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# Approximate Solutions to Boundary Value Problems of Higher Order by the Modified Decomposition Method

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**Abstract**—In this paper. we present an efficient numerical algorithm for approximate solutions of higher-order boundary value problems with two-point boundary conditions. A modified form of the Adomian decomposition method will be implemented to construct such solutions. The approach provides the solution in the form of a rapidly convergent series. The analysis is accompanied by numerical examples. The obtained results demonstrate reliability and efficiency of the proposed scheme. 

**Keywords**—Higher-order boundary value problems, Modified Adomian method, Series solutions.

## 1. INTRODUCTION

When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability sets in as ordinary convection, the ordinary differential equation is sixth order [1]. When this instability sets in as overstability, it is modelled by an eighth-order ordinary differential equation [1].

Now suppose that a uniform magnetic field is also applied across the fluid [1] in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary value problem; when instability sets in as overstability, it is modelled by a $12^{\text{th}}$ order boundary value problem. For more details about the occurrences of high-order boundary value problems, see [2–4]. An eighth-order differential equation occurring in torsional vibration of uniform beams was investigated by [5]. A class of characteristic-value problems of high order (as high as 24) are known to arise in hydrodynamic and hydromagnetic stability [2–4].

The literature of numerical analysis contains little on the solution of the high-order boundary value problems [3,4]. Research in this direction may be considered in its early stages. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey in a book by Agarwal [6], though no numerical methods are contained therein for solving boundary value problems of higher order.

Recently, the boundary value problems of higher order have been investigated because of both their mathematical importance and their potential for applications in hydrodynamic and hydromagnetic stability. Baldwin [7] applied global phase-integral methods for solving BVPs of sixth order. However, numerical methods of solution were introduced implicitly by Chawla and

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Katti [8], although the authors focused their attention on fourth-order BVPs. Computational results have also been obtained by [4] for special nonlinear boundary value problems of order $2m$ by using finite-difference methods. In a later work [1], octic splines solutions of linear eighth-order boundary value problems were implemented and the obtained results produced improvements over finite differences method. The spline function values at the midknots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations.

Generally speaking, a considerable amount of interest [1–8] was directed towards the use of finite differences methods and the spline solutions to handle boundary value problems of higher order.

The present work is motivated by the desire to obtain numerical solutions to higher-order boundary value problems with a better accuracy level. The goal of this study can be achieved by implementing a modified form of Adomian decomposition method [9–14]. In recent years, the Adomian decomposition method has been used in obtaining approximate solutions to a wide class of differential and integral equations. The method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be used directly without using restrictive assumptions.

As noted previously, our analysis for this work depends mainly on a modified form of the Adomian decomposition method established recently by [11–13]. This newly emerged technique will be implemented to higher-order boundary value problems in a straightforward manner. The modified technique provides a qualitative improvement over standard Adomian method although it introduces a slight change in the formulation of Adomian recursive relation. The reason for this improvement rests on the fact that the technique accelerates the convergence of the solution and facilitates the formulation of Adomian polynomials. The efficiency of the scheme gives it much wider applicability.

It is worth mentioning that several authors have treated many concepts related to the Adomian method such as the convergence concept and comparisons with other existing numerical techniques. The most significant works about convergence have been carried out by Cherruault *et al.* [15–18] by using fixed-point theorems or by substituting results in function series. Adomian [9] discussed the convergence concept on differential and integral equations. Eugene [19] proved the convergence by applying the decomposition method to the reaction-convection-diffusion equation which characterizes the dispersion of a chemically reactive material. In [20], Tonningen found that the decomposition method is easy to program in engineering problems and provides immediate and convergent solutions without any need for linearization or discretization.

However, relatively few papers deal with the comparison of the Adomian decomposition method and other existing numerical techniques. Bellomo and Monaco [21] conducted a useful study between the decomposition method and the perturbation algorithm and formally showed the efficiency and accuracy of the decomposition method compared to the tedious work required by perturbation techniques. In [22], the strong performance of the decomposition method over Picard's method has been emphasized. Recently, a comparison between Adomian decomposition method and Taylor series method has been carried out by Wazwaz [23] by using linear and nonlinear problems, and the study showed that the decomposition method is easy to use and produces reliable results with few iterations, whereas the Taylor series method suffers from certain computational difficulties.

