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When is a smash product semiprime? A partial answer ☆

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Abstract

It is an open question whether the smash product of a semisimple Hopf algebra and a semiprime module algebra is semiprime. In this paper we show that the smash product of a commutative semiprime module algebra over a semisimple cosemisimple Hopf algebra is semiprime. In particular we show that the central H-invariant elements of the Martindale ring of quotients of a module algebra form a von Neumann regular and self-injective ring whenever A is semiprime. For a semiprime Goldie PI H-module algebra A with central invariants we show that $A\#H$ is semiprime if and only if the H-action can be extended to the classical ring of quotients of A if and only if every non-trivial H-stable ideal of A contains a non-zero H-invariant element. In the last section we show that the class of strongly semisimple Hopf algebras is closed under taking Drinfeld twists. Applying some recent results of Etingof and Gelaki we conclude that every semisimple cosemisimple triangular Hopf algebra over a field is strongly semisimple.

1. Introduction

It is an important open question in the theory of Hopf algebra actions whether the smash product $A\#H$ of a semisimple Hopf H and a semiprime left H-module algebra A, is semiprime (see [17, Question 4.4.7]).

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Fisher and Montgomery had proved an analogous result for group rings (see [11]) and Cohen and Montgomery for duals of group rings (see [6]). Attempts had been made to tackle this question often by restricting the class of Hopf algebras (see for example [18]).

In order to give a partial answer to the semiprimness question, we will restrict the class of module algebras rather than the class of Hopf algebras. In particular we will show that the question has a positive answer for commutative module algebras in characteristic 0. The main step is to show that the subring of central H-invariant elements of the Martindale ring of quotients is von Neumann regular. The result follows applying a theorem of S. Zhu which says that a commutative module algebra is an integral extension of its invariants if the Hopf algebra involved is semisimple and cosemisimple.

In general one might ask what are necessary or sufficient conditions for a smash product to be semiprime. A very important necessary condition is the existence of non-trivial H-invariant elements in non-zero H-stable ideals of the module algebra. A sufficient condition is the ability of extending the H-action on a semiprime Goldie module algebra to its classical ring of quotients. We will see in Theorem 4.4 that for semiprime Goldie PI module algebras with central invariants those conditions are equivalent to the smash product being semiprime. In the final section we show that the class of strongly semisimple Hopf algebras is closed under Drinfeld twists. Applying finally a recent result of Etingof and Gelaki, we can also conclude that triangular semisimple cosemisimple Hopf algebras are strongly semisimple and satisfy the property that their smash product with a semiprime module algebra is semiprime.

All rings are supposed to be associative and have a unit element unless otherwise stated. Throughout the text R will denote a commutative ring, H a Hopf algebra over R with antipode S, counit ε and comultiplication Δ. We will make use of the so-called Sweedler-notation $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$ for the comultiplication of an $h \in H$. A left H-module algebra A is an R-algebra in the category of left H-modules. The smash product of A and H is an R-algebra with underlying R-module $A \otimes_R H$ and denoted by $A \# H$. The multiplication of two elements $a \# h$ and $b \# g$ in $A \# H$ is defined to be equal to

$$(a \# h)(b \# g) := \sum_{(h)} a(h_1 b) \# h_2 g.$$

We emphasis that A is a cyclic left $A \# H$-module and $\mathrm{End}_{A\#H}(A) \simeq A^H$. This allows to study A, A^H and $A \# H$ in module-theoretic terms.

We refer to all unexplained Hopf-algebraic terms to [17] and [20], to all ring-theoretic terms to [13] and to all module-theoretic terms to [23].

2. Separability of smash products

Many results on group actions are stated in terms of algebras over rings rather than in terms of algebras over fields. Throughout the paper we will consider Hopf algebras over a commutative ring R. Just when applying deeper results on Hopf algebras over fields we will assume that R is a field. In the case of a base ring R the adequate analogue of a semisimple Hopf algebra (over a field) is a Hopf algebra that is separable over R.

2.1. We will shortly recall the definition of separability in non-commutative ring extensions (see [12]).

Definition. Let $S \subseteq T$ be any ring extension. T is called *separable* over S if there exists an idempotent

$$\omega := \sum_{i=1}^{n} x_i \otimes y_i \in T \otimes_S T \quad \text{such that} \quad \sum_{i=1}^{n} x_i y_i = 1 \quad \text{and} \quad t\omega = \omega t$$

holds for all $t \in T$. We refer to ω as the *separability idempotent* of T over S.

Here we consider $T \otimes_S T$ as a $T - T$-bimodule via $t(x \otimes y) = tx \otimes y$ and $(x \otimes y)t = x \otimes yt$ for all $t \in T$ and $x \otimes y \in T \otimes_S T$.

2.2. Separable extensions are in particular semisimple extensions (see [12]). An extension $S \subseteq T$ is called semisimple if every exact sequence of left T-modules, which splits as a sequence of left S-modules, splits. Hence if H is a Hopf algebra over some field k such that $k \subseteq H$ is separable, H must be a semisimple ring. (Note that 'semisimple ring' shall always mean 'semisimple artinian ring'.) We will see soon that the converse is true as well.

2.3. Recall the submodule of left integrals in a Hopf algebra H:

$$\int_l := \{ t \in H \mid \forall h \in H\colon\ ht = \varepsilon(h)t \}.$$

Right integrals are defined analogously. It is known that $\int_l \neq 0$ in case H is finitely generated and projective as R-module (see [19]). The author was unable to find a reference for the following (maybe known) result which gives a criterium for $A \# H$ to be separable over A. Note that no hypothesis on H as a module over R is needed.

Proposition. *Let H be a Hopf algebra over R and let A be a left H-module algebra. Assume that there exists a left or right integral t in H with $\varepsilon(t)1_A$ invertible in A. Then $A \# H$ is separable over A.*

Proof. Let t be a right integral in H such that $\varepsilon(t)1_A$ is invertible in A and let $z \in A$ be its inverse. A straightforward calculation shows that z is a central H-invariant element of A. As in [5, Theorem 1.11] one shows that

$$\omega := \sum_{(t)} \big[1 \# S(t_1)\big] \otimes [z \# t_2] \in A \# H \otimes_A A \# H$$

is a separability idempotent for $A \# H$.

