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Gagliardo–Nirenberg inequalities on manifolds

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ABSTRACT

We prove Gagliardo–Nirenberg inequalities on some classes of manifolds, Lie groups and graphs.

Contents

1. Introduction

Cohen, Meyer and Oru [5], Cohen, Devore, Pentrushev and Xu [4], proved the following Gagliardo–Nirenberg type inequality

$$\|f\|_{1^*} \leqslant C \big\| |\nabla f| \big\|_1^{\frac{n-1}{n}} \|f\|_{B^{-(n-1)}_{\infty,\infty}}^{\frac{1}{n}} \tag{1.1}$$

for all $f \in W_1^1(\mathbb{R}^n)$ $(1^* = \frac{n}{n-1})$. The proof of (1.1) is involved and based on wavelet decompositions, weak type $(1,1)$ estimates and interpolation results.

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Using a simple method relying on weak type estimates and pseudo-Poincaré inequalities, Ledoux [14] obtained the following extension of (1.1). He proved that for $1 \leqslant p < l < \infty$ and for every $f \in W_p^1(\mathbb{R}^n)$

$$\|f\|_l \leqslant C \big\| |\nabla f| \big\|_p^{\theta} \|f\|^{1-\theta}_{B^{\frac{\theta}{\theta-1}}_{\infty,\infty}} \tag{1.2}$$

where $\theta = \frac{p}{l}$ and $C > 0$ only depends on l, p and n.

In the same paper, he extended (1.2) to the case of Riemannian manifolds. If $p = 2$ he observed that (1.2) holds without any assumption on M. If $p \neq 2$ he assumed that the Ricci curvature is non-negative and obtained (1.2) with $C > 0$ only depending on l, p when $1 \leqslant p \leqslant 2$ and on l, p and n when $2 < p < \infty$.

He also proved that a similar inequality holds on $\mathbb{R}^n$, Riemannian manifolds with non-negative Ricci curvature, Lie groups and Cayley graphs, replacing the $B^{\frac{\theta}{\theta-1}}_{\infty,\infty}$ norm by the $M^{\frac{\theta}{\theta-1}}_{\infty}$ norm (see definitions below).

Note that these two versions of Gagliardo–Nirenberg inequalities extend the classical Sobolev inequality in the Euclidean case:

$$\|f\|_{p^*} \leqslant C \big\| |\nabla f| \big\|_p \tag{1.3}$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ holds on $\mathbb{R}^n$ for every $f \in W_p^1(\mathbb{R}^n)$ and for every $1 \leqslant p < n$.

In the Riemannian case it is not generally true that (1.2) or (1.1) imply (1.3), without additional assumptions on the manifold (cf. Proposition 3.2 below). On the other hand we will now show examples of Riemannian manifolds where (1.3) holds independently of (1.2). It is clear that (1.3) holds on a compact Riemannian n-manifold M. As an example of complete non-compact Riemannian manifold satisfying (1.3), we can consider a complete Riemannian n-manifold M with non-negative Ricci curvature. If there exists $v > 0$ such that for all $x \in M$, $\mu(B(x, 1)) \geqslant v$, then M satisfies (1.3). Here $\mu(B(x, 1))$ is the Riemannian volume of the open ball $B(x, 1)$. For more general cases where we have (1.3) for some p's depending on the hypotheses, see [17]. Note that if (1.3) holds for some $1 \leqslant p < n$, then it holds for all $p \leqslant q < n$ (see [17, Chapter 3]).

We have also non-linear versions of Gagliardo–Nirenberg inequalities proved by Rivière and Strzelecki [16,19]. They got for every $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla f|^{p+2} \leqslant C \|f\|^2_{\mathrm{BMO}} \int_{\mathbb{R}^n} \big|\nabla^2 f\big|^2 |\nabla f|^{p-2}. \tag{1.4}$$

They applied this inequality and obtained a regularity property for solutions of non-linear elliptic equations of type

$$-\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) = G(x, u, \nabla u)$$

where G grows as $|\nabla u|^p$.

Recently, Martin and Milman [15] developed a new symmetrization approach to obtain the Gagliardo–Nirenberg inequalities (1.2) and, therefore the Sobolev inequalities (1.3) in $\mathbb{R}^n$. They also proved a variant of (1.4). The method of [15] to prove (1.2) is different from that of Ledoux. It relies essentially on an interpolation result for Sobolev spaces and pseudo-Poincaré inequalities in the Euclidean case.

In this paper, we prove analogous results on Riemannian manifolds, Lie groups and graphs, with some additional hypotheses on these spaces. For this purpose, we will adapt Martin and Milman's method. The only difficulty is that interpolation results for Sobolev spaces on metric measured spaces were *only* known in the euclidean case. We overcome this problem using our interpolation theorems on Riemannian manifolds [3]. We use the characterization of the K-functional of interpolation of non-homogeneous Sobolev spaces from [3] and prove in Theorem 4.1 below, a characterization of a variant of the K-functional for homogeneous Sobolev spaces. The statements in [3] require doubling property and Poincaré type inequalities.

More precisely we obtain in the case of Riemannian manifolds.