It is interesting to note that the special $2m$-order BVP used by [4] contained the boundary conditions at even-order derivatives. For comparison reasons, we will apply our technique using these types of boundary conditions. In fact, our proposed technique can handle any boundary value problem with a set of boundary conditions defined at any order derivatives.

Without loss of generality, in what follows, we will examine the $2m$-order boundary value problems with boundary conditions given at even-order derivatives as applied by [4], whereas examples of orders $2m$ and $2m+1$ will be examined in our analysis.

## 2. ANALYSIS OF THE METHOD

Consider the special $2m$-order BVP of the form

$$y^{(2m)}(x) = f(x, y), \qquad 0 < x < b, \tag{1}$$

with boundary conditions

$$y^{(2j)}(0) = \alpha_{2j}, \quad y^{(2j)}(1) = \beta_{2j}, \qquad j = 0, 1, 2, \dots, (m-1). \tag{2}$$

It is interesting to point out that $y(x)$ and $f(x, y)$ are assumed real and as many times differentiable as required for $x \in [0, b]$ and $\alpha_{2j}$ and $\beta_{2j}, j = 0, 1, 2, \dots, (m-1)$ are real finite constants [4]. Moreover, the constants $\alpha_{2j}$, $j = 0, 1, 2, \dots, (m-1)$ describe the even-order derivatives at the boundary $x = 0$. The book by Agarwal [6] contains theorems which explain the conditions for existence and uniqueness of solutions of higher-order boundary value problems, though no numerical methods are contained therein.

In an operator form, equation (1) becomes

$$Ly = f(x, y), \tag{3}$$

where the differential operator $L$ is given by

$$L = \frac{d^{2m}}{dx^{2m}}. \tag{4}$$

The inverse operator $L^{-1}$ is therefore considered a $2m$-fold integral operator defined by

$$L^{-1}(.) = \underbrace{\int_0^x}_{(2m)\text{ times}} (.) \underbrace{dx}_{(2m)\text{ times}} . \tag{5}$$

Operating with $L^{-1}$ on (3), it then follows

$$y(x) = \sum_{j=0}^{2m-1} \alpha_j \frac{1}{(j)!} x^j + L^{-1}(f(x, y)), \tag{6}$$

where $\alpha_{2k+1}, k = 0, 1, 2, \dots, (m-1)$ are constants that describe the boundary conditions at odd-order derivatives, defined by

$$\alpha_1 = y'(0), \qquad \alpha_3 = y'''(0), \qquad \alpha_5 = y^{(v)}(0), \dots, \alpha_{(2m-1)} = y^{(2m-1)}(0), \tag{7}$$

and will be determined later by using the boundary conditions at $x = b$. The other constants $\alpha_0, \alpha_2, \alpha_4$, and $\alpha_{2m-2}$ describe the boundary conditions at even-order derivatives and are prescribed in (2).

The Adomian decomposition method expresses the solution $y(x)$ of (1) by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8}$$

so that the components $y_n(x)$ will be determined recurrently. Moreover, the method defines the nonlinear function $f(x, y)$ by an infinite series of polynomials

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where $A_n$ are the so-called Adomian polynomials that can be derived for various classes of nonlinearity according to specific algorithms set by Adomian [9,10]. A new algorithm for calculating these polynomials was established by Wazwaz [14].

Substituting (8) and (9) into (6) yields

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{j=0}^{2m-1} \alpha_j \frac{1}{(j)!} x^j + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{10}$$

To determine the components $y_n(x)$, $n \geq 0$, we first identify the zero$^{\text{th}}$ component $y_0(x)$ by all terms that arise from the boundary conditions at $x = 0$ and from integrating the source term if it exists. Second, the remaining components of $y(x)$ can be determined in a way such that each component is determined by using the preceding components. In other words, the method introduces the recursive relation

$$\begin{aligned} y_0(x) &= \sum_{j=0}^{2m-1} \alpha_j \frac{1}{(j)!} x^j, \\ y_{k+1}(x) &= L^{-1}(A_k), \qquad k \geq 0, \end{aligned} \tag{11}$$

for the determination of the components $y_n(x)$ of $y(x)$. The series solution of $y(x)$ follows immediately with the constants $\alpha_{2j+1}$, $j = 0, 1, 2, \dots, (m-1)$ are as yet undetermined.

An important point to be made here is that we can elegantly determine the components $y_n(x)$ as far as we like to enhance the accuracy of the approximation. The $n$-term approximant

$$\phi_n = \sum_{k=0}^{n-1} y_k \tag{12}$$

can be used to approximate the solution.

