = -a + O(a^2), \quad a_2 = -a + O(a^2). \quad (1.11) \]

And by the definition of \( h^\pm \) we have clearly

\[ a_1 = \begin{cases} h^+(a), & a > 0, \\ (h^+)^{-1}(a), & a < 0 \end{cases} \quad a_2 = \begin{cases} (h^-)^{-1}(a), & a > 0, \\ h^-(a), & a < 0 \end{cases} \quad (1.12) \]

where \((h^\pm)^{-1}\) denote the inverse of \( h^\pm \). See Fig. 1.2.
Lemma 1.3.

(i) The function $F(a)$ has a positive (or negative) zero near $a = 0$ if and only if the successor function $d(a)$ has a positive (or negative) zero near $a = 0$.

(ii) If (1.15) and (1.6) hold for $0 \leq a \ll 1$ and $0 \leq -a \ll 1$ respectively, then we have

$$F(a) = -V_k a^k + O(a^{k+1}), \quad \text{for } 0 \leq a \ll 1,$$

and

$$F(a) = (-1)^{k+1} V_k a^k + O(a^{k+1}), \quad \text{for } 0 \leq -a \ll 1.$$

Proof. (i) For any small $a > 0$, by (1.13) and (1.4)

$$F(a) = 0 \iff h^+(a) = (h^-)^{-1}(a) \iff h^-(h^+(a)) = a \iff d(a) = 0.$$

For $a < 0$, the proof is similar.

(ii) By (1.12), for $a > 0$, we have

$$a = h^-(a_2), \quad a_1 = h^+(a) = h^+(h^-(a_2)).$$

Then by (1.13), (1.4), (1.6) and (1.11)

$$F(a) = a_1 - a_2 - h^+(h^-(a_2)) - a_2 = d(a_2) = \left[\frac{V_k}{(h^+(0))} a_2^k + O(a_2^{k+1})\right].$$

Noting that $(h^+)'(0) = a_1'(0) = -1$, by (1.11) we get

$$F(a) = \frac{V_k}{(-1)^{k-1}}(-1)^k a^k + O(a^{k+1}) = -V_k a^k + O(a^{k+1}).$$

For $a < 0$, we have $a_2 = h^-(h^+(a))$ by (1.12). Then similarly by (1.13), (1.4), (1.5) and (1.11), we have

$$F(a) = a_1 - a_2 - a_1 - h^-(h^+(a_1)) = -d(a_1) = -\left[V_k a_1^k + O(a_1^{k+1})\right]$$

$$= -\left[(-1)^k V_k a^k + O(a^{k+1})\right] = (-1)^{k+1} V_k a^k + O(a^{k+1}).$$

This ends the proof. □

It is clear that if (1.14) holds then the origin is a stable (respectively, unstable) focus of order $k$ as $V_k < 0$ (respectively, $V_k > 0$).

We now suppose $H_0^\pm(y)$ are polynomials. Then we can write

$$H_0^\pm(y) = \lambda^\pm(y^2 + H_1^\pm(y) + H_2^\pm(y)).$$

(1.15)
where $\lambda^\pm > 0$ and

$$H_1^\pm(y) = \sum_{j=1}^{k^\pm} h_j^\pm y^{2j+1}, \quad H_2^\pm(y) = \sum_{j=1}^{l^\pm} r_j^\pm y^{2j+2}, \quad l^\pm = k^\pm \text{ or } k^\pm - 1.$$ 

It is easy to see that

$$\deg H_0^+ = \begin{cases} 2k^+ + 1, & l^+ = k^+ - 1, \\ 2k^+ + 2, & l^+ = k^+ \end{cases}, \quad \deg H_0^- = \begin{cases} 2k^- + 1, & l^- = k^- - 1, \\ 2k^- + 2, & l^- = k^- \end{cases}.$$ 

Our main results can be stated as follows.

**Theorem 1.1.** Consider system (1.7) satisfying (1.15) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Let $k_0 = \min\{k^+, k^-\}$, $l = \max\{l^+, l^-\}$. Then for any $\varepsilon_0 > 0$ there are $r_j^\pm \in (-\varepsilon_0, \varepsilon_0)$ for $1 \leq j \leq l^\pm$ and $h_j^\pm \in (-\varepsilon_0, \varepsilon_0)$ for $1 \leq j \leq k^\pm$ such that the system (1.7) has at least $k_0 + l - 1$ small amplitude limit cycles near the origin surrounding the focus at the origin. Further, there are at most $|(\deg H_0^+ - 1)(\deg H_0^- - 1) - 2l/2|$ limit cycles which surround the origin.

Under the conditions of the above theorem, the origin is always a generalized singular point. If we consider a perturbation of system (1.7) of the form

$$\dot{x} = \widetilde{H}_y, \quad \dot{y} = -\widetilde{H}_x, \quad x \neq 0,$$

then the origin may be no longer singular. In this case, similar to (1.15) we have

$$\widetilde{H}^\pm(0, y) = H_0^\pm(y) + \varepsilon_2^\pm y = \lambda^\pm(y^2 + h_0^\pm y + H_1^\pm(y) + H_2^\pm(y)).$$

where $h_0^\pm = \varepsilon_2^\pm / \lambda^\pm$. For (1.16) we have

**Theorem 1.2.** Consider system (1.16) satisfying (1.17) and (1.18) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Let $k_0 = \min\{k^+, k^-\}$, $l = \max\{l^+, l^-\}$. Then for any $\varepsilon_0 > 0$ there are $r_j^\pm \in (-\varepsilon_0, \varepsilon_0)$ for $1 \leq j \leq l^\pm$ and $h_j^\pm \in (-\varepsilon_0, \varepsilon_0)$ for $0 \leq j \leq k^\pm$ such that the system (1.6) has at least $k_0 + l$ small amplitude limit cycles near the origin. Further, there are at most $|(\deg H_0^+ - 1)(\deg H_0^- - 1) - 2l/2|$ limit cycles which intersect the $y$-axis.

**Remark 1.1.** If $\deg H_0^+ = \deg H_0^- = m$, then $k_0 = k^+ = k^-$, $l = l^+ = l^-$. Hence, when $l = k_0 - 1$ (respectively, $l = k_0$) we have $m = 2k_0 + 1$ (respectively, $m = 2k_0 + 2$). Therefore $k_0 + l = m - 2$.

**Theorem 1.3.** Consider system (1.7) satisfying (1.15) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Let $k = \max\{k^+, k^-\}$. If one of the conditions below is satisfied:

1. $H_1^+(y) \equiv 0$;
2. $H_1^-(y) \equiv 0$;
3. $H_2^+(y) \equiv H_2^-(y)$. 

then there exists an \( \varepsilon_0 > 0 \) such that for all \( r_j \in (-\varepsilon_0, \varepsilon_0) \) for \( 1 \leq j \leq \ell \) and \( h_j \in (-\varepsilon_0, \varepsilon_0) \) for \( 1 \leq j \leq k \), the maximal number of limit cycles of system (1.7) near the origin is \( k - 1 \). In other words, the system (1.7) has Hopf cyclicity \( k - 1 \) at the origin.

**Remark 1.2.** If we consider system (1.16), we can obtain \( k \) limit cycles near the origin.

The above theorems are proved in Section 3, and as a preliminary some lemmas are first proved in Section 2. In the last section some concrete systems are studied showing our main results are sharp.

### 2. Preliminary lemmas

Let

\[
H_0(y) = y^2 + \sum_{j=1}^{\ell} r_j y^{2j+2} + \sum_{j=1}^{k} h_j y^{2j+1} = y^2 (1 + u(y) + v(y)),
\]

where

\[
u(y) = \sum_{j=1}^{\ell} r_j y^{2j}, \quad v(y) = \sum_{j=1}^{k} h_j y^{2j-1},
\]

and \( \ell = k - 1 \) or \( k \). Then \( \deg H_0 = 2k + 1 \) as \( \ell = k - 1 \), or \( 2k + 2 \) as \( \ell = k \). We will take \( r_j \) and \( h_j \) as small parameters.

Consider the equation \( H_0(y) = H_0(a) \) on the region \( y < 0 < a \). The equation is equivalent to

\[
\varphi(y) + \varphi(a) = 0, \quad y < 0 < a,
\]

where

\[
\varphi(y) = y \left( 1 + u(y) + v(y) \right)^{1/2}.
\]

Note \( \varphi \in C^\omega \) near the origin. The implicit function theorem implies that a unique \( C^\omega \) function of the form

\[
y = -a + \sum_{j \geq 2} \mu_j a^j = -a \left( 1 - \sum_{j \geq 2} \mu_j a^{j-1} \right)
\]

exists which satisfies (2.2). We want to know how the coefficients \( \mu_j \) in (2.4) depend on the coefficients \( r_j \) and \( h_j \) in (2.1). For the purpose, we first study the function \( \varphi \). We have

**Lemma 2.1.** The function \( \varphi \) has the form

\[
\varphi(y) = y + \frac{1}{2} \sum_{j \geq 1} \tilde{r}_j + q_j y^{2j+1} + \frac{1}{2} \sum_{j \geq 1} \tilde{h}_j y^{2j},
\]

where
Now (2.5) follows easily.