For a left integral t with $\varepsilon(t)$ invertible in A we set $t' := S(t)$. Since t' is a right integral and $\varepsilon(t') = \varepsilon(t)$ we can argue as above and conclude that $A \# H$ is separable over A. □

2.4. Letting H act trivially on R by setting $hr := \varepsilon(h)r$ for all $h \in H$ and $r \in R$, R becomes a left H-module algebra and $R \# H \simeq H$. Proposition 2.3 shows that H is separable over R if and only if there exists a left or right integral t in H with $\varepsilon(t)$ invertible in R. While the sufficiency follows from the proposition, the necessity follows because if H is separable over R the H-linear map $\varepsilon : H \to R$ splits as right H-modules. Hence there exists an H-linear $\sigma : R \to H$ such that $\varepsilon(\sigma(1)) = 1$. The element $t := \sigma(1)$ is our right integral.

In particular $A \# H$ is separable over A for every left H-module algebra A whenever H is separable over R Thus for instance whenever H is a semisimple Hopf algebra over a field.

Note that this fact holds without assuming any additional hypothesis on H as a module over R. On the other hand it is well known that a separable R-algebra H must be finitely generated as R-module if H is projective as R-module.

2.5. In case of a group ring $H = R[G]$ with G a finite group. The submodule of left and right integrals $\int_l$ is spanned by the element $t := \sum_{g \in G} g$. For an R-algebra A where G acts on, $A \# G$ is equal to the skew group ring of A and G. Proposition 2.3 says that $A \# G$ is separable over A provided $\varepsilon(t) = |G|$ is invertible in A.

2.6. Since separable extensions $S \subseteq T$ are semisimple extensions, every left T-module that is projective as left S-module is also projective as left T-module (see [24, 28.5]). In particular any separable extension of a semisimple artinian ring is itself semisimple artinian. Our next lemma shows that the analogue statement for flat modules and von Neumann regular rings is also true by [24, 20.12]:

Lemma. *Suppose T is separable over a subring S. Then every left T-module that is flat as left S-module is also flat as left T-module.*

Hence a separable extension of a von Neumann regular ring is itself von Neumann regular.

2.7. Combining Lemma 2.6 and Proposition 2.3 we get the following important Corollary which generalises a result of Cohen and Fischman that says that $A \# H$ is semisimple whenever A and H are semisimple (see [4, Theorem 6]).

Corollary. *Let H be an R-Hopf algebra and A a left H-module algebra, such that there exists a left or right integral t in H with $\varepsilon(t)$ invertible in A. If A is von Neumann regular, then $A \# H$ is von Neumann regular. If A is semisimple artinian, then $A \# H$ is semisimple artinian.*

2.8. A first application of the corollary above will allow us to show that whenever the H-action can be extended to the left maximal ring of quotients $Q^l_{\max}(A)$ of a left non-singular H-module algebra A the smash product $A \# H$ must also be left non-singular and moreover its left maximal ring of quotients is isomorphic to $Q^l_{\max}(A) \# H$.

Recall the definition of the left maximal ring of quotients. Let S be any ring and denote by $E(S)$ its injective hull in S-Mod. Define the left maximal ring of quotients of S as the S-submodule

$$Q^l_{\max}(S) := \{m \in E(S) \mid \forall f \in \mathrm{End}_S((S)):\ f(S) = 0 \Rightarrow f(m) = 0\}$$

of $E(S)$. Let $B := \mathrm{End}_{\mathrm{End}_S(E(S))}(E(S))$ be the biendomorphism ring of $E(S)$. The evaluation map $\Psi : B \to Q^l_{\max}(S)$ with $\varphi \mapsto \varphi(1)$ is an isomorphism of abelian groups and induces a ring structure on $Q^l_{\max}(S)$. Hence one might identify $Q^l_{\max}(S)$ with the biendomorphism ring of the injective hull of S.

2.9. Recall that a submodule N of a module M is called *dense* whenever

$$\mathrm{Hom}(L/N, M) = 0 \quad \text{for all } N \subseteq L \subseteq M.$$

$Q^l_{\max}(S)$ can also be seen as the maximal extension E of S such that S is dense in E.

Lemma. *Let $S \subseteq T$ be a ring extension such that $\mathrm{Hom}_S(T/S, T) = 0$ and ${}_ST$ is injective. Then $T \simeq Q^l_{\max}(S)$ as rings.*

Proof. Let L be an S-submodule of T containing S. By injectivity of T, every homomorphism $f : L/S \to T$ can be extended to an homomorphism $\bar{f} : T/S \to T$ which is zero by hypothesis. Thus S is dense in T. By [13, 13.11] there exists an injective ring homomorphism $g : T \hookrightarrow Q^l_{\max}(S)$ such that $g(s) = s$ for all $s \in S$. Hence g is left S-linear and by injectivity of T, $\mathrm{Im}(g)$ is a direct summand of $Q^l_{\max}(S)$ containing the essential submodule S. Thus g must be surjective and must be an isomorphism of rings. □

2.10. In the following theorem we will apply Corollary 2.7 and Lemma 2.9 to show that $Q^l_{\max}(A \# H) \simeq Q^l_{\max}(A) \# H$ is von Neumann regular. Using Johnson's Theorem that states that a ring S is left non-singular if and only if its left maximal ring of quotients $Q^l_{\max}(S)$ is von Neumann regular we can conclude that $A \# H$ is left non-singular.