As noted previously in the Introduction, a reliable modified form to the decomposition method has been introduced recently by [11–13]. The modified technique proposes a slight change in the definition of the components $y_0(x)$ and $y_1(x)$ in (11). While this slight variation is rather simple, it produces a remarkable tool for numerical applications.

The variation we propose in [11–13] is that only a part of $y_0(x)$ in (11) be assigned to $y_0(x)$, whereas the remaining part of $y_0(x)$ in (11) be assigned to the component $y_1(x)$ among other terms. As a result, we formulate a new recursive relation, to replace relation (11), given in the form

$$\begin{aligned} y_0(x) &= \alpha_0, \\ y_1(x) &= \sum_{j=1}^{2m-1} \alpha_j \frac{1}{(j)!} x^j + L^{-1}(A_0), \\ y_{k+1}(x) &= L^{-1}(A_k), \qquad k \geq 1. \end{aligned} \tag{13}$$

The main reason for this modification is that Adomian polynomials $A_n$ depend heavily on $y_0(x)$. The choice of $y_0(x)$ to contain minimal number of terms has a strong influence on facilitating the computational behavior of $A_n$.

Relation (13) will enable us to determine the components $y_n(x)$, $n \geq 0$, recurrently, and as a result, the series solution of $y(x)$ is readily obtained with the constants $\alpha_{2j+1}$, $j = 0, 1, 2, \dots, (m-1)$ are as yet undetermined. It is interesting to note that the accuracy of the approximation can be dramatically improved by simply determining the components $y_n(x)$ as far as we like. The $n$-term approximant

$$\phi_n = \sum_{k=0}^{n-1} y_k \tag{14}$$

can be used to approximate the solution. As stated before in the Introduction, the convergence concept has been established by many studies in [15–20].

Our aim now is to determine approximations to $\alpha_{2k+1}$, $k = 0, 1, 2, \ldots, (m-1)$. Imposing the boundary conditions at $x = b$ on the approximant $\phi_n$ leads to an algebraic system of equations. This system needs only be solved to obtain approximations to the constants $\alpha_{2k+1}$, $k = 0, 1, 2, \ldots, (m-1)$. Having determined these constants, the numerical solution of the $2m$-order boundary value problem follows immediately upon substituting the resulting components in (8).

## 3. NUMERICAL RESULTS

In the examples that follow, the modified technique will be tested by discussing three boundary-value problems of ninth-order, tenth-order, and $12^{\text{th}}$-order, respectively. In the first example, boundary conditions at any-order derivative are used. However, boundary conditions at even-order derivatives are given in the last two examples.

EXAMPLE 1. We first consider the linear ninth-order BVP

$$y^{(ix)}(x) = -9e^x + y(x), \qquad 0 < x < 1, \tag{15}$$

subject to the boundary conditions

$$\begin{aligned} y^{(j)}(0) &= (1-j), \qquad & j &= 0, 1, 2, 3, 4, \\ y^{(j)}(1) &= -je, \qquad & j &= 0, 1, 2, 3. \end{aligned} \tag{16}$$

The theoretical solution for this problem is

$$y(x) = (1-x)e^x. \tag{17}$$

In an operator form, equation (15) becomes

$$Ly = -9e^x + y(x), \qquad 0 < x < 1. \tag{18}$$

Applying $L^{-1}$, a nine-fold integral operator, on both sides of (18), and using the boundary conditions at $x = 0$, yields

$$\begin{aligned} y(x) = 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 \\ -9L^{-1}\left(e^x\right) + L^{-1}(y(x)), \end{aligned} \tag{19}$$

noting that

$$A = y^{(v)}(0), \qquad B = y^{(vi)}(0), \qquad C = y^{(vii)}(0), \qquad D = y^{(viii)}(0) \tag{20}$$

are constants that will be approximated later by using the boundary conditions at $x = 1$. Substituting the decomposition series (8) for $y(x)$ into (19) gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) = 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 \\ -9L^{-1}(e^x) + L^{-1}\left(\sum_{n=0}^{\infty} y_n(x)\right). \end{aligned} \tag{21}$$

As stated before, the modified decomposition technique admits the use of the newly developed recursive relation

