On the Maximal Subgroups of Finite Simple Groups*

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1. Introduction

In the course of their proof of the solvability of groups of odd order, W. Feit and J. G. Thompson [1] establish many deep properties of the maximal subgroups of a minimal simple group $G$ of odd order. Perhaps the most important of these results is the following: if for some prime $p$, $G$ possesses an elementary abelian subgroup of order $p^3$, then there exists a unique maximal subgroup $M$ of $G$ containing a given Sylow $p$-subgroup $\Psi$ of $G$; furthermore if $\Omega$ is any proper subgroup of $G$ such that $\Omega \cap \Psi$ possesses an elementary abelian subgroup of order $p^3$, then $\Omega$ is necessarily a subgroup of $M$.

In our study of finite groups $G$ with dihedral Sylow 2-subgroups, we require the identical result for certain odd primes $p$ dividing the order of the centralizer of an involution in $G$. On the basis of this theorem, we can show that the unique maximal subgroup $M$ of $G$ containing a given Sylow $p$-subgroup actually contains the entire centralizer of some involution in $G$ and has no normal subgroups of index 2. We are then able to apply to both $M$ and $G$ formulas for the order of an arbitrary group with dihedral Sylow 2-subgroup having no normal subgroups of index 2, as developed by means of character

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theory in Part III of [2], and to conclude that the index of \( \mathfrak{M} \) in \( G \) is at most 5. This leads at once to a contradiction if \( G \) is assumed to be simple and of order greater than 60. Since we can describe in advance the set \( \tau \) of odd primes dividing the order of the centralizer of an involution to which the preceding argument is applicable, an appropriate generalization of the theorem of Feit and Thompson will imply that the set \( \tau \) is empty. This has the effect of greatly simplifying the structure of the centralizer of an involution in \( G \), and puts us in a position to employ arithmetic methods to complete the classification of groups with dihedral Sylow 2-subgroups. These results will be described in detail in a subsequent paper.

Feit and Thompson studied minimal simple groups. This assumption that all proper subgroups are solvable thoroughly pervades the portion of their work which we shall generalize in the present paper. In order to effect this generalization it is necessary to point out the various ways in which this assumption is used and to devise appropriate axioms that are valid for other classes of simple groups. The axioms which we introduce hold, for example, in simple groups whose proper subgroups involve only solvable groups or the simple groups \( PSL(2, q) \), \( q \) odd, or \( A_\tau \). We have made preliminary studies to determine that these axioms will hold, say, for simple groups with proper subgroups that also involve the groups \( PSL(3, q) \), \( q \) odd, and we have attempted to formulate them in such a way that they may hold in even more general circumstances.

For example, one of the most important properties of solvable groups used by Feit and Thompson is the following, which is a direct consequence of Theorem B of Hall and Higman [5]: if \( S \) is a solvable group of odd order and if \( \Psi \) is a Sylow \( p \)-subgroup of the second term \( O_{p'}(S) \) of the ascending \( p \)-series of \( S \), then any \( p \)-subgroup \( \Psi' \) of \( S \) which normalizes \( \Psi \) and satisfies the commutator identity \( [\Psi, \Psi', \Psi] = 1 \) is contained in \( \Psi \). If \( S \) has even order, the same conclusion holds if \( p > 5 \), but for \( p = 3 \) a refinement of this statement must be made. This conclusion is not true in general for nonsolvable groups. The condition is essentially an assertion about \( p \)-modular representations, and is closely related to the question of which finite groups admit faithful \( p \)-modular representations in which some \( p \)-element has a quadratic minimal polynomial. We introduce the concept of \( p \)-stability to describe a group \( S \) which satisfies an appropriate generalization of the above condition, and we demand that certain proper subgroups \( \mathfrak{S} \) of the simple group \( G \) be \( p \)-stable.

If \( G \) is a minimal simple group, the centralizer \( C(\Psi) \) of any \( p \)-subgroup \( \Psi \neq 1 \) of \( G \) is solvable. Obviously if \( G \) possesses proper nonsolvable subgroups, \( C(\Psi) \) may be nonsolvable. It may happen, however, that whenever \( C(\Psi) \) is nonsolvable, then \( C(\Psi^+) \) is solvable, where \( \Psi^+ \) is a Sylow \( p \)-subgroup of \( O_{p'}.p(\Psi C(\Psi)) \). If this is the case for some odd prime \( p \), we say that \( G \) is
It turns out that many of the proofs of Feit and Thompson can be extended to simple groups which are p-constrained. The importance of this condition for the dihedral Sylow 2-subgroup problem rests upon the following fact: if p is an odd prime dividing \(|C(T)|\) for some involution T of G, then either G is p-constrained or the centralizer of a Sylow p-subgroup of C(T) is nonsolvable.

The conditions of p-stability and p-constraint lie at the heart of our generalization of the theorem of Feit and Thompson, even though additional restrictions must be placed upon G in order to extend their arguments completely. The proof of their theorem in the minimal simple case is divided into two essentially independent parts, the proofs for each case being quite distinct. This subdivision is based upon the following possibilities: if \(\mathfrak{P}\) denotes a Sylow p-subgroup of G, then \(\mathfrak{P}\) may or may not normalize some nonidentity p'-subgroup of G.

In order to generalize their proof in the first case, we require conditions on the set of primes which are equivalent to p in the sense of Feit-Thompson. A prime q is said to be equivalent to p provided there exists a solvable subgroup of G which contains an elementary abelian subgroup of both orders \(p^3\) and \(q^3\). Basically our condition is that for a certain well-defined subset \(\tau_0\) of the primes equivalent to p, G must be q-stable and q-constrained for each q in \(\tau_0\), and q and r must be equivalent for each pair of primes q, r in \(\tau_0\).

The proof for minimal simple groups in the second case depends in part upon the fact that one is able to control, by means of Theorem B of Hall-Higman, the weak closure in \(\mathfrak{P}\) with respect to G of certain normal subgroups of \(\mathfrak{P}\). In order to achieve the same results under more general conditions, we are forced to place further limitations upon the manner in which the p-subgroups of G are involved in the proper subgroups of G. The precise description of these conditions gives rise to the related concepts of a p-restricted and p-reductive group G. Under the assumption that \(\mathfrak{P}\) normalizes no nontrivial p'-subgroup of G, we are able to extend the theorem of Feit and Thompson to a simple group G which is p-stable, p-constrained, p-restricted, and p-reductive.

Definitions of all the terms to be used in the paper and a complete statement of our principal results will be given in Section II.

Finally we should like to acknowledge our deep thanks to Walter Feit and John Thompson for making available to us advanced copies of their proof of the solvability of groups of odd order. We are especially indebted to John Thompson for innumerable suggestions and ideas relating to the proofs and general content of the present paper.
II. Definitions and Statements of the Main Results

We shall refer to the paper "Solvability of Groups of Odd Order" by Walter Feit and John G. Thompson as F.T. It will be assumed that the reader is familiar with the appropriate portions of F.T.—primarily Chapters II and IV, and certain sections of Chapter I. Wherever possible we shall follow the notation and terminology of F.T. In general, we shall not redefine here any term which has been introduced in F.T. A list of most of the basic terms will be found in Chapter I, Section 2, of F.T.

We first introduce the additional concepts which we shall need for this paper.

**Definition 1.** If $G$ is a group, $S(G)$ shall denote the maximum normal solvable subgroup of $G$. Clearly $S(G)$ is unique and contains all solvable normal subgroups of $G$. We also define $O_p(G)$ to mean $O_p(S(G))$ with similar interpretations for $O_{p'}(G), O_{p''}(G)$, etc. We denote by $\pi(G)$ the set of primes dividing $|S(G)|$.

Theorem B of Hall-Higman (to which we shall hereafter refer simply as (B)) is used by Feit and Thompson in essentially three distinct, but closely related, ways. Corresponding to each of these, we introduce a general concept for arbitrary groups, which we call respectively $p$-stability, $p$-restriction, and $p$-reduction.

**Definition 2.** Let $G$ be a group, $p$ a prime in $\pi(G)$, $\mathfrak{A}$ an $S_p$-subgroup of $O_{p''}(G)$, and $\mathfrak{A}_0$ a subgroup of $\mathfrak{A}$ such that (i) $O_p(G) \mathfrak{A}_0 \lhd G$ and (ii) $O_p(G) = 1$, where $G = NB(\mathfrak{A}_0)/C_G(\mathfrak{A}_0)$. We shall say that $G$ is $p$-stable provided the following condition holds for any such subgroup $\mathfrak{A}_0$:

(a) If $\mathfrak{A}$ is a $p$-subgroup of $G$ which normalizes $\mathfrak{A}_0$ and satisfies the commutator identity $\gamma^{2[\mathfrak{A}_0]} = 1$, then $G \subseteq C_S(\mathfrak{A}_0)$.

Since $N_S(\mathfrak{A}_0)/C_S(\mathfrak{A}_0)$ is represented faithfully as a subgroup of $\text{Aut} \, \mathfrak{A}_0$, the question of whether a particular group $G$ is $p$-stable can easily be reduced to the following question concerning $\overline{G}$: if $\overline{G}$ is represented faithfully on a vector space $\mathfrak{B}$ over $GF(p)$, does every $p$-element of $\overline{G}$ have a minimal polynomial of degree greater than 2? We are thus led in a natural way to the following auxiliary concept. A group $\mathfrak{R}$ of linear transformations of a vector space over a field of characteristic $p$ and satisfying $O_p(\mathfrak{R}) = 1$ will be said to be $p$-stable if all its $p$-elements have minimal polynomials of degree greater than 2.

In the applications of this concept, we shall be led to consider certain groups which may appear as the group $\overline{G}$ in the definition of $p$-stability. For these groups, it is necessary to show that any group of transformations of a vector space over a field of characteristic $p$ which faithfully represents $\overline{G}$ is $p$-stable. In case $\overline{G}$ is solvable, (B) shows this to be true for odd primes with the exception of the case $p = 3$ and $\overline{G}$ involves the group $SL(2, 3)$. We
can also show that this is true if $\mathcal{S}$ is isomorphic to $PGL(n, q)$, $n > 1$, or to one of its normal subgroups, provided that $p$ is odd and $p$ does not divide $q$. It is also true if $\mathcal{S}$ is isomorphic to $PSL(2, p')$. It is anticipated that the proof of $p$-stability for the groups $PSL(n, q)$ can be extended to many more classes of known simple groups, and perhaps all.

Remark. In the proofs of Theorems A, B, and C which are stated at the end of this section, the condition of $p$-stability is used only when $\pi(p) = 3$. In this case, the definition can be put in the following simpler form, using Lemma 1.2.3 of [3]:

(a') If $C_p(\Psi)$ is solvable and $\Psi$ is a $p$-subgroup of $\mathcal{S}$ which normalizes $\Psi$ and satisfies the commutator identity $\gamma^2\Psi\Psi^2 = 1$, then $\Psi \leq \Psi$.

The full condition for $p$-stability is required in the proofs of Theorems D and E to determine the weak closure of certain $p$-subgroups of $\mathcal{S}$.

We turn now to the concept of $p$-restriction. Since we require this notion only for groups $\mathcal{S}$ in which $O_p(\mathcal{S}) = 1$, we shall limit the definition to such groups for simplicity of presentation, although the definition can easily be modified to cover groups in which $O_p(\mathcal{S}) \neq 1$.

**Definition 3.** Let $\mathcal{S}$ be a group such that $p \in \pi(\mathcal{S})$, and $O_p(\mathcal{S}) = 1$. Let $\Psi$ be a nonidentity subgroup of $Z(O_p(\mathcal{S}))$ which is normal in $\mathcal{S}$ and such that $O_p(\mathcal{S}/C_\Psi(\mathcal{S})) = 1$. Then $\mathcal{S}$ is said to be $p$-restricted if the following conditions hold for any such subgroup $\Psi$:

(a) If $\Psi^*$ is an $Sp(\Psi)$-subgroup of $\mathcal{S}$ and if $\Psi^* \subseteq C_\Psi(\mathcal{S})$, then $\Psi^*$ possesses a normal subgroup $\Psi_0^*$ such that $\Psi^*/\Psi_0^*$ is cyclic, and if $\Psi_0 = C_\Psi(\Psi_0^*)$, then $\gamma^2\Omega_1(\Psi_0) \Psi_0^{*2} \neq 1$.

(b) The same conclusion as in (a) if $\Psi^*$ is any abelian $p$-subgroup of $\mathcal{S}$ which is not contained in $C_\Psi(\mathcal{S})$.

Furthermore, if (a) and (b) hold for a particular subgroup $\Psi$ of $Z(O_p(\mathcal{S}))$ satisfying all these conditions, we shall say that $\mathcal{S}$ is $p$-restricted with respect to $\Psi$.

Since $\mathcal{S}/C_\Psi(\mathcal{S})$ is represented faithfully on $\Omega(\Psi)$, regarded as a vector space over $GF(p)$, the question of whether $\mathcal{S}$ is $p$-restricted can be reduced, as with $p$-stability, to a question about linear groups. Thus we shall say that a group $\mathcal{R}$ of linear transformations of a vector space $\Psi$ over a field of characteristic $p$ and satisfying $O_p(\mathcal{R}) = 1$ is $p$-restricted if, whenever $\Psi^* \neq 1$ is either an $Sp$-subgroup or an abelian $p$-subgroup of $\mathcal{R}$, there exists a subgroup $\Psi_0^*$ of $\Psi^*$ such that $\Psi^*/\Psi_0^*$ is cyclic, and such that the minimal polynomial of a generator of $\Psi^*/\Psi_0^*$ on $\Psi_0 = C_\Psi(\Psi_0^*)$ has degree greater than 2. Note that since $\Psi_0^*$ acts trivially on $\Psi_0$, the group $\Psi^*/\Psi_0^*$ is represented on $\Psi_0$.

In order to make this concept meaningful, we shall give a number of examples of $p$-restricted linear groups. Suppose first that $\mathcal{R}$ is solvable, and $p$
is odd. Since $O_p(\mathfrak{H}) = 1$ by assumption, Lemma 1.2.3 of [5] implies that $\mathfrak{P}^*$ normalizes, but does not centralize, some $q$-subgroup of $O_p(\mathfrak{H})$. If $\mathfrak{H}$ is a $q$-group of minimal order, normalized but not centralized, by $\mathfrak{P}^*$, then it follows from (3.11) of F.T. that $\mathfrak{H}$ is a special $q$-group and that $Z(\mathfrak{P}^*/\mathfrak{P}_1^*)$ is cyclic, where $\mathfrak{P}_1^* = \ker (\mathfrak{P}^* \to \text{Aut} \mathfrak{H})$. If $\mathfrak{P}_1 = C_{\mathfrak{H}}(\mathfrak{P}_1^*)$, one can easily show that $\mathfrak{H}$ is faithfully represented on $\mathfrak{P}_1$ (compare Lemma 3.7 below). Hence $\mathfrak{H}^*/\mathfrak{P}_1^*$ is faithfully represented on $\mathfrak{P}_1$. It follows now from (B) that every $p$-element of $\mathfrak{P}^*/\mathfrak{P}_1^*$ has minimal polynomial on $\mathfrak{P}_1$ of degree greater than 2, except possibly if $p = 3$ and $\mathfrak{H}$ is an extraspecial 2-group.

Assume first that $\mathfrak{P}^*$ is abelian. In this case, $\mathfrak{P}^*/\mathfrak{P}_1^* = Z(\mathfrak{P}^*/\mathfrak{P}_1^*)$ is cyclic. Hence if we set $\mathfrak{P}_0^* = \mathfrak{P}_1^*$ and $\mathfrak{P}_0 = \mathfrak{P}_1$, we see that $\mathfrak{P}_0^*$ satisfies the required conditions, with a single possible exception for $p = 3$. On the other hand, if $\mathfrak{P}^*$ is nonabelian, a more involved argument is needed to determine the exact conditions under which the required subgroup $\mathfrak{P}_0^*$ can be constructed. This argument will establish the following general result for solvable linear groups: $\mathfrak{H}$ is $p$-restricted if either $|\mathfrak{H}|$ is odd or if $p$ is not a Fermat prime and $p$ is odd. A proof of this result will be given in the above-mentioned paper on groups with dihedral $S_p$-subgroups.

We can also offer examples of nonsolvable $p$-restricted groups of transformations. We treat only the case that $\mathfrak{P}^*$ is abelian in our examples. When $\mathfrak{H}$ is a $p$-stable group of transformations with cyclic $S_p$-subgroups, it suffices to take $\mathfrak{P}_0^* = 1$ to obtain the desired conclusion. Examples of such groups are $\text{PSL}(2, q)$ for $p$ not dividing $q$ or for $p = q$, $p$ odd. When $q = p^t$, $t > 1$, and $p$ odd, it can be shown that $\text{PSL}(2, p^t)$ is not $p$-restricted even though it is $p$-stable.

In treating elementary $p$-subgroups of $\text{PSL}(n, q)$ we are forced to consider rather the corresponding groups $\text{GL}(n, q)$ in order to apply inductive arguments. This may be done since all the irreducible representations of $\text{PSL}(n, q)$ can be obtained from those of $\text{GL}(n, q)$. The procedure is to observe that the abelian subgroups of $\text{GL}(n, q)$ corresponding to elementary $p$-subgroups of $\text{PSL}(n, q)$ can be imbedded in a direct product of $\text{GL}(m, q)$, $m < q$, and a $p$-group. The desired result follows from Lemma 3.7 below and induction applied to the group $\text{GL}(m, q)$. There are some exceptional cases to be treated with this type of argument. We do not give the details.

We now introduce the concept of a $p$-reductive group $\mathfrak{S}$. As with $p$-restriction, we limit ourselves to the case $O_p(\mathfrak{S}) = 1$.

**Definition 4.** Let $\mathfrak{S}$ be a group such that $O_p(\mathfrak{S}) = 1$ and $p \in \pi_*(\mathfrak{S})$. Let $\mathfrak{P}$ be an $S_p$-subgroup of $\mathfrak{S}$, and let $\mathfrak{B}$ be a subgroup of $\mathfrak{P}$ such that (i) $\mathfrak{B}$ is generated by elementary subgroups $\mathfrak{B}_i$, $i = 1, 2, \ldots, s$, and (ii) $\mathfrak{B} = V(\text{cl}_{\mathfrak{P}}(\mathfrak{B}); \mathfrak{P})$. Let $\mathfrak{B}_0$ be a normal subgroup of $\mathfrak{P}$ contained in $O_p(\mathfrak{S})$ and set $\mathfrak{L} = C_{\mathfrak{S}}(\mathfrak{B}_0)$. Furthermore, suppose that $\mathfrak{S} \neq 2N(\mathfrak{B})$. 


Under these hypotheses, we shall say that \( \mathcal{S} \) is \( p \)-reductive provided there exists, for each such pair of subgroups \( \mathcal{X} \) and \( \mathcal{Y} \), a subgroup \( \mathcal{R} \) of \( \mathcal{S} \) which satisfies the following conditions:

(a) \( \mathbb{O}(\mathcal{S}) \subset \mathcal{R} \).
(b) If \( \mathcal{X} \) is the largest normal subgroup of \( \mathcal{R} \) which centralizes \( \mathcal{X} \), and if \( \mathcal{R}_i \) is the inverse image of \( \mathbb{O}(\mathcal{R}/\mathcal{X}) \) in \( \mathcal{R} \), then \( \mathcal{R}_i \subset \mathcal{R} \) and \( | \mathcal{R}_i/\mathcal{X}_i \cap \mathcal{R}_i | = p \) for some \( i = 1, 2, \ldots, s \).
(c) \( \mathcal{Y}_i = \mathcal{X}_i \cap \mathcal{R}_i \) is an \( S_p \)-subgroup of \( \mathcal{R}_i \).
(d) \( \mathcal{R}_i = N_{\mathcal{R}}(\mathcal{Y}_i) \) is a \( p \)-stable group.

In one situation in the proof of Theorem D (see Lemma 10.4) the concepts of \( p \)-stability and \( p \)-restriction are not sufficient. In order to complete the proof, it is necessary to replace a given group \( \mathcal{S} \) occurring in the argument by a particular subgroup \( \mathcal{R} \) which contains a configuration similar to that stipulated by the concept of \( p \)-restriction. Thus conditions (b) and (d) are the critical conditions in the definition of \( p \)-reduction. For example, if \( \mathcal{S} \) is solvable, one can show that \( \mathcal{S} \) is \( p \)-reductive under the same assumptions and by essentially the same methods as were described above in showing that \( \mathcal{S} \) is \( p \)-restricted.

Finally we shall carry over the notion of \( p \)-stability, \( p \)-restriction and \( p \)-reduction to an arbitrary simple group by making the following definition:

**Definition 5.** A simple group \( \mathcal{G} \) will be called (i) \( p \)-stable, (ii) \( p \)-restricted, (iii) \( p \)-reductive for some odd prime \( p \) provided every subgroup \( \mathcal{S} \) of \( \mathcal{G} \) which satisfies the following conditions is (i) \( p \)-stable, (ii) \( p \)-restricted, (iii) \( p \)-reductive:

(a) \( p \in \pi(\mathcal{S}) \).
(b) \( \mathbb{O}_p(\mathcal{S}) = 1 \) in the case of \( p \)-restriction and \( p \)-reduction.
(c) If \( \mathcal{Y} \) is an \( S_p \)-subgroup of \( \mathbb{O}_p(\mathcal{S}) \), then an \( S_p \)-subgroup of \( \mathcal{S} \) is an \( S_p \)-subgroup of \( N_\mathcal{G}(\mathcal{Y}) \).