Theorem 1.1. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. Moreover, assume that M satisfies the pseudo-Poincaré inequalities (P'_q) and (P'_∞). Consider $\alpha < 0$. Then, there exists $C > 0$ such that for every $f \in (W_q^1 + W_\infty^1) \cap B^\alpha_{\infty,\infty}$ with $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$, we have*

$$|f|^{q**\frac{1}{q}}(s) \leqslant C |\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s) \|f\|^{\frac{1}{1+|\alpha|}}_{B^\alpha_{\infty,\infty}}. \tag{1.5}$$

Above and from now on, $|f|^{q**\frac{1}{q}}$ means $(|f|^{q**})^{\frac{1}{q}}$. Recall that for every $t > 0$

$$f^*(t) = \inf\big\{\lambda;\ \mu\big(\big\{|f| > \lambda\big\}\big) \leqslant t\big\},$$

$$f^*(\infty) = \inf\big\{\lambda;\ \mu\big(\big\{|f| > \lambda\big\}\big) < \infty\big\}$$

and

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds.$$

Using this symmetrization result we prove

Theorem 1.2. *Let M be a complete Riemannian manifold satisfying the hypotheses of Theorem* 1.1. *Then* (1.2) *holds for all* $q \leqslant p < l < \infty$.

Corollary 1.3. *Let M be a Riemannian manifold with non-negative Ricci curvature. Then* (1.2) *holds for all* $1 \leqslant p < l < \infty$.

This corollary is exactly what Ledoux proved [14]. We obtain further generalizations:

Corollary 1.4. *Consider a complete Riemannian manifold M satisfying* (D), (P_1) *and assume that there exists* $C>0$ *such that for every* $x, y \in M$ *and* $t>0$

$$\left|\nabla_x p_t(x,y)\right| \leqslant \frac{C}{\sqrt{t}\mu(B(y,\sqrt{t}))}. \tag{G}$$

Then inequality (1.2) *holds for all* $1 \leqslant p < l < \infty$.

Note that a Lie group of polynomial growth satisfies the hypotheses of Corollary 1.4 (see [7]). Hence it verifies (1.2) for all $1 \leqslant p < l < \infty$.

Another example of a space satisfying the hypotheses of Corollary 1.4 is given by taking a Galois covering manifold of a compact manifold whose deck transformation group has polynomial growth (see [9]). We can also take the example of a Cayley graph of a finitely generated group (see [6,17]).

We also get the following corollary.

Corollary 1.5. *Let M be a complete Riemannian manifold satisfying* (D) *and* (P_2). *Then* (1.2) *holds for all* $2 \leqslant p < l < \infty$.

Note that (P'_2) is always satisfied. Hence, by Ledoux's method, inequality (1.2) with $p=2$ needs no assumption on M (see [14]). So our results are only interesting when $p \neq 2$.

Local version: Let M be a complete Riemannian manifold satisfying a local doubling property (D_{loc}) and a local Poincaré inequality $(P_{q\,\mathrm{loc}})$—we restrict our definitions to small balls. Moreover assume that M admits a local version of pseudo-Poincaré inequalities $(P'_{q\,\mathrm{loc}})$, $(P'_{\infty\,\mathrm{loc}})$: by $(P'_{r\,\mathrm{loc}})$ we mean

$$\left\|f-e^{t\Delta}f\right\|_r \leqslant Ct^{\frac{1}{2}}\left(\|f\|_r+\left\||\nabla f|\right\|_r\right).$$

In this context, the following local version of (1.2) holds: for every $q \leqslant p < l < \infty$ and $f \in W^1_p$

$$\|f\|_l \leqslant C\left(\|f\|_p+\left\||\nabla f|\right\|_p\right)^{\theta}\|f\|^{1-\theta}_{B^{\frac{\theta}{\theta-1}}_{\infty,\infty}}. \tag{1.6}$$

In the following theorem, we show a variant of Theorem 1.1 replacing the Besov norm by the Morrey norm. In the Euclidean case, the Morrey space is strictly smaller than the Besov space. Therefore, the following Theorem 1.6 (resp. Corollary 1.7) is weaker than Theorem 1.1 (resp. Theorem 1.2). In contrast, on Riemannian manifolds, the Besov and Morrey spaces are not comparable in general.

Theorem 1.6. *Let M be a complete non-compact Riemannian manifold satisfying* (D) *and* (P_q) *for some* $1 \leqslant q < \infty$. *Consider* $q \leqslant p < \infty$ *and* $\alpha < 0$. *Then, for every* $f \in (W^1_q + W^1_\infty) \cap M^\alpha_\infty$ *we have*

$$|f|^{q**\frac{1}{q}}(s) \leqslant C|\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s)\|f\|^{\frac{1}{1+|\alpha|}}_{M^\alpha_\infty}.$$

Corollary 1.7. *Under the hypotheses of Theorem* 1.6, *let* $q_0=\inf\{q \in [1,\infty[\colon\ (P_q)\text{ holds}\}$ *and consider* $q_0 < p < l < \infty$.[1] *Then, for every* $f \in W^1_p$, *we have*

$$\|f\|_l \leqslant C\left\||\nabla f|\right\|^{\theta}_p\|f\|^{1-\theta}_{M^{\frac{\theta}{\theta-1}}_\infty}. \tag{1.7}$$

[1] If $q_0=1$, we allow $1 \leqslant p < l < \infty$.