Using where

\[ \tilde{h}_j = h_j - \frac{1}{2} \sum_{i=1}^{j-1} h_i r_{j-i} + p_j (h_1, \ldots, h_{j-1}, r_1, \ldots, r_{j-1}), \quad j \geqslant 2, \]

\[ p_j = O(|h_1, \ldots, h_{j-1}|^2 + |h_1, \ldots, h_{j-1}| \cdot |r_1, \ldots, r_{j-1}|^2), \quad j \geqslant 1, \]

\[ r_j = 0 \text{ for } j \geqslant l + 1, \quad h_j = 0 \quad \text{for } j \geqslant k + 1. \]  

(2.6)

**Proof.** Note that

\[ (1 + u + v)^2 = 1 + \frac{1}{2} u (1 + P(u)) + \frac{1}{2} v \left( 1 - \frac{1}{2} u + Q(u, v) \right), \]

where \( P(u) = O(u), \) \( Q(u, v) = O(|v| + |u|^2) \) are \( C^\infty \) functions near the origin. We have by (2.3)

\[ \varphi(y) = y \left( 1 + \varphi_1(y) + \varphi_2(y) \right), \]

where

\[ \varphi_1(y) = \frac{1}{2} u(y) (1 + P(u(y))), \quad \varphi_2(y) = \frac{1}{2} v(y) \left( 1 - \frac{1}{2} u(y) + Q(u(y), v(y)) \right). \]

Using

\[ u^n(y) = (r_1 y^2 + r_2 y^4 + \cdots + r_j y^{2j})^n \]

\[ = r_1^n y^{2n} + n r_1^{n-1} r_2 y^{2n+2} + \cdots + r_j^n y^{2nl}, \]

it is easy to see that

\[ \varphi_1(y) = \frac{1}{2} \tilde{r}_1 y^2 + \frac{1}{2} \sum_{j=2}^{l} \sum_{j=1}^{j} \left( r_j + O(|r_1, \ldots, r_{j-1}|^2) \right) y^{2j} + \frac{1}{2} \sum_{j=l+1}^{j} O(|r_1, \ldots, r_l|^2) y^{2j} = \frac{1}{2} \sum_{j=1}^{l} \tilde{r}_j y^{2j}, \]

where each \( \tilde{r}_j \) is a polynomial in \( r_1, \ldots, r_l. \) Similarly, we have

\[ \varphi_2(y) = \frac{1}{2} v(y) - \frac{1}{4} u(y) v(y) + O(\left| u(y) \right| \left| v(y) \right| + \left| v(y) \right|^2) \]

\[ = \frac{1}{2} h_1 y + \frac{1}{2} \sum_{j=2}^{l} \left( h_j - \frac{1}{2} \sum_{i=1}^{j-1} h_i r_{j-i} \right) y^{2j-1} \]

\[ + \frac{1}{2} \sum_{j=2}^{l} p_j (h_1, \ldots, h_{j-1}, r_1, \ldots, r_{j-1}) y^{2j-1} + \frac{1}{2} \sum_{j=1}^{l} O(|h_1, \ldots, h_j|^2) y^{2j} \]

\[ = \frac{1}{2} \sum_{j=1}^{l} \tilde{h}_j y^{2j-1} + \frac{1}{2} \sum_{j=1}^{l} q_j y^{2j}. \]

Now (2.5) follows easily. \( \square \)
By (2.4) we can prove in the same way

**Lemma 2.2.** Let \( y \) satisfy (2.4). Then for \( m \geq 1 \)

\[
y^m = (-1)^m a^m \left[ 1 - m \left( \mu_2 a + \sum_{j \geq 2} \tilde{\mu}_{j+1,m} a^j \right) \right],
\]

where each \( \tilde{\mu}_{j+1,m} \) is a polynomial in \( \mu_2, \ldots, \mu_j \) and has the form

\[
\tilde{\mu}_{j+1,m} = \mu_{j+1} + O(\|\mu_2, \ldots, \mu_j\|^2), \quad j \geq 2.
\]  

(2.7)

By Lemma 2.2 we have

\[
y^{2m} + a^{2m} = 2a^{2m} - 2m \left( \mu_2 a^{2m+1} + \sum_{j \geq 2} \tilde{\mu}_{j+1,2m} a^{2m+j} \right),
\]

\[
y^{2m+1} + a^{2m+1} = (2m + 1) \left( \mu_2 a^{2m+2} + \sum_{j \geq 2} \tilde{\mu}_{j+1,2m+1} a^{2m+j+1} \right).
\]  

(2.8)

Hence, by (2.8)

\[
\frac{1}{2} h_1(y^2 + a^2) + \frac{1}{2} \sum_{j \geq 2} \tilde{h}_j(y^{2j} + a^{2j}) = h_1 a^2 - h_1 \mu_2 a^3 - h_1 \sum_{j \geq 2} \tilde{\mu}_{j+1,2} a^{j+2}
\]

\[
+ \sum_{j \geq 2} \tilde{h}_j a^{2j} - \sum_{j \geq 2} \tilde{h}_j \left( \mu_2 a^{2j+1} + \sum_{i \geq 2} \tilde{\mu}_{i+2,2j} a^{2j+i} \right)
\]

\[
= h_1 a^2 - h_1 \mu_2 a^3 + (\tilde{h}_2 - h_1 \tilde{\mu}_{3,2}) a^4 - (h_1 \tilde{\mu}_{4,2} + 2 \mu_2 \tilde{h}_2) a^5
\]

\[
+ (\tilde{h}_3 - 2 \tilde{h}_2 \tilde{\mu}_{3,4} - h_1 \tilde{\mu}_{5,2}) a^6 + \ldots
\]

\[
= h_1 a^2 - h_1 \mu_2 a^3 + \sum_{j \geq 1} A_j^{(1)} a^{2j+2} + \sum_{j \geq 1} B_j^{(1)} a^{2j+3},
\]  

(2.9)

where by (2.7)

\[
A_j^{(1)} = \tilde{h}_{j+1} + O(\|h_1, \ldots, h_j, |\mu_2, \ldots, \mu_{2j+1}\|),
\]

\[
B_j^{(1)} = O(\|h_1, \ldots, h_{j+1}, |\mu_2, \ldots, \mu_{2j+2}\|).
\]  

(2.10)

Similarly,

\[
\frac{1}{2} \sum_{j \geq 1} \tilde{r}_j(y^{2j+1} + a^{2j+1}) = \frac{1}{2} \sum_{j \geq 1} \left[ \tilde{r}_j(2j + 1) \left( \mu_2 a^{2j+2} + \sum_{i \geq 2} \tilde{\mu}_{i+1,2j+1} a^{2j+i+1} \right) \right]
\]

\[
= 3 \frac{r_1 \mu_2 a^4}{2} + 3 \frac{r_1 \tilde{\mu}_{3,3} a^5}{2} + \left( \frac{5}{2} \tilde{r}_2 \mu_2 + 3 \frac{1}{2} \tilde{\mu}_{4,3} \right) a^6
\]
\[
+ \left( \frac{5}{2} \tilde{r}_2 \tilde{\mu}_3, 3 \tilde{r}_1 \tilde{\mu}_5, 3 \right) a^7 + \left( \frac{7}{2} \tilde{r}_3 \mu_2 + \frac{5}{2} \tilde{r}_2 \tilde{\mu}_4, 5 + \frac{3}{2} \tilde{r}_1 \tilde{\mu}_6, 3 \right) a^8 + \cdots
= \sum_{j \geq 1} A_j^{(2)} a^{2j+2} + \sum_{j \geq 1} B_j^{(2)} a^{2j+3},
\] (2.11)

where
\[
A_j^{(2)} = \sum_{i=1}^{j} \left( i + \frac{1}{2} \right) \tilde{r}_i \tilde{\mu}_2 (j-i) + 2, 2i+1, \quad \tilde{\mu}_2, 2j+1 = \mu_2, \quad j \geq 1,
\]
\[
B_j^{(2)} = \sum_{i=1}^{j} \left( i + \frac{1}{2} \right) \tilde{r}_i \tilde{\mu}_2 (j-i) + 3, 2i+1 = O(|r_1, \ldots, r_j : |\mu_3, \ldots, \mu_2, j+1|).
\] (2.12)

Further,
\[
\frac{1}{2} \sum_{j \geq 1} q_j (y^{2j+1} + a^{2j+1}) = \sum_{j \geq 1} A_j^{(3)} a^{2j+2} + \sum_{j \geq 1} B_j^{(3)} a^{2j+3},
\] (2.13)

where
\[
A_j^{(3)} = O(|q_1, \ldots, q_j : |\mu_2, \ldots, \mu_2, j|),
\]
\[
B_j^{(3)} = O(|q_1, \ldots, q_j : |\mu_2, \ldots, \mu_2, j+1|).
\] (2.14)

By (2.2) and (2.5) we have
\[
y + a + \frac{1}{2} \sum_{j \geq 1} \tilde{h}_j (y^{2j} + a^{2j}) + \frac{1}{2} \sum_{j \geq 1} \tilde{r}_j (y^{2j+1} + a^{2j+1}) + \frac{1}{2} \sum_{j \geq 1} q_j (y^{2j+1} + a^{2j+1}) = 0.
\]