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= -\frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 \\ &\quad - 9L^{-1}(e^x) + L^{-1}(y_0(x)), \\ y_{k+1}(x) &= L^{-1}(y_k(x)), \qquad k \geq 1. \end{aligned} \tag{22}$$

This gives

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= -\frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 \\ &\quad - 9L^{-1}(e^x) + L^{-1}(y_0(x)) \\ &= -\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{120}Ax^5 + \frac{1}{720}Bx^6 + \frac{1}{5040}Cx^7 + \frac{1}{40320}Dx^8 \\ &\quad - \frac{1}{45360}x^9 - \frac{1}{403200}x^{10} - \frac{1}{4435200}x^{11} - \frac{1}{53222400}x^{12} + \cdots, \\ y_2(x) &= L^{-1}(y_1(x)), \\ &= -\frac{1}{39916800}x^{11} - \frac{1}{239500800}x^{12} + \cdots, \\ y_3(x) &= L^{-1}(y_2(x)) \\ &= \frac{1}{13!}x^{13} + \cdots. \end{aligned} \tag{23}$$

In view of (23), the approximation of $y(x)$ is given by

$$\begin{aligned} y(x) &= 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{120}Ax^5 + \frac{1}{720}Bx^6 + \frac{1}{5040}Cx^7 \\ &\quad + \frac{1}{8!}Dx^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots, \end{aligned} \tag{24}$$

where the constants $A, B, C$, and $D$ are as yet undetermined. This can be achieved by imposing the boundary conditions at $x = 1$ on the four-term approximant $\phi_4$ where

$$\phi_4 = \sum_{k=0}^{k=3} y_k, \tag{25}$$

to obtain the system

$$\begin{bmatrix} \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{8!} \\ \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} \\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{2849503}{68428800} \\ -e + \frac{9072821}{3628800} \\ -2e + \frac{5445427}{1209600} \\ -3e + \frac{259883}{51840} \end{bmatrix}. \tag{26}$$

Solving this system gives

$$\begin{aligned} A &= -3.999992, \\ B &= -5.00017, \\ C &= -5.9985, \\ D &= -7.005. \end{aligned} \tag{27}$$

Consequently, the series solution is given by

$$
\begin{aligned}
y(x) = 1 &- \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - 0.03333326667x^5 - 0.006944680556x^6 \\
&- 0.001190178571x^7 - 0.000173735119x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} \\
&- \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots .
\end{aligned} \tag{28}
$$

As stated before, the series solution (42) is convergent. In addition, using the ratio test implies that the series converges for every $x$.

Table 1 below shows the numerical results for Example 1. Examining the errors obtained by using the proposed modified decomposition method [11–13] shows the high accuracy obtained.

Table 1. Numerical results for Example 1.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.00000000 | 1.0000000000 | 0.000000 |
| 0.1 | 0.99465383 | 0.9946538264 | −2.0E−10 |
| 0.2 | 0.97712221 | 0.9771222066 | −2.0E−10 |
| 0.3 | 0.94490117 | 0.9449011654 | −2.0E−10 |
| 0.4 | 0.89509482 | 0.8950948186 | −2.0E−10 |
| 0.5 | 0.82436064 | 0.8243606355 | −2.0E−10 |
| 0.6 | 0.72884752 | 0.7288475206 | −6.0E−10 |
| 0.7 | 0.60412581 | 0.6041258131 | −1.0E−9 |
| 0.8 | 0.44510819 | 0.4451081876 | −2.0E−9 |
| 0.9 | 0.24596031 | 0.2459603145 | −3.4E−9 |
| 1.0 | 0.00000000 | 0.0000000000 | 0.000000 |

$*$ error = analytic solution − numerical solution.

EXAMPLE 2. We next consider the nonlinear tenth-order BVP

$$
y^{(x)}(x) = e^{-x}y^2(x), \qquad 0 < x < 1, \tag{29}
$$

subject to the boundary conditions

$$
y^{(2i)}(0) = 1, \quad y^{(2i)}(1) = e, \qquad i = 0, 1, 2, 3, 4. \tag{30}
$$

The exact solution for this problem is

$$
y(x) = e^x. \tag{31}
$$

Equation (29) may be written in an operator form by

$$
Ly = e^{-x}y^2(x), \qquad 0 < x < 1. \tag{32}
$$

Note that the inverse operator $L^{-1}$ here is a tenfold integral operator. Operating with $L^{-1}$ on both sides of (32), and using the boundary conditions at $x = 0$ yields