Theorem. *Let H be a Hopf algebra over R with H_R finitely generated and projective. Let A be a left H-module algebra, such that there exists a left or right integral $t \in H$ with $\varepsilon(t)1_A$ invertible in A. Assume that the H-action extends to the left maximal ring of quotients $Q^l_{\max}(A)$. If A is left non-singular, then $A \# H$ is left non-singular and $Q^l_{\max}(A \# H) \simeq Q^l_{\max}(A) \# H$.*

Proof. By hypothesis A is left non-singular. Hence by Johnson's Theorem the maximal ring of quotients $Q := Q^l_{\max}(A)$ of A is von Neumann regular and equals $E(A)$ the injective hull of A. In particular Q is injective as A-module. The invertibility of $\varepsilon(t)$ in A (and hence in Q) implies the separability of $Q \# H$ over Q by Proposition 2.3. From Corollary 2.7 we know that $Q \# H$ is von Neumann regular. Applying the exact functor $- \otimes_R H$ to the exact sequence

$$0 \to A \to Q \to Q/A \to 0$$

we get $Q\#H/A\#H \simeq (Q/A)\otimes_R H$ as left A-modules. Since ${}_RH$ is a direct summand of a free module R^k with $k \geqslant 1$ and since A is dense in Q, we get:

$$\begin{aligned}\operatorname{Hom}_{A\#H}(Q\#H/A\#H, Q\#H) &\subseteq \operatorname{Hom}_{A-}\big((Q/A)\otimes_R H, Q\otimes_R H\big)\\ &\subseteq \operatorname{Hom}_{A-}\big((Q/A)^k, Q^k\big) = 0.\end{aligned}$$

Hence $\operatorname{Hom}_{A\#H}(Q\#H/A\#H, Q\#H) = 0$. Since $A\#H$ is separable over A, we can also conclude that $Q\#H$ is an injective left $A\#H$-module, as Q and $Q\#H$ are injective left A-modules. By Lemma 2.9, $Q^l_{\max}(A\#H) \simeq Q\#H$ and by Johnson's Theorem (see [13, 13.36]) $A\#H$ is left non-singular. □

2.11. The question whether the H-action of a semisimple Hopf algebra can be extended to the maximal ring of quotients of a module algebra is still open. A claim that this is always possible was made in [22] but its proof is not complete as was confirmed by the author of [22].

3. Commutative semiprime module algebras

Consider the subring $M_H(A)$ of $\operatorname{End}_R(A)$ generated by the H-action on A and by the left and right multiplications of elements of A:

$$M_H(A) := \big\langle \{L_a, R_a, L_h \mid a \in A,\ h \in H\}\big\rangle \subseteq \operatorname{End}_R(A),$$

where L_a and R_a denotes the left and right multiplication with $a \in A$, respectively, and L_h denotes the H-action of the element h on A. A is a cyclic faithful $M_H(A)$-module whose submodules are precisely the H-stable two-sided ideals of A. If A is commutative then $M_H(A) \simeq A\#H/\operatorname{Ann}_{A\#H}(A)$.

3.1. A module algebra A is called *H-semiprime* if A does not contain any non-trivial nilpotent H-stable ideals.

Lemma. *The following statements are equivalent for an H-stable ideal I of an H-semiprime module algebra A.*

(a) $\operatorname{l.ann}_A(I) = 0$;
(b) *I is an essential $M_H(A)$-submodule of A*;
(c) *I is a dense $M_H(A)$-submodule of A.*

Proof. (a) ⇒ (b). Let J be an H-stable ideal of A. Since the left annihilator of I is zero, $J\cap I \supseteq JI \neq 0$ shows that I is essential $M_H(A)$-submodule of A.

(b) ⇒ (c). Let J be an H-stable ideal of A containing I and let $f : J \to A$ be $M_H(A)$-linear such that $I \subseteq \operatorname{Ker}(f)$. Then $K := f(J)\cap I$ is nilpotent since

$$K^2 \subseteq f(J)I = f(I) = 0.$$

As A is H-semiprime $K = 0$ and as I is essential $f = 0$. Hence $\mathrm{Hom}_{M_H(A)}(J/I, A) = 0$ shows that I is dense in A.

(c) $\Rightarrow$ (a). Let J denote the left annihilator of I. Since for all $h \in H, x \in J$ and $y \in I$ we have:

$$(hx)y = \sum_{(h)} h_1\big(x\big(S(h_2)y\big)\big) = 0,$$

J is an H-stable ideal of A. Since A is H-semiprime, $I \cap J = 0$. Let $\pi : J \oplus I \to J$ be the projection, then $\pi \in \mathrm{Hom}_{M_H(A)}((J \oplus I)/I, A) = 0$. Hence I has zero left annihilator. □

3.2. Recall that a module N is called M-generated if it is an epimorphic image of a direct sum of copies of M. The self-injective hull $\widehat{M}$ of a module M is the largest M-generated submodule of its injective hull $E(M)$. The endomorphism of the self-injective hull of a module whose essential submodules are dense is known to be von Neumann regular and self injective (see [24, 11.2]). Applying this module-theoretic fact to our situation Lemma 3.1 shows that the endomorphism ring T of the self-injective hull $\widehat{A}$ of A as $M_H(A)$-module is von Neumann regular and self-injective. We will construct an isomorphism between T and the subring of central H-invariant elements of the Martindale ring of quotients of A.

3.3. Let $\mathcal{F}$ denote the set of ideals of A with zero left and right annihilator. The *right Martindale ring of quotients* of A is

$$Q(A) := \varinjlim\big\{\mathrm{Hom}_{-A}(I, A) \mid I \in \mathcal{F}\big\}.$$

Alternatively one might construct $Q(A)$ as follows: define an equivalence relation on $\bigcup_{I\in\mathcal{F}} \mathrm{Hom}_{-A}(I, A)$ by letting $f : I \to A$ to be equivalent to $g : J \to A$ if there exists a $K \in \mathcal{F}$ such that $K \subseteq I \cap J$ and $f_{|K} = g_{|K}$. Note that the equivalence class of the zero map contains all maps f that vanish on some ideal in $\mathcal{F}$. Addition is defined by $[f] + [g] := [f + g : I \cap J \to A]$ while multiplication is set to be $[f][g] := [fg : JI \to A]$ where fg denotes the composition map $a \mapsto f(g(a))$.