It might seem that condition (c) is superfluous. Indeed, if \( \mathcal{G} \) has dihedral \( S_p \)-subgroups and if all proper subgroups of \( \mathcal{G} \) have composition factors which are known simple groups, then it can be shown that every subgroup \( \mathcal{S} \) of \( \mathcal{G} \) for which \( p \in \pi(\mathcal{S}) \) is, in fact, \( p \)-stable. There are, however, cogent reasons for imposing this additional requirement. First of all, it at least serves to emphasize which subgroups of \( \mathcal{G} \) play a critical role in our generalization of the results on minimal simple groups. But there exists a more important consideration, which can be illustrated by the following possibility. It may happen that a simple group \( \mathcal{G} \) contains a subgroup \( \mathcal{S}_1 \) such that \( \mathcal{Y}_1 = \mathbb{O}_p(\mathcal{S}_1) \) is abelian of type \( (p, p) \), \( \mathcal{S}_1/\mathcal{Y}_1 \cong SL(2, p) \) and \( \mathcal{Y}_1 \not\subseteq \mathcal{Z}(\mathcal{S}_1) \). Since \( \mathcal{S}_1 \) is not a \( p \)-stable group, it would follow at once, without condition
(c) that $\mathfrak{G}$ is not $p$-stable. On the other hand, we anticipate that at least in some circumstances it will be possible to prove the following assertion: if $\mathfrak{B}$ is a maximal $p$-subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$ and normalized by $\mathfrak{D}$, then $\mathfrak{H} = N(\mathfrak{B})$ is a $p$-stable group. In other words, one may be able to show that non-$p$-stable subgroups of $\mathfrak{G}$ can always be imbedded in $p$-stable subgroups. As our definition indicates, this would be entirely sufficient for our purposes. The same remarks apply for $p$-restriction and $p$-reduction.

We next introduce the concept of $p$-constraint.

**Definition 6.** We call a group $\mathfrak{G}$ $p$-constrained if for any $p$-subgroup $\mathfrak{B} \neq 1$ of $\mathfrak{G}$, $C(\mathfrak{B}^*)$ is solvable, where $\mathfrak{B}^*$ is an $S_p$-subgroup of $O_{p^e}(\mathfrak{B}C(\mathfrak{B}))$.

**Remarks.** As pointed out in the Introduction, if $\mathfrak{G}$ is a group with dihedral Sylow 2-subgroups, then either $\mathfrak{G}$ is $p$-constrained or $C(\mathfrak{B})$ is nonsolvable, where $\mathfrak{B}$ is an $S_p$-subgroup of the centralizer of an involution of $\mathfrak{G}$, in which case $\mathfrak{G}$ centralizes some four-subgroup of $\mathfrak{G}$. Because of the fundamental role which the four-subgroups of $\mathfrak{G}$ play in the study of groups with dihedral Sylow 2-subgroups, this characterization of the non-$p$-constrained primes turns out to be an extremely effective one.

There is a reason to believe that the concept of $p$-constraint will also be effective in studying simple groups $\mathfrak{G}$ all of whose proper subgroups (or at least certain critical ones) possess at most one non-abelian simple composition factor. On the other hand, it appears that the usefulness of this concept will diminish as the number of possible non-abelian composition factors of the proper subgroups of $\mathfrak{G}$ is allowed to increase. These observations may serve to point up the possible extent and limitations of the applicability of our results in their present form.

**Definition 7.** Let $\mathfrak{G}$ be a simple group and $p$ an odd prime in $\pi(\mathfrak{G})$. We say that $\mathfrak{G}$ is weakly $p$-tame provided:

(I) $\mathfrak{G} \cap N_3(p)$ is nonempty.

(II) $\mathfrak{G}$ is $p$-stable.

(III) $\mathfrak{G}$ is $p$-constrained.

(IV) Let $\mathfrak{D}$ be a proper subgroup of $\mathfrak{G}$ such that $p \notin \pi(\mathfrak{D})$, and assume that $\mathfrak{D}$ contains an element $\mathfrak{A}$ of $\mathfrak{G} \cap N_3(p)$. Then every element of $\mathfrak{N}_\mathfrak{D}(\mathfrak{A})$ is contained in $S(\mathfrak{D})$.

**Definition 8.** Let $\mathfrak{G}$ be a simple group which is weakly $p$-tame for some prime $p$. We say that $\mathfrak{G}$ is $p$-tame provided:

(V) If $\mathfrak{B}$ is an $S_p$-subgroup of $\mathfrak{G}$ and $\mathfrak{Q}$ is a nonidentity element of $\mathfrak{N}(\mathfrak{B}; q)$, then $p \in \pi(S(\mathfrak{N}(\mathfrak{Q})))$. 

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**MAXIMAL SUBGROUPS OF FINITE SIMPLE GROUPS**

175
Note that condition (V) holds trivially if $\mathcal{N}(\Psi)$ is trivial.

**Definition 9.** Let $\mathfrak{G}$ be a simple group. We say that $\mathfrak{G}$ is strongly $p$-tame provided $\mathfrak{G}$ is $p$-tame and

(VI) $\mathfrak{G}$ is $p$-restricted and $p$-reductive.

As in F.T. we shall write $p \sim q$ if $\mathfrak{G}$ possesses a $\{p, q\}$-subgroup which contains abelian subgroups of both types $(p, p, p)$ and $(q, q, q)$.

**Definition 10.** Let $\mathfrak{G}$ be a simple group and $\tau$ a set of primes in $\pi(\mathfrak{G})$. We say that $\mathfrak{G}$ is $\tau$-tame provided:

(VII) $\mathfrak{G}$ is $p$-tame for each $p$ in $\tau$.

(VIII) $p \sim q$ for every pair of primes $p, q$ in $\tau$.

**Remark.** Suppose $\mathfrak{G}$ is a simple group with dihedral Sylow 2-subgroups and that the structure of the proper subgroups of $\mathfrak{G}$ is known. Then if $\mathfrak{S}$ is a proper subgroup of $\mathfrak{G}$ for which $p \notin \pi(\mathfrak{S})$ and which contains an abelian subgroup on three or more generators, one can show that $\mathfrak{S}/S(\mathfrak{S})$ is isomorphic to a subgroup of $\text{PGL}(2, p^m)$ containing $\text{PSL}(2, p^m)$ with $m \geq 3$. That $\mathfrak{G}$ satisfies condition (IV) in the definition of weakly $p$-tame follows readily from this result. Furthermore, condition (V) follows from condition (IV) and certain results of Section IV which can be established on the basis of weak $p$-tameness. We anticipate that it will be possible to verify conditions (IV) and (V) in other classification problems.

In order to state our main results, we need one auxiliary concept. In the proof of the theorem of Feit and Thompson for minimal simple groups, the extended Sylow theorems for solvable groups play a vital part in determining the structure of the maximal subgroups of $\mathfrak{G}$. Since the proper subgroups of an arbitrary simple group are, in general, nonsolvable, the extended Sylow theorems cannot be applied to the subgroups of $\mathfrak{G}$. In order to get around this difficulty, it is necessary for us to extend the meaning of an $S$-subgroup of $\mathfrak{G}$. This is incorporated into the following definition.

**Definition 11.** Let $\mathfrak{G}$ be a group and $\sigma$ a set of primes in $\pi(\mathfrak{G})$. A subgroup $\mathfrak{S}$ of $\mathfrak{G}$ will be called an $S_\sigma$-subgroup provided:

(i) $\mathfrak{S}$ contains an $S_p$-subgroup of $\mathfrak{G}$ for each $p$ in $\sigma$.

(ii) $p \in \pi(\mathfrak{S})$ for each $p$ in $\sigma$.

(iii) An $S_\sigma$-subgroup of $S(\mathfrak{S})$ is normal in $\mathfrak{S}$.

If $\mathfrak{G}$ possesses an $S_\sigma$-subgroup we shall say that $\mathfrak{G}$ satisfies $E_\sigma$.

If $\mathfrak{S}$ is solvable, the definition implies that $\mathfrak{S}$ contains an $S_\sigma$-subgroup of $\mathfrak{G}$ as a normal subgroup. We remark that assumption (V) plays an essential
role in the proof of \( E'_r \) for a simple group \( \mathcal{G} \) which is \( \tau \)-tame for some set of primes \( \tau \).

We are finally in a position to state our main results. It will be helpful first to recall from F.T. the meanings of a few essential terms. If \( \mathcal{G} \) is a group, \( \pi_3 = \pi_3(\mathcal{G}) \) denotes the set of primes \( p \) in \( \pi(\mathcal{G}) \) for which \( \mathcal{F}\mathcal{C}.N_3(\mathcal{P}) \) is nonempty and \( \mathcal{U}(\mathcal{P}) \) is nontrivial, while \( \pi_4 = \pi_4(\mathcal{G}) \) denotes the set of primes \( p \) in \( \pi(\mathcal{G}) \) for which \( \mathcal{F}\mathcal{C}.N_4(\mathcal{P}) \) is nonempty but \( \mathcal{U}(\mathcal{P}) \) is trivial, where \( \mathcal{P} \) denotes an \( S_p \)-subgroup of \( \mathcal{G} \). Furthermore, if \( \mathcal{G} \) is a group, \( p \) a prime in \( \pi_3 \cup \pi_4 \), and \( \mathcal{P} \) an \( S_p \)-subgroup of \( \mathcal{G} \), then

\[
\mathcal{A}(\mathcal{P}) = \{ \mathcal{P}_0 \mid \mathcal{P}_0 \subseteq \mathcal{P}, \mathcal{P}_0 \text{ contains an element of } \mathcal{F}\mathcal{C}.N_3(\mathcal{P}) \}.
\]

\[
\mathcal{A}_i(\mathcal{P}) = \{ \mathcal{P}_0 \mid \mathcal{P}_0 \subseteq \mathcal{P}, \mathcal{P}_0 \text{ contains a subgroup } \mathcal{P}_1 \text{ of type } (p, p) \text{ such that } C_{\mathcal{P}_1}(P) \in \mathcal{A}_{i-1}(\mathcal{P}) \text{ for each } P \text{ in } \mathcal{P}_1, i = 2, 3, 4.
\]

In this paper we shall prove the following results:

**Theorem A.** Let \( \mathcal{G} \) be a simple group which is \( \tau \)-tame for some set of primes \( \tau \) in \( \pi_3 \). Then \( \mathcal{G} \) satisfies \( E'_r \).

**Theorem B.** Let \( \mathcal{G} \) be a simple group which is \( \tau \)-tame for some set of primes \( \tau \) in \( \pi_3 \). Then for at least one prime \( p \) in \( \tau \), there exists a subgroup \( \mathcal{P} \) of type \( (p, p) \) which is normal in an \( S_p \)-subgroup \( \mathcal{P} \) of \( \mathcal{G} \) such that \( \mathcal{P} \) centralizes every element of \( \mathcal{U}(\mathcal{P}; r) \) for \( r \) in \( \tau - p \).

**Theorem C.** Let \( \mathcal{G} \) be a simple group which is \( p \)-tame for some prime \( p \) in \( \pi_3 \). Let \( \mathcal{P} \) be an \( S_p \)-subgroup of \( \mathcal{G} \) and suppose there exists a normal subgroup \( \mathcal{B} \) of \( \mathcal{P} \) of type \( (p, p) \) such that \( \mathcal{B} \) centralizes every element of \( \mathcal{U}(\mathcal{P}; q) \) for all but at most one prime \( q \). Then \( \mathcal{G} \) possesses a unique subgroup \( \mathcal{M} \) containing \( \mathcal{P} \) which is maximal subject to the condition \( p \in \pi_s(\mathcal{M}) \).

**Remark.** In the applications, if \( \mathcal{G} \) is a simple group which is \( \tau \)-tame for a set of primes \( \tau \) in \( \pi_3 \), the prime \( p \) and the subgroup \( \mathcal{B} \) of \( \mathcal{P} \) will be taken to satisfy the conditions of Theorem B. Under this assumption, it suffices to verify the hypothesis of Theorem C for primes \( q \) which are not in \( \tau \).

**Theorem D.** Let \( \mathcal{G} \) be a simple group which is strongly \( p \)-tame for some prime \( p \) in \( \pi_4 \), and let \( \mathcal{P} \) be an \( S_p \)-subgroup of \( \mathcal{G} \). Then \( \mathcal{G} \) possesses a unique subgroup \( \mathcal{M} \) containing \( \mathcal{P} \) which is maximal subject to the condition \( p \in \pi_s(\mathcal{M}) \).

**Theorem E.** Let \( \mathcal{G} \) be a simple group which satisfies the hypothesis of either Theorem C or D, and let \( \mathcal{M} \) be the unique subgroup of \( \mathcal{G} \) maximal subject to \( \mathcal{P} \subseteq \mathcal{M} \) and \( p \in \pi_s(\mathcal{M}) \). Then for each \( i = 1, 2, 3, 4 \) and each element \( \mathcal{P}_0 \) of \( \mathcal{A}_i(\mathcal{P}) \), \( \mathcal{M} \) is the unique subgroup of \( \mathcal{G} \) containing \( \mathcal{P}_0 \) which is maximal subject
to the condition \( p \in \pi_\psi(\mathfrak{M}) \). In particular, \( \mathcal{N}_\psi(\mathfrak{M}) \) contains every subgroup of \( \psi \) of type \((p, p, p)\), so that \( \mathfrak{M} \) is the unique maximal such subgroup containing a subgroup of \( \psi \) of type \((p, p, p)\).

**Remark.** Our generalization of the work of Feit and Thompson consists of introducing certain axioms and then adapting their proofs to this axiomatic approach. Thus Theorems A-E are extensions of corresponding statements which are proved in F.T. for minimal simple groups of odd order. Except for some preliminary lemmas which are established in the next section, the same is true of all the results in this paper. Unless there is a particular reason, we shall not refer explicitly to the lemma or theorem of F.T. from which a given result of this paper is derived.

### III. Preliminary Lemmas

We begin with several lemmas which we shall need.

**Lemma 3.1.** Let \( \mathfrak{G} \) be a group, and \( \mathfrak{D}, \mathfrak{D}_1 \) two normal subgroups of \( \mathfrak{G} \) such that \( \mathfrak{D} \leq \mathfrak{D}_1 \) and \( \mathcal{C}_{\mathfrak{D}_1}(\mathfrak{D}) \leq \mathfrak{D} \). If \( \mathcal{C}_{\mathfrak{G}}(\mathfrak{D}) \) is nonsolvable, then \( \mathcal{C}_{\mathfrak{G}}(\mathfrak{D}_1) \) is nonsolvable.

**Proof.** Set \( \mathfrak{C} = \mathcal{C}_{\mathfrak{G}}(\mathfrak{D}) \). By assumption \( \mathfrak{C} \cap \mathfrak{D}_1 \leq \mathfrak{D} \), so also \( \mathfrak{C} \lhd \mathfrak{G} \). Hence \( \gamma \mathfrak{D}_1 \mathfrak{C} \leq \mathfrak{D}_1 \cap \mathfrak{C} \leq \mathfrak{D} \), so that \( \gamma^2 \mathfrak{D}_1 \mathfrak{C}^2 = 1 \). It follows that \( \mathfrak{C} \) stabilizes the chain \( 1 \leq \mathfrak{D}_1 \leq \mathfrak{D} \). Hence if \( \mathfrak{R} = \ker(\mathfrak{C} \rightarrow \text{Aut}\mathfrak{D}_1) \), a theorem of P. Hall (4, Lemma 3, p. 791) implies that \( \mathfrak{C}/\mathfrak{R} \) is nilpotent. Since \( \mathfrak{C} \) is nonsolvable by assumption, so is \( \mathfrak{R} \). But \( \mathfrak{R} = \mathcal{C}_{\mathfrak{G}}(\mathfrak{D}_1) \), proving the lemma.

**Lemma 3.2.** Let \( \mathfrak{G} \) be a group and \( \mathfrak{D}_1 \) a normal subgroup of \( \mathfrak{G} \). Then the conclusion of Lemma 3.1 holds with \( \mathfrak{D} = \mathcal{F}(\mathfrak{D}_1) \) and \( \mathfrak{D}_1 = \mathcal{S}(\mathfrak{D}_1) \).

**Proof.** Since \( \mathcal{F}(\mathfrak{D}_1) = \mathcal{F}((\mathfrak{S})(\mathfrak{D}_1)) \), and \( \mathcal{F}((\mathfrak{S})(\mathfrak{D}_1)) \) contains its own centralizer in \( \mathcal{S}(\mathfrak{D}_1) \), the lemma follows.

**Lemma 3.3.** Let \( \mathfrak{G} \) be a simple group which is \( p \)-constrained for some prime \( p \). Let \( \mathfrak{G} \) be a proper subgroup of \( \mathfrak{G} \) such that \( p \in \pi_\psi(\mathfrak{G}) \) and such that an \( S_p \)-subgroup \( \psi \) of \( \mathfrak{G} \) has index at most \( p \) in an \( S_p \)-subgroup of \( \mathfrak{G} \). Then \( \mathcal{C}_{\mathfrak{G}}(\psi \cap \mathcal{O}_{p', p}(\mathfrak{G})) \) is solvable.

**Proof.** Assume the lemma is false. Set \( \psi_1 = \psi \cap \mathcal{O}_{p', p}(\mathfrak{G}) \). Then by assumption \( \mathcal{C}_{\mathfrak{G}}(\psi_1) \) is not solvable. Since \( \mathfrak{G} = \mathcal{O}_{p', p}(\mathfrak{G}) \mathcal{N}_{\mathfrak{G}}(\psi_1) \) by Sylow's theorem, we may assume without loss that \( \psi_1 \lhd \mathfrak{G} \). Let \( \psi_1^+ \) be an \( S_p \)-subgroup of \( \mathcal{O}_{p', p}(\mathcal{N}(\psi_1)) \). Since \( \mathfrak{G} \) is \( p \)-constrained, \( \mathcal{C}(\psi_1^+) \) is solvable. This forces \( \psi_1^+ \supset \psi_1 \), otherwise \( \mathcal{C}_{\mathfrak{G}}(\psi_1) \) would be solvable, contrary to assumption.
Since \( \mathfrak{P} \leq N(\mathfrak{P}_1) \) and \( \mathfrak{P} \) has index at most \( p \) in an \( S_p \)-subgroup of \( \mathfrak{G} \), we have \( |\mathfrak{P}^* : \mathfrak{P}_1| = p \). Set \( \mathfrak{R} = N(\mathfrak{P}^*) \). Then \( N(\mathfrak{P}_1) = O_{p'}(N(\mathfrak{P}_1)) (\mathfrak{R} \cap N(\mathfrak{P}_1)) \), and it follows that \( \mathfrak{C} = C(\mathfrak{P}^*) \) is nonsolvable. Let \( \mathfrak{C}^* = \ker (\mathfrak{C} \rightarrow \text{Aut} \mathfrak{P}^*) \). Since \( |\mathfrak{P}^* : \mathfrak{P}_1| = p \), \( \mathfrak{C}^* \) is cyclic, and hence \( \mathfrak{C}^* \) is nonsolvable. But \( \mathfrak{C}^* = C(\mathfrak{P}_1^*), \) and \( C(\mathfrak{P}_1^*) \) is solvable. This contradiction completes the proof.

**Lemma 3.4.** Let \( \mathfrak{G} \) be a simple group which is weakly \( p \)-tame for some prime \( p \). Let \( \mathfrak{G} \) be a proper subgroup of \( \mathfrak{G} \) such that \( p \in \pi(\mathfrak{G}) \). Let \( \mathfrak{P}_1 \) be an \( S_p \)-subgroup of \( \mathfrak{G} \) and assume that \( \mathfrak{P}_1 \) is an \( S_p \)-subgroup of \( N(\mathfrak{P}_1 \cap O_{p'}(\mathfrak{G})) \). Then if \( \mathfrak{P} \) is an \( S_p \)-subgroup of \( \mathfrak{G} \) containing \( \mathfrak{P}_1 \), every element of \( \mathcal{F}(\mathcal{N}_p'(\mathfrak{P})) \) lies in \( O_{p'}(\mathfrak{G}) \).

In particular, the lemma holds if \( \mathfrak{P}_1 \) is an \( S_p \)-subgroup of \( \mathfrak{G} \).

**Proof.** Let \( \mathfrak{P}_0 = \mathfrak{P}_1 \cap O_{p'}(\mathfrak{G}) \). If \( \mathfrak{P} \in \mathcal{F}(\mathcal{N}_p'(\mathfrak{P})) \), set \( \mathfrak{P}_1 = \mathfrak{P} \cap \mathfrak{P}_0 \) and \( \mathfrak{P}_0 = \mathfrak{P} \cap \mathfrak{P}_0 \). If \( \mathfrak{P}_0 \subset \mathfrak{P}_1 \), then also \( \mathfrak{P}_0 \subset \mathfrak{P}_1 \) since \( \mathfrak{P} \cap N(\mathfrak{P}_0) \supset \mathfrak{P}_0 \), and \( \mathfrak{P}_1 \) is an \( S_p \)-subgroup of \( N(\mathfrak{P}_1) \). But \( \gamma \mathfrak{P}_1 \mathfrak{P}_1 = 1 \) since \( \mathfrak{P} \) is an abelian normal subgroup of \( \mathfrak{P} \). Since \( \mathfrak{G} \) is \( p \)-stable, it will suffice to show that \( C_{\mathfrak{G}}(\mathfrak{P}_0) \) is solvable, for then the definition of \( p \)-stability will imply that \( \mathfrak{P}_1 \leq \mathfrak{P}_0 \), whence \( \mathfrak{P}_0 = \mathfrak{P}_1 = \mathfrak{P} \leq \mathfrak{P}_0 \), as asserted.