Substituting (2.4), (2.9), (2.11) and (2.13) into the above gives
\[
\sum_{j \geq 2} \mu_j a^j + h_1 a^2 - h_1 \mu_2 a^3 + \sum_{j \geq 1} A_j a^{2j+2} + \sum_{j \geq 1} B_j a^{2j+3} = 0,
\]

where
\[
A_j = A_j^{(1)} + A_j^{(2)} + A_j^{(3)}, \quad B_j = B_j^{(1)} + B_j^{(2)} + B_j^{(3)}.
\]

It implies that
\[
\mu_2 = -h_1, \quad \mu_3 = h_1 \mu_2 = -h_1^2,
\]
\[
\mu_{2j+2} = -A_j, \quad \mu_{2j+3} = -B_j, \quad j \geq 1.
\] (2.15)
Thus, we have

\[ A_j = \tilde{h}_{j+1} + \sum_{i=1}^{j} \left( i + \frac{1}{2} \right) \tilde{r}_i \tilde{\mu}_{2(j-i)+2,2i+1} + O(|h_1, \ldots, h_j| \cdot |\mu_2, \ldots, \mu_{2j+1}|) \]

\[ = h_{j+1} - \frac{1}{2} \sum_{i=1}^{j} h_ir_{j+1-i} + \sum_{i=1}^{j} \left( i + \frac{1}{2} \right) \tilde{r}_i \tilde{\mu}_{2(j-i)+2,2i+1} + O(|h_1, \ldots, h_{j-1}|^3 + |h_1, \ldots, h_{j-1}| \cdot |r_1, \ldots, r_{j-1}|^2 + |h_1, \ldots, h_j| \cdot |\mu_2, \ldots, \mu_{2j+1}|) \]

By (2.10), (2.12), (2.14) and (2.6), for \( j \geq 1 \) we have

\[ B_j = O(|h_1, \ldots, h_{j-1}| \cdot |\mu_2, \ldots, \mu_{2j+2}|) + O(|r_1, \ldots, r_j| \cdot |\mu_3, \ldots, \mu_{2j+1}|) + O(|q_1, \ldots, q_j| \cdot |\mu_2, \ldots, \mu_{2j+1}|) \]

\[ = O(|h_1, \ldots, h_{j+1}| \cdot |\mu_2, \ldots, \mu_{2j+2}| + |r_1, \ldots, r_j| \cdot |\mu_3, \ldots, \mu_{2j+1}|) \]

(2.16)

Thus, by (2.15), (2.16) and (2.7)

\[ \mu_{2j+2} = -h_{j+1} + \frac{1}{2} \sum_{i=1}^{j} h_ir_{j+1-i} + O(|h_1, \ldots, h_{j-1}|^3 + |h_1, \ldots, h_{j-1}| \cdot |r_1, \ldots, r_{j-1}|^2 + |r_1, \ldots, r_j| \cdot |\mu_2, \ldots, \mu_{2j+1}|) \]

(2.17)

for \( j = 1, \ldots, k - 1 \),

\[ \mu_{2j+2} = \frac{1}{2} \sum_{i=1}^{j} h_ir_{j+1-i} - \sum_{i=1}^{k} \left( i + \frac{1}{2} \right) \tilde{r}_i \tilde{\mu}_{2(j-i)+2,2i+1} + O(|h_1, \ldots, h_k| \cdot |\mu_2, \ldots, \mu_{2j+1}| + |h_1, \ldots, h_{j-1}|^3 + |h_1, \ldots, h_{j-1}| \cdot |r_1, \ldots, r_{j-1}|^2) \]

(2.18)

for \( j = k, \ldots, 2k - 1 \), and

\[ \mu_{2j+3} = O(|h_1, \ldots, h_{j+1}| \cdot |\mu_2, \ldots, \mu_{2j+2}| + |r_1, \ldots, r_j| \cdot |\mu_3, \ldots, \mu_{2j+1}|) \]

(2.19)

for \( j \geq 1 \). Noting that \( \mu_2 = -h_1, \mu_3 = -h_1^2 \), by (2.17), (2.18), (2.19), (2.6) and (2.7) it follows

\[ \mu_{2j+2} = \begin{cases} 
- h_{j+1} + O(|h_1, \ldots, h_j|^2 + |h_1, \ldots, h_j| \cdot |r_1, \ldots, r_j|), & j = 1, \ldots, k - 1, \\
\frac{1}{2} \sum_{i=1}^{j} h_ir_{j+1-i} - \sum_{i=1}^{k} \left( i + \frac{1}{2} \right) r_i \mu_{2(j-i)+2} + O(|h_1, \ldots, h_k|^2 + |h_1, \ldots, h_k| \cdot |r_1, \ldots, r_k|^2), & j = k, \ldots, 2k - 1,
\end{cases} \]

(2.20)

and

\[ \mu_{2j+3} = O(|h_1, \ldots, h_{j+1}|^2), \quad j \geq 1. \]

(2.21)

Thus, we have
Theorem 2.1. Let (2.1) hold with \( l = k - 1 \) or \( k \). Then Eq. (2.2) defines a unique function given by (2.4) where \( \mu_2 = -h_1 \), \( \mu_3 = -h_1^2 \), and the coefficients \( \mu_j \) for \( j \geq 4 \) satisfy (2.20) and (2.21), where \( r_k = 0 \) as \( l = k - 1 \).

3. Proof of the main results

Proof of Theorem 1.1. Suppose (1.15) holds. For convenience, let

\[
r_j^+ = 0, \quad j \geq l^+ + 1, \quad h_j^- = 0, \quad j \geq k^- + 1.
\]

(3.1)

Then (1.15) can be rewritten as

\[
H_0^\pm(y) = \lambda^\pm\left(y^2 + \sum_{j \geq 1} r_j^\pm y^{2j+2} + \sum_{j \geq 1} h_j^\pm y^{2j+1}\right).
\]

(3.2)

Noting that \( k = \max\{k^+, k^-\} \), \( l = \max\{l^+, l^-\} \) and \( l^+ = k^+ \) or \( l^+ = k^+ - 1 \) we easily obtain \( l = k \) or \( l = k - 1 \). Without loss of generality, we suppose \( k = k^+ \) and \( k_0 = k^- \) in the following.

Now applying Theorem 2.1 we obtain two functions of form (2.4) as follows

\[
y = -a + \sum_{j \geq 2} \mu_j^\pm a^j \equiv \psi^\pm(a)
\]

(3.3)

such that \( H_0^\pm(y) = H_0^\pm(a) \) for \( 0 < |a| \ll 1 \). Let

\[
a_1 = \psi^+(a), \quad a_2 = \psi^-(a), \quad F(a) = a_1 - a_2 = \sum_{j \geq 2} \mu_j a^j.
\]

(3.4)

where \( \mu_j = \mu_j^+ - \mu_j^- \), \( j \geq 2 \). By (2.15) and (2.20), we have

\[
\mu_2^\pm = -h_1^\pm,
\]

\[
\mu_{2j+2}^\pm = -h_{j+1}^\pm + O(|h_1^\pm, \ldots, h_j^\pm|^2 + |h_1^\pm, \ldots, h_j^\pm| \cdot |r_1^\pm, \ldots, r_j^\pm|),
\]

for \( j = 1, \ldots, k - 1 \).

(3.5)

Then

\[
\mu_2 = h_1^- - h_1^+,
\]

\[
\mu_{2j+2} = h_{j+1}^- - h_{j+1}^+ + O(|h_1^\pm, \ldots, h_j^\pm|^2 + |h_1^\pm, \ldots, h_j^\pm| \cdot |r_1^\pm, \ldots, r_j^\pm|),
\]

for \( j = 1, \ldots, k - 1 \).

(3.6)

Note that

\[
\frac{\partial (\mu_2, \mu_4, \ldots, \mu_{2k})}{\partial (h_1^+, \ldots, h_k^+)} \bigg|_{h=0, \sigma=0} = \begin{vmatrix} -1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1 \end{vmatrix} = (-1)^k \neq 0,
\]

where
Then by the implicit function theorem for $\sigma$ and $h$ small we can solve from (3.6)

\[ h_1^+ = h_1^- - \mu_2, \]
\[ h_{j+1}^+ = h_{j+1}^- - \mu_{2j+2} + O\left(|h_1^-, \ldots, h_j^-|^2 + |\mu_2, \mu_4, \ldots, \mu_{2j}|^2 + |h_1^+, \ldots, h_j^+| + |\mu_2, \mu_4, \mu_{2j}, r_j^+, \ldots, r_j^+|\right) = \Psi_{j+1}(\mu_2, \mu_4, \ldots, \mu_{2j+2}, h_1^-, \ldots, h_j^+, r_1^+, \ldots, r_j^+) \in C^\omega, \quad j = 1, \ldots, k - 1, \quad (3.7) \]

which implies that $\mu_2, \mu_4, \ldots, \mu_{2k}$ can be taken as free parameters.