Ledoux [14] showed that (1.7) holds on any unimodular Lie group equipped with a left invariant Riemannian metric and the associated Haar measure. Once again, this is due to the fact that his method uses essentially the pseudo-Poincaré inequalities (P''_p), which hold on such a group for all $1 \leqslant p \leqslant \infty$ (see [17]). With our method, we only get the local version of (1.7), namely the analog of (1.6). However notice that we prove (1.7) in its full strength for Lie groups of polynomial growth.

Let us compare our result with Ledoux's one. Our hypotheses are stronger, we assume in addition of the pseudo-Poincaré inequality—which is the only assumption of Ledoux—(D) and (P_q) but recover most of his examples. Moreover we obtain Corollary 1.4 which gives us more examples as we have seen in the introduction. For instance, on Lie groups, Ledoux only mentioned in his paper the Morrey version while Corollary 1.4 yield (1.2) on Lie groups with polynomial growth for every $1 \leqslant p < l < \infty$. We get also the interpolation of his inequality (1.2). Since it is not known if the pseudo-Poincaré inequalities interpolate, his method gives (1.2) (resp. (1.7)) for the same exponent p of pseudo-Poincaré inequality. With our method, we get (1.2) (resp. (1.7)) for every $p \geqslant q$.

We also give another symmetrization inequality which had been used in [15] to prove Gagliardo–Nirenberg inequalities with a Triebel–Lizorkin condition.

Theorem 1.8. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. Moreover, assume that M satisfies the pseudo-Poincaré inequalities (P'_q) and (P'_∞). Consider $\alpha < 0$. Then there is $C > 0$ such that for every $f \in W^1_q + W^1_\infty$ with $f^*(\infty) = 0$, $|\nabla f|^*(\infty) = 0$ and satisfying $(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f(.)|) \in L_q + L_\infty$:*

$$|f|^{q**\frac{1}{q}}(s) \leqslant C |\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s) \Big[\Big(\sup_{t>0} t^{-\frac{\alpha}{2}} \big|P_t f(.)\big| \Big)^{q**\frac{1}{q}}(s) \Big]^{\frac{1}{1+|\alpha|}}, \quad s > 0. \tag{1.8}$$

We finish with the following non-linear Gagliardo–Nirenberg theorem.

Theorem 1.9. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. Moreover, assume that M satisfies (P'_q) and (P'_∞). Let $p \geqslant \max(2, q)$. Then for every $f \in C_0^\infty(M)$*

$$\int_M |\nabla f|^{p+1}\, d\mu \leqslant C \|f\|_{B^{-1}_{\infty,\infty}} \int_M \big|\nabla^2 f\big|^2 |\nabla f|^{p-2}\, d\mu.$$

To the knowledge of the author this inequality is new in the case of manifolds.

The paper is organized as follows. In Section 2, we give the definitions on a Riemannian manifold of Besov and Morrey spaces, Sobolev spaces, doubling property, Poincaré and pseudo-Poincaré inequalities. In Section 3, we show how to obtain under our hypotheses Ledoux's inequality (1.2) and different Sobolev inequalities. Section 4 is devoted to prove Theorem 1.1 and Theorem 1.6. In Section 5 we give another symmetrization inequality. Finally we prove Theorem 1.9 in Section 6.

2. Preliminaries

Throughout this paper C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exist two constants $C_1, C_2 > 0$ such that $C_1 u \leqslant v \leqslant C_2 u$.

Let M be a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leqslant p \leqslant +\infty$. Let $P_t = e^{t\Delta}$, $t \geqslant 0$, be the heat semigroup on M and p_t the heat kernel.

2.1. Besov and Morrey spaces

For $\alpha < 0$, we introduce the Besov norm

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t f\|_\infty < \infty$$

for measurable functions f such that this makes sense and say $f \in B^\alpha_{\infty,\infty}$ (we shall not try here to give the most general definition of the Besov space).

Lemma 2.1. *We have for every $f \in B^\alpha_{\infty,\infty}$*

$$\|f\|_{B^\alpha_{\infty,\infty}} \sim \sup_{t>0} t^{-\frac{\alpha}{2}} \big\|P_t(f - P_t f)\big\|_\infty. \tag{2.1}$$

Proof. It is clear that $\sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty \leqslant (1 + 2^{\frac{\alpha}{2}}) \|f\|_{B^\alpha_{\infty,\infty}}$. On the other hand

$$t^{-\frac{\alpha}{2}} P_t f = t^{-\frac{\alpha}{2}} (P_t f - P_{2t} f) + 2^{\frac{\alpha}{2}} (2t)^{-\frac{\alpha}{2}} P_{2t} f.$$

By taking the supremum over all $t > 0$, we get

$$\|f\|_{B^{\alpha}_{\infty,\infty}} \leqslant \sup_{t>0} t^{-\frac{\alpha}{2}} \left\| P_t(f - P_t f)\right\|_{\infty} + 2^{\frac{\alpha}{2}} \|f\|_{B^{\alpha}_{\infty,\infty}}.$$

Thus, $\|f\|_{B^{\alpha}_{\infty,\infty}} \leqslant \frac{1}{1-2^{\frac{\alpha}{2}}} \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_{\infty}$. □

For $\alpha < 0$, the Morrey space M^{α}_{∞} is the space of locally integrable functions f for which the Morrey norm

$$\|f\|_{M^{\alpha}_{\infty}} := \sup_{r>0,\, x\in M} r^{-\alpha} |f_{B(x,r)}| < \infty$$

where $f_B := \fint_B f\, d\mu = \frac{1}{\mu(B)} \int_B f\, d\mu$.