Let \( \mathfrak{R} = N(\mathfrak{P}_0) \). Now \( \mathfrak{R} = O_{p'}(\mathfrak{G}) (\mathfrak{R} \cap \mathfrak{G}) \) by Sylow's theorem. Since \( \mathfrak{P}_1 \) is an \( S_p \)-subgroup of \( \mathfrak{R} \) by assumption, it follows that \( \mathfrak{P}_0 \) is an \( S_p \)-subgroup of \( O_{p'}(\mathfrak{R}) \). But \( C(\mathfrak{P}_0) \subset \mathfrak{R} \), and hence an \( S_p \)-subgroup of \( O_{p'}(C(\mathfrak{P}_0)) \) is contained in \( O_{p'}(\mathfrak{R}) \). We conclude that \( \mathfrak{P}_0 \) is an \( S_p \)-subgroup of \( O_{p'}(\mathfrak{P}_0 C(\mathfrak{P}_0)) \). Since \( \mathfrak{G} \) is \( p \)-constrained, it follows that \( C(\mathfrak{P}_0) \) is solvable, and hence that \( C_{\mathfrak{G}}(\mathfrak{P}_0) \) is solvable.

The following lemma is essentially a restatement of Lemma 1.2.3 of [5].

**Lemma 3.5.** Let \( \mathfrak{G} \) be a group such that \( p \in \pi(\mathfrak{G}) \), let \( \mathfrak{P} \) be an \( S_p \)-subgroup of \( O_{p'}(\mathfrak{G}) \), and assume that \( C_{\mathfrak{G}}(\mathfrak{P}) \) is solvable. If \( \mathfrak{D} \) is any \( p' \)-subgroup of \( \mathfrak{G} \) such that \( \gamma \mathfrak{D} \mathfrak{P} \) is a \( p' \)-group, then \( \mathfrak{D} \leq O_{p'}(\mathfrak{G}) \).

**Proof.** Set \( \mathfrak{F} = \mathfrak{P}/O_{p'}(\mathfrak{G}) \) and let \( \mathfrak{P} \), \( \mathfrak{P} \) be the images of \( \mathfrak{D} \), \( \mathfrak{P} \), respectively, in \( \mathfrak{F} \). Since \( C_{\mathfrak{G}}(\mathfrak{P}) \) is solvable, \( C_{\mathfrak{F}}(\mathfrak{P}) \) is solvable, and consequently \( C_{\mathfrak{F}}(\mathfrak{P}) \leq S(\mathfrak{F}) \). Since \( \gamma \mathfrak{D} \mathfrak{P} \) is a \( p' \)-group, \( \mathfrak{F} \) centralizes \( \mathfrak{P} \), and hence \( \mathfrak{F} \leq S(\mathfrak{F}) \). But by Lemma 1.2.3 of [5], \( \mathfrak{P} \) contains its own centralizer in \( S(\mathfrak{F}) \). Hence \( \mathfrak{F} = 1 \), and \( \mathfrak{D} \leq O_{p'}(\mathfrak{G}) \).

We need two lemmas concerning automorphisms of \( p \)-groups.

**Lemma 3.6.** Let \( \mathfrak{P} \) be a \( p \)-group of class 2 and exponent \( p \). Let \( \mathfrak{P} \) be a \( p ' \)-subgroup of automorphisms of \( \mathfrak{P} \) such that \( O_{p'}(\mathfrak{P}) = 1 \) and \( [O_{p'}(\mathfrak{P}), \mathfrak{Z}(\mathfrak{P})] = 1 \). Let \( X \) be a \( p \)-element of \( \mathfrak{G} \). Assume \( p \) is odd, and that \( p \) is not a Fermat prime if \( [X, O_{p'}(\mathfrak{G})] \) is a nonabelian 2-group. Then \( C_{\mathfrak{G}}(X) \) is noncyclic.
Proof. The proof is by induction on $|\mathfrak{B}|$. Without loss we may assume that $\mathfrak{B} = \langle O_{p'}(\mathfrak{B}), X \rangle$. Suppose first that $|Z(\mathfrak{B})| = p$, in which case $\mathfrak{B}$ is extraspecial. Hence $\mathfrak{B} = \mathfrak{B}/Z(\mathfrak{B})$ may be regarded as a vector space of even dimension over $GF(p)$ on which $\mathfrak{B}$ is represented. Now (B) asserts that $X$ is represented on $\mathfrak{B}$ with minimal polynomial of degree $|X|$. Hence as $|X|$ is odd, $X$ is not represented indecomposably on $\mathfrak{B}$. Therefore $C_{\mathfrak{B}}(X)$ is noncyclic.

Hence we may assume that $|Z(\mathfrak{B})| > p$. Since $\langle Z(\mathfrak{B}), X \rangle$ is a $p$-group, $C_{Z(\mathfrak{B})}(X)$ contains a subgroup $\Psi_0$ of order $p$. In this case we set $\mathfrak{B} = \mathfrak{B}/\Psi_0$. Since $O_{p'}(\mathfrak{B})$ centralizes $Z(\mathfrak{B})$ by hypothesis, $\mathfrak{B}$ is represented on $\mathfrak{B}$. Let $\mathfrak{B} = Z(\mathfrak{B})$, and suppose first that $O_{p'}(\mathfrak{B})$ centralizes $\mathfrak{B}$. Then $\mathfrak{B} = \Psi_0$; otherwise $O_{p'}(\mathfrak{B})$ stabilizes the chain $\mathfrak{B} \supset Z(\mathfrak{B}) \supset \Psi_0$, in which case $O_{p'}(\mathfrak{B})$ centralizes $\mathfrak{B}$ by Lemma 8.1 of F.T., contrary to the fact that $O_{p'}(\mathfrak{B}) \not\subseteq \operatorname{Aut} \mathfrak{B}$ and $O_{p'}(\mathfrak{B}) \neq 1$. Hence $\mathfrak{B} \supset \mathfrak{B}$ and consequently $c(\mathfrak{B}) = 2$. Since $O_{p'}(\mathfrak{B})$ centralizes $Z(\mathfrak{B})$ and since $|\mathfrak{B}| < |\mathfrak{B}|$, we may apply the lemma by induction to $\mathfrak{B}$ to obtain that $C_{\mathfrak{B}}(X)$ is noncyclic. Denoting now by $\mathfrak{B}$ the inverse image of $\mathfrak{B}$ in $\mathfrak{B}$, it follows exactly as in the preceding paragraph that $C_{\mathfrak{B}}(X)$ is noncyclic.

So assume finally that $O_{p'}(\mathfrak{B})$ does not centralize $\mathfrak{B}$. Now the image $\mathfrak{B}_0$ of $Z(\mathfrak{B})$ in $\mathfrak{B}$ is contained in $\mathfrak{B}$, and since $|\mathfrak{B}| > p$, $\mathfrak{B}_0 \neq 1$. Since $O_{p'}(\mathfrak{B})$ centralizes $Z(\mathfrak{B})$, we see that $\mathfrak{B}_0 = C_{\mathfrak{B}}(O_{p'}(\mathfrak{B})) \neq 1$. But $X$ normalizes $\mathfrak{B}_0$, and consequently $C_{\mathfrak{B}}(X) \neq 1$. On the other hand, $X$ also normalizes $\mathfrak{B}_2 = [O_{p'}(\mathfrak{B}), \mathfrak{B}]$, and $\mathfrak{B}_2 \neq 1$. Thus $C_{\mathfrak{B}}(X) \neq 1$. Now $\mathfrak{B}$, being elementary, can be regarded as a vector space over $GF(p)$. It follows therefore from the complete reducibility of the representation of $O_{p'}(\mathfrak{B})$ on $\mathfrak{B}$ that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = 1$. We conclude from this that $C_{\mathfrak{B}}(X)$ is noncyclic, and it follows once again as in the preceding cases that $C_{\mathfrak{B}}(X)$ is noncyclic.

The following lemma is due to John Thompson.

**Lemma 3.7.** Let $x \times y$ be a group of automorphisms of a $p$-group $\mathfrak{B}$, where $x$ is a $p$-group and $y$ is a $p'$-group, and assume that $C_{\mathfrak{B}}(x) \subseteq C_{\mathfrak{B}}(y)$. Then $y = 1$.

**Proof.** Suppose false, and let $\mathfrak{B}$ be a $p'$-group of minimal order which satisfies the hypotheses, but not the conclusion of the lemma. Set $\mathfrak{R} = D(\mathfrak{B})$. Since $C_{\mathfrak{B}}(x) \subseteq C_{\mathfrak{B}}(y)$, the minimality of $\mathfrak{B}$ implies that $y$ acts trivially on $\mathfrak{R}$. Hence $C_{\mathfrak{B}}(y) \supseteq \mathfrak{R}$. We shall show that equality holds. If $\mathfrak{B}_1 = y \mathfrak{B}$, then $\mathfrak{B}_1$ is invariant under both $x$ and $y$. Since $C_{\mathfrak{B}_1}(x) \subseteq C_{\mathfrak{B}_1}(y)$, the assumption $\mathfrak{B}_1 \subseteq \mathfrak{B}$ forces $y$ to act trivially on $\mathfrak{B}_1$. But $\mathfrak{B}_1 = y \mathfrak{B}_1$ by Lemma 8.11 of F.T., and consequently $\mathfrak{B}_1 = 1$. Thus $y$ centralizes $\mathfrak{B}$, whence $y = 1$.
contrary to our choice of $\mathfrak{P}$. Hence $\mathfrak{P}_1 = \mathfrak{P}$; and it follows from the complete reducibility of $\mathfrak{P}$ on $\mathfrak{P}/\mathfrak{Q}$ that $\mathfrak{C}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{P}) - 1$. Hence $\mathfrak{C}_{\mathfrak{P}}(\mathfrak{P}) \subseteq \mathfrak{Q}$, and $\mathfrak{C}_{\mathfrak{P}}(\mathfrak{P}) - \mathfrak{Q}$ follows.

Now let $\mathfrak{H}$ be the inverse image in $\mathfrak{P}$ of $\mathfrak{C}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{Q})$. Then $\mathfrak{Q} \subseteq \mathfrak{H}$ and $\mathfrak{H}$ is $\mathfrak{P}$-invariant. Since $\gamma \mathfrak{H} \subseteq \mathfrak{H}$, we have $\gamma^2 \mathfrak{H} \mathfrak{Q}^2 = 1$. Since $\gamma \mathfrak{Q} \mathfrak{Q}^2 = 1$, we also have $\gamma \mathfrak{Q} \mathfrak{Q}^2 = 1$. But then $\gamma \mathfrak{Q} \mathfrak{Q}^2 = 1$ by (3.1) of F.T., and consequently $\mathfrak{Q} \mathfrak{H} \subseteq \mathfrak{C}_{\mathfrak{Q}}(\mathfrak{Q}) \subseteq \mathfrak{C}_{\mathfrak{P}}(\mathfrak{Q}) = \mathfrak{Q}$. Thus $\mathfrak{H} \mathfrak{Q} \subseteq \mathfrak{C}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{Q}) = 1$, contrary to the fact that $\mathfrak{H} \mathfrak{Q} \neq 1$.

IV. The Transitivity Theorem

The next five sections are devoted to a proof of Theorem A. An essential step in the proof is a generalization of the transitivity theorem of Feit and Thompson (Theorem 17.1 of F.T.).

**Theorem 1.** Let $\mathfrak{G}$ be a simple group which is weakly $p$-tame for some prime $p$, let $\mathfrak{P}$ be an $S_p$-subgroup of $\mathfrak{G}$, and let $\mathfrak{Q} \in \mathcal{K}_3(\mathfrak{P})$. Then $\mathfrak{C}(\mathfrak{Q})$ acts transitively on the maximal elements of $\mathcal{N}(\mathfrak{Q}; \mathfrak{P})$ for any prime $q$.

The proof depends upon the following lemma:

**Lemma 4.1.** If $\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, then every element of $\mathcal{N}(\mathfrak{H}; \mathfrak{Q})$ is contained in $\mathfrak{Q}_p(\mathfrak{H})$.

**Proof.** Suppose the lemma false. Among all subgroups $\mathfrak{S}$ of $\mathfrak{G}$ which violate the lemma, choose $\mathfrak{S}$ so that $|\mathfrak{S} \cap \mathfrak{Q}|$ is maximal, subject to this condition, assume $|\mathfrak{S}|_p$ is maximal, and subject to these conditions, assume that $|\mathfrak{S}|_p$ is minimal. Let $Q$ be an element of $\mathcal{N}(\mathfrak{S}; \mathfrak{Q})$ such that $Q \subseteq \mathfrak{C}_{\mathfrak{P}}(\mathfrak{S})$. First of all, if $p \notin \pi(\mathfrak{S})$, $Q \leq S(\mathfrak{S})$ by condition (IV) in the definition of weakly $p$-tame groups. But in this case, $S(\mathfrak{S}) = \mathfrak{Q}_p(\mathfrak{S})$, and hence $Q \subseteq \mathfrak{Q}_p(\mathfrak{S})$. Thus $p \in \pi(\mathfrak{S})$. Hence if $\mathfrak{P}_1$ is an $S_{p'}$-subgroup of $\mathfrak{Q}_{p'}(\mathfrak{S})$ normalized by $\mathfrak{P} \cap \mathfrak{S}$, we have $\mathfrak{P}_1 \neq 1$. Set $\mathfrak{H}_1 = N_{\mathfrak{G}}(\mathfrak{P}_1)$. Since $\mathfrak{H}_1 \subseteq \mathfrak{Q}_p(\mathfrak{S})$, by Sylow's theorem, $\mathfrak{H}_1$ must contain an element of $\mathcal{N}(\mathfrak{Q}; \mathfrak{P})$ which does not lie in $\mathfrak{Q}_p(\mathfrak{S})$, otherwise $\mathfrak{Q}_p(\mathfrak{S})$ would contain every element of $\mathcal{N}(\mathfrak{Q}; \mathfrak{P})$. So then $\mathfrak{H}_1 = \mathfrak{H}$ by the minimality of $|\mathfrak{S}|_p$. Thus $\mathfrak{H} \subseteq N_{\mathfrak{G}}(\mathfrak{P}_1)$. Since $\mathfrak{Q}_p(\mathfrak{N}(\mathfrak{P}_1)) \cap \mathfrak{H} \subseteq \mathfrak{Q}_p(\mathfrak{S})$, it follows that the lemma is violated in $\mathfrak{N}(\mathfrak{P}_1)$. But now $|\mathfrak{S}|_p = |\mathfrak{N}(\mathfrak{P}_1)|_p$ by the maximality of $|\mathfrak{S}|_p$, and hence $\mathfrak{P}_1$ is an $S_{p'}$-subgroup of $\mathfrak{Q}_{p'}(\mathfrak{N}(\mathfrak{P}_1))$. Since $\mathfrak{G}$ is $p$-constrained, this implies that $\mathfrak{C}(\mathfrak{S}_1)$ is solvable.

Let $\mathfrak{B}$ be an element of $\mathcal{N}(\mathfrak{P})$ with $\mathfrak{B} \cap \mathfrak{Q}$. Let $B \in \mathfrak{B}$, and set $\mathfrak{Q}_0 = \mathfrak{C}_{\mathfrak{Q}}(B)$, $\mathfrak{P}_0 = \mathfrak{C}_{\mathfrak{P}}(B)$. Suppose that $\mathfrak{Q}_0 \subseteq \mathfrak{Q}(\mathfrak{C})$. Since $\mathfrak{P}_0 \subseteq \mathfrak{C}(\mathfrak{B})$, it follows that $\gamma \mathfrak{Q}_0 \mathfrak{P}_0$ is a $p$-group. But $\gamma \mathfrak{Q}_0 \mathfrak{P}_0$ is a $p$-group, and hence $\mathfrak{Q}_0$ centralizes $\mathfrak{P}_0$. Set $\mathfrak{H} = \mathfrak{H}/\mathfrak{Q}_0(\mathfrak{S})$ and let $\mathfrak{H}_0, \mathfrak{P}_0, \mathfrak{B}$, be the images of $\mathfrak{Q}_0, \mathfrak{P}_0, \mathfrak{B}$.
respectively, in $\mathbb{S}$. Since $C(\mathbb{S})$ is solvable, $C(\mathbb{S}) \subseteq O_{\mu}(\mathbb{S})$ by Lemma 1.2.3 of [6], and hence $C(\mathbb{S}) \subseteq \mathbb{S}$. Thus $\mathbb{S}$ is faithfully represented as a subgroup of $\text{Aut} \ \mathbb{S}$. We apply Lemma 3.7 to the group $\mathbb{S} \cong \mathbb{B}$ acting on $\mathbb{S}$. Since $\mathbb{S}$ centralizes $\mathbb{S} \cong C(\mathbb{B})$, we conclude that $\mathbb{S} \subseteq \mathbb{B}$, and hence $\mathbb{S} \subseteq C(\mathbb{B}) \subseteq O_{\mu}(\mathbb{S})$. Now $\mathbb{S}$ is generated by its subgroups $C(\mathbb{B})$ with $\mathbb{B}$ in $\mathbb{S}$. Since $\mathbb{S} \subseteq O_{\mu}(\mathbb{S})$, it follows from the preceding argument that $C(\mathbb{B}) \subseteq O_{\mu}(C(\mathbb{B}))$ for some $\mathbb{B}$ in $\mathbb{S}$. But $C(\mathbb{B})$ is invariant under $\mathbb{S}$ and $\mathbb{S} \subseteq C(\mathbb{B})$. Hence the lemma is violated in $\mathbb{S}$. Since $\mathbb{S} \cap C(\mathbb{B})$ has index at most $p$ in $\mathbb{S}$, we conclude from our maximal choice of $\mathbb{S} \cap \mathbb{S}$ that $\mathbb{S} \cap \mathbb{S}$ has index at most $p$ in $\mathbb{S}$. Let $\mathbb{S}$ be an $S_p$-subgroup of $\mathbb{S}$ containing $\mathbb{S} \cap \mathbb{S}$. Now $Z(\mathbb{S}) = \mathbb{S} \subseteq \mathbb{S} \cap \mathbb{S}$ since $\mathbb{S}$ is $p$-stable, and $C(\mathbb{S})$ is solvable, it follows that $Z(\mathbb{S}) \subseteq \mathbb{S}$. Hence if $\mathbb{S}_1 = C(\mathbb{S}(\mathbb{S}))$, we have $\mathbb{S} = \mathbb{S}_1(\mathbb{S} \cap O_{\mu}(\mathbb{S}))$. Suppose $\mathbb{S}_1 \subseteq O_{\mu}(\mathbb{S})$, where $\mathbb{S} = N(\mathbb{S})$. Then $\mathbb{S}_1$ is a $p'$-group; and in particular, $\mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group. Since $\mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group, it follows that $\mathbb{S}_1 \subseteq O_{\mu}(\mathbb{S})$, $\mathbb{S}_1 \cap \mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ and $\mathbb{S}_1 \cap \mathbb{S}_1(\mathbb{S} \cap \math{S})$ is a $p'$-group. Since $\mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group, it follows from Lemma 3.5 that $\mathbb{S}_1 \cap \mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ and $\mathbb{S}_1 \cap \mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group. Since $\mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group, it follows from Lemma 3.5 that $\mathbb{S}_1 \cap \mathbb{S}_1(\mathbb{S} \cap \mathbb{S})$ is a $p'$-group. Applying Lemma 3.5 once again, we conclude that $\mathbb{S} \subseteq O_{\mu}(\mathbb{S})$. The lemma follows.

We turn now to the proof of Theorem 1. Denote by $2_1, 2_2, \ldots, 2_t$ the sets of transitivity of the maximal elements of $\mathcal{N}(\mathbb{S}; q)$ under the action of $C(\mathbb{S})$. We shall derive a contradiction from the assumption $t \geq 2$. Clearly this assumption implies that 1 is not a maximal element of $\mathcal{N}(\mathbb{S}; q)$.

We first show that if $\mathbb{S}_1$ and $\mathbb{S}_2$ are maximal elements of $\mathcal{N}(\mathbb{S}; q)$ with $\mathbb{S}_1$ in $\mathcal{N}_i$ and $\mathbb{S}_2$ in $\mathcal{N}_j$, $i \neq j$, then $\mathbb{S}_1 \cap \mathbb{S}_2 = 1$. Assume false. Among all pairs $(i, j)$ with $1 \leq i < j \leq t$ for which there exist elements $\mathbb{S}_1$, $\mathbb{S}_2$ in $\mathcal{N}_i, \mathcal{N}_j$ respectively with nontrivial intersection, choose $i, j$ and $\mathbb{S}_1$, $\mathbb{S}_2$ so that $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$ has maximal order. Since $\mathbb{S}_1, \mathbb{S}_2$ are maximal elements of $\mathcal{N}(\mathbb{S}; q)$, $\mathbb{S}$ is necessarily a proper subgroup of both $\mathbb{S}_1$ and $\mathbb{S}_2$. Set $\mathbb{S} = N(\mathbb{S})$ and let $\mathbb{S}_i' = N(\mathbb{S}_i) = i \neq 1, 2$. Then $\mathbb{S}_i', \mathbb{S}_2', \mathbb{S}_2$, and $\mathbb{S}$ are contained in $\mathbb{S}$ and $\mathbb{S}$ normalizes $\mathbb{S}_i$, $i = 1, 2$. It follows therefore from the preceding lemma that $\mathbb{S}_i' \subseteq O_{\mu}(\mathbb{S})$, $i = 1, 2$. But now by $D_{\mu,q}$ in $O_{\mu}(\mathbb{S})$ $\mathbb{S}_i'$ is contained in an $\mathbb{S}$-invariant $S_\mu$-subgroup $\mathbb{S}_i^{\mu}$ of $O_{\mu}(\mathbb{S})$, $i = 1, 2$. Since $\mathbb{S}$ is an
$S_p$-subgroup of $O_{p'}(\mathcal{V})$ if $Q_1^+$ and $Q_2^+$ are necessarily conjugate by an element $D$ in $C(\mathfrak{H}) \cap O_{p'}(\mathcal{V})$. Thus $Q_1^+ = Q_2^{+D}$ and $D \in C(\mathfrak{H})$. Let $Q_3$ be a maximal element of $\mathcal{V}(\mathfrak{H}; q)$ containing $Q_1^+$. Since $Q_3 \cap Q_1 \subseteq Q_1 \cap \mathcal{H}$, $Q_3 \in \mathcal{L}_1$. Since $Q_3 \cap Q_2^{+D} \subseteq Q_2^{+D} \cap \mathcal{H}$, and $D \in C(\mathfrak{H})$, $Q_3 \in \mathcal{L}_1$. Thus $i = j$, a contradiction.