In order to extend the H-action on A to some subring of $Q(A)$, Miriam Cohen considered the subset $\mathcal{F}_H$ of H-stable ideals belonging to $\mathcal{F}$ and constructed the following ring:

$$Q_0(A) := \varinjlim\big\{\mathrm{Hom}_{-A}(I, A) \mid I \in \mathcal{F}_H\big\}.$$

We will refer to the elements of $Q_0(A)$ as equivalence classes in the above sense. Moreover $Q_0(A)$ is a subring of $Q(A)$. The H-action on A extends to $Q_0(A)$ by letting an element $h \in H$ act on $f : I \to A$ by $(h \cdot f) : I \to A$ with

$$(h \cdot f)(x) := \sum_{(h)} h_1 f\big(S(h_2)x\big) \quad \text{for all } x \in I.$$

One checks as in [3, Theorem 18] that $Q_0(A)$ becomes a left H-module algebra with this action.

3.4. It is well know that the extended centroid of a semiprime algebra is a von Neumann regular self-injective ring (see [23, 32.1(2)]). We are now in position to show that the subring of central H-invariant elements $Z(Q_0)^H := Z(Q_0) \cap Q_0^H$ of the right Martindale ring of quotients of a semiprime module algebra is von Neumann regular and self-injective.

Proposition. *Let H be a Hopf algebra over R and let A be a left H-semiprime module algebra with right Martindale ring of quotients Q_0. Let T be the endomorphism ring of the self-injective hull $\widehat{A}$ of A as $M_H(A)$-module. Assume that*

- *A is commutative, or*
- *A is semiprime, or*
- *H has a bijective antipode.*

Then

$$\psi : T \to Z(Q_0)^H \quad \text{with } f \mapsto [f : I_f \to A]$$

is a ring isomorphism where $I_f := f^{-1}(A) \cap A$. Moreover $Z(Q_0)^H$ is a von Neumann regular self-injective ring.

Proof. Let $\widehat{A}$ denote the self-injective hull of A as $M_H(A)$-module and let T denote the endomorphism ring of $\widehat{A}$ as $M_H(A)$-module. For each endomorphism $f \in T$ define $I_f := f^{-1}(A) \cap A$. Since pre-images of essential submodules are essential, I_f is an essential $M_H(A)$-submodule of A. By Lemma 3.1, I_f has zero left annihilator. If A is commutative or semiprime I_f has also zero right annihilator and belongs to $\mathcal{F}_H$. If the antipode of H is bijective then the right annihilator J of I_f is also an H-stable ideal since for all $h \in H$, $x \in J$ and $y \in I_f$ we have:

$$x(hy) = \sum_{(h)} h_2\big(\big(S^{-1}(h_1)x\big)y\big) = 0.$$

As $I_f \cap J$ is a nilpotent H-stable ideal and as A is H-semiprime $I_f \cap J$ must be equal to the zero submodule. I_f being an essential $M_H(A)$-submodule implies that J is zero. Thus also in this case I_f belongs to $\mathcal{F}_H$.

We will show that ψ is a ring homomorphism. Let $f, g \in T$. Note that $I_f I_g \in \mathcal{F}_H$ and $I_f I_g \subseteq I_{fg}$. Thus

$$\psi(f)\psi(g) = [f : I_f \to A][g : I_g \to A] = [fg : I_f I_g \to A] = [fg : I_{fg} \to A] = \psi(fg).$$

This shows that ψ is a ring homomorphism. Assume $\psi(f) = 0$ for some $f \in T$. Then there exists an $J \in \mathcal{F}_H$ with $J \subseteq I_f$ and $f(J) = 0$. Hence $f \in \mathrm{Hom}_{M_H(A)}(I_f/J, A) = 0$ as J is dense by Lemma 3.1. This shows that ψ is injective. On the other hand ψ is also

surjective. Let $[q : I \to A] \in Z(Q_0)^H$. First note that q is $M_H(A)$-linear: Let $a \in A$ then $[q][L_a] = [L_a][q]$ implies the existence of an ideal $J \in \mathcal{F}_H$ with $J \subseteq I$ and

$$q' := qL_a - L_a q \in \mathrm{Hom}_{-A}(I/J, A).$$

Since J has zero left annihilator and $q'(I/J)J = 0$ we can conclude $q' = 0$. This shows

$$q(ax) = qL_a(x) = L_a q(x) = aq(x)$$

for all $x \in I$. Hence q is a left A-linear.

Note that since $q \in Q_0^H$ for all $h \in H$: $h \cdot q = \varepsilon(h)q$. Let $h \in H$. For all $x \in I$ we have:

$$\begin{aligned} q(hx) &= \sum_{(h)} \varepsilon(h_1) q(h_2 x) = \sum_{(h)} h_1 \cdot q(h_2 x) = \sum_{(h)} h_1 q\big(S(h_2)h_3 x\big) \\ &= \left[\sum_{(h)} h_1 \varepsilon(h_2)\right] q(x) = hq(x). \end{aligned}$$

This shows the H-linearity of q. Since q is by definition right A-linear we have shown that q is an $M_H(A)$-linear map.

By injectivity of $\widehat{A}$, $q : I \to A$ can be extended to an $M_H(A)$-linear map $\bar{q} \in T$. This extension is unique since $\mathrm{Hom}_{M_H(A)}(\widehat{A}/I, \widehat{A}) = 0$. Moreover $\psi(\bar{q}) = [q]$ as $I \subseteq I_{\bar{q}}$ and $\bar{q}_{|I} = q$. This shows that ψ is surjective and we have established an isomorphism of rings between $Z(Q_0)^H$ and T which is von Neumann regular and self-injective by [24, 11.2]. □

3.5. Our main result follows now easily from the preceding paragraphs.

Theorem. *Let H be a Hopf algebra over R such that H_R is flat and let A be a commutative semiprime left H-module algebra. Assume that there exists a left or right integral $0 \neq t \in H$ such that $\varepsilon(t)$ is not a zero divisor in A. Then $A \# H$ is semiprime provided A is integral over A^H.*

Proof. Denote by Q_0 the right Martindale ring of quotients of the module algebra A. Assume $\varepsilon(t)$ is invertible in A. Let $\tilde{A} := \langle A, Q_0^H \rangle \subseteq Q_0$ be the subalgebra of Q_0 generated by A and Q_0^H. Obviously $\tilde{A}$ is a left H-module algebra. Since $\tilde{A}$ is a subalgebra of the right Martindale ring of quotients Q of A which is commutative and semiprime, also $\tilde{A}$ is commutative and semiprime. By hypothesis A is an integral extension of A^H. Hence $\tilde{A}$ is integral over Q_0^H. To see this note that $A^H \subseteq Q_0^H$ and let $aq \in \tilde{A}$. There exists a monic polynomial

$$f(X) = \sum_{i=0}^{n} r_i X^i \in A^H[X]$$

with $f(a) = 0$. Define the monic polynomial

$$\tilde{f}(X) := \sum_{i=0}^{n} r_i q^{n-i} X^i \in Q_0^H[X].$$

Then $\tilde{f}(aq) = f(a)q^n = 0$ shows that every element of the form aq of $\tilde{A}$ is integral over Q_0^H. Since the set of integral elements is closed under sums, we get $\tilde{A}$ is integral over Q_0^H.

