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Applications of combinatorics to statics—a second survey

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To the memory of Professor Zdeněk Frolík.

Abstract

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Some recent results are presented, concerning the algorithmic aspects of 2-dimensional generic rigidity, and 1-story buildings as tensegrity frameworks. Most of these results were obtained after the completion of the first survey (Recski, 1984) for a 'Winter School' organized by the late Professor Z. Frolík. Results in Sections 3 and 4 of the first survey are used throughout.

1. On the algorithmic aspects of generic planar rigidity

Infinitesimal rigidity (or briefly rigidity) of generic planar bar-and-joint frameworks is one of the best developed areas of the geometry of rigid structures. A large number of research and survey papers is available (the list of references is probably incomplete) but a systematic summary of the available algorithms is missing.

Rigidity of a bar-and-joint framework (in any dimension) can be formulated as a rank condition of a matrix. In case of generic frameworks, however, the entries of this matrix are of form $x_j^i - x_k^i$ (x_j^i stands for the ith coordinate of the jth joint) where these letters are algebraic symbols only. If the size of such a matrix is $O(n)$, then the time complexity of determining its rank is not $O(n^3)$ like for matrices with real entries, but at least $O(n!)$. Thus the decision problem GR(G, d): 'Decide whether a graph G is generic rigid in the d-dimensional space', is clearly in NP (the positive answer can be verified in, say, $O(n^3)$ steps if a rigid embedding of G into the d-dimensional space is presented), but finding such an embedding seems to be hard.

Laman's celebrated theorem stated that GR(G, 2) is in co-NP (a negative answer can be verified in $O(n)$ steps if a violating subset is presented). The corresponding question is open for $d \geq 3$. Finally, Lovász and Yemini proved that

GR(G, 2) is in P. Based on their results, the following problems can all be solved in polynomial time.

(1) Decide whether G is generic rigid in the plane.

(2) If yes and $e > 2n - 3$, find a set X of $e - 2n + 3$ edges so that $G - X$ is still generic rigid.

(3) Decide whether G is generic independent in the plane.

(4) If yes and $e < 2n - 3$, find a set Y of $2n - 3 - e$ edges so that $G \cup Y$ is still generic independent.

(5) If $e = 2n - 3$ and G is not generic rigid then find a pair of sets X, Y with minimum cardinality so that $(G - X) \cup Y$ is generic rigid.

We shall see that the weighted versions of problems (2), (4) and (5) are also polynomially solvable, i.e., if weights are associated to the edges, the minimum (or maximum) weight subsets X and/or Y with the required property can also be obtained.

The polynomial time complexity of all these problems follows from the following observations since Theorem (a) allows us to use the greedy algorithm if an 'independence oracle' can be realized in polynomial time complexity and Theorem (b) serves for exactly this purpose, using [9–11].

Theorem (a). *Let E be the edge set of the complete graph on n vertices. Call a subset X of E 'good' if the corresponding graph (with n vertices and $|X|$ edges) is generic independent. Then the 'good' subsets are the independent subsets of a matroid over E.*

Theorem (b). *A graph is generic independent if and only if doubling any edge of it results in a graph which is the union of two forests.*

Theorem (a) was first given by Crapo [8] and Theorem (b) by Lovász and Yemini, [16]. Corollary 4 of this latter article furthermore states the following.

Corollary (c). *A graph is generic rigid if and only if*

$$\sum_{i=1}^{k} (2|V(G_i)| - 3) \geq 2\,|V(G)| - 3$$

holds for every system of subgraphs G such that $G_1 \cup \cdots \cup G_k = G$.

As the authors pointed out, Corollary (c) is essentially a submodular function minimization problem but the 'ellipsoid method' (see Grötschel, Lovász and Schrijver [14] is not needed here. A graph is generic rigid if and only if its edge set is spanning (i.e., has maximum rank) in the matroid defined in Theorem (a).

A combination of Theorems (a) and (b) trivially gives the following combinatorial characterization of minimum generic rigid graphs.

Theorem (d). *A graph G is generic rigid and independent if and only if* $e = 2n - 3$ *and doubling any edge of G results in a graph which is the disjoint union of two spanning trees.*

However, it is not true that a graph is generic rigid if and only if doubling any edge of it results in a graph which contains two edge-disjoint spanning trees. For example, consider the 3×1 square grid and add two diagonals to the first and to the third square each.

2. One-story buildings as tensegrity frameworks

Consider a 1-story building, with the vertical bars fixed to the earth via joints. If each of the four external vertical walls consists of a diagonal, the four corners of the roof become fixed. Hence questions related to the infinitesimal rigidity of a one-story building are reduced [4] to those related to the infinitesimal rigidity of a 2-dimensional square grid of size $k \times l$ where the corners are pinned down. Then the minimum number of necessary diagonal rods for infinitesimal rigidity was proved to be $k + l - 2$ [4] and the minimum systems correspond to asymmetric 2-component forests [7]. Recall that if no joints are pinned down then the $k \times l$ square grid as a planar framework requires at least $k + l - 1$ diagonal rods and the minimum systems correspond to the spanning trees.