Let $\mu_s, s \geq 2$ be the first nonzero coefficient in (3.4). Then

\[ a_1 - a_2 = \sum_{j \geq s} \mu_j a^j = a^s (\mu_s + O(a)) . \]

It follows from Definition 1.4 and Lemma 1.3 that the origin is a focus of order $s$ of system (1.7). If the origin is stable, then $a_1 > a_2$ for $0 < |a| \ll 1$, see Fig. 1.2. Hence,

\[ \mu_s a^s > 0, \quad \text{for} \ a \neq 0 \]

which implies

\[ \mu_s > 0, \quad s = 2j, \quad \text{for some} \ j \in \mathbb{N}. \]

Similarly, if the origin is unstable, then

\[ \mu_s < 0, \quad s = 2j, \quad \text{for some} \ j \in \mathbb{N}. \]

Thus, we have

\[ \mu_2 = \mu_3 = \cdots = \mu_{2j} = 0 \quad \Rightarrow \quad \mu_{2j+1} = 0, \quad j \geq 1. \quad (3.8) \]

By (2.21), we have

\[ \mu_{2j+1}^\pm = O\left(|h_1^\pm, \ldots, h_j^\pm|^2\right), \quad \text{for} \ j \geq 1. \]

Hence, by (3.7),

\[ \mu_{2j+1} = \mu_{2j+1}^+ - \mu_{2j+1}^- = O\left(|h_1^+, \ldots, h_j^+|^2 + |h_1^-, \ldots, h_j^-|^2\right) = O\left(|h_1^-, \ldots, h_j^-|^2 + |\mu_2, \mu_4, \ldots, \mu_{2j}|^2\right), \quad \text{for} \ j = 1, \ldots, k, \]
\[ \mu_{2j+1} = O\left(|h_1^-, \ldots, h_k^-|^2 + |\mu_2, \mu_4, \ldots, \mu_{2k}|^2\right), \quad \text{for} \ j \geq k + 1. \quad (3.9) \]

Then by (2.15) and (3.9), it can be seen that $\mu_{2j+1} \in C^\omega$ in $(\mu_2, \mu_4, \ldots, \mu_{2k})$ for each $j \geq 1$. It follows from (3.8) that
\[
\mu_3 = O(\mu_2),
\mu_5 = O(|\mu_2, \mu_4|),
\vdots
\mu_{2k+1} = O(|\mu_2, \mu_4, \ldots, \mu_{2k}|).
\]

Further, by (2.20) we have
\[
\mu_{2k+2}^\pm = -\left(\frac{3}{2}r_1^\pm \mu_2^\pm + \cdots + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{k} h_i^\pm r_{k+1-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right),
\]
\[
\mu_{2k+4}^\pm = -\left(\frac{3}{2}r_1^\pm \mu_2^\pm + \frac{5}{2} r_2^\pm \mu_2^\pm + \cdots + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{k+1} h_i^\pm r_{k+2-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right),
\]
\vdots
\[
\mu_{4k}^\pm = -\left(\frac{3}{2}r_1^\pm \mu_2^\pm + \cdots + \frac{2k-1}{2} r_{k-1}^\pm \mu_2^\pm + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{2k-1} h_i^\pm r_{2k-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right).
\]

By (3.5), it is obvious that
\[
|\mu_2^\pm, \mu_4^\pm, \ldots, \mu_{2k}^\pm| = O\left(|h_1^\pm, \ldots, h_k^\pm|\right).
\]

Then substituting the first \( j \) equations of (3.11) into the \((j+1)\)th one for \( j = 1, \ldots, k - 1 \) we obtain from (3.11)
\[
\mu_{2k+2}^\pm = -\left(\frac{3}{2}r_1^\pm \mu_2^\pm + \cdots + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{k} h_i^\pm r_{k+1-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right),
\]
\[
\mu_{2k+4}^\pm = -\left(\frac{5}{2}r_2^\pm \mu_2^\pm + \frac{3}{2} r_2^\pm \mu_2^\pm + \cdots + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{k+1} h_i^\pm r_{k+2-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right),
\]
\vdots
\[
\mu_{4k}^\pm = -\left(\frac{5}{2}r_2^\pm \mu_2^\pm + \cdots + \frac{2k-1}{2} r_{k-1}^\pm \mu_2^\pm + \frac{2k+1}{2} r_k^\pm \mu_2^\pm\right) + \frac{1}{2}\sum_{i=1}^{2k-1} h_i^\pm r_{2k-i}^\pm \\
+ O\left(|h_1^\pm, \ldots, h_k^\pm|^2 + |h_1^\pm, \ldots, h_k^\pm| \cdot |r_1^\pm, \ldots, r_k^\pm|^2\right).
\]
Let us suppose
\[ \mu_2 = \mu_4 = \cdots = \mu_{2k} = 0 \]  
(3.13)
which leads to \( \mu_{2j}^+ = \mu_{2j}^- \) for \( j = 1, \ldots, k \). Then by (3.7), we obtain
\[ h_1^+ = h_1^- , \]
\[ h_{j+1}^+ = h_{j+1}^- + O \left( \frac{h_1^-}{h_1^+} , \ldots , \frac{h_j^-}{h_j^+} \right)^2 + \frac{\mu_2}{h_1^+} \cdot \left| r_1^+ , \ldots , r_j^\pm \right| , \]
for \( j = 1, \ldots, k - 1 \).  
(3.14)

By (2.15) and (2.20), we have
\[ \mu_2^- = -h_1^- , \]
\[ \mu_{2j+2}^- = -h_{j+1}^- + O \left( \frac{h_1^-}{h_1^+} , \ldots , \frac{h_j^-}{h_j^+} \right)^2 + \frac{\mu_1}{h_1^+} \cdot \left| r_1^+ , \ldots , r_j^\pm \right| , \]
for \( j = 1, \ldots, k - 1 \).  
(3.15)

Then by (3.12), (3.14) and (3.15), we have
\[ \mu_{2k+2} = 2(r_1^- - r_1^+)h_k^- + 3(r_2^- - r_2^+)h_{k-1}^- + \cdots + (k + 1)(r_k^- - r_k^+)h_1^- \]
\[ + O \left( \frac{h_1^-}{h_1^+} , \ldots , \frac{h_k^-}{h_k^+} \right)^2 + \frac{\mu_2}{h_1^+} \cdot \left| r_1^+ , \ldots , r_k^\pm \right| , \]
\[ \mu_{2k+4} = 3(r_2^- - r_2^+)h_k^- + \cdots + (k + 1)(r_k^- - r_k^+)h_2^- \]
\[ + O \left( \frac{h_1^-}{h_1^+} , \ldots , \frac{h_k^-}{h_k^+} \right)^2 + \frac{\mu_1}{h_1^+} \cdot \left| r_1^+ , \ldots , r_k^\pm \right| , \]
\[ \vdots \]
\[ \mu_{4k} = (k + 1)(r_k^- - r_k^+)h_k^- \]
\[ + O \left( \frac{h_1^-}{h_1^+} , \ldots , \frac{h_k^-}{h_k^+} \right)^2 + \frac{\mu_1}{h_1^+} \cdot \left| r_1^+ , \ldots , r_k^\pm \right| . \]
(3.16)

We want to solve some \( r_j^+ - r_j^- \) from (3.16). For the purpose, let
\[ r_j^\pm = \varepsilon \delta_j^\pm, \quad h_j^- = \varepsilon^2 \beta_j, \quad \delta_j = \delta_j^+ - \delta_j^- , \quad \mu_{2k+2j} = \gamma_j \cdot \varepsilon^3, \quad j = 1, \ldots, k, \]  
(3.17)
with \( \varepsilon > 0 \) small. Then (3.16) becomes
\[ \gamma_1 = 2\delta_1 \beta_k + 3\delta_2 \beta_{k-1} + \cdots + i\delta_{i-1} \beta_{k+2-i} + \cdots + (k + 1)\delta_k \beta_1 + O(\varepsilon) , \]
\[ \gamma_2 = 3\delta_2 \beta_k + \cdots + i\delta_{i-1} \beta_{k+3-i} + \cdots + (k + 1)\delta_k \beta_2 + O(\varepsilon) , \]
\[ \vdots \]
\[ \gamma_{k-1} = k\delta_{k-1} \beta_k + (k + 1)\delta_k \beta_{k-1} + O(\varepsilon) , \]
\[ \gamma_k = (k + 1)\delta_k \beta_k + O(\varepsilon) . \]  
(3.18)
Noting that $h_j = 0$ or $\beta_j = 0$ for $j \geq k - 1$ and $r_j^\pm = 0$ or $\delta_j^\pm = 0$ for $j \geq l^\pm + 1$ which implies $\delta_j = 0$ for $j \geq l + 1$, (3.18) can be rewritten as

$$
\gamma_1 = (k + 2 - k^-)\delta_{k+1-k^-} - \beta_{k^-} + (k + 3 - k^-)\delta_{k+2-k^-} - \beta_{k^-} - 1 + \cdots + (l + 1)\delta_l \beta_{k-l+1} + O(\varepsilon),
$$

$$
\gamma_2 = (k + 3 - k^-)\delta_{k+2-k^-} - \beta_{k^-} + \cdots + (l + 1)\delta_l \beta_{k-l+2} + O(\varepsilon),
$$

$$
\vdots
$$

$$
\gamma_{k^- - (k-l)} = (l + 1)\delta_l \beta_{k^-} + O(\varepsilon).
$$

Let $\beta_{k^-} \neq 0$. Then from (3.19)

$$
\text{rank } \frac{\partial(\gamma_1, \gamma_2, \ldots, \gamma_{k^- - (k-l)})}{\partial(\delta_{k+1-k^-}, \delta_{k+2-k^-}, \ldots, \delta_l)} \bigg|_{\varepsilon=0} = k^- - (k-l).
$$

We can solve from (3.19)

$$
\delta_j = L_j(\gamma_{k^- - (k-j)}, \ldots, \gamma_{k^- - (k-l)}) + O(\varepsilon), \quad j = k + 1 - k^- \ldots, l,
$$

where $L_j$ is linear in its variables $\gamma_{k^- - (k-j)}, \ldots, \gamma_{k^- - (k-l)}$. Thus, $\gamma_1, \ldots, \gamma_{k^- - (k-l)}$ can be taken as free parameters.