By Proposition 3.4, Q_0^H is von Neumann regular. Recall that a commutative ring is von Neumann regular if and only if it is semiprime and every prime ideal is maximal. Since $Q_0^H \subseteq \tilde{A}$ is an integral extension, the height of a prime ideal P in $\tilde{A}$ is equal to the height of the prime ideal $P \cap Q_0^H$ (see for example [8, 9.2]) every prime ideal of $\tilde{A}$ is maximal and therefore $\tilde{A}$ is von Neumann regular.

Since $\varepsilon(t)1_A$ is invertible in A, it is also invertible in $\tilde{A}$. By Corollary 2.7, $\tilde{A} \# H$ is von Neumann regular. Let $I \subseteq A \# H$ be an ideal with $I^2 = 0$. Then $\tilde{I} := I(Q_0^H \# 1)$ is an ideal of $\tilde{A} \# H$. Since $Q_0^H \# 1$ is central in $\tilde{A} \# H$ we get $\tilde{I}^2 = 0$. As $\tilde{A} \# H$ is von Neumann regular, hence semiprime, we have $\tilde{I} = 0$. Since ${}_R H$ is flat, $A \# H$ is a subring of $\tilde{A} \# H$ and thus $I = 0$. This shows $A \# H$ does not contain a non-trivial nilpotent ideal and must be semiprime.

In case $\varepsilon(t)1_A$ is not invertible in A but a non-zero divisor, we can localise A by the powers of $\varepsilon(t)1_A$ and obtain a semiprime commutative module algebra $A[\varepsilon(t)^{-1}]$. Thus $A[\varepsilon(t)^{-1}] \# H = A \# H[\varepsilon(t)^{-1} \# 1]$ is semiprime and so must be also $A \# H$. □

3.6. S. Zhu showed that a commutative H-module algebra A is an integral extensions of its invariants whenever H is a finite dimensional Hopf algebra over a field k such that $\mathrm{char}(k) \nmid \dim(H)$ and $S^2 = \mathrm{id}$ (see [26, Theorem 2.1]). Etingof and Gelaki proved in [9] that a finite dimensional Hopf algebra H satisfies $\mathrm{char}(k) \nmid \dim(H)$ and $S^2 = \mathrm{id}$ if and only if H is semisimple and cosemisimple. Combining Zhu's and Etingof and Gelaki's result with Theorem 3.5 we obtain the following corollary.

Corollary. *Let H be a semisimple cosemisimple Hopf algebra over a field and let A be a commutative semiprime H-module algebra. Then $A \# H$ is semiprime.*

It is well known, that a semisimple Hopf algebra over a field of characteristic 0 is also cosemisimple.

4. Semiprime Goldie PI module algebras

Assume that the smash product $A \# H$ of a module algebra A and a semisimple Hopf algebra H is semiprime. Then every non-zero H-stable left ideal of A contains a non-zero H-invariant element. In this section we will show that this necessary condition is also a sufficient condition for semiprime Goldie PI module algebras with central invariants. More generally we will show that the H-action on such a module algebra can be extended to its

classical ring of quotients in case every non-zero H-stable left ideal contains a non-zero H-invariant element.

4.1. A module M is called *retractable* if $\mathrm{Hom}(M, N) \neq 0$ for all non-zero submodules N of M (see [25]). Recall that one has an R-linear isomorphism $I^H \simeq \mathrm{Hom}_{A\#H}(A, I)$ for all H-stable left ideals I of A. Hence the existence of non-trivial H-invariant elements in non-zero H-stable left ideals can be expressed as A being a retractable $A \# H$-module.

Lemma. *Let M be a retractable left R-module whose endomorphism ring is semisimple. Then M is a semisimple artinian R-module. If moreover R is a PI-ring, then M is finitely generated over its endomorphism ring.*

Proof. Let N be a non-zero submodule of M. By hypothesis there exists a non-trivial idempotent $e \in S := \mathrm{End}_R(M)$ such that $\mathrm{Hom}_R(M, N) = Se$. Thus $M = Me \oplus M(1-e)$ implies $N = Ne \oplus (N \cap M(1-e))$. Hence

$$\begin{aligned}\mathrm{Hom}_R\big(M, N \cap M(1-e)\big) &= \mathrm{Hom}_R(M, N) \cap \mathrm{Hom}_R\big(M, M(1-e)\big)\\ &= Se \cap S(1-e) = 0\end{aligned}$$

implies by hypothesis $N \cap M(1-e) = 0$, i.e., N is a direct summand of M. This shows that M is a semisimple R-module. As $\mathrm{End}(M)$ is artinian, M is artinian.

Write $M = \bigoplus_{i=1}^{k} E_i^{n_i}$ with pairwise non-isomorphic simple R-modules E_i and k, $n_i \geqslant 1$. Set $P_i := \mathrm{Ann}_R(E_i)$. Then $S = \bigoplus_{i=1}^{k} M_{n_i}(\Delta_i)$ where $\Delta_i = \mathrm{End}_R(E_i)$. Assume that R is a PI-ring. By Kaplansky's Theorem (see [16, 13.3.8]) there exists $m_i \geqslant 1$ such that R/P_i is isomorphic to the full matrix ring $M_{m_i}(\Delta_i)$ and E_i is a finite-dimensional Δ_i-vector space. Hence $E_i^{n_i}$ and also M are finitely generated over their endomorphism rings. □

4.2. Applying the above lemma to the module algebra situation we will see, that the H-action on a semiprime Goldie PI module algebra whose non-zero H-stable ideals contain non-zero central H-invariant elements can be extended to its ring of quotients.