$$
\begin{aligned}
y(x) = 1 + Ax &+ \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\
&+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + L^{-1}\left(e^{-x}y^2(x)\right),
\end{aligned} \tag{33}
$$

where the constants

$$
A = y'(0), \qquad B = y^{(iii)}(0), \qquad C = y^{(v)}(0), \qquad D = y^{(vii)}(0), \qquad E = y^{(ix)}(0) \tag{34}
$$

are to be determined. Substituting the decomposition assumption (8) for the solution $y(x)$ and the polynomials representation (9) for the nonlinear term $y^2(x)$ into (33) gives

$$\sum_{n=0}^{\infty} y_n(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}B\,x^3 + \frac{1}{4!}\,x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + L^{-1}\left(e^{-x}\sum_{n=0}^{\infty} A_n\right), \tag{35}$$

where $A_n$ are the so-called Adomian polynomials that represent the nonlinear term $y^2(x)$. We point out that Adomian polynomials can be generated for all classes of nonlinearity according to specific formulae set by [9,10]. A new algorithm for calculating Adomian polynomials for nonlinear operators has been established recently by [14].

To determine the components $y_n(x)$, $n \geq 0$, the modified decomposition method [11–13] introduces the recursive relation

$$\begin{aligned} y_0(x) &= 1,\\ y_1(x) &= Ax + \frac{1}{2}x^2 + \frac{1}{6}B\,x^3 + \frac{1}{24}\,x^4 + \frac{1}{120}Cx^5 + \frac{1}{720}x^6 + \frac{1}{5040}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9\\ &\quad + L^{-1}\left(e^{-x}A_0\right),\\ y_{k+1}(x) &= L^{-1}(e^{-x}A_k),\ k \geq 1. \end{aligned} \tag{36}$$

It is useful to list the first few Adomian polynomials $A_n$ for the nonlinear operator $F(y) = y^2(x)$. Following the analysis of [9] or [14] yields

$$\begin{aligned} A_0 &= F(y_0)\\ &= y_0^2(x),\\ A_1 &= y_1(x)F'(y_0)\\ &= 2y_0(x)y_1(x),\\ A_2 &= y_2F'(y_0) + \frac{y_1^2}{2!}F''(y_0)\\ &= 2y_0(x)y_2(x) + y_1^2(x), \end{aligned} \tag{37}$$

and so on for other polynomials.

Inserting (37) into (38) yields

$$\begin{aligned} y_0(x) &= 1,\\ y_1(x) &= Ax + \frac{1}{2}x^2 + \frac{1}{6}B\,x^3 + \frac{1}{24}\,x^4 + \frac{1}{120}Cx^5 + \frac{1}{720}x^6 + \frac{1}{5040}Dx^7 + \frac{1}{8!}x^8 + \frac{E}{9!}x^9\\ &\quad + L^{-1}(e^{-x}A_0)\\ &= Ax + \frac{1}{2}x^2 + \frac{1}{6}Bx^3 + \frac{1}{24}x^4 + \frac{1}{120}Cx^5 + \frac{1}{720}x^6 + \frac{1}{5040}Dx^7\\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} - \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \cdots,\\ y_2(x) &= L^{-1}(e^{-x}A_1)\\ &= \frac{2}{11!}Ax^{11} + \left(-\frac{4}{12!}A + \frac{1}{239500800}\right)x^{12} + \cdots,\\ y_3(x) &= L^{-1}(e^{-x}A_2)\\ &= \frac{1}{13!}x^{13} + \cdots. \end{aligned} \tag{38}$$

Consequently, the approximation of $y(x)$ is given by

$$\begin{aligned} y(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \left(\frac{1}{19958400}A - \frac{1}{39916800}\right)x^{11} \\ &+ \left(-\frac{1}{119750400}A + \frac{1}{159667200}\right)x^{12} + \cdots . \end{aligned} \tag{39}$$

It remains to determine approximations to the constants $A, B, C, D$, and $E$. This can be achieved by imposing the boundary conditions at $x = 1$ on the four-term approximant $\phi_4$ derived from (39). Accordingly, we obtain the algebraic system

$$\begin{bmatrix} \frac{23950081}{23950080} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\ \frac{1}{226800} & 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\ \frac{1}{3360} & 0 & 1 & \frac{1}{3!} & \frac{1}{5!} \\ \frac{1}{90} & 0 & 0 & 1 & \frac{1}{3!} \\ \frac{1}{3!} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} e - \frac{246379361}{159667200} \\ e - \frac{5599523}{3628800} \\ e - \frac{20737}{13340} \\ e - \frac{123}{80} \\ e - \frac{35}{24} \end{bmatrix}. \tag{40}$$