Now choose $Q_4 \in \mathcal{L}_1$ and $Q_5 \in \mathcal{L}_2$. Since $\mathfrak{H} \in \mathcal{P}(\mathcal{H})$, $\mathcal{H}$ contains an elementary subgroup $E$ of order $p^3$. Hence there exists an element $E$ in $C(\mathfrak{H})$ such that $Q_4 = C_{Q_4}(E) \neq 1$, $i = 1, 2$. Set $\mathcal{E} = C(E)$. Now $\mathcal{E}$ normalizes $Q_4$ and $Q_4^{+D} \subseteq \mathcal{E}$. It follows therefore from Lemma 4.1 that $Q_4^{+D} \subseteq O_{p'}(\mathfrak{H})$, $i = 1, 2$. As above, this implies that there exists an element $D$ in $C(\mathfrak{H})$ such that $Q_4^{+D}$ and $Q_5^{+D}$ lie in an $\mathfrak{H}$-invariant $S_p$-subgroup $\mathcal{L}_1^+$ of $O_{p'}(\mathfrak{H})$. If $Q_3$ denotes a maximal element of $\mathcal{V}(\mathfrak{H}; q)$ containing $Q_1^+$, it follows that $Q_3 \cap Q_4 \neq 1$ and that $Q_3 \cap Q_5^{+D} \neq 1$. But then $Q_3 \in \mathcal{L}_1$ and $Q_3 \in \mathcal{L}_2$ by the preceding argument. This contradiction completes the proof of the theorem.

The following three corollaries of the transitivity theorem are proved exactly as in F.T. Let $\mathfrak{G}$ be a simple group which is weakly $p$-tame for some prime $p$, let $\mathcal{G}$ be an $S_p$-subgroup of $\mathfrak{G}$, and let $\mathfrak{H} \in \mathcal{P}(\mathcal{H})$.

**Corollary 4.1.** If $Q$ is a maximal element of $\mathcal{V}(\mathfrak{H}; q)$, then

$$N(\mathfrak{H}) = (N(Q) \cap N(\mathfrak{H})) C(\mathfrak{H}),$$

and $N(Q)$ contains an $S_p$-subgroup of $N(\mathfrak{H})$.

**Corollary 4.2.** $\mathcal{G}$ normalizes some maximal element $Q$ of $\mathcal{V}(\mathfrak{H}; q)$. Furthermore, if $G \in \mathfrak{G}$ and $\mathfrak{H}^G \subseteq \mathcal{G}$, then $\mathfrak{H}^G = \mathfrak{H}^N$ for some $N$ in $N(Q)$.

**Corollary 4.3.** If $p \in \pi_1$, then $\mathcal{V}(\mathfrak{H})$ is trivial.

Theorem 1 and Lemma 4.1 imply the following additional corollary:

**Corollary 4.4.** A maximal element of $\mathcal{V}(\mathfrak{H}; q)$ is a maximal element of $\mathcal{V}(\mathfrak{H}; q)$.

We can obtain one additional consequence of Theorem 1 which depends only upon the assumption that $\mathfrak{G}$ is weakly $p$-tame.

**Lemma 4.2.** Let $\mathfrak{G}$ be a simple group which is weakly $p$-tame, for some prime $p$. Let $\mathcal{G}$ be an $S_p$-subgroup of $\mathfrak{G}$, let $\mathfrak{H} \in \mathcal{P}(\mathcal{H})$, and let $Q$ be a maximal element of $\mathcal{V}(\mathfrak{H}; q)$ which is normalized by $\mathcal{G}$. Set $\mathfrak{H} = N(Q)$ and $\mathfrak{H}_1 = N(Z(\mathfrak{H}))$, where $\mathfrak{H} = V(ccl_0(\mathfrak{H}); \mathcal{G})$. Then $\mathfrak{H}_1 = O_{p'}(\mathfrak{H}_1)(\mathfrak{H}_1 \cap \mathfrak{H})$.

**Proof.** It will suffice to show that $C(\mathfrak{H}) \subseteq O_{p'}(\mathfrak{H}_1)$, for then the proof of Lemma 17.3 of F.T. will apply without change. Since $Z(\mathfrak{H})$ centralizes
\[
\begin{align*}
\mathfrak{A} \subseteq \Psi \text{ and } \forall \in S^*G_{\mathcal{C}}(\Psi), Z(\Psi) \subseteq \mathfrak{A} \text{ and hence } C(\mathfrak{A}) \subseteq \mathfrak{A}. \text{ By Lemma 3.4, } \mathfrak{A} \subseteq O_{p',p}(\mathfrak{A}). \text{ Now } C(\mathfrak{A}) = \mathfrak{A} 	imes \mathfrak{D}, \text{ where } \mathfrak{D} \text{ is a } p'\text{-group by (3.10) of F.T. Since } \Psi \text{ normalizes } \mathfrak{D}, \gamma^\Psi \mathfrak{D} \text{ is a } p'\text{-group. Furthermore Lemma 3.3 implies that } C(\mathfrak{A}, (\Psi) \text{ is solvable, where } \Psi_1 = \Psi \cap O_{p',p}(\mathfrak{A}). \text{ Since } \gamma^\Psi \mathfrak{D} \text{ is a } p'\text{-group, it follows from Lemma 3.5 that } \mathfrak{D} \subseteq O_{p'}(\Psi_1). \text{ Thus } C(\mathfrak{A}) \subseteq O_{p',p}(\Psi_1), \text{ as required.}
\end{align*}
\]

V. Consequences of the Transitivity Theorem

A number of striking consequences of the transitivity theorem are established in F.T. by exploiting this result in conjunction with the transfer theorem of P. Hall (3, Theorem 14.4.1, p. 211). We shall now derive the corresponding results for any simple group which is \(p\)-tame for some prime \(p\). Throughout this section \(\mathfrak{G}\) will denote a simple group which is \(p\)-tame.

We let \(\mathfrak{S}\) be an \(S_p\)-subgroup of \(\mathfrak{G}\), \(\forall \in \mathfrak{S}\), \(\Psi_1\) an element of \(S^*G_{\mathcal{C}}(\Psi), \mathfrak{S}\) a maximal element in \(\mathcal{N}(\mathfrak{G}; q)\) which is normalized by \(\Psi, \mathfrak{N} = N(\mathfrak{S}) \text{ and } \mathfrak{R}_1 = N(Z(\Psi)), \text{ where } \Psi = V(C_{\mathfrak{G}}(\Psi); \Psi)\).

**Lemma 5.1.** If \(\mathfrak{A}^G \subseteq \Psi\) for some \(G\) in \(\mathfrak{G}\), then \(\mathfrak{A}^G = \mathfrak{A}^N\) for some element \(N\) in \(N(\mathfrak{S}) \cap N(\Psi)\).

**Proof.** By Corollary 4.2, \(\mathfrak{A}^G = \mathfrak{A}^X, X \in \mathfrak{R}\). Since \(\mathfrak{G}\) is \(p\)-tame, it follows from (V) that \(p \in \pi(\mathfrak{R})\), so that \(\mathfrak{A} \subseteq O_{p',p}(\mathfrak{R})\) by Lemma 3.4. If \(\Psi_1 = \Psi \cap O_{p',p}(\mathfrak{R})\), then \(\forall = O_{p'}(\mathfrak{R}) N_{\Psi}(\Psi_1)\) by Sylow's theorem. Since \(\mathfrak{A} \subseteq \Psi_1\), the first assertion of the proof implies that \(\Psi \subseteq \Psi_1\). Thus \(N_{\Psi}(\Psi_1) \subseteq N_{\Psi}(\Psi)\), and we conclude that \(\forall = O_{p'}(\Psi) N_{\Psi}(\Psi)\). Now the proof of Lemma 17.1 of F.T. applies without change.

**Lemma 5.2.**

\[ \mathfrak{R}_1 = O_{p}(\mathfrak{R}_1). \]

**Proof.** The proof is identical with that of Lemma 17.2 of F.T., inasmuch as we have established Lemma 5.1 and have shown that \(\forall = O_{p'}(\Psi) N_{\Psi}(\Psi)\), these being the two critical results which are needed, in addition to the Hall transfer theorem.

**Lemma 5.3.**

\[ \forall = O_{p}(\mathfrak{R}) \]

**Proof.** By Lemma 4.2, \(\mathfrak{R}_1 = O_{p'}(\mathfrak{R}_1) (\mathfrak{R}_1 \cap \forall)\). Since \(\mathfrak{R}_1\) possesses no
normal subgroups of index $p$ by the preceding lemma, it follows at once
from this equality that $\mathbb{G}$ possesses no normal subgroups of index $p$.

**Lemma 5.4.** If $\mathbb{G}$ is a subgroup of $\mathbb{H}$ which contains $\mathbb{B}$, then $N(\mathbb{G}) \subseteq \mathbb{B}$. The proof is identical with that of Lemma 17.4 of F.T.

**Lemma 5.5.** If $\mathbb{B}$ is a proper subgroup of $\mathbb{H}$ which contains $\mathbb{B}$ and such that $p \in \pi(\mathbb{H})$, then $\mathbb{B} \subseteq O_{p'.p}(\mathbb{H})$.

**Proof.** Since we have already shown that $\mathbb{B} \subseteq O_{p'.p}(\mathbb{H})$, the proof of Lemma 17.5 of F.T. applies without change.

**Lemma 5.6.** Let $\mathbb{B}$ be a proper subgroup of $\mathbb{H}$ and $\mathbb{B}_1$ an $S_p$-subgroup of $\mathbb{B}$, and assume that $p \in \pi(\mathbb{B})$. Then if $\mathbb{B}$ is an $S_p$-subgroup of $\mathbb{H}$ containing $\mathbb{B}_1$ and $\mathbb{B} \in \mathcal{C}(\mathbb{B}, \mathbb{B}_1)$, $\mathbb{B}_1 \subseteq O_{p'.p}(\mathbb{B})$.

**Proof.** Suppose false. Choose $\mathbb{B}$ to violate the lemma so that $|\mathbb{B}|_p$ is maximal, and subject to this condition, make $|\mathbb{B}|_p$ minimal. Set $\mathbb{B}_0 = \mathbb{B}_1 \cap O_{p'.p}(\mathbb{B})$. Since $\mathbb{B} = O_{p'.p}(\mathbb{B}) \cap O_{p'.p}(\mathbb{B}_0)$, the minimality of $\mathbb{B}$ implies that $\mathbb{B} = N(\mathbb{B}_0)$, whence $\mathbb{B}_0 \leq \mathbb{B}$. If $\mathbb{B}_1$ were not an $S_p$-subgroup of $N(\mathbb{B}_0)$, then by our maximal choice of $\mathbb{B}$, $\mathbb{B}_1 \subseteq O_{p'.p}(N(\mathbb{B}_0))$. Since $\mathbb{B} \subseteq N(\mathbb{B}_0)$, $\mathbb{B}_1 \subseteq O_{p'.p}(\mathbb{B})$ would follow. Hence $\mathbb{B}_1$ is an $S_p$-subgroup of $N(\mathbb{B}_0)$, so that by Lemma 3.4, $\mathbb{B} \subseteq \mathbb{B}_0$. But then $N(\mathbb{B}_0)$, and consequently also $\mathbb{B}$, lies in $\mathbb{B}_1$ by Lemma 5.4. Since $\mathbb{B}_1 \subseteq \mathbb{B}$, Lemma 5.5 implies that $\mathbb{B}_1 \subseteq O_{p'.p}(\mathbb{B}_1)$. Since $\mathbb{B} \subseteq \mathbb{B}_1$, $\mathbb{B}_1 \subseteq O_{p'.p}(\mathbb{B})$, and the lemma follows.

**VI. A Sufficient Condition for $E_{p^q}$**

In this section we shall establish the following generalization of Theorem 19.1 of F.T. for primes in $\pi_q$.

**Theorem 2.** Let $\mathbb{G}$ be a simple group which is $(p, q)$-tame for two primes $p$, $q$ in $\pi_q$. If an $S_p$-subgroup of $\mathbb{G}$ centralizes every $q$-subgroup which it normalizes and if an $S_q$-subgroup of $\mathbb{G}$ centralizes every $p$-subgroup of $\mathbb{G}$ which it normalizes, then $\mathbb{G}$ satisfies $E_{p^q}$.

We proceed by contradiction. The following lemmas are proved under the assumption that Theorem 2 is false.

**Lemma 6.1.** If $\mathbb{B} \in \mathcal{B}(p)$, then $\mathbb{B}$ centralizes every element of $\mathcal{V}(\mathbb{B}; q)$.

**Proof.** Assume the lemma false, and choose $\mathbb{Q}$ of minimal order violating the conclusion. If $\mathbb{B}_0 = \ker(\mathbb{B} \rightarrow \text{Aut} \mathbb{G})$, (3.11) of F.T. implies that $\mathbb{B}_0 \neq 1$. 

Let $\mathfrak{B}$ be an $S_p$-subgroup of $\mathfrak{N}(\mathfrak{B})$, let $\mathfrak{C} = C(\mathfrak{B}_0)$, and let $\mathfrak{B}_0 = C(\mathfrak{B}) \cap \mathfrak{B}$. Then $\mathfrak{B}_0 : \mathfrak{B}_0 < p$ and $\mathfrak{B}_0 \subseteq \mathfrak{C}$. Let $\mathfrak{B}^*$ be an $S_p$-subgroup of $\mathfrak{C}$ containing $\mathfrak{B}_0$. If $\mathfrak{B}^* = \mathfrak{B}_0$, then $C_{\mathfrak{C}}(\mathfrak{B}^* \cap O_{p'.p}(\mathfrak{C}))$ is solvable by Lemma 3.3. Since $\mathfrak{B}$ centralizes $\mathfrak{B}^*$, $\mathfrak{B} \subseteq \mathfrak{B}^* \cap O_{p'.p}(\mathfrak{C})$ by Lemma 1.2.3 of [5]. On the other hand, if $\mathfrak{B}^* \supsetneq \mathfrak{B}_0$, then $\mathfrak{B}^*$ is an $S_p$-subgroup of $\mathfrak{C}$ and $\mathfrak{B} \subseteq \mathfrak{B}^*$ in this case as well. But now our minimal choice of $\mathfrak{B}$ forces $\mathfrak{B} \subseteq O_{p'.p}(\mathfrak{C})$.

By Lemma 8.9 of F.T., $\mathfrak{B}$ is contained in an element $\mathfrak{A}$ of $\mathfrak{F}(\mathfrak{N}(\mathfrak{B}))$, and $\mathfrak{B} \subseteq \mathfrak{C}$. By $D_{p,q}$ in $O_{p'.q}(\mathfrak{C})$, $\mathfrak{A}$ normalizes some $S_q$-subgroup $\mathfrak{Q}_1$ of $O_{p'.q}(\mathfrak{C})$. Hence by Theorem 1, Corollary 4.2, and the hypotheses of the present theorem, $\mathfrak{B}$ centralizes $\mathfrak{Q}_1$. Thus $\mathfrak{B}$ centralizes every $q$-subgroup of $O_{p'.q}(\mathfrak{C})$ which it normalizes, so that, in particular, $\mathfrak{B}$ centralizes $\mathfrak{Q}$, a contradiction.

**Lemma 6.2.** If $\mathfrak{A} \in \mathfrak{W}(p)$ and $\mathfrak{B} \in \mathfrak{W}(q)$ and $\mathfrak{R}$ is a subgroup of $\mathfrak{G}$ containing $\mathfrak{A}$ and $\mathfrak{B}$, then neither $p$ nor $q$ is in $\pi_4(\mathfrak{R})$.

**Proof.** Suppose the lemma false and let $\mathfrak{R}$ be a subgroup of $\mathfrak{G}$ which violates the lemma. Set $\sigma = \{p, q\}$. Since $p$ or $q$ is in $\pi_4(\mathfrak{R})$, either $O_{p,q}(\mathfrak{R}) \supseteq O_{p,q}(\mathfrak{R})$ or $O_{p,q}(\mathfrak{R}) \supseteq O_{p,q}(\mathfrak{R})$. For definiteness, assume that $O_{p,q}(\mathfrak{R}) \supseteq O_{p,q}(\mathfrak{R})$. Among all subgroups $\mathfrak{R}$ which violate the lemma and in which $O_{p,q}(\mathfrak{R}) \supseteq O_{p,q}(\mathfrak{R})$, choose $\mathfrak{R}$ to maximize $|\mathfrak{R}|_p$ and subject to this condition, minimize $|\mathfrak{R}|_q$. Let $\mathfrak{A}$, $\mathfrak{B}$ be elements of $\mathfrak{W}(p)$ and $\mathfrak{W}(q)$ respectively which lie in $\mathfrak{R}$. Let $\mathfrak{D}$ be an $S_{p,q}$-subgroup of $S(\mathfrak{R})$ and set $\mathfrak{R}_1 = N_{\mathfrak{R}}(\mathfrak{D})$. Since $\mathfrak{R} = S(\mathfrak{R})$, $\mathfrak{R}_1$ by $C_{p,q}$ in $S(\mathfrak{R})$, $\mathfrak{R}_1$ contains an $S_p$-subgroup and an $S_q$-subgroup of $\mathfrak{R}$, and hence contains a conjugate of $\mathfrak{A}$ and a conjugate of $\mathfrak{B}$. Thus the lemma is violated in $\mathfrak{R}_1$. Since $O_{p,q}(\mathfrak{R}_1) \supseteq O_{p,q}(\mathfrak{R}_1)$, it follows from our minimal choice of $|\mathfrak{R}|_p$ that $\mathfrak{R}_1 = \mathfrak{R}$. Thus $\mathfrak{D} \subset \mathfrak{R}$ and $O_{p,q}(\mathfrak{D}) \neq 1$. Furthermore, if $\mathfrak{S} = O_{p,q}(\mathfrak{D})$, our maximal choice of $|\mathfrak{R}|_p$ implies that $|\mathfrak{R}|_p = |\mathfrak{S}|_p$. Since $\mathfrak{R} \subseteq \mathfrak{S}$, it follows that an $S_{p,q}$-subgroup $\mathfrak{F}_1$ of $\mathfrak{R}$ is an $S_p$-subgroup of $\mathfrak{S}$. But $\mathfrak{S}$ is $p$-constrained, and hence $C(\mathfrak{F}_1 \cap O_{p',q}(\mathfrak{S}))$ is solvable. Since $\mathfrak{R} \subseteq \mathfrak{S}$,

$$\mathfrak{F}_1 \cap O_{p',q}(\mathfrak{S}) \supseteq \mathfrak{F}_1 \cap O_{p',q}(\mathfrak{S})$$

and consequently $C(\mathfrak{F}_1 \cap O_{p',q}(\mathfrak{S}))$ is solvable. Therefore $C(\mathfrak{S})$ is solvable, and we conclude from Lemma 3.2 that $C_\mathfrak{R}(F(\mathfrak{D}))$ is solvable.

We shall use this last result to show that $<\mathfrak{A}, \mathfrak{B}>$ is solvable, so assume the contrary. By the preceding lemma, $\mathfrak{B}$ centralizes $O_{p}(\mathfrak{D})$, and hence $\mathfrak{B} \subseteq \mathfrak{C} = C_\mathfrak{R}(O_{p}(\mathfrak{D}))$. Now $\mathfrak{C} \not\subseteq \mathfrak{R}$, and hence $\mathfrak{L} = \mathfrak{C} \cap \mathfrak{B}$ is a group. Again by the preceding lemma, $\mathfrak{L}^\mathfrak{U}$ centralizes $O_{p}(\mathfrak{D})$. Thus $\mathfrak{X} = \mathfrak{L}^\mathfrak{U} \cap \mathfrak{C}$ centralizes $F(\mathfrak{D}) = O_{p}(\mathfrak{D}) \times O_{q}(\mathfrak{D})$, and consequently $\mathfrak{X}$ is solvable. On the other hand, $\mathfrak{L}^\mathfrak{U} \not\subseteq \mathfrak{L}$ and $\mathfrak{B} \subseteq \mathfrak{L}$, forcing $\mathfrak{L}^\mathfrak{U}$ to be nonsolvable; otherwise $\mathfrak{L}^\mathfrak{U} \mathfrak{B}$ would be solvable, and hence $<\mathfrak{A}, \mathfrak{B}>$ would be solvable, contrary to our present
assumption. But \( \mathfrak{L} = \mathfrak{L} \) and \( \mathfrak{C} \subseteq \mathfrak{L} \), whence \( \mathfrak{L}^{1/\mathfrak{L}} \) is a \( p \)-group. Thus \( \mathfrak{L} \) is not solvable. This contradiction shows that \( \mathfrak{A} \) and \( \mathfrak{B} \) generate a solvable subgroup of \( \mathfrak{R} \).