Proposition. *Let H be a Hopf algebra over R with H_R finitely generated and let A be a semiprime Goldie PI H-module algebra with classical ring of quotients $Q_{\mathrm{cl}}(A)$. If every non-zero H-stable ideal of A contains a non-zero central H-invariant element, then the H-action on A can be extended to $Q_{\mathrm{cl}}(A)$ and $Q_{\mathrm{cl}}(A)$ is equal to the central localisation $A[\mathcal{C}^{-1}]$ of regular elements $\mathcal{C}$ of the subring $Z(A)^H$ of A.*

Proof. Let $Z(A)^H := Z(A) \cap A^H$ and let $\mathcal{C}$ denote the set of regular elements of $Z(A)^H$. The elements of $\mathcal{C}$ form an Ore set in A and are also regular elements of A since $\mathrm{Ann}_A(c)^H = 0$ implies $\mathrm{Ann}_A(c) = 0$ for all $c \in \mathcal{C}$. Denote by $\tilde{A} := A[\mathcal{C}^{-1}]$ the localisation of A by $\mathcal{C}$. Note that A is a subring of $\tilde{A}$ and the map $I \mapsto (I \cap A)$ from ideals of $\tilde{A}$ to ideals of A is injective. In particular $\tilde{A}$ is semiprime. Since $\tilde{A}$ is a central extension of the

PI-ring A, $\tilde{A}$ is PI by [16, 13.1.11]. By [21, 6.1.1], $\tilde{A} \otimes \tilde{A}^{\mathrm{op}}$ is a PI-ring and hence its factor ring

$$\tilde{A} \otimes \tilde{A}^{\mathrm{op}} / \operatorname{Ann}_{\tilde{A}\otimes\tilde{A}^{\mathrm{op}}}\big(\tilde{A}\big) \simeq M\big(\tilde{A}\big) := \big\langle \big\{ L_x, R_x \mid x \in \tilde{A} \big\} \big\rangle \subseteq \operatorname{End}_R\big(\tilde{A}\big)$$

is a PI-ring. The H-action on A extends trivially to $\tilde{A}$ by letting an element $h \in H$ act on an element ac^{-1} as $(h \cdot a)c^{-1}$. Since H is finitely generated as R-module, $M_H(\tilde{A})$ is a finite extension of $M(\tilde{A})$ and therefore also a PI-ring by [16, 13.4.9]. Note that

$$\operatorname{End}_{M_H(\tilde{A})}\big(\tilde{A}\big) \simeq Z\big(\tilde{A}\big)^H \simeq Z(A)^H\big[\mathcal{C}^{-1}\big] \simeq Q_{\mathrm{cl}}\big(Z(A)^H\big)$$

is semisimple artinian. Moreover let I be a non-trivial H-stable ideal of $\tilde{A}$. Then $I \cap A$ is a non-trivial H-stable ideal of A and contains a non-trivial central H-invariant element. Using the isomorphism

$$\operatorname{Hom}_{M_H(\tilde{A})}\big(\tilde{A}, I\big) \simeq I \cap Z\big(\tilde{A}\big)^H \neq 0$$

we see that $\tilde{A}$ is a retractable module over the PI-ring $M_H(\tilde{A})$ having a semisimple artinian endomorphism ring isomorphic to $Z(\tilde{A})^H$. By Lemma 4.1, $\tilde{A}$ is finitely generated over $Z(\tilde{A})^H$ and is therefore left and right artinian. Being semiprime artinian makes $\tilde{A}$ a semisimple artinian ring and since A is a left order in $\tilde{A}$ we can conclude that $\tilde{A}$ is equal to the classical ring of quotients of A. Thus $Q_{\mathrm{cl}}(A) = \tilde{A}$ is finitely generated over $Z(Q_{\mathrm{cl}}(A))^H$. □

Examples of H-module algebras A with the property that every non-zero H-stable ideal contains a non-zero central H-invariant element are Hopf–Galois extensions with central invariants (see [7]). If the extension A/A^H is H^*-Galois then A is a generator in $A \# H^*$-Mod. Hence $I^H \simeq \operatorname{Hom}_{A\#H^*}(A, I) \neq 0$ for all H-stable left ideals I of A. So-called Hopf PI triple considered in [2] are examples of Hopf–Galois extensions: Let ϵ be a primitive lth root of unity of 1 and let $H = \mathcal{O}_\epsilon(SL_2(\mathbb{C}))$ be the quantized coordinate ring of $SL_2(\mathbb{C})$ as defined in [2]. Then there exists an l^3-dimensional Hopf algebra $\overline{H}$ such that H is a left $\overline{H}^*$-module algebra. The subring of invariants $Z_0 := H^{\overline{H}^*} = \mathcal{O}(SL_2(\mathbb{C})$ is central and H/Z_0 is a Hopf–Galois extension (see [2, III.4.6]).

4.3. In case there do not exist non-trivial H-stable ideals we obtain the following corollary from the previous proposition.

Corollary. *Let H be a Hopf algebra over R with H_R finitely generated. Any semiprime Goldie PI H-module algebra that is H-simple is finite dimensional over $Z(A)^H$ and equals its classical ring of quotients.*

Proof. Since A is H-simple $Z(A)^H$ is a field. Thus by Proposition 4.2, $Q_{\mathrm{cl}}(A) = A$ and $\dim_{Z(A)^H}(A)$ is finite. □

4.4. Assume that A is a semiprime Goldie PI H-module algebra with central invariants. We can now prove the main result of this section showing that the ability of extending the H-action to the classical left ring of quotients of a semiprime Goldie PI H-module algebra A with central invariants is equivalent to $A\#H$ being semiprime.