2.2. Sobolev spaces on Riemannian manifolds

Definition 2.2. (See [2].) Let M be a C^{∞} Riemannian manifold of dimension n. Write E^1_p for the vector space of C^{∞} functions φ such that φ and $|\nabla\varphi| \in L_p$, $1 \leqslant p < \infty$. We define the non-homogeneous Sobolev space W^1_p as the completion of E^1_p for the norm

$$\|\varphi\|_{W^1_p} = \|\varphi\|_p + \left\| |\nabla\varphi| \right\|_p.$$

We denote W^1_{∞} for the set of all bounded Lipschitz functions on M.

Proposition 2.3. *(See [2].) Let M be a complete Riemannian manifold. Then C^{∞}_0 is dense in W^1_p for $1 \leqslant p < \infty$.*

Definition 2.4. Let M be a C^{∞} Riemannian manifold of dimension n. For $1 \leqslant p \leqslant \infty$, we define $\dot{E}^1_p$ to be the vector space of distributions φ with $|\nabla\varphi| \in L_p$, where $\nabla\varphi$ is the distributional gradient of φ. It is well known that the elements of $\dot{E}^1_p$ are in $L_{p,\mathrm{loc}}$. We equip $\dot{E}^1_p$ with the semi norm

$$\|\varphi\|_{\dot{E}^1_p} = \left\| |\nabla\varphi| \right\|_p.$$

Definition 2.5. We define the homogeneous Sobolev space $\dot{W}^1_p$ as the quotient space $\dot{E}^1_p/\mathbb{R}$.

Remark 2.6. For all $\varphi \in \dot{E}^1_p$, $\|\overline{\varphi}\|_{\dot{W}^1_p} = \||\nabla\varphi|\|_p$.

2.3. Doubling property and Poincaré inequalities

Definition 2.7 *(Doubling property).* Let (M, d, μ) be a Riemannian manifold. Denote by $B(x,r)$ the open ball of center $x \in M$ and radius $r > 0$. One says that M satisfies the doubling property (D) if there exists a constant $C_d > 0$ such that for all $x \in M$, $r > 0$ we have

$$\mu\big(B(x, 2r)\big) \leqslant C_d \mu\big(B(x,r)\big). \tag{D}$$

Observe that if M satisfies (D) then

$$\operatorname{diam}(M) < \infty \quad \Leftrightarrow \quad \mu(M) < \infty \quad \big(\text{see [1]}\big).$$

Definition 2.8 *(Poincaré inequality).* A complete Riemannian manifold M admits a Poincaré inequality (P_q) for some $1 \leqslant q < \infty$ if there exists a constant C such that for all $f \in C^{\infty}_0$ and for every ball B of M of radius $r > 0$, we have

$$\left(\fint_B |f - f_B|^q\, d\mu\right)^{\frac{1}{q}} \leqslant Cr\left(\fint_B |\nabla f|^q\, d\mu\right)^{\frac{1}{q}}. \tag{P_q}$$

Remark 2.9. Since C^{∞}_0 is dense in W^1_q, if M admits (P_q) for all $f \in C^{\infty}_0$ then (P_q) holds for all $f \in W^1_q$. In fact, by Theorem 1.3.4 in [11], M admits (P_q) for all $f \in \dot{E}^1_q$.

The following recent result from Keith and Zhong [12] improves the exponent of Poincaré inequality.

Theorem 2.10. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_q), for some $1 < q < \infty$. Then there exists $\epsilon > 0$ such that (X, d, μ) admits (P_p) for every $p > q - \epsilon$.*

Definition 2.11 *(Pseudo-Poincaré inequality for the heat semigroup).* A Riemannian manifold M admits a pseudo-Poincaré inequality for the heat semigroup (P'_q) for some $1 \leqslant q < \infty$ if there exists a constant C such that for all $f \in C_0^\infty$ and all $t > 0$, we have

$$\|f - P_t f\|_q \leqslant C t^{\frac{1}{2}} \big\||\nabla f|\big\|_q. \tag{P'_q}$$

M admits a pseudo-Poincaré inequality (P'_∞) if there exists $C > 0$ such that for every bounded Lipschitz function f we have

$$\|f - P_t f\|_\infty \leqslant C t^{\frac{1}{2}} \big\||\nabla f|\big\|_\infty. \tag{P'_∞}$$

Remark 2.12. Again by density of C_0^∞ in W_q^1, if M admits (P'_q) for some $1 \leqslant q < \infty$ for all $f \in C_0^\infty$ then M admits (P'_q) for all $f \in W_q^1$.

Let $2 < p \leqslant \infty$. Consider the following condition: there exists $C > 0$ such that for every $t > 0$

$$\big\||\nabla e^{t\Delta}|\big\|_{p \to p} \leqslant \frac{C}{\sqrt{t}}. \tag{G_p}$$

Lemma 2.13. *(See [8].) Let M be a complete Riemannian manifold M satisfying (D) and the Gaussian heat kernel upper bound, that is, there exist $C, c > 0$ such that for every $x, y \in M$ and $t > 0$*

$$p_t(x, y) \leqslant \frac{C}{\mu(B(y, \sqrt{t}))} e^{-c\frac{d^2(x,y)}{t}}. \tag{2.3}$$

Then (G) holds if and only if (G_∞) holds.