Therefore \( \langle \mathfrak{A}, \mathfrak{B} \rangle \) contains an \( S_{p,q} \)-subgroup \( \mathfrak{D}^* \), and consequently a conjugate of \( \mathfrak{A} \) and a conjugate of \( \mathfrak{B} \) lies in \( \mathfrak{D}^* \). Thus there exists an element \( \mathfrak{A} \) in \( \mathfrak{A}(\mathfrak{p}) \) and an element \( \mathfrak{B} \) in \( \mathfrak{B}(\mathfrak{q}) \) such that \( \langle \mathfrak{A}, \mathfrak{B} \rangle \) is a \( p,q \)-group. If \( \mathfrak{A}, \mathfrak{B} \) are chosen in such a way that \( \langle \mathfrak{A}, \mathfrak{B} \rangle \) is a \( p,q \)-group of minimal order, it follows as in the first part of the proof of Lemma 19.2 of F.T. that 
\[
\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{A} \times \mathfrak{B}.
\]

Let \( \mathfrak{A} = \mathbf{N}(\mathfrak{A}) \) and let \( \mathfrak{D} \) now denote an \( S_{p,q} \)-subgroup of \( \mathfrak{S}(\mathfrak{A}) \). As above, we may assume that \( \mathfrak{B} \subseteq \mathfrak{Y}_1 = \mathbf{N}_{\mathfrak{R}}(\mathfrak{D}) \). Furthermore, by \( C_{\mathfrak{p},\mathfrak{q}} \) in \( \mathfrak{S}(\mathfrak{R}) \), \( \mathfrak{Y}_1 \) contains an \( S_{p} \)-subgroup \( \mathfrak{G} \) of \( \mathfrak{S} \). By our hypothesis, \( \mathfrak{S} \) centralizes \( \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}) \), and hence \( \mathbf{O}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{R}) = \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}) \times \mathbf{O}_{\mathfrak{q}}(\mathfrak{D}) \). Now \( \mathfrak{B} \) centralizes \( \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}) \) by the preceding lemma and \( \mathbf{C}_{\mathfrak{p}}(\mathbf{O}_{\mathfrak{p}}(\mathfrak{D})) \) is solvable by Lemma 3.3, so that \( \mathfrak{B} \subseteq \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}) \) by Lemma 3.5. Thus \( \mathfrak{S} \) centralizes \( \mathfrak{D} \). Hence if \( \mathfrak{M} = \mathbf{N}_{\mathfrak{R}}(\mathfrak{S}) \), \( \mathfrak{M} \) contains \( \mathfrak{D} \) as well as an \( S_{\mathfrak{p}} \)-subgroup \( \mathfrak{Q} \) of \( \mathfrak{S} \). Let \( \mathfrak{D}_1 \) be an \( S_{\mathfrak{p},\mathfrak{q}} \)-subgroup of \( \mathfrak{S}(\mathfrak{M}) \) and let \( \mathfrak{M}_1 = \mathbf{N}_{\mathfrak{M}}(\mathfrak{D}_1) \). We may assume that \( \mathfrak{D} \) and \( \mathfrak{Q} \) lie in \( \mathfrak{M}_1 \). It follows as above that \( \mathbf{O}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{D}_1) = \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}_1) \times \mathbf{O}_{\mathfrak{q}}(\mathfrak{D}_1) \). Since \( \mathfrak{S} \) is \( \mathfrak{q} \)-constrained and \( \mathfrak{S} \) centralizes \( \mathbf{O}_{\mathfrak{p}}(\mathfrak{D}_1) \), Lemma 3.3 and Lemma 1.2.3 of [5] imply that \( \mathfrak{D} \subseteq \mathbf{O}_{\mathfrak{p}}(\mathfrak{M}_1) \). But \( \mathfrak{Q} \) normalizes some \( S_{\mathfrak{p}} \)-subgroup of \( \mathbf{O}_{\mathfrak{p}}(\mathfrak{M}_1) \), which without loss we may assume to be \( \mathfrak{D} \). Since \( \mathfrak{Q} \) centralizes every \( \mathfrak{p} \)-subgroup it normalizes, \( \mathfrak{Q} = \mathfrak{G} \times \mathfrak{Q} \), and \( \mathfrak{S} \) satisfies \( E_{\mathfrak{p},\mathfrak{q}} \). Since we are proceeding by contradiction, we accept this lemma.

**Lemma 6.3.** If \( \mathfrak{A} \in \mathfrak{A}(\mathfrak{p}) \), either \( \mathbf{C}(\mathfrak{A}) \) is a \( q \)-group or an \( S_{\mathfrak{q}} \)-subgroup of \( \mathbf{C}(\mathfrak{A}) \) has order \( q \) and centralizes no element \( \mathfrak{B} \) of \( \mathfrak{A}(\mathfrak{q}) \).

**Proof.** Let \( \mathfrak{E} \) be an \( S_{\mathfrak{p}} \)-subgroup of \( \mathbf{C}(\mathfrak{A}) \), and suppose \( \mathfrak{E} \neq 1 \). By Lemma 6.2, no element of \( \mathfrak{E} \) centralizes any \( \mathfrak{B} \in \mathfrak{A}(\mathfrak{q}) \). Let \( \mathfrak{Q} \) be an \( S_{\mathfrak{p}} \)-subgroup of \( \mathfrak{S} \) containing \( \mathfrak{E} \) and let \( \mathfrak{F} \in \mathfrak{B}(\mathfrak{Q}) \). Then \( \mathbf{C}_{\mathfrak{Q}}(\mathfrak{B}) \) is of index 1 or \( q \) in \( \mathfrak{Q} \), and is disjoint from \( \mathfrak{E} \). It follows that \( |\mathfrak{E}| = q \).

Let \( \mathfrak{G} \) be an \( S_{\mathfrak{p}} \)-subgroup of \( \mathfrak{S} \), let \( \mathfrak{V} \in \mathfrak{C} \mathfrak{C} \mathfrak{A} \mathfrak{L} \mathfrak{A}_{\mathfrak{p}}(\mathfrak{B}) \), and set \( \mathfrak{M}_1 = \mathbf{N}(\mathfrak{Z}(\mathfrak{B})) \), where \( \mathfrak{Z} = \mathbf{V}(\mathfrak{C} \mathfrak{C} \mathfrak{A} \mathfrak{L} \mathfrak{A}_{\mathfrak{p}}(\mathfrak{A}); \mathfrak{B}) \). Then by Lemma 5.5, \( \mathfrak{Z} \leq \mathbf{O}_{\mathfrak{p},\mathfrak{p}}(\mathfrak{M}_1) \). We now prove

**Lemma 6.4.** Assume that \( \mathfrak{B} \) is an \( S_{\mathfrak{p}} \)-subgroup of \( \mathbf{O}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{A}_1) \) and that \( \mathfrak{B} \) possesses no noncyclic characteristic abelian subgroups. Let \( \mathfrak{L} \) be a subgroup of \( \mathfrak{A} \) containing \( \mathfrak{Z}(\mathfrak{B}) \) and let \( \mathfrak{S} \) be a subgroup of \( \mathfrak{S} \) such that \( \mathfrak{L} \subseteq \mathfrak{S} \) and \( p \in \pi_\mathfrak{S}(\mathfrak{S}) \). Then \( \mathfrak{S} \leq \mathbf{O}_{\mathfrak{p},\mathfrak{p}}(\mathfrak{S}) \).

**Proof.** Assume the lemma false, and choose \( \mathfrak{S} \) to violate the lemma in such a way that \( |\mathfrak{S}|_p \) is maximal, and subject to this condition, minimize
Let $\mathcal{S}$ be an $S_p$-subgroup of $O_p'(\mathfrak{G})$ normalized by $\mathfrak{G}$, and set $\mathcal{S}_1 = N_{\mathfrak{S}}(\mathcal{S})$. Since $\mathcal{S} = O_p'(\mathfrak{G})$ by Sylow's theorem, $\mathfrak{S} \not\leq O_p'(\mathfrak{G})$. But then $\mathcal{S}_1 = \mathcal{S}$ by our minimal choice of $|\mathcal{S}|_p$. Thus $\mathfrak{S}_1 = O_p(\mathfrak{G})$.

Now let $\mathfrak{R} = N(\mathfrak{S}_1)$. Then we must have $|\mathfrak{R}|_p = |\mathcal{S}|_p$; otherwise $\mathfrak{S} \leq O_p'(\mathfrak{G})$ by the maximality of $|\mathcal{S}|_p$. Since $\mathfrak{S} \leq \mathfrak{R}$, this would imply that $\mathfrak{S} \leq O_p'(\mathfrak{G})$, contrary to our choice of $\mathfrak{S}$. Hence if $\mathfrak{R}$ is an $S_p$-subgroup of $\mathfrak{G}$ such that $\mathfrak{R} \cap \mathfrak{S}$ is an $S_p$-subgroup of $\mathfrak{S}$ containing $\mathfrak{S}$, we conclude from Lemma 3.4 that $\mathfrak{R}$ contains every element of $\mathcal{S}$. And thus $\mathfrak{R} \leq \mathfrak{R}_1$. We conclude from the maximality of $|\mathcal{S}|_p$ that $\mathfrak{R} \leq \mathcal{S}$.

It will suffice to prove that $\mathfrak{R} \leq \mathfrak{R}_1$; for assume this to be the case. Then $\mathfrak{R} = \mathfrak{R}_1$ for some $\mathfrak{R}$ in $\mathfrak{R}_1$, and we may suppose that $\mathfrak{R} = \mathfrak{R}_1$. Thus $\mathfrak{R} = \mathfrak{R}_1$, and hence $\mathfrak{R}_1 = \mathfrak{R}_1 \cap \mathfrak{R}_1 = \mathfrak{R}_1$. But then $\mathfrak{R} \leq \mathfrak{R}_1$. Since $\mathfrak{R} \leq O_p'(\mathfrak{G})$, we conclude that $\mathfrak{R} \leq O_p'(\mathfrak{G})$, contrary to our choice of $\mathfrak{R}$.

Let $\mathfrak{D} = \mathfrak{R} \cap \mathfrak{R}_1$, and set $\mathfrak{R} = N(\mathfrak{D})$. We need only show that $\mathfrak{R} \leq \mathfrak{R}_1$, for this will force $\mathfrak{D} = \mathfrak{R}$, otherwise $\mathfrak{R} \cap \mathfrak{R}$ would contain $\mathfrak{D}$ properly and would lie in $\mathfrak{R}_1$, contrary to our choice of $\mathfrak{D}$.

Now $\mathfrak{D} \leq \mathfrak{R}_1$ for some $\mathfrak{D}$ in $\mathfrak{R}_1$, and $\mathfrak{R}_1 \cap O_p'(\mathfrak{R}_1) = \mathfrak{R}_1$. Since $\mathfrak{R} \leq O_p'(\mathfrak{R}_1)$, we have $\mathfrak{R} \leq \mathfrak{R}_1$. Furthermore by definition of $\mathfrak{D}$, $\mathfrak{D}$ has index $p$ in $\mathfrak{R}_1$, so that $\mathfrak{D}$ has index at most $p$ in $\mathfrak{R}_1$. But then $\mathfrak{R}_1$ normalizes $\mathfrak{D}$, and hence lies in $\mathfrak{R}_1$. By Lemma 5.6, $\mathfrak{R}_1 \subseteq O_p'(\mathfrak{R}_1)$, and consequently $\mathfrak{R} = O_p'(\mathfrak{R}) \cap \mathfrak{R}_1$. Since $\mathfrak{R} \subseteq O_p'(\mathfrak{R}) \cap \mathfrak{R}_1$, it remains only to show that $O_p'(\mathfrak{R}) \subseteq \mathfrak{R}_1$. By (3.5) of F.T., $\mathfrak{R}$ and $\mathfrak{R}_1$ are each the central product of an extraspecial group and a cyclic group. Since $\mathfrak{R} \subseteq \mathfrak{R}_1$, and $\mathfrak{R}_1 \subseteq \mathfrak{R}_1$, we have $Z(\mathfrak{R}) \leq Z(\mathfrak{R}_1)$. Now $Z(\mathfrak{R}) \leq Z(\mathfrak{R}_1)$ by hypothesis. Since $\mathfrak{R}$ has index $p$ in $\mathfrak{R}_1$, it follows from the structure of $\mathfrak{R}$ that $Z(\mathfrak{R})$ is noncyclic. But $\mathfrak{R}$ has index $p$ in $\mathfrak{R}_1$ and $\mathfrak{R}$ is not the central product of a cyclic group and an extraspecial group. We must therefore have $Z(\mathfrak{R}) \leq \mathfrak{R}_1$. But then $Z(\mathfrak{R}) \leq Z(\mathfrak{R}_1)$. Since $O_p'(\mathfrak{R})$ centralizes $Z(\mathfrak{R})$, we conclude that $O_p'(\mathfrak{R}) \subseteq C(Z(\mathfrak{R})) \subseteq \mathfrak{R}_1$, thus completing the proof of the lemma.

Now let $\mathfrak{B}$ be an $S_p$-subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ an element of $S^C\mathcal{N}_3(\mathfrak{B})$. Set $\mathfrak{M}_1 = N(\mathfrak{M})$, where $\mathfrak{M} = V(ccl_{\mathfrak{G}}(\mathfrak{M}; \mathfrak{B}))$. If $\mathfrak{B}$ is not an $S_p$-subgroup of $O_p'(\mathfrak{M}_1)$ or if $\mathfrak{M}$ contains a noncyclic characteristic abelian subgroup, define $\mathcal{F}(\mathfrak{M}; \mathfrak{B})$ to be the set of subgroups of $\mathfrak{M}$ of type $(p, p)$. In the contrary case, define $\mathcal{F}(\mathfrak{M}; \mathfrak{B})$ to be the set of subgroups of $\mathfrak{M}$ of type $(p, p)$ together with all subgroups of $\mathfrak{M}$ of type $(p, p)$ containing $O_1(\mathfrak{M})$. Define $\mathcal{F}(\mathfrak{B})$ to be the set of all subgroups $\mathfrak{M}$ such that $\mathfrak{M} \in \mathcal{F}(\mathfrak{M}; \mathfrak{B})$ for some $\mathfrak{B}$ in $S^C\mathcal{N}_3(\mathfrak{B})$. Finally define $\mathcal{F}(p)$ to be the set of all subgroups $\mathfrak{M}$ such that $\mathfrak{M} \in \mathcal{F}(\mathfrak{M}; \mathfrak{B})$ for some $S_p$-subgroup $\mathfrak{B}$ of $\mathfrak{G}$.

**Lemma 6.5.** If $\mathfrak{M} \in \mathcal{F}(p)$, then $\mathfrak{M}$ centralizes every element of $\mathcal{V}(\mathfrak{M}; q)$. 
Proof. Assume false, and let \( \mathcal{Q} \) be an element of minimal order which is normalized, but not centralized by \( \mathcal{T} \). Then \( \mathcal{Q} = \ker(\mathcal{T} \to \text{Aut}(\mathcal{Q})) \neq 1 \) by (3.11) of F.T. Suppose first that \( \mathcal{Q} \subseteq \mathfrak{A} \) for some element \( \mathfrak{A} \) of \( \mathcal{C}(\mathcal{Q}) \). Set \( \mathcal{C} = C(\mathcal{Q}) \), so that \( \mathcal{Q} \) and \( \mathfrak{A} \) are contained in \( \mathcal{C} \). By Lemma 5.6, \( \mathfrak{A} \subseteq \mathcal{O}_{p',p}^{\mathcal{C}}(\mathcal{C}) \), and hence \( \mathcal{Q} \subseteq \mathcal{O}_{p',p}^{\mathcal{C}}(\mathcal{C}) \). But then \( \mathcal{T} \) does not centralize \( \mathcal{Q} \cap \mathcal{O}_{p'}^{\mathcal{C}}(\mathcal{C}) \), whence \( \mathcal{Q} \subseteq \mathcal{O}_{p'}^{\mathcal{C}}(\mathcal{C}) \) by our minimal choice of \( \mathcal{Q} \). Now Corollary 4.1 and the hypothesis of Theorem 2 imply that \( \mathfrak{A} \), and hence also \( \mathcal{T} \), centralizes an \( S_q \)-subgroup of \( \mathcal{O}_{p'}^{\mathcal{C}}(\mathcal{C}) \). It follows at once that \( \mathcal{T} \) centralizes every \( q \)-subgroup of \( \mathcal{O}_{p'}^{\mathcal{C}}(\mathcal{C}) \) which it normalizes. Thus \( \mathcal{T} \) centralizes \( \mathcal{Q} \), a contradiction.

Assume next that \( \mathcal{T} \) is a subgroup of \( \mathfrak{V} \) of type \((p, p)\) containing \( \mathcal{O}_{1}(\mathcal{Z}(\mathfrak{V})) \), where \( \mathfrak{V} = V(\text{cc}(\mathfrak{V}); \mathfrak{W}) \), \( \mathfrak{W} \) is an \( S_p \)-subgroup of \( \mathfrak{W} \), \( \mathfrak{V} \subseteq \mathcal{O}_{p'}^\mathfrak{W}(\mathfrak{W}) \), and if \( \mathfrak{A} = N(\mathcal{Z}(\mathfrak{V})) \), then \( \mathfrak{V} \) is an \( S_{p'} \)-subgroup of \( \mathcal{O}_{p'}^\mathfrak{W}(\mathfrak{W}) \) and \( \mathfrak{V} \) is the central product of an extraspecial group and a cyclic group. In this case, \( \mathcal{C}(\mathcal{Q}) \) contains a subgroup \( \mathcal{U} \) such that \( \mathcal{U} : \mathcal{U} = p \), \( \mathcal{Z}(\mathfrak{V}) \subseteq \mathcal{U} \), and \( \mathcal{T} \subseteq \mathcal{U} \). Since \( \mathcal{Z}(\mathfrak{V}) \subseteq \mathcal{Z}(\mathfrak{W}) \), the hypotheses of Lemma 6.4 are satisfied. Since \( \mathcal{U} \subseteq \mathcal{C} \), we conclude that \( \mathcal{U} \subseteq \mathcal{O}_{p',p}^{\mathcal{C}}(\mathcal{C}) \). Since \( \mathcal{T} \subseteq \mathcal{U} \), we again have \( \mathcal{Q} \subseteq \mathcal{O}_{p',p}^{\mathcal{C}}(\mathcal{C}) \).

It will suffice to show that \( \mathfrak{V} = \mathcal{O}_{1}(\mathcal{Z}(\mathfrak{V})) \) centralizes an \( \mathfrak{W} \)-invariant \( S_q \)-subgroup of \( \mathcal{O}_{p'}^{\mathfrak{W}}(\mathfrak{W}) \), for then \( \mathfrak{V} \) will centralize every \( q \)-subgroup of \( \mathcal{O}_{p'}^{\mathfrak{W}}(\mathfrak{W}) \) which it normalizes. Since \( \mathcal{C} \subseteq \mathcal{F} \), this will imply that \( \mathfrak{V} \) centralizes \( \mathcal{Q} \) and hence that \( \mathcal{Q} \subseteq \mathfrak{A} = N(\mathfrak{V}) \). But then \( \mathfrak{W} \subseteq \mathcal{O}_{p',p}(\mathfrak{V}) \), and consequently \( \mathcal{Q} \subseteq \mathcal{O}_{p',p}(\mathfrak{V}) \). Since \( \mathfrak{V} \subseteq \mathfrak{A} \), \( \mathfrak{V} \) centralizes an \( S_q \)-subgroup of \( \mathcal{O}_{p',p}(\mathfrak{V}) \), and hence so does \( \mathcal{T} \). We conclude once again that \( \mathcal{T} \) centralizes \( \mathcal{Q} \), contrary to our choice of \( \mathcal{Q} \).

Let then \( \mathcal{Q}_1 \) be an \( S_{q'} \)-subgroup of \( \mathcal{O}_{p'}^{\mathfrak{W}}(\mathfrak{W}) \) normalized by \( \mathfrak{V} \), and set \( \mathfrak{A}_1 = \mathfrak{A} \cap \mathcal{Q}_1 \). Since \( \mathfrak{V} : \mathfrak{A}_1 = p \), \( \mathfrak{A} : \mathfrak{A}_1 \leq p \), and hence \( \mathfrak{A}_1 \) is noncyclic. But now every subgroup of \( \mathfrak{A}_1 \) of type \((p, p)\) lies in \( \mathcal{F}(p) \), so that by the first case of the proof, every subgroup of \( \mathfrak{A}_1 \) of type \((p, p)\) centralizes \( \mathcal{Q}_1 \). Thus \( \mathcal{O}_{1}(\mathfrak{A}_1) \) centralizes \( \mathcal{Q}_1 \). Since \( \mathcal{C} \subseteq \mathfrak{A}_1 \), \( \mathfrak{V} \) centralizes \( \mathcal{Q}_1 \), and the lemma is proved.

Lemma 6.6. If \( \mathfrak{A} \in \mathcal{F}(p) \) and \( \mathfrak{B} \in \mathcal{W}(q) \), and \( \mathfrak{R} \) is a subgroup of \( \mathfrak{W} \) containing \( \mathfrak{A} \) and \( \mathfrak{B} \), then neither \( p \) nor \( q \) is in \( \pi_s(\mathfrak{R}) \).

Proof. The proof is essentially identical to that of Lemma 6.2. If the lemma is false, and if \( \mathfrak{R} \) is chosen minimal as in Lemma 6.2, the same argument shows that \( \mathfrak{R} = \mathfrak{A} \times \mathfrak{B} \), since now \( \mathfrak{A} \) centralizes every \( q \)-subgroup which it normalizes by the preceding lemma. But then \( \mathcal{C}(\mathfrak{B}) \) contains the noncyclic subgroup \( \mathfrak{A} \), contrary to Lemma 6.3 with \( p \) and \( q \) interchanged.