Theorem. *Let H be a Hopf algebra over R with H_R finitely generated and projective. Let A be a semiprime Goldie PI H-module algebra with central invariants such that there exists a left or right integral t with $\varepsilon(t)1_A$ invertible in A. Then the following statements are equivalent*:

(a) *Every essential left ideal of A contains a regular H-invariant element.*
(b) *The H-action on A extends to the classical left ring of quotients $Q_{\mathrm{cl}}(A)$.*
(c) *$A\#H$ is semiprime.*
(d) *Every H-stable left ideal of A contains a non-zero H-invariant element.*

Then $Q_{\mathrm{cl}}(A) = A[\mathcal{C}^{-1}]$ and $Q_{\mathrm{cl}}(A\#H) = A\#H[\mathcal{C}^{-1}\#1]$, where $\mathcal{C}$ denotes the set of regular elements of A^H.

Proof. Let $\mathcal{C}$ denote the set of regular elements of A^H.

(a) $\Rightarrow$ (b). Consider $\tilde{A} := A[\mathcal{C}^{-1}]$ and let I be an essential left ideal of $\tilde{A}$. Then $I \cap A$ is an essential left ideal of A and contains an element of $\mathcal{C}$. Hence $I = \tilde{A}$ shows that $\tilde{A}$ has no proper essential submodules and must be semisimple artinian. Since A is a right order in $\tilde{A}$ we obtain that $\tilde{A} = Q_{\mathrm{cl}}(A)$. The H-action can be extended trivially to $\tilde{A}$.

(b) $\Rightarrow$ (c). Let $\mathcal{D}$ denote the set of regular elements of A. The H-action on A can be extended to the classical left ring of quotients $Q_{\mathrm{cl}}(A) = A[\mathcal{D}^{-1}]$ by hypothesis. Since A is a semiprime Goldie PI-algebra, $Q_{\mathrm{cl}}(A)$ is semisimple artinian. By Corollary 2.7, $Q_{\mathrm{cl}}(A)\#H$ is semisimple artinian since $\varepsilon(t)1_{Q_{\mathrm{cl}}(A)}$ is invertible in $Q_{\mathrm{cl}}(A)$. As A is a left and right order in $Q_{\mathrm{cl}}(A)$ every element of $Q_{\mathrm{cl}}(A)$ can be written in the form $d^{-1}a$ with $d \in \mathcal{D}$ and $a \in A$. Hence $A\#H$ is a left order in $Q_{\mathrm{cl}}(A)\#H$. Thus by Goldie's Theorem $A\#H$ is semiprime and $Q_{\mathrm{cl}}(A\#H) \simeq Q_{\mathrm{cl}}(A)\#H$.

(c) $\Rightarrow$ (d). Note that $a \mapsto a\#t$ is an injective $A\#H$-linear map from A to $A\#H$. Assume that $A\#H$ is semiprime and let I be a non-zero H-stable left ideal of A. Then $0 \neq (I\#t)^2 = I(t\cdot I)\#t$ shows $I^H \supseteq t\cdot I \neq 0$.

(d) $\Rightarrow$ (a). By Proposition 4.2, $\tilde{A} = A[\mathcal{C}^{-1}]$ equals $Q_{\mathrm{cl}}(A)$ and is semisimple artinian. Let I be an essential left ideal of A. Then $I[\mathcal{C}^{-1}]$ is an essential left ideal of the semisimple ring $\tilde{A}$ and therefore improper. Thus $I[\mathcal{C}^{-1}] = \tilde{A}$ implies that there exist $a \in I$ and $c \in \mathcal{C}$ such that $ac^{-1} = 1$. Equivalently $a = c \in I \cap \mathcal{C}$ shows that I contains a regular H-invariant element. □

4.5. Note that condition (d) of 4.4 says that for every left ideal I in the filter $\mathcal{F}$ of essential left ideals of A and for every $h \in H$ there exists an essential left ideal $I' \in \mathcal{F}$ such that $hI' \subseteq I$. Montgomery had termed H-actions with this property $\mathcal{F}$-continuous and had shown in [17] that this condition is sufficient for extending the H-action to the ring of quotients with respect to the filter $\mathcal{F}$. We see that under the assumptions of 4.4 the $\mathcal{F}$-continuity of the H-action is also a necessary condition.

4.6. Combining Theorems 3.6 and 4.4 we obtain the following corollary for Hopf actions on integral domains.

Corollary. *Let H be a semisimple, cosemisimple Hopf algebra over a field k and let A be a left H-module algebra that is an integral domain. Then the quotient field Q of A equals $A[\mathcal{C}^{-1}]$ where $\mathcal{C} := A^H \setminus \{0\}$. The H-action extends to Q and $Q^H \subseteq Q$ is a finite field extension. $A \# H$ is a semiprime Goldie PI-algebra with classical ring of quotient isomorphic to $Q \# H$.*

4.7. A classical result of Bergman and Isaac asserts, that a ring A with group action G such A is $|G|$-torsionfree is nilpotent whenever A^G is nilpotent. As a kind of Hopf-algebraic analogue Bahturin and Linchenko showed in [1] that every left H-module algebra A (possibly without unit) is nilpotent whenever A^H is nilpotent if and only if every left H-module algebra A (possibly without unit) is PI whenever A^H is PI if and only if $T(H)/\langle \int_l \rangle$ has finite dimension, where H is a finite dimensional Hopf algebra over a field of characteristic 0, $T(H)$ denotes the tensor algebra of H and $\langle \int_l \rangle$ the ideal of $T(H)$ generated by the left integrals in H. They also show that under those equivalent conditions above H must be semisimple. Whether every semisimple Hopf algebra fulfills one of the above properties is still open.

Combining Bahturin and Linchenko's result with Theorem 4.4 we can conclude the following: If H is a finite dimensional Hopf algebra over a field k of characteristic 0 such that $T(H)/\langle \int_l \rangle$ is finite dimensional and if A is a semiprime Goldie left H-module algebra with central invariants then one can extended the H-action to $Q_{\mathrm{cl}}(A)$, $Q_{\mathrm{cl}}(A)$ is equal to the localisation of A by the regular elements of A^H and $A \# H$ is semiprime with classical ring of quotients equal to $Q_{\mathrm{cl}}(A) \# H$.

5. Drinfeld twists of strongly semisimple Hopf algebras

We finish the paper by showing that Cohen's question has a positive answer if H is semisimple cosemisimple triangular.