Further, by (3.8), (3.9), (3.17) and (3.19) it can be seen that $\mu_{2j+3} \in C^0$ ($j \geq k$) in $\gamma_1, \ldots, \gamma_{k^- - (k-l)}$. Thus, we have under (3.13)

$$
\mu_{2k+2j+1} = O(\varepsilon^4|\gamma_1, \ldots, \gamma_j|), \quad j = 1, \ldots, k^- - (k-l).
$$

Hence it follows from (3.4) and (3.17) that

$$
F(a)|_{\mu_2=\cdots=\mu_{2k}=0} = \gamma_1\varepsilon^3a^{2k+2} + O(\varepsilon^4|\gamma_1|a^{2k+3} + \gamma_2\varepsilon^3a^{2k+4} + O(\varepsilon^4|\gamma_1, \gamma_2|)a^{2k+5} + \cdots
$$

$$
+ \gamma_{k^- - (k-l)}\varepsilon^3a^{2k^-+2l} + O(\varepsilon^4|\gamma_1, \ldots, \gamma_{k^- - (k-l)}|)a^{2k^-+2l+1}
$$

$$
+ O(a^{2k^-+2l+2})
$$

$$
= \varepsilon^3\left[\gamma_1 a^{2k+2}(1 + \varepsilon P_1(a)) + \gamma_2 a^{2k+4}(1 + \varepsilon P_2(a)) + \cdots
$$

$$
+ \gamma_{k^- - (k-l)} a^{2k^-+2l}(1 + \varepsilon P_{k^- - (k-l)}(a))\right] + O(a^{2k^-+2l+2}),
$$

where $P_j(a) = O(a) \in C^0$, $j = 1, 2, \ldots, k^- - (k-l)$.

Since $\gamma_1, \ldots, \gamma_{k^- - (k-l)}$ can be taken as free parameters, we can take them such that $\gamma_{k^- - (k-l)} \neq 0$ and

$$
|\gamma_1| \ll |\gamma_2| \ll \cdots \ll |\gamma_{k^- - (k-l)}|, \quad \gamma_j^k a_j^{1+} < 0, \quad j = 1, \ldots, k^- - (k-l) - 1.
$$

Then under (3.13) for small $\varepsilon > 0$ there exist $k^- - (k-l) - 1$ positive zeros of $F(a)$ near $a = 0$. 

Next, let $\mu_{2j} \neq 0$, $j = 1, \ldots, k$. Then by (3.4), (3.10) and (3.17) we have

$$F(a) = \mu_2 a^2 + O(\mu_2) a^3 + \mu_4 a^4 + O(\mu_2, \mu_4) a^5 + \cdots + \mu_{2k} a^{2k}$$

$$+ O((\mu_2, \mu_4, \ldots, \mu_{2k}) a^{2k+1} + (\gamma_1 \epsilon^3 + O((\mu_2, \ldots, \mu_{2k}) a^{2k+2} + O(a^{2k+3}))$$

$$= \mu_2 (1 + Q_1(a)) a^2 + \mu_4 (1 + Q_2(a)) a^4 + \cdots + \mu_{2k} (1 + Q_k(a)) a^{2k}$$

$$+ (\gamma_1 \epsilon^3 + O((\mu_2, \ldots, \mu_{2k})) a^{2k+2} + O(a^{2k+3}),$$

where $Q_j(a) = O(a) \in \mathbb{C}^\omega$, $j = 1, \ldots, k$. We now take $\mu_2, \mu_4, \ldots, \mu_{2k}$ to satisfy

$$|\mu_2| \ll |\mu_4| \ll \cdots \ll |\mu_{2k}| \ll |\gamma_1 \epsilon^3|, \quad \mu_{2j} \mu_{2j+2} < 0,$$

$$j = 1, 2, \ldots, k-1, \quad \mu_{2k} \gamma_1 < 0 \quad (3.21)$$

which ensures that $F(a)$ has $k$ more simple smaller positive zeros in $a$. Totally, we obtain $k^- + l - 1$ positive zeros of $F(a)$ under (3.20) and (3.21). Then by Lemma 1.3 we can obtain $k^- + l - 1$ limit cycles of (1.7). Then the first part of Theorem 1.1 follows. For its second part, we use the following [8].

**Bezout’s Theorem.** Let $P_i$ be polynomials in the variables $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of degree $d_i$ for $i = 1, \ldots, n$. Consider the following system of polynomial equations

$$P_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n.$$

If the number of solutions of this system, counted with their multiplicities, is finite, then it is bounded by the product $d_1 \cdots d_n$.

Consider the system of equations

$$H^-_0(y) = H^-_0(x), \quad H^+_0(y) = H^+_0(x)$$

on the region $y < 0 < x$. From the above discussion, the system (1.7) has a limit cycle surrounding the origin if and only if the above system has a solution $(x, y)$ with $y < 0 < x$. Let

$$P_1(x, y) = \frac{H^-_0(y) - H^-_0(x)}{y - x}, \quad P_2(x, y) = \frac{H^+_0(y) - H^+_0(x)}{y - x}.$$

Then the above system is equivalent to

$$P_1(x, y) = 0, \quad P_2(x, y) = 0 \quad (3.22)$$

on the region $y < x$. Now noting the facts: (1) $\deg P_1 = \deg H^-_0 - 1$, $\deg P_2 = \deg H^+_0 - 1$; (2) if $(x, y)$ is a solution of (3.22), the so is $(y, x)$; (3) by (3.4) $(0, 0)$ is a solution of (3.22) with multiplicity at least 2, then the second part of Theorem 1.1 follows from Bezout’s Theorem.

The proof of Theorem 1.1 is completed. □

**Proof of Theorem 1.2.** Next we prove Theorem 1.2 based on the conclusion of Theorem 1.1. From the above proof we can suppose that the system (1.7) has $k_0 + l - 1$ limit cycles near the origin such that $F(a) = \mu_2 a^2 + O(a^3)$ with $\mu_2 > 0$. We need to find one more limit cycle near the origin for some small $h^+_0$. Consider the system of equations
\[ \tilde{H}^+(0, \tilde{a}_1) = \tilde{H}^+(0, a), \quad \tilde{H}^-(0, \tilde{a}_2) = \tilde{H}^-(0, a) \]

on the region \((a - \tilde{a}_1)(a - \tilde{a}_2) > 0\). Let \(\tilde{F}(a) = \tilde{a}_1 - \tilde{a}_2\). Since any limit cycle surrounding the origin has two intersection points with the y-axis it suffices to find small \(h_0^\pm\) such that the function \(\tilde{F}\) has two different zeros near \(a = 0\). In fact, by (1.18) we have

\[ \tilde{H}^\pm(0, y) - \tilde{H}^\pm(0, a) = (y - a)[y + a + h_0^\pm + h_1^\pm (y^2 + ya + a^2) + r_1^\pm(y + a)(y^2 + a^2) + \cdots] \]

which implies

\[ \tilde{a}_{1,2} = h_0^\pm + O(h_0^\pm) + (-1 + O(h_0^\pm))a + (\mu_2^\pm + O(h_0^\pm))a^2 + O(a^3). \]

Hence,

\[ \tilde{F}(a) = h_0^+ - h_0^- + O(|h_0^+, h_0^-|^2) + O(|h_0^+, h_0^-|)a + (\mu_2 + O(|h_0^+, h_0^-|))a^2 + O(a^3). \]

Now it is clear that \(\tilde{F}(a)\) has two zeros with one positive, the other negative, as \(h_0^+ - h_0^- + O(|h_0^+, h_0^-|^2) < 0\). Then the first part of Theorem 1.2 follows. The second part is just similar to the second part of Theorem 1.1. This completes the proof of Theorem 1.2. \(\square\)

**Proof of Theorem 1.3.** We now prove Theorem 1.3. Suppose first \(H_0^- (y) = 0\). In this case, by (1.15), \(H_0^- (y)\) is even in \(y\). Hence the equation \(H_0^- (a_2) = H_0^- (a)\) with \(a_2a < 0\) defines the solution \(a_2 = -a\). Therefore, by (3.3) and (3.4) we have

\[ F(a) = a_1 - a_2 = \sum_{j \geq 2} \mu_j^+ a^j = \sum_{j \geq 2} \mu_j a^j. \]