Once again let \( \mathfrak{B} \) be an \( S_{p'} \)-subgroup of \( \mathfrak{W} \), \( \mathfrak{A} \) an element of \( \mathcal{C}(\mathfrak{W}) \) and \( \mathfrak{A}_1 = N(\mathfrak{B}(\mathfrak{W})) \), where \( \mathfrak{B} = V(\text{cc}(\mathfrak{W}); \mathfrak{W}) \). Let \( \mathfrak{V} \) be an \( S_{p'} \)-subgroup of \( \mathfrak{A}_1 \) and \( \mathfrak{A} \) an \( S_{q'} \)-subgroup of \( \mathfrak{W} \) containing \( \mathfrak{A} \).
Lemma 6.7. Suppose \( \Omega \) contains an elementary subgroup \( \mathcal{C} \) of order \( q^4 \). Then the following conditions hold:

(i) \( p \equiv 1 \pmod{q} \).

(ii) Some element \( Y \) in \( \mathcal{C} \) centralizes an element of \( \mathcal{F}(p) \).

(iii) If \( E \in \mathcal{C} \) and \( E \) centralizes an element of \( \mathcal{F}(p) \), then \( \mathcal{C}(E) \) does not contain an elementary subgroup of order \( q^4 \).

(iv) \( \mathcal{C}(Z(\mathcal{C})) \) is contained in \( \mathcal{C} \) and has order \( q \).

Proof. Let \( \mathcal{D} \) be an \( S_{p,q} \)-subgroup of \( S(\mathfrak{A}) \) normalized by \( \mathcal{D} \) and set \( \mathfrak{G} = N_{S(1)}(\mathcal{D}) \). Since \( \mathfrak{G} \) contains an \( S_{p,q} \)-subgroup of \( \mathfrak{A} \), we may assume without loss that \( \Omega \subseteq \mathfrak{G} \). Set \( \mathfrak{D}_1 = \mathfrak{G} \cap O_{p,q}(\mathfrak{A}) \). Since \( \mathcal{D} \) centralizes \( O_{p,q}(\mathfrak{G}) \) by the hypothesis of the theorem, \( O_{p,q}(\mathfrak{G}) = O_{p,q}(\mathfrak{G}) \times O_{p,q}(\mathfrak{G}) \), and hence \( \mathfrak{D}_1 = O_{p,q}(\mathfrak{G}) \). Finally let \( \mathfrak{D} \in \mathcal{H}(\mathcal{C}) \), and set \( \mathfrak{B}_1 = \mathcal{C}_\mathfrak{D}(\mathfrak{C}) \). Since \( \mathfrak{C} \) is elementary of order \( q^3 \), \( \mathfrak{B}_1 \) is noncyclic.

Suppose first that \( \mathfrak{D}_1 \) contains a noncyclic characteristic abelian subgroup \( \mathfrak{C} \). Then \( \mathfrak{C} \) is contained in some element of \( L(\mathfrak{C},N_{g}(\mathfrak{D})) \), and consequently every subgroup of \( \mathfrak{C} \) of type \( (p, p) \) lies in \( \mathcal{F}(\mathfrak{D}) \). But then \( \mathcal{C}_\mathfrak{D}(E) \) must be cyclic for each \( E \in \mathcal{C} \), otherwise \( \mathcal{C}(E) \) would contain an element of \( \mathcal{F}(\mathfrak{D}) \) and an element of \( \mathfrak{D}(\mathfrak{C}) \) for some \( E \in \mathcal{C} \), contrary to Lemma 6.6. Since \( \mathfrak{D}_1 \) normalizes each \( \mathcal{C}_\mathfrak{D}(E) \), this is possible only if \( p \equiv 1 \pmod{q} \). Thus (i) holds. Furthermore, since \( \mathfrak{C} \) is elementary of order \( q^3 \), \( \mathcal{C}_\mathfrak{D}(Y) \) is noncyclic for some \( Y \in \mathcal{C} \). Thus (ii) also holds in this case. Assume next that \( \mathfrak{D}_1 \) contains no noncyclic characteristic abelian subgroups, in which case \( \mathfrak{D}_1 \) is the central product of a cyclic group and an extraspecial group. Since \( \mathfrak{A} \in \mathcal{H}(\mathcal{C},N_{g}(\mathfrak{D})) \) and \( \mathfrak{G} \subseteq \mathfrak{D}_1 \), \( \mathfrak{A} \) is a maximal normal abelian subgroup of \( \mathfrak{D}_1 \). Now \( \mathfrak{G} \subseteq \mathfrak{D}_1 \), by Lemma 5.6, and hence either \( \mathfrak{G} = \mathfrak{D}_1 \) or \( \mathfrak{Z}(\mathfrak{D}) \) is noncyclic. Furthermore, \( \mathfrak{Z}(\mathfrak{D}) \subseteq \mathfrak{G} \), so that every subgroup of \( \mathfrak{Z}(\mathfrak{D}) \) of type \( (p, p) \) lies in \( \mathcal{F}(\mathfrak{D}) \). But \( \mathfrak{C} \) normalizes \( \mathfrak{Z}(\mathfrak{D}) \). Hence if \( \mathfrak{Z}(\mathfrak{D}) \) is noncyclic, (i) and (ii) follow at once as in the preceding case. So assume finally that \( \mathfrak{Z}(\mathfrak{D}) \) is cyclic, in which case \( \mathfrak{G} = \mathfrak{D}_1 \). In this case, it follows from the definition of \( \mathcal{F}(\mathfrak{D}) \) that every subgroup of \( \mathfrak{D} \) of type \( (p, p) \) containing \( \Omega(\mathfrak{Z}(\mathfrak{D})) \) is in \( \mathcal{F}(\mathfrak{D}) \). Set \( \mathfrak{C}_0 = \ker(\mathfrak{C} \to \Aut(\mathfrak{Z}(\mathfrak{D})) \). Since \( \mathfrak{Z}(\mathfrak{D}) \) is cyclic and \( \mathfrak{C} \) is elementary, \( |\mathfrak{C}/\mathfrak{C}_0| \leq q \), and if equality holds, then \( p \equiv 1 \pmod{q} \). Since \( \mathfrak{C}_0 \) is noncyclic, we have \( \mathcal{C}_\mathfrak{D}(Y) \neq 1 \) for some \( Y \) in \( \mathfrak{C}_0 \), where \( \mathfrak{D} = \mathfrak{B}/\mathfrak{Z}(\mathfrak{D}) \). But then \( \mathcal{C}_\mathfrak{D}(Y) \) contains a subgroup \( \mathfrak{H} \) of type \( (p, p) \) with \( \Omega(\mathfrak{Z}(\mathfrak{D})) \subset \mathfrak{H} \). Thus \( \mathfrak{H} \in \mathcal{F}(\mathfrak{D}) \) and (ii) holds. Furthermore, if \( p \equiv 1 \pmod{q} \), the element \( Y \) can be taken in \( \mathfrak{C}_0 \), in which case \( \mathcal{C}(Y) \) contains \( \mathfrak{H} \) and \( \mathfrak{D} \), contrary to Lemma 6.6. We conclude that (i) and (ii) hold in all cases.

In order to prove (iii), we first show that \( \Omega \) does not contain an elementary subgroup \( \Sigma \) of order \( q^4 \). For then \( \mathcal{C}_\Sigma(\mathfrak{D}) \) would contain an elementary subgroup of order \( q^3 \) which we could take for \( \mathcal{C} \). But then by (ii), \( \mathcal{C}(Y) \) would contain \( \mathfrak{D} \) as well as an element of \( \mathcal{F}(\mathfrak{D}) \), contrary to Lemma 6.6.
Suppose now that (iii) is false, and let $E$ be an element of $E_#$ such that $C(E)$ contains an element of $\mathcal{F}(\mathfrak{R})$ as well as an elementary subgroup of order $q^4$. Set $\sigma = \{p, q\}$. Among all subgroups $\mathfrak{R}$ of $\mathfrak{G}$ which contain an element of $\mathcal{F}(p)$ and an elementary subgroup of order $q^4$, and for which $q \in \pi_\mathfrak{R}(\mathfrak{R})$, choose $\mathfrak{R}$ so that $|\mathfrak{R}|_\sigma$ is maximal, and subject to this condition, minimize $|\mathfrak{R}|_\sigma$. Then by the minimality of $|\mathfrak{R}|_\sigma$, an $S_{p,q}$-subgroup $\mathfrak{R}$ of $\mathfrak{S}(\mathfrak{R})$ is normal in $\mathfrak{R}$. Without loss we may assume that $\mathfrak{E} \cap \mathfrak{R}$ is an $S_q$-subgroup of $\mathfrak{R}$. If $\mathfrak{R}$ is a $q$-group, then $\mathfrak{E} \cap \mathfrak{R}$ is an $S_q$-subgroup of $\mathfrak{N}(\mathfrak{R})$ by our maximal choice of $\mathfrak{R}$. But then $\mathfrak{R}$ contains every element of $\mathcal{F}_\mathfrak{E} \mathcal{N}_3(\mathfrak{E})$ by Lemma 3.4. Thus $\mathfrak{R}$ contains an element of $\mathcal{F}(p)$ and an element of $\mathfrak{H}(q)$, contrary to Lemma 6.6. Thus $\mathfrak{R}$ is not a $q$-group.

Let $\mathfrak{E}$ be an element of $\mathcal{F}(p)$ which lies in $\mathfrak{R}$. Then by Lemma 6.5, $\mathfrak{E}$ centralizes $\mathfrak{O}_\mathfrak{R}(\mathfrak{E}) = \mathfrak{O}_\mathfrak{E}(\mathfrak{R})$. If $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ were noncyclic, some element $X \neq 1$ of $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ would centralize an element of $\mathfrak{H}(\mathfrak{E})$, and Lemma 6.6 would be violated in $\mathfrak{C}(X)$. Hence $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ is cyclic. Since $q < p$ by (i), it follows that $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ centralizes an $S_p$-subgroup of $\mathfrak{R}$. But then $\mathfrak{O}_\mathfrak{E}(\mathfrak{R}) = \mathfrak{O}_\mathfrak{E}(\mathfrak{R}) \times \mathfrak{O}_\mathfrak{E}(\mathfrak{R})$. Now let $\mathfrak{H}$ be an $S_p$-subgroup of $\mathfrak{G}$ containing an $S_p$-subgroup of $\mathfrak{R}$. Then, $\mathfrak{H} \cap \mathfrak{R}$ is an $S_p$-subgroup of $\mathfrak{N}(\mathfrak{O}_\mathfrak{E}(\mathfrak{R}))$ by our maximal choice of $|\mathfrak{R}|_\sigma$. Since $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ is an $S_p$-subgroup of $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$, it follows from Lemma 3.4 that $\mathfrak{O}_\mathfrak{E}(\mathfrak{R})$ contains every element of $\mathcal{F}_\mathfrak{E} \mathcal{N}_3(\mathfrak{H})$. Hence if $\mathfrak{H} = \mathfrak{H}^G$, $G \in \mathfrak{G}$, $\mathfrak{H} = \mathfrak{H}^G \subseteq \mathfrak{O}_\mathfrak{E}(\mathfrak{R})$. Set $\mathfrak{H} = \mathfrak{V}(\mathfrak{O}_\mathfrak{E}(\mathfrak{H}); \mathfrak{H})$, and let $\mathfrak{R}_1 = \mathfrak{N}(\mathfrak{Z}(\mathfrak{H}))$. By Lemma 5.4, $\mathfrak{R} \subseteq \mathfrak{R}_1$. Thus $\mathfrak{R}_1$ contains an elementary subgroup of order $q^4$. Since $\mathfrak{R}_1 = \mathfrak{R}_{\mathfrak{H}}$, $\mathfrak{R}_1$ contains an elementary subgroup of the same order. But we have shown above that $\mathfrak{R}_1$ contains no such subgroup. This completes the proof of (iii).

Finally we prove (iv). Let $Y$ be an element of $E_#$ satisfying (ii). If $\Omega_1 = \Omega_1(\mathfrak{Z}(\mathfrak{E}))$ were not contained in $\mathfrak{E}$, then $\langle \mathfrak{E}, \Omega_1 \rangle$ would contain an elementary subgroup of order $q^4$, which would lie in $\mathfrak{C}(Y)$. Since $\mathfrak{C}(Y)$ also contains an element of $\mathcal{F}(\mathfrak{Y})$, this would contradict (iii). Thus $\Omega_1 \subseteq \mathfrak{E}$. Furthermore, we must have $|\Omega_1| = p$, otherwise $\Omega_1$ would contain an element of $\mathfrak{H}(\mathfrak{E})$, and $\Omega_1$ would contain elements of both $\mathfrak{H}(\mathfrak{Y})$ and $\mathfrak{H}(\mathfrak{E})$, contrary to Lemma 6.3. Thus (iv) holds, and the lemma is proved.

We shall now show that the hypotheses of Lemma 6.7 are satisfied whenever $p > q$.

**Lemma 6.8.** If $p > q$, then $\mathfrak{R}_1$ contains an elementary subgroup of order $q^3$.

**Proof.** Since $p \sim q$ by the hypothesis of Theorem 2, there exists a $(p, q)$-subgroup of $\mathfrak{G}$ which contains subgroups of both types $(p, p, p)$ and $(q, q, q)$. Set $\sigma = \{p, q\}$. Among all subgroups $\mathfrak{H}$ of $\mathfrak{G}$ which contain subgroups of both types $(p, p, p)$ and $(q, q, q)$ and such that $p \in \pi_\mathfrak{H}(\mathfrak{H})$, choose $\mathfrak{H}$ so that $|\mathfrak{H} \cap \Omega_1(\mathfrak{H})|$ is a maximum, $\mathfrak{H}$ ranging over the elements of $\mathcal{F}_\mathfrak{E} \mathcal{N}_3(\mathfrak{H})$. Then by the maximality of $|\mathfrak{H} \cap \Omega_1(\mathfrak{H})|$, $\mathfrak{H}$ contains every element of $\mathfrak{E}_\mathfrak{H}(\mathfrak{H})$.
subject to this condition maximize $|\mathfrak{J}|_\rho$, and subject to these conditions, minimize $|\mathfrak{J}|_\rho$.

Let $\mathfrak{D}$ be an $S_{\rho,\rho}$-subgroup of $S(\mathfrak{H})$. Since $\mathfrak{H} = S(\mathfrak{H})N_\rho(\mathfrak{D})$, the minimality of $|\mathfrak{J}|_\rho$ implies that $\mathfrak{H} = N_\rho(\mathfrak{D})$. Thus $\mathfrak{D} \triangleleft \mathfrak{H}$. Let $\mathfrak{P}_1$ be an $S_\rho$-subgroup of $\mathfrak{H}$ and $\mathfrak{P}$ an $S_{\rho,\rho}$-subgroup of $\mathfrak{H}$ containing $\mathfrak{P}_1$. By taking a suitable conjugate of $\mathfrak{P}$, if necessary, we may assume without loss that $\mathfrak{P} = \mathfrak{P}_1$.

Suppose first that $\mathfrak{P}_1$ centralizes $O_p(\mathfrak{H}) = O_p(\mathfrak{D})$, in which case $O_{\rho,\rho}(\mathfrak{D}) = O_{\rho}(\mathfrak{H}) \times O_{\rho}(\mathfrak{D})$. Set $\mathfrak{D}_1 = N(O_{\rho}(\mathfrak{D}))$. Then maximality of $|\mathfrak{J}|_\rho$ implies that $\mathfrak{P}_1$ is an $S_{\rho,\rho}$-subgroup of $\mathfrak{H}_1$. Since $O_{\rho}(\mathfrak{D})$ is an $S_{\rho,\rho}$-subgroup of $O_{\rho,\rho}(\mathfrak{P})$, it follows from Lemma 3.4 that $\mathfrak{P}_1$ contains every element of $\mathfrak{H}(\rho,\rho)(\mathfrak{D})$, and in particular, that $\mathfrak{D}_1 \subseteq \mathfrak{P}_1$. But then $\mathfrak{D}_1 \subseteq \mathfrak{D}_1$ by Lemma 5.4. Since $\mathfrak{D}$ contains a subgroup of type $(\rho, \rho, \rho)$, the lemma follows in this case.

So assume that $\mathfrak{P}_1$ does not centralize $O_p(\mathfrak{D})$. Since $\rho > \rho$, this implies that $O_p(\mathfrak{D})$ is noncyclic. If $\mathfrak{P}_1$ contained a subgroup $\mathfrak{H}$ of $\mathfrak{H}(\rho,\rho)$, then $\mathfrak{H}$ would centralize $O_p(\mathfrak{D})$ by Lemma 6.5. Since $O_p(\mathfrak{D})$ is noncyclic, some element $Q$ of $O_p(\mathfrak{D})$ centralizes an element of $\mathfrak{H}(\rho)$. But then $C(Q)$ would contain an element of $\mathfrak{H}(\rho)$ and an element of $\mathfrak{H}(\rho)$, contrary to Lemma 6.6. Thus $O_p(\mathfrak{D})$ contains order 1 or $\pmod{\rho}$ for all $\mathfrak{D}$ in $\mathfrak{H}(\rho,\rho)$.

Now let $\mathfrak{E} \in \mathfrak{H}(\rho)$ and set $\mathfrak{P}_2 = C_{\rho,\rho}(\mathfrak{E})$. Since $\mathfrak{P}_1$ contains a subgroup $\mathfrak{E}_1$ of type $(\rho, \rho, \rho)$, $\mathfrak{E}_2 = \mathfrak{E}_1 \cap \mathfrak{P}_2$ is noncyclic. If $\mathfrak{E}_2$ does not centralize $O_p(\mathfrak{D})$, let $\mathfrak{Q}$ be a subgroup of $O_p(\mathfrak{D})$ of minimal order which is normalized, but not centralized by $\mathfrak{E}_2$ and set $\mathfrak{E}_0 = \ker ((\mathfrak{E}_2 \to \text{Aut} \mathfrak{Q}))$. Then by (3.11) of F.T., $\mathfrak{E}_0 \neq 1$. On the other hand, if $\mathfrak{E}_2$ centralizes $O_p(\mathfrak{D})$, set $\mathfrak{E}_0 = \mathfrak{E}_2$. Then $C(\mathfrak{E}_0)$ contains $\mathfrak{E}_1$, and correspondingly $\mathfrak{Q}$ or $O_p(\mathfrak{D})$. Since $\mathfrak{Q}$ or $O_p(\mathfrak{D})$ admits a nontrivial $p$-automorphism and since $\rho > \rho$, it follows from Lemma 8.5 of F.T. that $\mathfrak{Q}$ or $O_p(\mathfrak{D})$ contains a subgroup of type $(\rho, \rho, \rho)$. But then $C(\mathfrak{E}_0)$ contains subgroups of both types $(\rho, \rho, \rho)$ and $(\rho, \rho, \rho)$, and $\rho \not\equiv p \pmod{\pi}(C(\mathfrak{E}_0))$.

Since $C(\mathfrak{E}_0)$ also contains the element $\mathfrak{E}$ of $\mathfrak{H}(\rho)$, we see that $C(\mathfrak{E}_0)$ is greater in our ordering than $\mathfrak{D}$, contrary to our maximal choice of $\mathfrak{D}$.

We can now complete the proof of Theorem 2 as follows. Assume $\rho > \rho$, and let the notation be as above. Then by Lemma 6.8, $\mathfrak{H}_1$ contains an elementary subgroup $\mathfrak{E}$ of order $\rho^3$, to which we can apply the results of Lemma 6.7. As in Lemma 6.7, $\mathfrak{D}$ is an $S_{\rho,\rho}$-subgroup of $\mathfrak{H}_1$, $\mathfrak{H} = N_\rho(\mathfrak{D})$, $\mathfrak{H}$ contains $\mathfrak{P}_1$ and $\mathfrak{E}$, and $\mathfrak{P}_1 = \mathfrak{P} \cap O_{\rho,\rho}(\mathfrak{D}) = O_{\rho}(\mathfrak{D})$. Furthermore, $\mathfrak{D}$ is an $S_{\rho,\rho}$-subgroup of $\mathfrak{H}$ containing $\mathfrak{E}$, $\mathfrak{P} = \mathfrak{H}(\rho)$, $\mathfrak{E}_1 = C_{\rho,\rho}(\mathfrak{E})$ has order $\rho^2$ and contains $\Omega_1 = \Omega_1(\mathfrak{H}(\rho))$. Finally the element $Y$ of $\mathfrak{E}$ which satisfies Lemma 6.7 (ii) is not contained in $\mathfrak{E}_1$.

Now $\Omega_1$ does not centralize $\mathfrak{P}_1$, otherwise Lemma 6.6 would be violated in $C(\Omega_1)$. Since $\mathfrak{P}_1$ is generated by its subgroups $C_{\rho,\rho}(E)$ with $E$ in $\mathfrak{E}_1$, we can find an element $E$ in $\mathfrak{E}_1$ such that $\Omega_1$ does not centralize $C_{\rho,\rho}(E)$. Then $C(E)$ contains $\mathfrak{H}$ and a $p$-subgroup which is normalized by $\mathfrak{E}$, but not
centralized by $\Omega_1$. Among all subgroups $R$ of $G$ containing $BE$ and a $p$-subgroup which is normalized by $E$, but not centralized by $\Omega_1$, and such that $q \in \pi_p(R)$, choose $R$ so that $|R|_q$ is maximal, and subject to this condition, minimize $|R|_q$. If $R$ denotes an $S_{p, q}$-subgroup of $S(R)$ normalized by $BE$, then $R \leq \Omega$ by $C_{p, q}$ in $S(R)$ and the minimality of $|R|_q$. Let $Q_1^*$ be an $S_{p, q}$-subgroup of $G$ such that $Q_1^* \cap R$ is an $S_{p, q}$-subgroup of $R$ and such that $BE \leq Q_1^*$. Set $Q_1^* = Q_1^* \cap O_{p, q}(R)$. Then $R = O_{p}(R)N_{p}(Q_1^*)$ by Sylow's theorem. Now $G$ centralizes $O_{p}(R)$ by Lemma 6.1. From this it follows that $N_{p}(Q_1^*)$ contains a $p$-subgroup which is normalized by $E$, and not centralized by $\Omega_1$; for if this were not the case, $R$ would not contain such a subgroup. Thus $R = N_{p}(Q_1^*)$ by the minimality of $|R|_q$, and consequently $Q_1^* \leq R$. On the other hand, $Q_1^* \cap R$ is an $S_{p, q}$-subgroup of $\Omega_1^*$ by the maximality of $|R|_q$. But now $Q_1^*$ contains an element $Q_1^*$ of $S_{p, q}(Q_1^*)$ by Lemma 3.4, and we conclude from Lemma 5.4 that $R \subseteq N_{p}(Q_1^*) = \mathfrak{N}(Q_1^*)$ where $\mathfrak{N} = \mathfrak{V}(\mathfrak{C}_p; Q_1^*)$. It follows at once from our maximal choice of $|R|_q$ that $Q_1^* \leq R$.