Lemma 2.14. *Let M be a complete Riemannian manifold. If the condition (G_p) holds for some $1 < p \leqslant \infty$ then M admits a pseudo-Poincaré inequality $(P'_{p'})$, p' being the conjugate of p ($\frac{1}{p} + \frac{1}{p'} = 1$).*

Proof. For $f \in C_0^\infty$, we have

$$f - e^{t\Delta} f = -\int_0^t \Delta e^{s\Delta} f\, ds.$$

Remark that (G_p) gives us that $\|\Delta e^{s\Delta} f\|_{p'} \leqslant \frac{C}{\sqrt{s}} \||\nabla f|\|_{p'}$. Indeed

$$\begin{aligned}
\big\|\Delta e^{s\Delta} f\big\|_{p'} &= \sup_{\|g\|_p = 1} \int_M \Delta e^{s\Delta} f g\, d\mu \\
&= \sup_{\|g\|_p = 1} \int_M f \Delta e^{s\Delta} g\, d\mu \\
&= \sup_{\|g\|_p = 1} \int_M \nabla f . \nabla e^{s\Delta} g\, d\mu \\
&\leqslant \big\||\nabla f|\big\|_{p'} \sup_{\|g\|_p = 1} \big\||\nabla e^{s\Delta} g|\big\|_p \\
&\leqslant \frac{C}{\sqrt{s}} \big\||\nabla f|\big\|_{p'}.
\end{aligned}$$

Therefore

$$\big\|f - e^{t\Delta} f\big\|_{p'} \leqslant C \big\||\nabla f|\big\|_{p'} \int_0^t \frac{1}{\sqrt{s}}\, ds = C\sqrt{t} \big\||\nabla f|\big\|_{p'}$$

which finishes the proof of the lemma. □

Definition 2.15 *(Pseudo-Poincaré inequality for averages).* A complete Riemannian manifold M admits a pseudo-Poincaré inequality for averages (P''_q) for some $1 \leqslant q < \infty$ if there exists a constant C such that for all $f \in C_0^\infty$ and for every ball B of M of radius $r > 0$, we have

$$\big\|f - f_{B(.,r)}\big\|_q \leqslant C r \big\||\nabla f|\big\|_q. \tag{P''_q}$$

Remark 2.16. (See Lemma 5.3.2 in [17].) If M is a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$, then it satisfies (P''_q). Hence (P''_q) holds for all $f \in \dot{E}^1_q$.

3. Ledoux's and Sobolev inequalities

Some comments about the proofs of Theorem 1.2, Corollaries 1.3, 1.4 and 1.5:

- Ledoux's inequality (1.2) in Theorem 1.2 follows from Theorem 1.1. We refer to [15] for details.
- For the proof of Corollary 1.3, remark that Riemannian manifolds with non-negative Ricci curvature satisfy (D) (with $C_d = 2^n$), (P_1). They also satisfy (P'_p) for all $1 \leqslant p \leqslant \infty$, where the constant C is numerical for $1 \leqslant p \leqslant 2$ and only depends on n for $2 < p \leqslant \infty$ (see [14]).
- In the proof of Corollary 1.4, the fact that M satisfies (D) and admits (P_1), hence (P_2), gives the Gaussian heat kernel upper bound (2.3). Since (G) holds, Lemma 2.13 asserts that (G_∞) holds too. Applying Lemma 2.14 it comes that M admits a pseudo-Poincaré inequality (P'_1). We claim that (P'_∞) holds on M. Indeed, (2.3) yields

$$\begin{aligned}\|f - e^{t\Delta} f\|_\infty &\leqslant \sup_{x\in M} \int_M \big|f(x) - f(y)\big| p_t(x, y)\, d\mu(y)\\ &\leqslant C\big\||\nabla f|\big\|_\infty \sup_{x\in M} \frac{1}{\mu(B(x,\sqrt{t}))} \int_M d(x, y) e^{-c\frac{d^2(x,y)}{t}}\, d\mu(y)\\ &\leqslant C\sqrt{t}\big\||\nabla f|\big\|_\infty \sup_{x\in M} \frac{1}{\mu(B(x,\sqrt{t}))} \int_M e^{-c'\frac{d^2(x,y)}{t}}\, d\mu(y)\\ &\leqslant C\sqrt{t}\big\||\nabla f|\big\|_\infty \sup_{x\in M} \frac{1}{\mu(B(x,\sqrt{t}))} \mu\big(B(x,\sqrt{t})\big)\\ &= C\sqrt{t}\big\||\nabla f|\big\|_\infty\end{aligned}$$

where the last estimate is a straightforward consequence of (D).

Remark 3.1. Under the hypotheses of Corollary 1.4, Theorem 1.6 and Theorem 1.9 also hold.

- Finally for Corollary 1.5, we know that (G_2) always holds on M. Then (P'_2) holds by Lemma 2.14. Moreover (D) and (P_2) yield (P'_∞) as we have just seen in the previous point.