Thus, by (3.7) we have that

\[ \mu_2 = \mu_4 = \cdots = \mu_{2k} = 0 \]

then

\[ h_1^+ = h_2^+ = \cdots = h_k^+ = 0 \]

which follows \(H_0^+ (-y) = H_0^+ (y)\) and hence \(a_1 = -a\) or \(\mu_{2k+j} = 0\) for all \(j \geq 1\). Thus, by (3.8)

\begin{align*}
F(a) &= \mu_2 a^2 + O(\mu_2) a^3 + \mu_4 a^4 + O(|\mu_2, \mu_4|) a^5 + \cdots + \mu_{2k} a^{2k} \\
&\quad + \sum_{j \geq 1} O(|\mu_2, \mu_4, \ldots, \mu_{2k}|) a^{2k+j} \\
&= \mu_2 (1 + P_1(a)) a^2 + \mu_4 (1 + P_2(a)) a^4 + \cdots + \mu_{2k} (1 + P_k(a)) a^{2k}.
\end{align*}

(3.23)

where \(P_j(a) = O(a) \in C^\omega, j = 1, \ldots, k\). Then following the idea in the proof of Theorem 1.3 in Han [4], we can prove that the maximal number of positive zeros of the function \(F(a)\) near \(a = 0\) is \(k - 1\), and this number can be reached by choosing suitable \(\mu_2, \mu_4, \ldots, \mu_{2k}\). The conclusion of Theorem 1.3 follows in the first case. The proof for the second case is just similar. For the third case, suppose
which implies that Then by (3.6) we have

\[ \mu_{2j} = \lambda_j + O(|h_1^{-}, \ldots, h_{j-1}^{-}|^2 + |\lambda_1, \ldots, \lambda_{j-1}|^2 + |h_1^{-}, \ldots, h_{j-1}^{-}| \cdot |\lambda_1, \ldots, \lambda_{j-1}| + |r_1, \ldots, r_{j-1}| \cdot |h_1^{-}, \ldots, h_{j-1}^{-}, \lambda_1, \ldots, \lambda_{j-1}|) \]  

(3.24)

for \( j = 1, \ldots, k \). By our assumption, we have \( H_0^+(y) \equiv H_0^-(y) \) or \( F(a) = 0 \) if \( \lambda_j = 0 \) for \( j = 1, \ldots, k \), which implies that \( \mu_j = 0 \) for all \( j \geq 2 \) as \( \lambda_j = 0 \) for \( j = 1, \ldots, k \). Thus, by (3.24) we have

\[ \mu_{2j} = \lambda_j + O(|\lambda_1, \ldots, \lambda_{j-1}|), \quad j = 1, \ldots, k, \]  

(3.25)

and

\[ \mu_{2k+j} = O(|\lambda_1, \ldots, \lambda_k|), \quad k \geq 1. \]

From (3.25) we can solve

\[ \lambda_j = \mu_{2j} + O(|\mu_2, \mu_4, \ldots, \mu_{2j-2}|), \quad j = 1, \ldots, k. \]

Hence

\[ \mu_{2k+j} = O(|\mu_2, \mu_4, \ldots, \mu_{2k}|), \quad j \geq 1. \]

Then by (3.10) we see that (3.23) remains true in this case, which yields the conclusion in the third case. This ends the proof of Theorem 1.3.

4. Applications

For some concrete systems we obtain more precise results.

Consider system (1.7) with

\[ H_0^+(y) = y^2 + \sum_{j=1}^{3} r^+ j y^{2j+2} + \sum_{j=1}^{3} h^+_j y^{2j+1}, \]

\[ H_0^-(y) = y^2 + \sum_{j=1}^{3} r^- j y^{2j+2} + \sum_{j=1}^{3} h^-_j y^{2j+1}. \]  

(4.1)

In this case, we have \( k^+ = 3, l^+ = 3 \). Thus, by Theorem 1.1 we can find 5 limit cycles near the origin. Is it possible to have more limit cycles locally? The following theorem gives a negative answer. To state the theorem, let us introduce conditions (A1), \ldots, (A6) and (B) as follows.

(A1) \( h_1^+ \neq h_1^- \),

(A2) \( h_1^+ = h_1^- \), \( h_2^+ \neq h_2^- + 2h_1^- (r_1^+ - r_1^-) \).

(A3) \( h_1^+ = h_1^- \), \( h_2^+ = h_2^- + 2h_1^- (r_1^+ - r_1^-) \), \( h_3^+ \neq h_3^- + (r_1^+ - r_1^-) (2h_2^- - 4h_1^- r_1^- - 3h_1^-^3) + 3h_1^- (r_2^+ - r_2^-) \).
(A4) \[ h^+_1 = h^-_1, \quad h^+_2 = h^-_2 + 2h^-_1 (r^+_1 - r^-_1), \]
\[ h^+_3 = h^-_3 + (r^+_1 - r^-_1) (2h^-_2 - 4h^-_1 r^-_1 - 3h^-_1^3) + 3h^-_1 (r^+_2 - r^-_2), \quad \tilde{\mu}_8 \neq 0. \]

(A5) \[ h^+_1 = h^-_1, \quad h^+_2 = h^-_2 + 2h^-_1 (r^+_1 - r^-_1), \]
\[ h^+_3 = h^-_3 + (r^+_1 - r^-_1) (2h^-_2 - 4h^-_1 r^-_1 - 3h^-_1^3) + 3h^-_1 (r^+_2 - r^-_2), \quad \tilde{\mu}_8 = 0, \quad \tilde{\mu}_{10} \neq 0. \]

(A6) \[ h^+_1 = h^-_1, \quad h^+_2 = h^-_2 + 2h^-_1 (r^+_1 - r^-_1), \]
\[ h^+_3 = h^-_3 + (r^+_1 - r^-_1) (2h^-_2 - 4h^-_1 r^-_1 - 3h^-_1^3) + 3h^-_1 (r^+_2 - r^-_2), \quad \tilde{\mu}_8 = 0, \quad \tilde{\mu}_{10} = 0, \quad \tilde{\mu}_{12} \neq 0. \]

and

(B) \[ h^+_1 = h^-_1, \quad h^+_2 = h^-_2, \quad h^+_3 = h^-_3, \quad r^+_3 = r^-_3, \quad r^+_2 = r^-_2, \quad r^+_1 = r^-_1, \]

where

\[ \tilde{\mu}_8 = 4h^-_1 (r^+_3 - r^-_3) + [(r^+_2 - r^-_2) (3h^-_2 - 6r^-_1 h^-_1 - 11h^-_1^3) + (r^+_1 - r^-_1) \]
\[ \times (2h^-_3 + 11h^-_1^5 + 28h^-_1^3 r^-_1 + 8h^-_1 r^-_1^2 - 4h^-_2 r^-_1 + 11h^-_1^2 h^-_2 - 6r^-_1 h^-_1)] \]
\[ \tilde{\mu}_{10} = (r^+_3 - r^-_3) (4h^-_2 + 82h^-_1^3 - 8h^-_1 r^-_1) + (r^+_2 - r^-_2) [3h^-_3 - 6r^-_1 h^-_2 \]
\[ + 12r^+_1 h^-_1^2 + 42h^-_1^2 h^-_2 - 75h^-_1^3 r^-_1 + 22h^-_1^3 r^+_1 - 219h^-_1^5 - 9r^-_2 h^-_1] \]
\[ + (r^+_1 - r^-_1) (219h^-_1 h^-_2 - 206h^-_1^2 h^-_2 - 13h^-_1 h^-_2^2 + 56h^-_1^3 r^-_1 - 22h^-_1^5 r^+_1 \]
\[ + 513h^-_1^5 r^-_1 - 56r^-_1 h^-_1^3 r^+_1 + 56h^-_1^3 h^-_2 + 22h^-_1^2 h^-_2 - 8h^-_1 h^-_2 r^-_1 + 41h^-_1 h^-_2 \]
\[ - 101h^-_1^3 r^-_2 + 12r^+_1 h^-_1^2 r^+_1 r^-_1 + 4r^+_1 - r^-_1 r^-_2 r^-_1 + 12r^+_1 h^-_1 (r^+_2 r^-_1 + r^-_1 r^+_2) \]
\[ - 2r^+_1 h^-_1^2 h^-_2 r^-_1 + 24h^-_1^2 r^-_1 - 4r^+_1 r^-_1 r^-_2)] \]
\[ \tilde{\mu}_{12} = (r^+_3 - r^-_3) (4h^-_1 h^-_1^2 - 12r^-_2 h^-_1 + 100h^-_1 h^-_1 - 8r^-_1 h^-_2 \]
\[ + 16r^+_1 h^-_1^2 - 672h^-_1^3 r^-_1) + (r^+_2 r^-_3 - r^-_2 r^-_1) (16r^-_1 h^-_1 - 192h^-_1^3 - 8h^-_1^2) \]
\[ - 12h^-_1 (r^+_2 r^-_3 - r^-_1 r^-_3)] \]
\[ + (r^+_2 - r^-_2) (33h^-_1 r^-_2 - 174h^-_1^3 r^-_2 + 69h^-_1 h^-_1^2 - 2593h^-_1^3 - 407h^-_1 h^-_2 h^-_1 \]
\[ + 165h^-_1 h^-_2 - 936h^-_1 h^-_1 h^-_2 + 44h^-_1^3 r^-_1^2 + 482h^-_1 h^-_1 r^-_1 + 798r^-_1 h^-_1^2 h^-_2) \]
\[ + (r^+_1 r^-_2 - r^-_1 r^-_2) (108h^-_1 r^-_1 - 72r^-_1^2 h^-_1 - 12r^-_3 - 24r^-_1 h^-_1 - 72h^-_1 h^-_2 \]
\[ + 36r^-_1 h^-_1) + 12h^-_2 (r^+_1 r^-_2 - r^-_1 h^-_1) \]
\[ + (r^+_2 - r^-_1) (4r^-_1 h^-_1^3 - 4713h^-_1^3 r^-_1) \]
\[ - 744r^-_1 h^-_1^3 - 190h^-_1^2 h^-_1 r^-_1 + 660r^-_1 h^-_1^3 r^-_2 \]
\[ - 220r^-_1 h^-_1 h^-_2 + 908r^-_1 h^-_1^2 h^-_2 - 3878h^-_1 r^-_1^2 + 3284h^-_1 r^-_1 h^-_2 - 5h^-_1^3 \]
Theorem 4.1. Let system (1.7) satisfy (4.1) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Then the origin is a focus of order at most 12, and it is of order 2k if and only if the above (Ak) holds, \(k = 1, 2, 3, 4, 5, 6\); and the origin is a center if and only if the above (B) holds. Thus, there can be at most 5 limit cycles near the origin.