Since $Q_1^* \leq R$, $Q_1^*$ centralizes $O_{p}(R) = O_{p}(R)$. Thus

$$O_{p, q}(R) = O_{p}(R) \times O_{q}(R).$$

We must have $\Omega_1 \not\subseteq O_{q}(R)$, otherwise $\Omega_1$ would centralize any $p$-subgroup of $R$ normalized by $E$. Now $Y$ does not centralize $G$, and hence $S = \langle Y, R \rangle$ is nonabelian of order $q^2$ with center $\Omega_1$. Since $\Omega_1 \not\subseteq O_{q}(R)$, it follows that $S \cap O_{q}(R) = 1$. Since $|S \cap E| = q^2$, we conclude that $|E \cap O_{q}(R)| \leq q$.

Now let $\mathfrak{V}_0$ be a $p$-subgroup of $R$ of minimal order which is normalized, but not centralized by $E$. Then $E$ acts trivially on $D(\mathfrak{V}_0)$ and irreducibly on $\mathfrak{V}_0 = \mathfrak{V}_0 \cdot D(\mathfrak{V}_0)$. Let $E_0 := \ker(E \to \Aut \mathfrak{V}_0)$. Then $\mathfrak{V} = E_0 \mathfrak{V}_0$ is cyclic. Now $E$ is represented faithfully and irreducibly on $\mathfrak{V}_0$ regarded as a vector space over $GF(p)$. Since $p = 1 \pmod{q}$ by Lemma 6.7 (i), the irreducible representations of $E$ on a vector space over $GF(p)$ are all one-dimensional. Hence $|\mathfrak{V}_0| = p$, and consequently $\mathfrak{V}_0$ is cyclic. Thus $|\mathfrak{V}_0| = p$ by our minimal choice of $\mathfrak{V}_0$.

Since $Q_1^*$ centralizes $O_{p}(R)$, $0 \not\subseteq O_{p}(R)$. On the other hand, since $E$ is $q$-constrained, and $O_{p}(R)$ is an $S_{p, q}$-subgroup of $O_{p, q}(R)$, $C_{p}(O_{p}(R)) \subseteq O_{p, q}(R)$. Thus $0$ is faithfully represented as a subgroup of $\Aut O_{p}(R)$. Let $\mathfrak{W}_0 = C(E_0) \cap O_{p}(R)$. Since $0_0E_0 = 0_0 \times E_0$, it follows from Lemma 3.7 that $0_0$ is faithfully represented as a subgroup of $\Aut 0_0$. Furthermore, since $E$ is abelian, $\mathfrak{W}_0$ is invariant under $E$. Now by Lemma 8.2 of F.T., $\mathfrak{W}_0$ possesses a characteristic subgroup $C^*$ of class at most 2 on which $0_0$ is represented as a group of automorphisms. Choose $E$ to be an $E$-invariant subgroup of $C^*$ of minimal order on which $0_0$ is represented faithfully. Then clearly $E = 0_0(E_0)$, and since $cl(E) \leq 2$ and $q$ is odd, $E$ is regular and thus has exponent $q$. Also if $cl(E) = 2$, $0_0$ centralizes $Z(E)$. 

MAXIMAL SUBGROUPS OF FINITE SIMPLE GROUPS 193
Suppose first that \( cl(\mathcal{C}) = 2 \). Let \( \mathcal{X} \) be a subgroup of order \( q \) in \( \mathcal{C} \). Apply Lemma 3.6 with \( \langle \mathcal{X}, \Psi_0 \rangle \) in the role of \( \mathcal{G} \) and \( \mathcal{C} \) in the role of \( \mathcal{H} \) to obtain that \( C_{\mathcal{C}}(\mathcal{X}) \) is noncyclic. But as \( \mathcal{C} \) centralizes \( \mathcal{C} \), \( C_{\mathcal{C}}(\mathcal{X}) = C_{\mathcal{C}}(\mathcal{C}) \), and hence \( C_{\mathcal{C}}(\mathcal{E}) \) is noncyclic. But \( \mathcal{C} \subseteq O_q(\mathcal{H}) \) and \( |\mathcal{C} \cap O_q(\mathcal{H})| = q' \). Since \( \mathcal{C} \) has exponent \( q \), it follows that \( \mathcal{E} \) is contained in an elementary subgroup of \( \mathcal{G} \) of order \( q^4 \), contrary to Lemma 6.7 (iii).

So assume that \( \mathcal{E} \) is abelian. As \( [\Psi_0, \mathcal{E}] \cap \mathcal{E} = 1 \), and hence \( \mathcal{C} \cap \mathcal{E} \neq [\Psi_0, \mathcal{E}] \). By our minimal choice of \( \mathcal{C} \), \( \mathcal{E} = [\Psi_0, \mathcal{E}] \), and consequently \( \mathcal{C} \cap [\Psi_0, \mathcal{E}] = 1 \). But \( \mathcal{C} \lhd \mathcal{G} \); hence \( C_{\mathcal{G}}(\mathcal{E}) \neq 1 \). It follows that \( \mathcal{E} \) is contained in an elementary subgroup of \( \mathcal{G} \) of order \( q^4 \), and Lemma 6.7 (iii) again yields a contradiction. This completes the proof of the theorem.

VII. AN \( \mathcal{E}^* \)-THEOREM

In this section we shall prove the following result, corresponding to Theorem 20.1 of F.T.

**Theorem 3.** Let \( \mathcal{G} \) be a simple group which is \( \{p, q\} \)-tame for two primes \( p \) and \( q \) in \( \pi_\mathcal{G} \). Then \( \mathcal{G} \) satisfies \( \mathcal{E}^*_{\mathcal{G}, q} \).

The proof is by contradiction.

**Lemma 7.1.** Either an \( \mathcal{S}_p \)-subgroup of \( \mathcal{G} \) normalizes, but does not centralize some \( \mathcal{Q} \)-subgroup of \( \mathcal{G} \), or an \( \mathcal{S}_q \)-subgroup of \( \mathcal{G} \) normalizes but does not centralize some \( \mathcal{Q} \)-subgroup of \( \mathcal{G} \).

**Proof.** Theorem 2 implies this lemma at once.

For definiteness, we assume that an \( \mathcal{S}_p \)-subgroup \( \mathcal{V} \) of \( \mathcal{G} \) normalizes, but does not centralize some nonidentity \( \mathcal{Q} \)-subgroup \( \Omega_1 \) of \( \mathcal{G} \), which we may take to be a maximal element in \( \mathcal{U}(\mathcal{V}; \mathcal{G}) \). Set \( \Omega_1 = N(\Omega_1) \), let \( \mathcal{D} \) be an \( S_{p, q} \)-subgroup of \( S(\Omega_1) \) normalized by \( \mathcal{V} \) and set \( \Omega = N_\mathcal{V}(\mathcal{D}) \). Since \( \Omega_1 = S(\Omega_1) \mathcal{V} \), \( \mathcal{D} \) is an \( S_{p, q} \)-subgroup of \( S(\Omega) \) such that \( \mathcal{S} \cap \mathcal{D} \) is permutable with \( \mathcal{V} \), and let \( \Omega \) be an \( S_q \)-subgroup of \( \mathcal{G} \) containing \( \Omega_1 \).

**Lemma 7.2.** \( O_p(\Omega) \neq 1 \).

**Proof.** Suppose \( O_p(\Omega) = 1 \). Then \( p \notin \pi(O_q(\Omega)) \), and hence by the maximal choice of \( \Omega_1 \), \( \Omega_1 \) is an \( S_q \)-subgroup of \( O_q(\Omega) \), and hence \( \Omega_1 \) contains every element of \( \mathcal{F} \mathcal{C} \mathcal{N}_3(\mathcal{Q}) \). Hence by Lemma 5.4, \( \Omega \subseteq \Omega_1 = N(\mathcal{Q}(\mathcal{V})) \), \( \mathcal{V} = \mathcal{V}(G(\mathcal{U}); \mathcal{Q}) \), \( \mathcal{U} \in \mathcal{F} \mathcal{C} \mathcal{N}_3(\mathcal{Q}) \). Thus \( \Omega_1 \) contains \( \mathcal{V} \) as well as \( \Omega \). Furthermore by condition (V) in the definition of \( p \)-tame, \( p \in \pi(\Omega_1) \). Thus
\(\mathfrak{N}_1\) possesses an \(S_{\nu,\sigma}\)-subgroup by \(C_{\nu,\sigma}\) in \(S(\mathfrak{N}_1)\). We conclude that \(6\) satisfies \(E_{\nu,\sigma}\). Since we are arguing by contradiction, the lemma follows.

Set \(\Psi_1 = O_{\nu}(6)\) and \(\mathfrak{M} = N(\Psi_1)\).

**Lemma 7.3.** Let \(\mathfrak{N}_0\) be a nontrivial abelian \(q\)-subgroup of \(6\) such that \(\mathfrak{R} = C(\mathfrak{N}_0)\) contains \(\Psi^*\Psi_1\). Then \(\Psi^*\) is an \(S_{\nu}\)-subgroup of \(\mathfrak{R}\).

**Proof.** We first show that \(\Psi^*\) is an \(S_{\nu}\)-subgroup of \(\mathfrak{M}\). If \(\Psi_0 = \Psi \cap O_{\nu}(\mathfrak{M})\), then \(\gamma_{\Gamma_1}^{\Psi_0}\) is a \(\sigma^\prime\)-group, and it follows from Lemmas 3.3 and 3.5 that \(\Gamma_1 \subseteq O_{\nu}(\mathfrak{M})\). Since \(\Psi_1\) is a maximal element in \(\forall(\Psi; q)\), \(\Psi_1\) is an \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\), so that \(\mathfrak{M} = O_{\nu}(\mathfrak{M}) N(\Omega_1)\) by Sylow's theorem. Since \(\Psi^*\) is an \(S_{\nu}\)-subgroup of \(N(\Omega_1)\) and \(\Psi^* \subseteq \mathfrak{M}\), the desired conclusion follows.

Let \(\Xi_1^* = \Xi^* \cap O_{\nu}(\mathfrak{M})\). We shall show that \(\Psi_1\) is a maximal element of \(\forall_{\mathfrak{M}}(\Omega_1^*; p)\). Since \(\Psi^* \cap \mathfrak{D}\) is an \(S_{\nu}\)-subgroup of \(\mathfrak{D}\), \(\Psi^* \cap \mathfrak{D}\) is an \(S_{\nu}\)-subgroup of \(S(\Omega_1^*)\); hence \(\Xi_1^* \subseteq \Psi^* \cap \mathfrak{D}\) and \(\Xi_1^*\) is an \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\).

Since \(\Psi_1 = O_{\nu}(\Psi) = O_{\nu}(\mathfrak{D})\), we see that \(\Psi_1 \in \forall_{\mathfrak{M}}(\Omega_1^*; p)\). Now let \(\Psi^*\) be a maximal element of \(\forall_{\mathfrak{M}}(\Omega_1^*; p)\) containing \(\Psi_1\) and set \(\Psi^* = \Psi^* \cap O_{\nu}(\mathfrak{M})\). Then \(\gamma^\Psi_1^* \subseteq \Psi_1^*\), and consequently \(\Psi^* = \Psi_1^* C_{\Psi^*}(\mathfrak{M})\).

Then \(\forall_{\mathfrak{M}}(\Omega_1^*; p) \subseteq O_{\nu}(\mathfrak{M})\) by Lemma 1.2.3 of [5], and we conclude that \(\Xi^* \subseteq O_{\nu}(\mathfrak{M})\). On the other hand, \(\Psi_1 \cap O_{\nu}(\mathfrak{M}) = O_{\nu}(\mathfrak{M})\). Furthermore, \(\Psi \cap O_{\nu}(\mathfrak{M})\) is an \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\) since \(\Psi\) is an \(S_{\nu}\)-subgroup of \(\mathfrak{M}\). This implies that \(\Psi^* = \Psi_1\) is an \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\). Thus \(\Psi_1^*\) is a maximal element of \(\forall_{\mathfrak{M}}(\Omega_1^*; p)\), as asserted.

This result has several consequences. First of all, set \(\Omega_2 = \Omega^* \cap O_{\nu,\sigma}(\mathfrak{D})\). Then \(\Omega_2\) is an \(S_{\nu}\)-subgroup of \(O_{\nu,\sigma}(\mathfrak{D})\). Since \(\Xi^*_1 = \Xi^* \cap O_{\nu,\sigma}(\mathfrak{D}) \subseteq O_{\nu,\sigma}(\mathfrak{D})\), we have \(\Omega^*_1 \subseteq \Omega_2\). Furthermore, \(\Psi_1 \in \forall_{\mathfrak{M}}(\Omega_2^*; p)\) since \(\Psi_1^* \subseteq \Omega_2\). It follows, therefore, from the preceding argument that \(\Psi_1\) is a maximal element of \(\forall_{\mathfrak{M}}(\Omega_1^*; p)\). Next let \(\Psi_2\) be a maximal element of \(\forall_{\mathfrak{M}}(\Omega_2^*; p)\) containing \(\Psi_1\).

Since \(\Omega_1 \subseteq O_{\nu}(\mathfrak{D})\), we have \(\Omega_1 = \Omega_2\). But we have shown above that \(\Omega_1 \subseteq O_{\nu}(\mathfrak{M})\). This implies that \(\Omega_1\) centralizes \(\Psi_2\), and hence that \(\Psi_2 \subseteq \Omega_1 = N(\Omega_2)\). Thus \(\Psi_2 \in \forall_{\mathfrak{M}}(\Omega_2^*; p)\), whence \(\Psi_2 = \Psi_1\). We conclude that \(\Psi_1\) is a maximal element of \(\forall_{\mathfrak{M}}(\Omega_2^*; p)\).

Since \(\Omega_2\) normalizes some \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\) containing \(\Psi_1\), it also follows that \(\Psi_2\) is an \(S_{\nu}\)-subgroup of \(O_{\nu}(\mathfrak{M})\).

We turn now to the proof of the lemma. Suppose false, and let \(\mathfrak{N}_0\) be a nontrivial \(q\)-subgroup of \(6\) such that \(\mathfrak{R} = C(\mathfrak{N}_0)\) contains \(\mathfrak{A}^*\Psi_1\) and such that \(\mathfrak{A}^*\) is not an \(S_{\nu}\)-subgroup of \(\mathfrak{R}\). Among all subgroups \(\Xi\) of \(\mathfrak{R}\) containing \(\mathfrak{A}^*\Psi_1\) and such that \(\mathfrak{A}^*\) is not an \(S_{\nu}\)-subgroup of \(\Xi\), choose \(\Xi\) in such a way that \(|S(\Xi)|_{q}\) is maximal, and subject to this condition, minimize \(|\Xi|_{q}\).

By the minimality of \(\Xi\), an \(S_{\nu,\sigma}\)-subgroup \(\mathfrak{R}\) of \(S(\Xi)\) is normal in \(\Xi\). Set \(\Xi_1 = O_{\nu}(\Xi)\) and \(\Xi_2 = O_{\sigma}(\Xi)\), so that \(\Xi_1 \times \Xi_2 = F(\mathfrak{N})\).

We first show that \(\Xi_1 \subseteq \Psi_1\). Suppose false. Then \(\Xi_1 \neq 1\). If \(\Xi_1^* = \Xi_2^* = 1\)
$N_{Z_1}(\Omega \cap \mathfrak{U}) = \Omega \cap \mathfrak{U}$ is a group. We let $Z_1 \cap \mathfrak{U}$ be a chief factor of $Z_1 \cap \mathfrak{U}$, and it follows that $Z_1 \cap \mathfrak{U}$ is a chief factor of $Z_1 \cap \mathfrak{U}$. Then $\mathfrak{V}$ centralizes $Z_1 \cap \mathfrak{U}$, and it follows that $Z_1 \cap \mathfrak{U}$ normalizes $\mathfrak{V}$. Thus $Z_1 \cap \mathfrak{U}$ is a maximal element of $N_{Z_1}(\mathfrak{U} \cap \mathfrak{V})$.

We shall show next that $Z_2 \subseteq \mathfrak{U}$. By definition, $\mathfrak{U}$ is an $S_\mathfrak{V}$-subgroup of $N(\mathfrak{U})$. Since $\mathfrak{U}$ is a $G$-subgroup of $N(\mathfrak{U})$, we have $C_{\mathfrak{U}}(\mathfrak{U}) \subseteq \mathfrak{U}$. Hence $[C_{\mathfrak{U}}(\mathfrak{U}), \mathfrak{V}] = \mathfrak{V}$ since $\mathfrak{V} \cap \mathfrak{U} = 1$. Thus $\mathfrak{V}$ centralizes $C_{\mathfrak{U}}(\mathfrak{U})$. Since $\mathfrak{V} = \mathfrak{U} \cap \mathfrak{V}$, we can therefore apply Lemma 3.7 to conclude that $\mathfrak{V}$ centralizes $Z_2$. But now $Z_2 \subseteq C(\mathfrak{V}) \subseteq N(\mathfrak{V})$, and consequently $Z_2 \cap \mathfrak{U}$ is a $G$-subgroup of $N(\mathfrak{V})$. Since we have shown above that $\mathfrak{U}$ is an $S_\mathfrak{V}$-subgroup of $N(\mathfrak{V})$, it follows that $Z_2 \subseteq \mathfrak{U}$.

We prove next that $\mathfrak{V} \subseteq \mathfrak{V}$. If $\mathfrak{V}$ is solvable, this clearly holds, for then $\mathfrak{V} = \mathfrak{V}$. So assume $\mathfrak{V}$ is nonsolvable, in which case $\mathfrak{U}$ is a proper subgroup of $\mathfrak{V}$. Since $\mathfrak{V} \subseteq \mathfrak{V}$, and since $\mathfrak{V} \cap \mathfrak{U} = 1$, it follows therefore from our choice of $\mathfrak{V}$ that $\mathfrak{U}$ is a $G$-subgroup of $\mathfrak{V}$. Hence $\mathfrak{U} = \mathfrak{U} \cap \mathfrak{V}$ is a $G$-subgroup of $\mathfrak{V}$. On the other hand, it follows from our maximal choice of $\mathfrak{V}$ that $\mathfrak{V}$ contains an $S_\mathfrak{V}$-subgroup $\mathfrak{V}$ of $\mathfrak{V}$. Since $\mathfrak{V} = \mathfrak{V}$, it follows now from Lemma 3.5 that $\mathfrak{V} \subseteq \mathfrak{V}$, as asserted.

We now prove that $\mathfrak{V} \subseteq \mathfrak{V}$. We have shown above that $\mathfrak{V} \subseteq \mathfrak{V}$ and that $\mathfrak{V}$ centralizes $\mathfrak{V}$. Hence $\mathfrak{V} \subseteq \mathfrak{V}$ since $\mathfrak{V} \cap \mathfrak{V} = 1$. Thus $\mathfrak{V} \subseteq \mathfrak{V} \cap \mathfrak{V}$. Since $\mathfrak{V} \subseteq \mathfrak{V}$ and since $\mathfrak{V} \cap \mathfrak{V} = 1$, it follows therefore from our choice of $\mathfrak{V}$ that $\mathfrak{V}$ is an $S_{\mathfrak{V}}$-subgroup of $\mathfrak{V}$. Since $\mathfrak{V} = \mathfrak{V}$, we can therefore apply Lemma 3.7 to conclude that $\mathfrak{V}$ centralizes $\mathfrak{V}$. But no $\mathfrak{V}$ is a $G$-subgroup of $\mathfrak{V}$, and consequently $\mathfrak{V} \subseteq \mathfrak{V}$, as asserted.

Finally consider $C_{\mathfrak{U}}(\mathfrak{U})$. Since $\mathfrak{U} \subseteq \mathfrak{V}$, $C_{\mathfrak{U}}(\mathfrak{U}) \subseteq \mathfrak{U} = N(\mathfrak{U})$. Now $\mathfrak{U}$, by its definition, is an $S_{\mathfrak{V}}$-subgroup of $\mathfrak{V}$. Since $\mathfrak{V}$ centralizes $\mathfrak{V}$, $\mathfrak{V} \subseteq \mathfrak{V} \cap \mathfrak{V} \cap C_{\mathfrak{U}}(\mathfrak{U})$. Since $\mathfrak{V} = \mathfrak{V}$, $\mathfrak{V}$ is a $G$-subgroup of $\mathfrak{V}$, and it follows at once that $\mathfrak{V} \subseteq \mathfrak{V} \cap \mathfrak{V} \cap C_{\mathfrak{U}}(\mathfrak{U})$. Hence $\mathfrak{V} \subseteq \mathfrak{V}$, as asserted.