3.1. The classical Sobolev inequality

Proposition 3.2. *Consider a complete non-compact Riemannian manifold satisfying the hypotheses of Theorem* 1.1 *and assume that* $1 \leqslant q < \nu$ *with* $\nu > 0$. *From* (1.2) *and under the heat kernel bound* $\|P_t\|_{q\to\infty} \leqslant Ct^{-\frac{\nu}{2q}}$, *one recovers the classical Sobolev inequality*

$$\|f\|_{q^*} \leqslant C\big\||\nabla f|\big\|_q$$

with $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{\nu}$. *Consequently, we get*

$$\|f\|_{p^*} \leqslant C\big\||\nabla f|\big\|_p$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$ *for* $q \leqslant p < \nu$.

Proof. Recall that $\|f\|_{B^\alpha_{\infty,\infty}} \sim \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty$. The pseudo-Poincaré inequality (P'_q), (1.2) and the heat kernel bound $\|P_t\|_{q\to\infty} \leqslant Ct^{-\frac{\nu}{2q}}$ yield

$$\|f\|_{q^*} \leqslant C\big\||\nabla f|\big\|_q^\theta \Big(\sup_{t>0} t^{-\frac{1}{2}} \|f - P_t f\|_q\Big)^{1-\theta} \leqslant C\big\||\nabla f|\big\|_q.$$

Thus we get (1.3) with $p = q < \nu$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{\nu}$. □

3.2. Sobolev inequalities for Lorentz spaces

For $1 \leqslant p \leqslant \infty$, $0 \leqslant r < \infty$ we note $L(p,r)$ the Lorentz space of functions f such that

$$\|f\|_{L(p,r)} = \left(\int_0^\infty \big(f^{**}(t) t^{\frac{1}{p}}\big)^r \frac{dt}{t}\right)^{\frac{1}{r}} < \infty$$

and

$$\|f\|_{L(p,\infty)} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty.$$

Consider a complete non-compact Riemannian manifold M satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. Moreover, assume that the following global growth condition

$$\mu(B) \geqslant Cr^\sigma \tag{3.1}$$

holds for every ball $B \subset M$ of radius $r > 0$ and for some $\sigma > q$ (remark that $\sigma \geqslant n$). Using Remark 4 in [10], we get

$$f^{**}(t) - f^*(t) \leqslant Ct^{\frac{1}{\sigma}} |\nabla f|^{q**\frac{1}{q}}(t) \tag{3.2}$$

for every $f \in \dot{E}^1_q$. We can write (3.2) as

$$f^{**}(t) - f^*(t) \leqslant \left[Ct^{\frac{1}{\sigma}} |\nabla f|^{q**\frac{1}{q}}(t)\right]^{1-\theta} \left(f^{**}(t) - f^*(t)\right)^\theta, \quad 0 \leqslant \theta \leqslant 1. \tag{3.3}$$

Take $\frac{1}{r} = \frac{1-\theta}{p*} + \frac{\theta}{l}$, $\frac{1}{m} = \frac{1-\theta}{m_0} + \frac{\theta}{m_1}$ with $0 \leqslant \theta \leqslant 1$, $\sigma \geqslant p > q$, $m_0 \geqslant q$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\sigma}$. Then from (3.3) and Hölder's inequality, we obtain the following Gagliardo–Nirenberg inequality for Lorentz spaces

$$\|f\|_{L(r,m)} \leqslant C \big\| |\nabla f| \big\|^{1-\theta}_{L(p,m_0)} \|f\|^\theta_{L(l,m_1)}. \tag{3.4}$$

We used also the fact that for $1 < p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$

$$\|f\|_{L(p,r)} \sim \left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^r \frac{dt}{t}\right]^{\frac{1}{r}}$$

to obtain the term $\||\nabla f|\|_{L(p,m_0)}$ (see [18, Chapter 5, Theorem 3.21]).

If we take $\theta = 0$ and $m_0 = m = p$, $r = p^*$, (3.4) becomes

$$\|f\|_{L(p^*,p)} \leqslant C \big\| |\nabla f| \big\|_p. \tag{3.5}$$

Noting that $p^* > p$, hence $\|f\|_{L(p^*,p^*)} \leqslant C\|f\|_{L(p^*,p)}$, (3.5) yields (1.3) with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\sigma}$ and $q < p \leqslant \sigma$. Using Theorem 2.10, we get (1.3) for every $q_0 < p \leqslant \sigma$ where $q_0 = \inf\{q \in [1,\infty[; (P_q) \text{ holds}\}$. If $q_0 = 1$, we allow $p = 1$.

Remark 3.3. 1. As we mentioned in the introduction, a Lie group of polynomial growth satisfies (D), (P_1). Moreover, for $n \in [d, D]$ we have $\mu(B) \geqslant cr^n$ for any ball B of radius $r > 0$, d being the local dimension and D the dimension at infinity. Therefore this subsection applies on such a group.

2. It has been proven [17] that under (D), (P''_q) and (3.1) with $\sigma > q$, the Sobolev inequality (1.3) holds for all $q \leqslant p < \sigma$. Since (D) and (P_q) yield (P''_q), we recover this result under our hypotheses. Besides, we are able to treat the limiting case $p = \sigma$.

4. Proof of Theorems 1.1, 1.6, 1.8 and 1.9

The main tool to prove Theorems 1.1 and 1.6 is the following characterization of a variant of the K-functional of real interpolation for the homogeneous Sobolev norm.