Proof. By Maple 13 we can give the coefficients in (3.4) and (3.3) as follows

\[
\mu_{2k} = \mu_{2k}^+ - \mu_{2k}^-, \quad k = 1, 2, 3, 4, 5, 6,
\]

where

\[
\mu_2^\pm = -h_1^\pm + \frac{1}{2} h_2^\pm + 2r_1^\pm h_1^\pm,
\]

\[
\mu_4^\pm = -h_2^\pm + 2h_1^\pm + 2r_1^\pm h_1^\pm,
\]

\[
\mu_6^\pm = -h_3^\pm - 9h_1^\pm + 19h_2^\pm + 11h_2^\pm + \frac{1}{2} h_3^\pm + 3r_2^\pm h_1^\pm - 4r_2^\pm h_1^\pm + 2r_1^\pm h_2^\pm,
\]

\[
\mu_8^\pm = -12r_1^\pm h_1^\pm + 93r_1^\pm h_1^\pm + 18h_3^\pm + 4r_3^\pm h_3^\pm + 51h_2^\pm + 165h_1^\pm h_3^\pm,
\]

\[
- 104h_2^\pm h_3^\pm - 112r_1^\pm h_3^\pm - 17h_2^\pm h_3^\pm + 43h_3^\pm h_2^\pm + 3r_2^\pm h_3^\pm + 3r_2^\pm h_3^\pm + 8r_3^\pm h_3^\pm,
\]

\[
- 4r_2^\pm h_3^\pm + 2r_1^\pm h_3^\pm,
\]

\[
\mu_{10}^\pm = 1407h_1^\pm r_1^\pm + 924h_1^\pm h_2^\pm + 441h_1^\pm h_2^\pm + 163h_1^\pm r_2^\pm + 192h_2^\pm h_1^\pm h_2^\pm + 141h_2^\pm r_2^\pm + 1700h_2^\pm r_2^\pm - 50h_2^\pm h_2^\pm h_2^\pm + 135h_2^\pm h_3^\pm r_1^\pm - 16r_2^\pm r_1^\pm h_1^\pm + 504r_2^\pm r_2^\pm h_3^\pm - 12r_2^\pm r_1^\pm h_2^\pm - 323h_1^\pm + 8h_2^\pm - 476r_2^\pm h_1^\pm h_2^\pm + 36r_2^\pm r_2^\pm h_1^\pm - 220h_3^\pm h_1^\pm + 520r_3^\pm h_3^\pm - 404h_2^\pm h_3^\pm + 82h_3^\pm h_3^\pm + 3r_2^\pm h_3^\pm - 9r_2^\pm h_3^\pm - 16r_2^\pm h_1^\pm h_2^\pm + 8r_3^\pm h_2^\pm + 4r_2^\pm h_3^\pm + 4r_2^\pm h_2^\pm,
\]

\[
\mu_{12}^\pm = -9r_2^\pm h_2^\pm + 32r_2^\pm h_2^\pm + 16r_2^\pm h_2^\pm + 8r_3^\pm h_3^\pm + 4130h_1^\pm r_2^\pm - 2241h_1^\pm h_2^\pm + 11976h_1^\pm h_2^\pm + 8053h_1^\pm h_2^\pm + 20720h_1^\pm h_2^\pm + 6168h_1^\pm h_2^\pm + 12586h_1^\pm h_2^\pm + 700h_1^\pm h_2^\pm + 480^2 h_1^\pm h_2^\pm - 2080^2 h_1^\pm h_2^\pm + 61r_2^\pm h_2^\pm - 34h_3^\pm h_2^\pm - 33h_3^\pm h_2^\pm + 1006h_1^\pm r_3^\pm + 4r_3^\pm h_3^\pm - 204r_2^\pm h_2^\pm h_2^\pm + 378r_1^\pm h_3^\pm h_1^\pm - 2188h_1^\pm h_2^\pm h_3^\pm.
\]
suppose the origin is a center. Then together with

Further, substituting (4.6) into By (3.8) we easily know that the origin is a focus of order 2k if and only if

\[ \mu_{2k} \neq 0, \quad \mu_{2j} = 0, \quad j \leq k - 1. \]  (4.4)

By the formulas of \( \mu^\pm_2, \mu^\pm_4 \) and \( \mu^\pm_6 \), it is easy to see that the conclusion is true for \( k = 1, 2, \) or 3. For \( k = 4 \), we can solve from the equations \( \mu_2 = \mu_4 = \mu_6 = 0 \)

\[
\begin{align*}
\quad h_1^+ &= h_1^- , \quad h_2^+ = h_2^- + 2h_1^- (r_1^+ - r_1^-), \\
\quad h_3^+ &= h_3^- + (r_1^+ - r_1^-)(2h_2^- - 4h_1^- r_1^- - 3h_1^-^3) + 3h_1^- (r_2^- - r_2^-).
\end{align*}
\]  (4.5)

Substituting (4.5) into \( \mu_8 \) in (4.3) we obtain

\[
\begin{align*}
\mu_8 \Big|_{\mu_2 = \mu_4 = \mu_6 = 0} &= 4h_1^- (r_3^+ - r_3^-) + [(r_2^+ - r_2^-)(3h_2^- - 6r_1^- h_1^- - 11h_1^-^3) \\
&\quad + (r_1^+ - r_1^-)(2h_3^- + 11h_1^-^5 + 28h_1^-^3 r_1^- + 8h_1^-^2 r_1^- - 4h_2^- r_1^- \\
&\quad - 11h_1^-^2 h_2^- - 6r_2^- h_1^-)] = \tilde{\mu}_8.
\end{align*}
\]

By (4.4) and (4.5) we see that the origin is a focus of order 8 if and only if (A4) holds. Similarly, the conclusion is true for \( k = 5 \) or 6.

Next, we prove that the origin is a center if and only if \( (B) \) holds. Clearly, if \( (B) \) holds, then \( H_0(y) = H_0^\prime (y) \) which implies \( a_1 = a_2 \) or \( F(a) \equiv 0 \), and hence, the origin is a center. Now let us suppose the origin is a center. Then \( a_1 = a_2 \). Hence \( \mu_{2k} = 0 \) for \( k = 1, 2, \ldots, 6 \). Thus, (4.5) holds together with \( \mu_8 = \mu_{10} = \mu_{12} = 0 \). There are several cases to consider.

\[ (i) \quad h_1^- \neq 0, \quad K \neq 0, \quad K = 12r_1^- h_1^- h_2^- + 52h_1^-^4 r_1^- - 17h_1^-^3 h_2^- - 18r_1^- h_1^-^2 + 6h_1^- h_1^- + 13h_1^-^6 - 6h_2^-^2. \]

From \( \tilde{\mu}_8 = 0 \) we can solve

\[
\begin{align*}
\quad r_3^+ &= r_3^- + \frac{1}{4h_1^-} [(r_2^- - r_2^-)^3 h_2^- - 6r_1^- h_1^- - 11h_1^-^3) + (r_1^- - r_1^-)(2h_3^- \\
&\quad + 11h_1^-^5 + 28h_1^-^3 r_1^- + 8h_1^-^2 r_1^- - 4h_2^- r_1^- - 11h_1^-^2 h_2^- - 6r_2^- h_1^-)].
\end{align*}
\]  (4.6)

Further, substituting (4.6) into \( \tilde{\mu}_{10} \) in (4.2), and then from \( \tilde{\mu}_{10} = 0 \) we can solve

\[
\begin{align*}
\quad r_2^+ &= r_2^- + \frac{1}{K} (r_1^- - r_1^-) [(48r_1^- h_1^-^3 h_2^- + 24r_1^- h_1^-^2 r_2^- - 16r_1^- h_1^- h_2^- - 78h_1^-^6 r_1^- \\
&\quad - 104r_1^-^2 h_1^-^4 + 44h_1^-^4 r_2^- + 8r_1^- h_2^-^2 - 4h_2^- h_3^- - 16r_3^- h_1^-^2 + 17h_1^-^5 h_2^- \\
&\quad - 4h_1^-^2 h_2^-^2 - 13h_1^-^8)].
\end{align*}
\]  (4.7)
Now we substitute (4.6) and (4.7) into $\bar{\mu}_{12}$ in (4.2). Then from $\bar{\mu}_{12} = 0$ we can solve

$$r_1^+ = r_1^-$$

which implies (B) easily.