Solving this algebraic system gives

$$\begin{aligned} A &= 1.00001436, \\ B &= 0.999858964, \\ C &= 1.001365775, \\ D &= 0.987457318, \\ E &= 1.093279434. \end{aligned} \tag{41}$$

This results in the series solution

$$\begin{aligned} y(x) &= 1 + 1.00001436x + \frac{1}{2}x^2 + 0.1666431607x^3 + \frac{1}{4!}x^4 \\ &+ 0.008344714791x^5 + \frac{1}{6!}x^6 + 0.000195924071x^7 \\ &+ \frac{1}{8!}x^8 + 3.013 \times 10^{-6}x^9 \\ &+ \frac{1}{10!}x^{10} + 2.51 \times 10^{-8}x^{11} - 2.087 \times 10^{-9}x^{12} + \cdots . \end{aligned} \tag{42}$$

Using the ratio test implies that the series converges for every $x$.

Table 2 below shows the exact values, numerical solutions, and the errors obtained by using the modified decomposition method. Table 2 also provides some numerical evidence which suggests that the performance of the modified decomposition method is promising.

We close our analysis by discussing a twelfth-order boundary value problem where the $12^{\text{th}}$-order is a function of a lower-order derivative.

EXAMPLE 3. Finally, we consider the nonlinear $12^{\text{th}}$-order BVP

$$y^{(xii)}(x) = 2e^x y^2(x) + y'''(x), \qquad 0 < x < 1, \tag{43}$$

subject to the boundary conditions

$$y^{(2i)}(0) = 1, \quad y^{(2i)}(1) = e^{-1}, \qquad i = 0, 1, 2, 3, 4, 5. \tag{44}$$

Table 2. Numerical results for Example 2.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 0.00000 |
| 0.1 | 1.105170918 | 1.10517233 | −1.41E−6 |
| 0.2 | 1.221402758 | 1.221405446 | −2.69E−6 |
| 0.3 | 1.349858808 | 1.349862509 | −3.70E−6 |
| 0.4 | 1.491824698 | 1.49182905 | −4.35E−6 |
| 0.5 | 1.648721271 | 1.648725849 | −4.58E−6 |
| 0.6 | 1.822118800 | 1.822123158 | −4.36E−6 |
| 0.7 | 2.013752707 | 2.013756415 | −3.71E−6 |
| 0.8 | 2.225540928 | 2.225543623 | −2.69E−6 |
| 0.9 | 2.459603111 | 2.459604528 | −1.42E−6 |
| 1.0 | 2.718281828 | 2.7182830 | 2.00E−9 |

* error = analytical solution − numerical solution

The exact solution for this problem is

$$y(x) = e^{-x}. \tag{45}$$

Proceeding as before, we set

$$Ly = y'''(x) + 2e^{x}y^{2}(x), \qquad 0 < x < 1. \tag{46}$$

Note that the inverse operator $L^{-1}$ here is a 12-fold integral operator. Operating with $L^{-1}$ on both sides of (46) gives

$$\begin{aligned} y(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 \\ + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + L^{-1}\left(y'''(x)\right) + L^{-1}\left(2e^{x}y^{2}(x)\right), \end{aligned} \tag{47}$$

where the constants

$$\begin{aligned} A &= y'(0), & B &= y^{(iii)}(0), & C &= y^{(v)}(0), \\ D &= y^{(vii)}(0), & E &= y^{(ix)}(0), & F &= y^{(xi)}(0) \end{aligned} \tag{48}$$

are to be determined. Substituting the decomposition assumption (8) for the solution $y(x)$ and the polynomials representation (9) for the nonlinear term $y^2(x)$ into (47) gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 \\ + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} \\ + L^{-1}\left(\sum_{n=0}^{\infty} y_n{'''}(x)\right) + L^{-1}\left(2e^{x}\sum_{n=0}^{\infty} A_n\right), \end{aligned} \tag{49}$$

where $A_n$ are Adomian polynomials as described above.