In what follows, every graph will be the subgraph of the complete bipartite graph $K_{k,l}$; the two subsets of the vertex set of $K_{k,l}$ will be denoted by A and B with respective cardinalities k and l. A 2-component forest with vertex sets V_1, V_2 of the components is called asymmetric if

$$|V_i \cap A| \neq \frac{k}{l} \cdot |V_i \cap B| .$$

Since real life rods are less reliable against compressions than against tensions, one might prefer using more diagonal rods, but under tension only. In order to study such questions, the concept of tensegrity frameworks has been introduced; the elements connecting the joints may be rods (which are rigid both under tension and compression), cables (used under tension only) and possibly struts as well (used under compression only). In this section we survey the results when the edges forming the square grid (or the 1-story building) are rods but the diagonals in the grid (in the roof, respectively) are cables.

Baglivo and Graver [1] recognized that a directed graph model is needed for the square grids with diagonal cables. The edges are directed 'up' or 'down' depending on the 'Northwest–Southeast' or 'Southwest–Northeast' direction of the corresponding diagonal cables. Then the tensegrity framework is infinitesimally rigid if and only if the corresponding directed graph is strongly connected. Their theorem is illustrated in Fig. 1 (cables are drawn by dashed lines): in case of the nonrigid system a deformation is presented in Fig. 2.

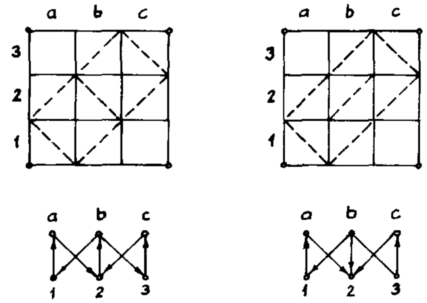

Fig. 1.

If we wish to use diagonal cables to make a square grid (with the four corners pinned down) infinitesimally rigid, the minimum number of the necessary cables was proved to be

$$f(k,l)=\begin{cases}0 & \text{if } k=l=1,\\ 4 & \text{if } k=l=2,\\ k+l-1 & \text{otherwise}\end{cases}$$

[6] and the minimum systems were characterized only recently [25]. Before presenting the general solution we mention that in a previous stage of our investigations we found the following partial result [21]:

Suppose that all the diagonal cables are parallel. Then a collection of $f(k, l)$ cables makes the grid infinitesimally rigid if and only if $|N(X)| > (l/k)|X|$ holds for every proper subset X of A, where $N(X)$ denotes the set of those vertices of B which are adjacent to at least one vertex of X. For example, the first framework in Fig. 3 is infinitesimally rigid, the second one is not. One can see this difference by letting X be the subset formed by the first five rows. Then $N(X)$ is of cardinality 13 in the first case and 12 in the second one, and $12 < (5\cdot 17)/7 < 13$.

This condition can be checked in polynomial time: If $k = l$ (which can be ensured by a straightforward transformation) then by a theorem of Hetyei [15] this is equivalent to the property that each edge is contained in a perfect matching.

The general solution of the problem [25] consists of two statements. Let G denote the directed graph determined by a system of diagonal cables. If G is not connected then the system is infinitesimally rigid if and only if G is an asymmetric 2-component graph with strongly connected components. If G is connected then

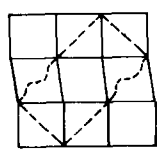

Fig. 2.

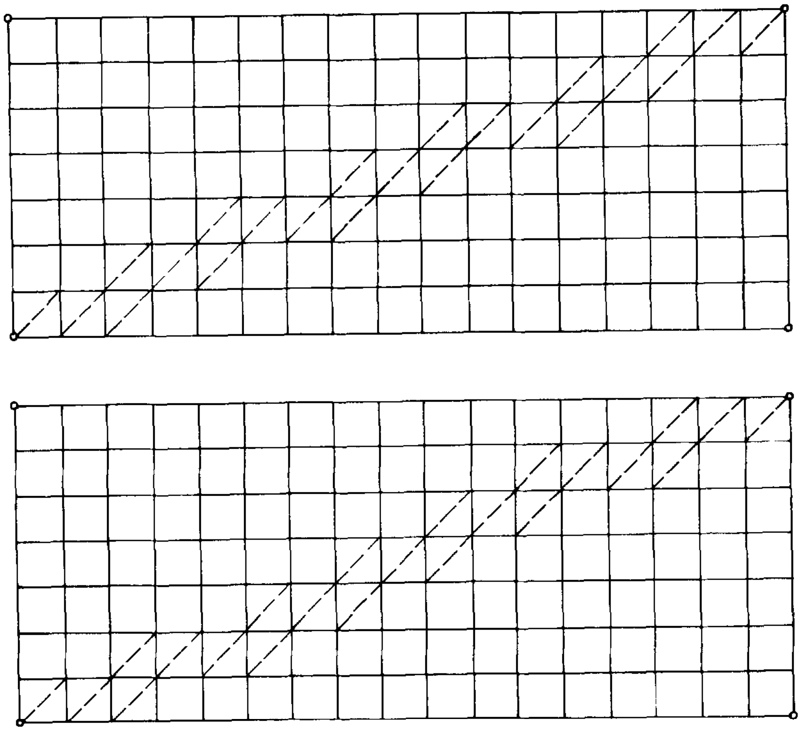

Fig. 3.

the system is infinitesimally rigid if and only if either

$$|N(X)| > \frac{l}{k}|X| \quad \text{for all proper subsets } X \text{ of } A$$

or

$$|N(Y)| > \frac{k}{l}|Y| \quad \text{for all proper subsets } Y \text{ of } B.$$

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