Theorem 4.1. *Let M be a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. For $f \in W^1_q + W^1_\infty$, consider the functional of interpolation K' defined as follows*:

$$K'(f,t) = K'\left(f,t,\dot{W}^1_q,\dot{W}^1_\infty\right) = \inf_{\substack{f=h+g \\ h\in W^1_q,\, g\in W^1_\infty}} \left(\big\||\nabla h|\big\|_q + t\big\||\nabla g|\big\|_\infty\right).$$

Let $f \in W^1_q + W^1_\infty$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. We have

$$K'\left(f,t^{\frac{1}{q}}\right) \sim t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t) \tag{4.1}$$

where the implicit constants do not depend on f and t.

Proof. Obviously

$$t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t) \leqslant K'\left(f,t^{\frac{1}{q}}\right)$$

for all $f \in W_q^1 + W_\infty^1$. We used the fact that $K(g, t^{\frac{1}{q}}, L_q, L_\infty) \sim t^{\frac{1}{q}} |g|^{q**\frac{1}{q}}(t)$ for every g and every $t > 0$. For the converse estimation, we distinguish three cases:

1. Let $f \in C_0^\infty$. For $t > 0$, we consider the Calderón–Zygmund decomposition given by Proposition 5.5 in [3] with $\alpha(t) = (\mathcal{M}(|\nabla f|^q))^{*\frac{1}{q}}(t) \sim (|\nabla f|^{q**})^{\frac{1}{q}}(t)$. We can write then $f = b + g$ with $\||\nabla b|\|_q \leqslant C\alpha(t) t^{\frac{1}{q}}$ and g Lipschitz with $\||\nabla g|\|_\infty \leqslant C\alpha(t)$ (see also the proof of Theorem 1.4 in [3]). Notice that since $f \in C_0^\infty$ one has in addition $b \in L_q$ and $g \in L_\infty$. Consequently, $b \in W_q^1$ and g is in W_∞^1. Therefore, we get (4.1).

2. Let $f \in W_q^1$. There exists a sequence $(f_n)_n$ such that for all n, $f_n \in C_0^\infty$ and $\|f - f_n\|_{W_q^1} \to 0$. Since $|\nabla f_n|^q \to |\nabla f|^q$ in L_1, it follows that $|\nabla f_n|^{q**}(t) \to |\nabla f|^{q**}(t)$ for all $t > 0$. We have seen in item 1. that for every n there is $g_n \in W_\infty^1$ such that $\||\nabla(f_n - g_n)|\|_q + t^{\frac{1}{q}} \||\nabla g_n|\|_\infty \leqslant C t^{\frac{1}{q}} (|\nabla f_n|^{q**})^{\frac{1}{q}}(t)$. Then

$$\big\||\nabla(f - g_n)|\big\|_q + t^{\frac{1}{q}} \big\||\nabla g_n|\big\|_\infty \leqslant \big\||\nabla(f - f_n)|\big\|_q + \big(\big\||\nabla(f_n - g_n)|\big\|_q + t^{\frac{1}{q}} \big\||\nabla g_n|\big\|_\infty\big)$$
$$\leqslant \epsilon_n + C t^{\frac{1}{q}} \big(|\nabla f_n|^{q**}\big)^{\frac{1}{q}}(t)$$

where $\epsilon_n \to 0$ when $n \to \infty$. We let $n \to \infty$ to obtain (4.1).

3. Let $f \in W_q^1 + W_\infty^1$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. Fix $t > 0$ and $p_0 \in M$. Consider $\varphi \in C_0^\infty(\mathbb{R})$ satisfying $\varphi \geqslant 0$, $\varphi(\alpha) = 1$ if $\alpha < 1$ and $\varphi(\alpha) = 0$ if $\alpha > 2$. Then put $f_n(x) = f(x)\varphi(\frac{d(x,p_0)}{n})$. Elementary calculations establish that f_n lies in W_q^1, hence $K'(f_n, t^{\frac{1}{q}}) \leqslant C t^{\frac{1}{q}} |\nabla f_n|^{q**\frac{1}{q}}(t)$. It is shown in [3] that

$$K\big(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1\big) \sim \left(\int_0^t |f|^{q*}(s)\,ds\right)^{\frac{1}{q}} + \left(\int_0^t |\nabla f|^{q*}(s)\,ds\right)^{\frac{1}{q}}.$$

We recall that $K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) := \inf_{f=h+g,\, h \in W_q^1,\, g \in W_\infty^1} \big(\|h\|_{W_q^1} + t\|g\|_{W_\infty^1}\big)$. All these ingredients yield

$$\begin{aligned} K'\big(f, t^{\frac{1}{q}}\big) &\leqslant K'\big(f - f_n, t^{\frac{1}{q}}\big) + K'\big(f_n, t^{\frac{1}{q}}\big) \\ &\leqslant K\big(f - f_n, t, W_q^1, W_\infty^1\big) + K'(f_n, t) \\ &\leqslant C\left(\int_0^t |f - f_n|^{q*}(s)\,ds\right)^{\frac{1}{q}} + C\left(\int_0^t |\nabla f - \nabla f_n|^{q*}(s)\,ds\right)^{\frac{1}{q}} + C\left(\int_0^t |\nabla f_n|^{q*}(s)\,ds\right)^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

Now we invoke the following theorem from [13, pp. 67–68] stated there in the Euclidean case. As the proof is the same, we state it in the more general case.