Definition. A Hopf algebra H over R is called *strongly semisimple* if for every H-semiprime left H-module algebra A the smash product $A \# H$ is semiprime.

Criterions for a Hopf algebra to be strongly semisimple are given in [18] but those criterions are hard to verify. Over a field, every commutative or cocommutative semisimple Hopf algebra is strongly semisimple. Moreover Montgomery and Schneider showed, that every semisimple Hopf algebra that admits a normal series H_i whose quotients H_{i+1}/H_i are either commutative or cocommutative, is strongly semisimple (see [18, 8.16]). Those Hopf algebras are called semi-solvable.

We will show that the class of strongly semisimple Hopf algebras is closed under Drinfeld twists. Applying a theorem of Etingof and Gelaki, that classifies all triangular semisimple cosemisimple Hopf algebras as Drinfeld Twists of group algebras, we

can conclude that all triangular semisimple cosemisimple Hopf algebras are strongly semisimple.

5.1. Recall the definition of Drinfeld twists for a Hopf algebra.

Definition. Let H be an Hopf algebra over R. A *Drinfeld Twist* for H is an invertible element $J \in H \otimes H$, such that

$$(J \otimes 1)(\Delta \otimes 1)(J) = (1 \otimes J)(1 \otimes \Delta)(J), \tag{1}$$

$$(\varepsilon \otimes 1)(J) = 1 = (1 \otimes \varepsilon)(J) \tag{2}$$

holds. We write formally $J =: \sum J^1 \otimes J^2$ and $J^{-1} =: Q =: \sum Q^1 \otimes Q^2$.

If H is a Hopf algebra over R with comultiplication Δ and antipode S, then $\Delta^J := J\Delta J^{-1}$ defines a new comultiplication on H with $\Delta^J(h) := J\Delta(h)J^{-1}$ for all $h \in H$. Let $U := \sum J^1 S(J^2)$ and $U^{-1} = \sum S(Q^1)Q^2$ and define a new map $S^J := USU^{-1}$ by $S^J(h) := US(h)U^{-1}$ for all $h \in H$. Then it has been shown in [14, 2.3.4] that Δ^J and S^J define a new Hopf algebra structure on H keeping the same multiplication, unit and counit. We denote the obtained Hopf algebra by H^J. Obviously $\Delta^J(h)J = J\Delta(h)$ for all $h \in H$.

Moreover it is not difficult to see that J^{-1} is a Drinfeld twist for H^J.

5.2. Having 'twisted' the comultiplication of H we can also 'twist' the multiplication of a left H-module algebra A such that A becomes a left H^J-module algebra.

Definition. Let A be a left H-module algebra with multiplication μ and let J be a Drinfeld twist for H. We define a new multiplication $\mu^J : A \otimes A \to A$ on A with

$$a \cdot_J b := \mu^J(a \otimes b) := \sum \big(Q^1 \cdot a\big)\big(Q^2 \cdot b\big) \quad \text{for all } a, b \in A.$$

It had been shown in [14, 2.3.8] that A^J with multiplication μ^J is a left H^J-module algebra. Moreover the smash products $A \# H$ and $A^J \# H^J$ are isomorphic R-algebras. This follows from a more general theorem by Majid (see [15, 2.9]).

5.3. Note that for every two elements $a, b \in A$ we have:

$$ab = \sum \big(J^1 \cdot a\big) \cdot_J \big(J^2 \cdot b\big).$$

In particular take any H^J-stable ideal I of A^J, then I is also an H-stable ideal of A. Moreover if I is nilpotent as an ideal of A^J, then it is also nilpotent as an ideal of A. This shows A^J is H^J-semiprime whenever A is H-semiprime. By the same argument applied to $A = (A^J)^{J^{-1}}$ we obtain A is H-semiprime whenever A^J is H^J-semiprime.

5.4. Combining the results of the last two paragraphs we can prove that the class of strongly semisimple Hopf algebras is closed under Drinfeld twists.

Corollary. *The class of strongly semisimple Hopf algebras is closed under Drinfeld twists.*

Proof. Let H be a strongly semisimple Hopf algebra and let J be a Drinfeld twist for H. Let A be a left H^J-module algebra, then $A^{J^{-1}}$ is a left H-module algebra by [14, 2.3.8]. If A is H^J-semiprime, then $A^{J^{-1}}$ is H-semiprime by 5.3. As noticed in 5.2 from [15, 2.9] follows

$$A^{J^{-1}} \# H = A^{J^{-1}} \# H^{J J^{-1}} \simeq A \# H^J.$$

Since H is strongly semisimple, $A^{J^{-1}} \# H$ and therefore $A \# H^J$ is semiprime. Hence H^J is strongly semisimple. □

5.5. A Hopf algebra is called *triangular*, if there exists an invertible element $\mathcal{R} \in H \otimes H$ with

$$(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$
$$\Delta^{\mathrm{cop}} = \mathcal{R}\Delta\mathcal{R}^{-1} \quad \text{and} \quad \mathcal{R}^{-1} = \tau(\mathcal{R})$$

where $\tau : H \otimes H \to H \otimes H$ is the isomorphism $x \otimes y \mapsto y \otimes x$. For $\mathcal{R} = \sum a_i \otimes b_i$ we set

$$\mathcal{R}_{13} := \sum a_i \otimes 1 \otimes b_i, \qquad \mathcal{R}_{23} := \sum 1 \otimes a_i \otimes b_i, \qquad \mathcal{R}_{12} := \sum a_i \otimes b_i \otimes 1.$$

P. Etingof and S. Gelaki classified in [10] semisimple cosemisimple triangular Hopf algebras over algebraically closed fields as Drinfeld twists of group rings. From this we obtain as a corollary:

Corollary. *All triangular semisimple cosemisimple Hopf algebras over an algebraically closed field are strongly semisimple.*

Proof. Let H be a semisimple cosemisimple triangular Hopf algebra over an algebraically closed field k. By Etingof and Gelaki's result [10, Corollary 6.2] there exists a group G and a Drinfeld twist $J \in k[G] \otimes k[G]$ such that $H \simeq k[G]^J$ as Hopf algebras. As $k[G]$ is strongly semisimple also H is strongly semisimple by 5.4. □

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