(ii) $h_1^- \neq 0$, $K = 0$. From $\bar{\mu}_8 = 0$ we can solve $r_3^+$ satisfying (4.6). Substituting (4.6) into $\bar{\mu}_{10}$ and then from $\bar{\mu}_{10} = 0$ we can solve

$$r_1^+ = r_1^-.$$  \hspace{1cm} (4.8)

Hence, substituting (4.6) and (4.8) into $\bar{\mu}_{12}$, from $\bar{\mu}_{12} = 0$ we can solve

$$r_2^+ = r_2^-$$

which implies (B).

(iii) $h_1^- = 0$, $h_2^- \neq 0$, $\bar{K} \neq 0$, $\bar{K} = h_3^- - 2r_1^- h_2^- h_3^- + 3r_2^- h_2^-$. From $\bar{\mu}_8 = 0$ we can solve

$$r_2^+ = \frac{1}{3h_2^-}(r_1^+ - r_1^-)(4r_1^- h_2^- - 2h_3^-) + r_2^-.$$ \hspace{1cm} (4.9)

Substituting (4.9) into $\bar{\mu}_{10}$ in (4.2), from $\bar{\mu}_{10} = 0$ we can solve

$$r_1^+ = r_1^- + 2h_2^- (r_3^+ - r_3^-)/\bar{K}.$$ \hspace{1cm} (4.10)

Further, substituting (4.9) and (4.10) into $\bar{\mu}_{12}$, from $\bar{\mu}_{12} = 0$ we can solve

$$r_3^+ = r_3^-.$$ \hspace{1cm} (4.11)

which implies (B) as before.

(iv) $h_1^- = 0$, $h_2^- \neq 0$, $\bar{K} = 0$. From $\bar{\mu}_8 = 0$ we can solve $r_2^+$ satisfying (4.9). Substituting (4.9) into $\bar{\mu}_{10}$, from $\bar{\mu}_{10} = 0$ we can solve

$$r_3^+ = r_3^-.$$ \hspace{1cm} (4.11)

Further, substituting (4.9) and (4.11) into $\bar{\mu}_{12}$ in (4.2) we can solve

$$r_1^+ = r_1^-$$

implying (B).

(v) $h_1^- = 0$, $h_2^- = 0$. From $\bar{\mu}_8 = 0$ we can solve

$$r_1^+ = r_1^-.$$ \hspace{1cm} (4.12)

Substituting (4.12) into $\bar{\mu}_{10}$, from $\bar{\mu}_{10} = 0$ we can solve

$$r_2^+ = r_2^-.$$ \hspace{1cm} (4.13)

Hence, substituting (4.12) and (4.13) into $\bar{\mu}_{12}$ in (4.2) we can solve
\[ r_3^+ = r_3^- \]

also implying (B). This ends the proof. \( \square \)

The above theorem suggests the following

**Conjecture.** Consider system (1.7) satisfying (1.15) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Let \( k = \max(k^+, k^-), l = \max(l^+, l^-) \). Then an upper bound of number of limit cycles of (1.7) near the origin is \( k + l - 1 \). If further \( k = k^+ = k^- \), then the lowest upper bound of number of limit cycles of (1.7) near the origin is \( k + l - 1 \).

If we take

\[
H_0^+(y) = y^2 + \sum_{j=1}^{3} r_j^+ y^{2j+2} + \sum_{j=1}^{3} h_j^+ y^{2j+1}.
\]

\[
H_0^-(y) = y^2 + r_1^- y^4 + h_1^- y^3, \quad h_1^- \neq 0
\]

then by Theorem 1.1 we can find 3 limit cycles near the origin. We will see that there are maximally 4 limit cycles in this case. In fact, noting \( h_2 = h_3 = r_2 = r_3 = 0 \) in (4.1), then (4.2) holds with \( h_2 = h_3 = r_2 = r_3 = 0 \) and the conditions (A1)-(A5) become

\[
(A'1) \quad h_1^+ \neq h_1^-.
\]

\[
(A'2) \quad h_2^+ = h_1^-,
\]

\[
(A'3) \quad h_2^+ = 2h_1^-(r_1^+ - r_1^-), \quad h_3^+ = 3r_2^+ h_1^- + (r_1^+ - r_1^-)(-3h_1^3 - 4h_1^- r_1^-),
\]

\[
(A'4) \quad h_2^+ = 2h_1^-(r_1^+ - r_1^-), \quad h_3^+ = 3r_2^+ h_1^- + (r_1^+ - r_1^-)(-3h_1^3 - 4h_1^- r_1^-),
\]

\[
(A'5) \quad h_2^+ = 2h_1^-(r_1^+ - r_1^-), \quad h_3^+ = 3r_2^+ h_1^- + (r_1^+ - r_1^-)(-3h_1^3 - 4h_1^- r_1^-),
\]

\[
r_3^+ = (r_1^- - r_1^+)(\frac{11}{4} h_1^{-4} + 7h_1^{-2} r_1^- + 2r_1^-) + \frac{11}{4} r_2^+ h_1^- + \frac{3}{2} r_2^+ r_1^-,
\]

\[
\mu_{10} \neq 0.
\]

And the origin cannot be a focus of order 12 since we have automatically \( \mu_{12} = 0 \) when \( \mu_2 = \mu_4 = \mu_6 = \mu_8 = \mu_{10} = 0 \). Further, by (4.5)-(4.8) and \( h_1^- \neq 0 \) in (4.14), one can see that \( \mu_2 = \mu_4 = \mu_6 = \mu_8 = \mu_{10} = 0 \) if and only if the following (B') holds:

\[
(B') \quad (B'1) \quad 4r_1^- + h_1^- \neq 0, \quad h_1^+ = h_1^-,
\]

\[
h_2^+ = 2h_1^-(r_1^+ - r_1^-),
\]

\[
h_3^+ = 2r_1^- h_1^-(r_1^+ - r_1^-), \quad r_3^+ = r_1^2(r_1^+ - r_1^-), \quad r_2^+ = (r_1^+ - r_1^-)(2r_1^- + h_1^-);
\]

or

\[
(B'2) \quad r_1^- = -\frac{1}{4} h_1^- 2, \quad h_1^+ = h_1^-,
\]

\[
h_2^+ = 0, \quad h_3^+ = 0, \quad r_3^+ = 0, \quad r_1^+ = r_1^-.
\]

In this case, we have
Theorem 4.2. Let system (1.7) satisfy (4.14) and one of the conditions (i), (ii) and (iii) in Lemma 1.2. Then the origin is a focus of order at most 10, and it is of order 2k if and only if (A_k) holds, k = 1, 2, 3, 4, 5, the origin is a center if and only if the above (B') holds. Thus, there can be at most 4 limit cycles near the origin.

Proof. The first part is obvious by the proof of Theorem 4.1. For the second part, if the origin is a center, then \( F(a) = 0 \), and hence \( \mu_{2k} = 0 \) for \( k = 1, 2, 3, 4, 5 \) which implies (B') by the proof of Theorem 4.1.

On the other hand, let (B') be satisfied. First, let (B'1) hold. Then

\[
H_0^+(y) = y^2 + \rho_1^+ y^4 + (r_1^+ - r_1^-)(2r_1^- + h_1^-)y^6 + r_1^- 2\rho_1^- y^8 + h_1^- y^3
+ 2h_1^- (r_1^+ - r_1^-)y^5 + 2r_1^- h_1^- (r_1^+ - r_1^-)y^7
= y^2 + h_1^- y^3 + r_1^- y^4 + (r_1^+ - r_1^-)(y^2 + h_1^- y^3 + r_1^- y^4)^2
= u(y) + (r_1^+ - r_1^-)u^2(y),
\]

where \( u(y) = y^2 + h_1^- y^3 + r_1^- y^4 \). Thus

\[
H_0^+(a) = H_0^+(a_1) \iff (a_1 - u(a))[1 + (r_1^+ - r_1^-)(u(a_1) + u(a))] = 0.
\]

By (3.3) and (3.4), we have

\[
u(a_1) + u(a) = 2a^2 + O(a^3)
\]

and then \( 1 + (r_1^+ - r_1^-)(u(a_1) + u(a)) \neq 0 \) for small \( r_1^+, r_1^- \) and a. Thus, noting \( u(y) = y^2 + h_1^- y^3 + r_1^- y^4 = H_0^-(y) \)

\[
H_0^+(a) = H_0^+(a_1) \iff u(a_1) = u(a) \iff H_0^-(a) = H_0^-(a_1).
\]

Then by Theorem 2.1, (3.3) and (3.4) it follows from (B'1) that \( a_1 = a_2 \) or \( F(a) \equiv 0 \), which means that the origin is a center of (1.7).

Next, let (B'2) hold. In this case we have \( H_0^-(y) = H_0^+(y) \) which directly implies \( F(a) = a_1 - a_2 = 0 \), or the origin is a center. This ends the proof. □

References