The components $y_n(x)$, $n \geq 0$ can be elegantly determined by using the recursive relation

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 \\ &\quad + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + L^{-1}\left(y_0{'''}\right) + L^{-1}\left(2e^{x}A_0\right), \\ y_{k+1}(x) &= L^{-1}\left(y_k{'''}\right) + L^{-1}\left(2e^{x}A_k\right), \qquad k \geq 1. \end{aligned} \tag{50}$$

Using $A_n$ as derived in (37) and proceeding as discussed above, we find

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{1}{239500800}x^{12} + \cdots, \\ y_2(x) &= \frac{1}{12!}Bx^{12} + \cdots. \end{aligned} \tag{51}$$

Combining the components computed in (51) gives the approximation

$$\begin{aligned} y(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} \\ &\quad + \left(\frac{2}{12!} + \frac{1}{12!}B\right)x^{12} + \cdots. \end{aligned} \tag{52}$$

We then follow the approach used before to evaluate the constants $A, B, C, D, E$, and $F$. Using the boundary conditions at $x = 1$ on the three-term approximant $\phi_3$ derived from (52) gives the algebraic system

$$\begin{bmatrix} 1 & \frac{79833601}{479001600} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} \\ 0 & \frac{3628801}{3628800} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\ 0 & \frac{1}{8!} & 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\ 0 & \frac{1}{6!} & 0 & 1 & \frac{1}{3!} & \frac{1}{5!} \\ 0 & \frac{1}{4!} & 0 & 0 & 1 & \frac{1}{3!} \\ 0 & \frac{1}{2!} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} e^{-1} - \frac{369569047}{239500800} \\ e^{-1} - \frac{1399883}{907200} \\ e^{-1} - \frac{31109}{20160} \\ e^{-1} - \frac{139}{90} \\ e^{-1} - \frac{19}{12} \\ e^{-1} - 2 \end{bmatrix}. \tag{53}$$

It follows that

$$\begin{aligned} A &= -0.9999983604, \\ B &= -1.000016174, \\ C &= -0.9998407313, \\ D &= -1.001558298, \\ E &= -0.9851011393, \\ F &= -1.132112472. \end{aligned} \tag{54}$$

The series solution is therefore given by

$$\begin{aligned} y(x) &= 1 - 0.9999983604x + \frac{1}{2!}x^2 - 0.1666693624x^3 + \frac{1}{4!}x^4 \\ &\quad - 0.008332006094x^5 + \frac{1}{6!}x^6 - 0.0001987218845x^7 \\ &\quad + \frac{1}{8!}x^8 - 2.715 \times 10^{-6}x^9 \\ &\quad + \frac{1}{10!}x^{10} - 2.836 \times 10^{-8}x^{11} + 2.087 \times 10^{-9}x^{12} + \cdots. \end{aligned} \tag{55}$$

The series solution converges for every value of $x$ in a parallel manner to that discussed in the preceding examples.

In Table 3, we show the results of the calculations and the errors obtained by using the approximant of (55).

Table 3. Numerical results for Example 3.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 0.00000 |
| 0.1 | 0.904837418 | 0.904837579 | −1.61E−7 |
| 0.2 | 0.818730753 | 0.818731060 | −3.07E−7 |
| 0.3 | 0.740818221 | 0.740818643 | −4.22E−7 |
| 0.4 | 0.670320046 | 0.670320543 | −4.97E−7 |
| 0.5 | 0.606530659 | 0.606531182 | −5.22E−7 |
| 0.6 | 0.548811636 | 0.548812133 | −4.97E−7 |
| 0.7 | 0.496585304 | 0.496585726 | −4.22E−7 |
| 0.8 | 0.449328964 | 0.449329271 | −3.07E−7 |
| 0.9 | 0.406569659 | 0.406569821 | −1.61E−7 |
| 1.0 | 0.367879441 | 0.367879441 | 2.00E−10 |

* error = analytical solution − numerical solution.

## 4. CONCLUDING REMARKS

The computations associated with the examples discussed above were performed by using Maple V. The existence and uniqueness of the solution is guaranteed according to a comprehensive analysis by Agarwal's book [6]. The proposed algorithm produced a rapidly convergent series.

There are two important points to make here. First, unlike the traditional techniques used by other numerical algorithms, where the solution $y(x)$ is defined at grid points only, the solution here is given in a series form. Second, the approach is implemented directly in a straightforward manner without using restrictive assumptions or linearization. We believe that the efficiency of the decomposition method gives it much wider applicability.

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