Theorem 4.2. *Let M be a measured space. Consider a sequence of measurable functions $(\psi_n)_n$ and g on M such that $\mu\{|g| > \lambda\} < \infty$ for all $\lambda > 0$ with $|\psi_n(x)| \leqslant |g(x)|$. If $\psi_n(x) \to \psi(x)$ μ-a.e. then $(\psi - \psi_n)^*(t) \to 0$ $\forall t > 0$.*

We apply this theorem three times:

(a) With $\psi_n = |f - f_n|^q$, $\psi = 0$ and $g = 2^q f^q$. Using the Lebesgue dominated convergence theorem we obtain $\int_0^t |f - f_n|^{q*}(s)\,ds \to 0$.

(b) With $\psi_n = |\nabla f - \nabla f_n|^q$, $\psi = 0$ and $g = C(|\nabla f|^q + |f|^q)$, where C only depends on q, since

$$\nabla f_n = \nabla f \mathbb{1}_{B(p_0,n)} + \left(\frac{1}{n} f \varphi'\left(\frac{d(x,p_0)}{n}\right) \nabla\big(d(x,p_0)\big) + \nabla f \varphi\left(\frac{d(x,p_0)}{n}\right)\right) \mathbb{1}_{B(p_0,n)^c}.$$

So again by the Lebesgue dominated convergence theorem we get $\int_0^t |\nabla f - \nabla f_n|^{q*}(s)\,ds \to 0$.

(c) With $\psi_n = |\nabla f_n|^q$, $\psi = |\nabla f|^q$ and $g = C(|\nabla f|^q + |f|^q)$, C only depending on q, so we get $\int_0^t |\nabla f_n|^{q*}(s)\,ds \to \int_0^t |\nabla f|^{q*}(s)\,ds$.

Passing to the limit in (4.2) yields $K'(f, t^{\frac{1}{q}}) \leqslant C t^{\frac{1}{q}} |\nabla f|^{q**\frac{1}{q}}(t)$ and finishes the proof. □

Proof of Theorem 1.1. Let $t > 0$, $f \in W_q^1 + W_\infty^1$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. As in [15], proof of Theorem 1.1(i), it is enough to prove that

$$|f - P_t f|^{q**\frac{1}{q}}(s) \leqslant C t^{\frac{1}{2}} |\nabla f|^{q**\frac{1}{q}}(s). \tag{4.3}$$

The main tool will be the pseudo-Poincaré inequalities (P'_q), (P'_∞) and Theorem 4.1.

Let $f \in W_q^1 + W_\infty^1$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. Assume that $f = h + g$ with $h \in W_q^1$, $g \in W_\infty^1$. We write

$$f - P_t f = (h - P_t h) + (g - P_t g).$$

Let $s > 0$. The pseudo-Poincaré inequalities (P'_q) and (P'_∞) yield

$$\|h - P_t h\|_q + s^{\frac{1}{q}} \|g - P_t g\|_\infty \leqslant C t^{\frac{1}{2}} \left(\big\| |\nabla h| \big\|_q + s^{\frac{1}{q}} \big\| |\nabla g| \big\|_\infty \right).$$

Since

$$K\left(f, s^{\frac{1}{q}}, L_q, L_\infty\right) \sim \left(\int_0^s \left(f^*(u)\right)^q du \right)^{\frac{1}{q}} = s^{\frac{1}{q}} |f|^{q**\frac{1}{q}}(s)$$

we obtain

$$\begin{aligned}
s^{\frac{1}{q}} |f - P_t f|^{q**\frac{1}{q}}(s) &\sim \inf_{\substack{f - P_t f = h' + g' \\ h' \in L_q,\, g' \in L_\infty}} \left(\|h'\|_q + s^{\frac{1}{q}} \|g'\|_\infty \right) \\
&\leqslant \inf_{\substack{f = h + g \\ h \in W_q^1,\, g \in W_\infty^1}} \left(\|h - P_t h\|_q + s^{\frac{1}{q}} \|g - P_t g\|_\infty \right) \\
&\leqslant C t^{\frac{1}{2}} \inf_{\substack{f = h + g \\ h \in W_q^1,\, g \in W_\infty^1}} \left(\big\| |\nabla h| \big\|_q + s^{\frac{1}{q}} \big\| |\nabla g| \big\|_\infty \right) \\
&= C t^{\frac{1}{2}} K'\left(f, s^{\frac{1}{q}}\right).
\end{aligned}$$

Applying Theorem 4.1, we obtain the desired inequality (4.3). □

Proof of Theorem 1.6. The proof of this theorem is similar to that of Theorem 1.1. Here the key ingredients are the pseudo-Poincaré inequality for averages (P''_q) that holds for all $f \in \dot{E}_q^1$. This pseudo-Poincaré inequality follows from (D) and the Poincaré inequality (P_q). We also make use of Theorem 4.1. □

Proof of Theorem 1.8. The proof goes as in [15], proof of Theorem 1.1(iii). □

Proof of Theorem 1.9. Same proof as that of Theorem 4 in [15] noting that $|g|^* = |g|^{q*\frac{1}{q}}$ and $|g|^* \leqslant |g|^{**}$. □

Remark 4.3. Let M be a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leqslant q < \infty$. Then Theorem 1.9 holds replacing the Besov norm $B_{\infty,\infty}^{-1}$ by the Morrey norm M_∞^{-1}. This can be proved using Theorem 1.6.

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