Now consider $C_{\mathfrak{U}}(\mathfrak{U})$. Since $\mathfrak{U} \subseteq \mathfrak{V}$, $C_{\mathfrak{U}}(\mathfrak{U}) \subseteq \mathfrak{V} = N(\mathfrak{V})$. Now $\mathfrak{V}$, by its definition, is an $S_{\mathfrak{V}}$-subgroup of $\mathfrak{V}$. Since $\mathfrak{V}$ centralizes $\mathfrak{V}$, $\mathfrak{V} \subseteq \mathfrak{V} \cap \mathfrak{V} \cap C_{\mathfrak{U}}(\mathfrak{U})$. Since $\mathfrak{V} = \mathfrak{V}$, $\mathfrak{V}$ is a $G$-subgroup of $\mathfrak{V}$, and it follows at once that $\mathfrak{V} \subseteq \mathfrak{V} \cap \mathfrak{V} \cap C_{\mathfrak{U}}(\mathfrak{U})$. Hence $\mathfrak{V} \subseteq \mathfrak{V}$, as asserted.
of F.T.: \(\mathcal{Q}_1\) is elementary, \(|\mathcal{Q}_1| > q\), \(\mathfrak{A}\cap \mathcal{Q}_1 = 1\), \(\mathfrak{A}\) centralizes \(\mathcal{Q}_1\), \(\mathfrak{A}\) contains \(Z(\mathcal{Q})\) and also contains an element of \(\mathcal{V}(\mathcal{Q})\). Since \(\gamma^2\mathcal{Q}_2\mathfrak{A}\mathfrak{A}^2 = 1\) \((\mathcal{Q}_2 = \mathcal{Q}\cap \mathcal{O}_{\gamma}(\bar{s}))\) and \(\mathfrak{A}\) is \(q\)-stable, it also follows that \(\mathfrak{A}\subseteq \mathcal{Q}_2\). We remark that the fact that \(|\mathcal{Q}_1| > q\) depends upon Lemma 5.3. This is the only place in the paper in which this lemma is used.

Since \(\mathfrak{A}\) centralizes \(\mathcal{Q}_1\) and \(\mathfrak{A} = \mathcal{Q}_1 = 1\), it follows that \(\mathfrak{A}\) is faithfully represented on \(\mathcal{Q}_1\). We proceed now as in F.T. to show that for each element \(Z\) in \(\mathcal{Q}(\mathcal{Q})^\#\), there exists a \(p\)-subgroup \(\mathfrak{A}(Z)\) in \(\mathcal{V}(\mathcal{Q}; p)\) which is not centralized by \(Z\). Since \(\mathfrak{A}\) is faithfully represented on \(\mathcal{Q}_1\), we can determine, for each \(Z\) in \(\mathcal{Q}(\mathcal{Q})^\#\), a subgroup \(\mathfrak{A}(Z)\) of \(\mathcal{Q}_1\) which is normalized, but not centralized by \(Z\), but which is centralized by an element \(E\) of \(\mathfrak{A}\), where \(E\) is contained in an element of \(\mathcal{V}(\mathcal{Q})\). Set \(C = C(E)\), and let \(C\) be an \(S_p\)-subgroup of \(C\) containing \(\mathcal{Q}_1\). Since \(|\mathcal{Q}_1 : C\cap \mathcal{Q}_1| = \gamma^2\mathcal{Q}_1(\mathcal{Q}_1)^2 = 1\).

If \(C_1 = \mathcal{Q}_1\cap \mathcal{O}_{\gamma}(\mathcal{Q}_1)\), then \(C_1(\mathcal{Q}_1)\) is solvable by Lemma 3.3. Since \(\mathfrak{A}\) is \(q\)-stable, it follows that \(Z\in \mathcal{O}_{\gamma}(\mathcal{Q}_1)\). This implies that \(Z\) does not centralize \(\mathfrak{A}(Z)\cap \mathcal{O}_{\gamma}(\mathcal{Q}_1)\). Now by our choice of \(E\), \(C\) contains \(\mathfrak{A}\) and it follows that an \(\mathfrak{A}\)-invariant \(S_{\gamma}\)-subgroup \(\mathfrak{A}(Z)\) of \(\mathcal{O}_{\gamma}(\mathcal{Q}_1)\) satisfies the required conditions.

But now Theorem I implies that if \(\mathfrak{A}\) is a maximal element of \(\mathcal{V}(\mathcal{Q}; p)\), then \(\mathcal{Q}(\mathcal{Q})^\#\) is faithfully represented on \(\mathfrak{A}\). Let \(\mathfrak{A}_1 = N(\mathfrak{A})\), let \(\mathfrak{A}\) be an \(S_{\gamma}\)-subgroup of \(S(\mathfrak{A}_1)\) normalized by \(\mathfrak{A}\) and set \(\mathfrak{A}_2 = N(\mathfrak{A}_1)\). It follows that \(\mathcal{O}_{\gamma}(\mathfrak{A}_2) = 1\), in contradiction to Lemma 7.2 with \(p\) and \(q\) interchanged.

VIII. PROOF OF THEOREM A

We need a few preliminary lemmas, the first of which will also be used in Section IX.

LEMMa 8.1. Let \(\mathfrak{A}\) be a simple group which is \(p\)-tame for some prime \(p\) in \(\pi_\mathfrak{a}\). Let \(\mathfrak{A}\) be an \(S_p\)-subgroup of \(\mathfrak{A}\) and let \(\mathfrak{A}\) be a proper subgroup of \(\mathfrak{A}\) containing \(\mathfrak{A}\) such that \(S(\mathfrak{A})\neq 1\). Suppose that \(\mathfrak{A}\) contains an \(S_{p}\)-subgroup of \(\mathfrak{A}\) for some prime \(q\neq p\). Then any \(\mathfrak{A}\)-invariant \(S_{p}\)-subgroup of \(\mathcal{O}_{\gamma}(\mathfrak{A})\) is a maximal element of \(\mathcal{V}(\mathfrak{A}; q)\).

Proof. Let \(\mathfrak{A} \in P\mathcal{V}(\mathfrak{A}; q)\). If \(\mathcal{V}(\mathfrak{A}; q)\) is trivial, then certainly \(\mathcal{V}(\mathfrak{A}; q)\) is trivial. Since \(\mathfrak{A}\) normalizes an \(S_{p}\)-subgroup of \(\mathcal{O}_{\gamma}(\mathfrak{A})\), the lemma clearly holds in this case. We may therefore assume that \(\mathcal{V}(\mathfrak{A}; q)\) is nontrivial. Furthermore, \(\mathcal{O}_{\gamma}(\mathfrak{A}_1) \neq 1\) for a prime \(r\) since \(S(\mathfrak{A})\neq 1\). It follows therefore from assumption (V) in the definition of \(p\)-tame groups that \(\mathfrak{A}_1 \cap S(\mathcal{N}(\mathcal{O}_{\gamma}(\mathfrak{A}))\neq 1\). Since \(\mathfrak{A} \subseteq \mathcal{N}(\mathcal{O}_{\gamma}(\mathfrak{A}))\), this implies that \(\mathfrak{A}_1 \cap S(\mathfrak{A})\neq 1\). Thus \(p \in \pi_\mathfrak{a}(\mathfrak{A})\).

We shall first show that \(q \in \pi(\mathcal{O}_{\gamma}(\mathfrak{A}))\), so assume the contrary. Set \(\mathfrak{A}_1 = \mathcal{V}(ccl(\mathfrak{A}; q); \mathfrak{A})\). Then \(\mathcal{C}_{\mathfrak{A}}(\mathfrak{A}) \subseteq \mathcal{C}_{\mathfrak{A}}(\mathfrak{A})\). Hence if \(\mathfrak{A}_0\) is an \(S_{q}\)-subgroup
of $C_{\Phi}(\mathfrak{g})$, then $\Sigma_0 \in \mathcal{U}_B(\mathfrak{g}; q)$, and $\Sigma_0 \subseteq O_v(\mathfrak{g})$ by Lemma 4.1. Thus by our assumption, $\Sigma_0 = 1$, and hence $q \notin \pi(C_{\Phi}(\mathfrak{g}))$. Furthermore, since $p \in \pi(\mathfrak{g})$, $\mathfrak{g} \subseteq O_p(\mathfrak{g})$ by Lemma 5.5. Since $q \notin \pi(O_v(\mathfrak{g}))$, it follows therefore from Sylow's theorem that $N(\mathfrak{g})$ contains an $S_\sigma$-subgroup $\Sigma$ of $\mathfrak{g}$. By hypothesis, $\Sigma$ is an $S_\sigma$-subgroup of $\mathfrak{g}$.

Now set $\mathfrak{h}_1 = N(\mathfrak{z}(\mathfrak{g}))$. Then $\mathfrak{g} \subseteq O_{p, v}(\mathfrak{h}_1)$ by Lemma 5.5, and consequently $\mathfrak{h}_1 = O_{p, v}(\mathfrak{h}_1) N(\mathfrak{g})$ by Sylow's theorem. Since $\Sigma$ normalizes $\mathfrak{g}$, $\Sigma \cap O_{p, v}(\mathfrak{h}_1) \subseteq C_p(\Sigma)$, and hence $\Sigma \cap O_{p, v}(\mathfrak{h}_1) = 1$. But $\Sigma$ is an $S_\sigma$-subgroup of $\mathfrak{h}_1$, and hence $\Sigma \cap O_{p, v}(\mathfrak{h}_1)$ is an $S_\sigma$-subgroup of $O_{p, v}(\mathfrak{h}_1)$. It follows that $q \notin \pi(O_{p, v}(\mathfrak{h}_1))$.

Finally let $\mathfrak{h}_1$ be a maximal element of $\mathcal{U}(\mathfrak{g}; q)$ normalized by $\mathfrak{g}$. Since $\mathcal{U}(\mathfrak{g}; q)$ is nontrivial, Corollary 4.2 implies that $\mathfrak{h}_1 \neq 1$. Furthermore, by Lemma 4.2, $\mathfrak{h}_1 = O_{p, v}(\mathfrak{h}_1)$ (where $\mathfrak{h} = N(\Sigma_1)$). Since $q \notin \pi(O_{p, v}(\mathfrak{h}_1))$, $\Sigma_1 \cap \mathfrak{y}$ contains an $S_\tau$-subgroup $\Sigma$ of $\mathfrak{h}_1$, and for the same reason, $\Sigma \cap O_{p, v}(\mathfrak{h}_1 \cap \mathfrak{y}) = 1$. Since $\Sigma \cap O_{p, v}(\mathfrak{h}) \subseteq O_{p, v}(\mathfrak{h}_1 \cap \mathfrak{y})$, it follows that $\Sigma \cap O_{p, v}(\mathfrak{h}) = 1$. But $\Sigma \cap O_{p, v}(\mathfrak{h})$ is an $S_\tau$-subgroup of $O_{p, v}(\mathfrak{h})$ as $\Sigma$ is an $S_\sigma$-subgroup of $\mathfrak{g}$. This contradicts the fact that $\Sigma_1 \subseteq O_{p, v}(\mathfrak{h})$ and $\Sigma_1 \neq 1$.

Thus $q \notin \pi(O_{p, v}(\mathfrak{h}))$, as asserted.

Set $\Sigma^* = \Sigma \cap O_{p, v}(\mathfrak{g})$. Then $\Sigma^* \neq 1$ and $\Sigma^*$ is an $S_\tau$-subgroup of $O_{p, v}(\mathfrak{g})$. Replacing $\Sigma$ by a conjugate, if necessary, we may assume that $\mathfrak{g}$ normalizes $\Sigma^*$. We shall now show that $\Sigma^*$ is, in fact, a maximal element of $\mathcal{U}(\mathfrak{g}; q)$. Set $\mathfrak{h} = N(\Sigma^*)$. Then $\mathfrak{g} = O_{p, v}(\mathfrak{h}) (\mathfrak{h} \cap \mathfrak{g})$ by Sylow's theorem, and $\mathfrak{h} \subseteq \mathfrak{g}$. Hence $\mathfrak{h} \cap O_{p, v}(\mathfrak{h}) \subseteq O_{p, v}(\mathfrak{g})$, and consequently $\mathfrak{h} \cap O_{p, v}(\mathfrak{h}) = \Sigma^*$. Since $\Sigma \cap O_{p, v}(\mathfrak{h})$ is an $S_\tau$-subgroup of $O_{p, v}(\mathfrak{h})$, we conclude that $\Sigma^*$ is an $S_\tau$-subgroup if $O_{p, v}(\mathfrak{h})$.

Now let $\Sigma_1^*$ be a maximal element of $\mathcal{U}(\mathfrak{g}; q)$ containing $\Sigma^*$. Then $\Sigma_1^* = N(\Sigma^*) \subseteq \mathfrak{g}$, and $\Sigma_1^*$ is invariant under $\Sigma$. Since $\mathfrak{h} \subseteq \mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$, $\Sigma_1^* \subseteq O_{p, v}(\mathfrak{h})$ by Lemma 4.1. Since $\Sigma^* \subseteq \Sigma_1^*$ and $\Sigma^*$ is an $S_\tau$-subgroup of $O_{p, v}(\mathfrak{h})$, we conclude at once that $\Sigma^* = \Sigma_1^*$ is a maximal element of $\mathcal{U}(\mathfrak{h}; q)$, and the lemma follows.

**Lemma 8.2.** Let $\mathfrak{g}$ be a simple group which is $(p, q)$-tame for two primes $p$ and $q$ in $\pi_3$, and let $\Sigma_1, \Sigma_2$ be two $S_{\mu_3, \beta}$-subgroups of $\mathfrak{g}$. If $\Sigma, \Sigma_1$ denote $S_{\mu_3, \beta}$-subgroups of $S(\mathfrak{g})$ and $S(\Sigma_1)$ respectively, then $F(\Sigma)$ and $F(\Sigma_1)$ are conjugate.

**Proof.** By definition of an $S_{\mu_3, \beta}$-subgroup, $\Sigma \lhd S(\mathfrak{g})$, $\Sigma_1 \lhd S(\Sigma_1)$, $\mu, q \in \pi(\Sigma)$, and $\mu, q \in \pi(\Sigma_1)$. Since $\Sigma \neq 1$, either $O_{\mu}(\Sigma) \neq 1$ or $O_{\mu}(\Sigma) \neq 1$. For definiteness, assume that $O_{\mu}(\Sigma) \neq 1$. Let $\Psi$ be an $S_\mu$-subgroup of $\mathfrak{g}$. Replacing $\Sigma_1$ by a suitable conjugate, we may assume without loss that $\Psi \subseteq \Sigma_1$. Since $O_{\mu}(\Sigma), O_{\mu}(\Sigma_1)$ are $S_\mu$-subgroups of $O_{\mu}(\mathfrak{g})$, $O_{\mu}(\Sigma_1)$ respectively, it follows from the preceding lemma that both are maximal elements of $\mathcal{U}(\mathfrak{g}; q)$. Hence by Corollary 4.4 and Theorem 1 there exists an element
G in $\mathfrak{S}$ such that $O_p(DG) = O_p(\mathfrak{S})$. Since $\mathfrak{P}G$ and $\mathfrak{P}$ are $S_p$-subgroups of $\mathfrak{S}$, and each is contained in $N(O_p(\mathfrak{S}))$, there exists an element $G_2$ in $N(O_p(\mathfrak{S}))$ such that $\mathfrak{P}G_2G = \mathfrak{P}$. Setting $G = G_1G_2$, we see that $O_p(DG) = O_p(\mathfrak{S})$ and that $\mathfrak{P} \subset S_p^G$. Thus without loss we may assume that $S_p^G = S_p$, and hence that $O_p(D) = O_p(\mathfrak{S})$.

Set $\mathfrak{K} = N(O_p(\mathfrak{S}))$. Then both $\mathfrak{S}$ and $S_p$ are contained in $\mathfrak{K}$. Let $\mathfrak{L}$ be an $S_p$-subgroup of $\mathfrak{S}$. Since $\mathfrak{S}$ is an $S_p$-subgroup of $\mathfrak{K}$ and $O_p(\mathfrak{S}) \subseteq O_p(\mathfrak{S})$ by Lemma 4.1. But by the preceding lemma applied to $\mathfrak{S}$, $O_p(\mathfrak{S})$ is a maximal element of $V(\mathfrak{S}; p)$. Since $\mathfrak{S}$ normalizes some $S_p$-subgroup of $O_p(\mathfrak{S})$, we conclude that $O_p(\mathfrak{S})$ is an $S_p$-subgroup of $O_p(\mathfrak{S})$. Similarly $S_p^G$ is an $S_p$-subgroup of $O_p(\mathfrak{S})$. Thus there exists an element $X$ in $O_p(\mathfrak{S})$ such that $O_p(DX) = O_p(D)$. But $F(\mathfrak{S}) = O_p(\mathfrak{S}) \times O_p(\mathfrak{S})$ and $F(\mathfrak{S}) = O_p(\mathfrak{S}) \times O_p(\mathfrak{S})$. Since $O_p(\mathfrak{S}) = O_p(\mathfrak{S})$ and $X$ normalizes $O_p(\mathfrak{S})$, $F(\mathfrak{S})$ completing the proof.

**Lemma 8.3.** Let $\mathfrak{S}$ be a simple group which is $\tau$-tame for some set of primes $\tau$ in $\pi_3$ consisting of at least two elements, and assume that $\mathfrak{S}$ satisfies $E'$. Let $\mathfrak{S}, \mathfrak{S}_1$ be two $S_p$-subgroups of $\mathfrak{S}$, and let $\mathfrak{S}, \mathfrak{S}_1$ be $S_p$-subgroups of $S(\mathfrak{S}), S(\mathfrak{S}_1)$ respectively. Then for any $p$ in $\tau$, $O_p(\mathfrak{S})$ is conjugate to $O_p(D)$ and $O_p(\mathfrak{S}_1)$ is conjugate to $O_p(D_1)$. Furthermore, $F(\mathfrak{S})$ is conjugate to $F(\mathfrak{S}_1)$.

**Proof.** If $\tau$ consists of two primes, the lemma follows immediately from Lemma 8.2. Hence we may assume that $\tau$ consists of at least 3 primes. We proceed by induction on $\tau$. If $\tau = \{p_1, p_2, \ldots, p_n\}$, set $\tau_i = \tau - p_i$. Suppose first that $F(\mathfrak{S})$, $F(\mathfrak{S}_1)$ are each $p_i$-groups. In this case, $O_p(\mathfrak{S}) = 1$ and $O_p(\mathfrak{S}_1) = 1$. Hence $O_p(\mathfrak{S}) = 1$ and $O_p(\mathfrak{S}_1) = 1$ for all $i = 2, 3, \ldots, n$. We may assume without loss that $\mathfrak{S}$ and $\mathfrak{S}_1$ both contain the same $S_p$-subgroup $\mathfrak{P}_1$ of $\mathfrak{S}$. Since $\mathfrak{S}$ is an $S_p$-subgroup of $\mathfrak{S}$ and $\mathfrak{S}$ normalizes $F(\mathfrak{S})$, it follows from Lemma 4.1 that $\mathfrak{S} \subseteq F(\mathfrak{S})$ and $\mathfrak{S} \subseteq F(\mathfrak{S})$. Hence $\mathfrak{S}$ and $\mathfrak{S}_1$ both lie in $\mathfrak{S}_1 = N(\mathfrak{S}(\mathfrak{S}))$, where $\mathfrak{S} = V(\mathfrak{S}(\mathfrak{S}), p_1, \mathfrak{S}_1)$, by Lemma 5.4. Let $\mathfrak{S}$ be an $S_p$-subgroup of $\mathfrak{S}$ for $p_1$. Since $\mathfrak{S}$ is an $S_p$-subgroup of $\mathfrak{S}$ and $\mathfrak{S}$ normalizes $F(\mathfrak{S})$, it follows from Lemma 4.1 that $\mathfrak{S}(\mathfrak{S}) \subseteq O_q(\mathfrak{S}_1)$. Since this holds for each $q$ in $\tau_1$, we see that $F(\mathfrak{S}) \subseteq O_q(\mathfrak{S}_1)$. But, by definition of $\mathfrak{S}_1$, $O_p(\mathfrak{S}_1) \subset O_p(\mathfrak{S})$. Since $F(\mathfrak{S}) = O_p(\mathfrak{S}_1)$ under the present assumptions, we conclude that $\mathfrak{S}_1 \cap O_p(\mathfrak{S}_1) = F(\mathfrak{S})$. Similarly $\mathfrak{S}_1 \cap O_p(\mathfrak{S}_1) = F(\mathfrak{S}_1)$, and hence $F(\mathfrak{S}) = F(\mathfrak{S}_1)$. Furthermore, $O_p(\mathfrak{S})$ and $O_p(\mathfrak{S}_1)$ lie in $O_p(\mathfrak{S})$ for all $i > 1$ by Lemma 4.1. On the other hand, it follows from Lemma 8.1 that $O_p(\mathfrak{S})$ and $O_p(\mathfrak{S}_1)$ are each $S_p$-subgroups of $O_p(\mathfrak{S})$. Thus $O_p(\mathfrak{S})$ and $O_p(\mathfrak{S}_1)$ are conjugate in $\mathfrak{S}_1$ for all $i > 1$. Since $O_p(\mathfrak{S}) = O_p(\mathfrak{S}_1) = 1$, the lemma follows in this case.

Suppose next that $F(\mathfrak{S})$ is a $p_1$-group, but that $F(\mathfrak{S}_1)$ is not. Let $q \in \pi(F(\mathfrak{S}_1))$, $q \neq p_1$. Then $V(\mathfrak{S}_1; q)$ is nontrivial. Since $\mathfrak{S}$ contains an $S_p$-subgroup of $\mathfrak{S}$, it follows from Lemma 8.1 that $O_p(\mathfrak{S})$ contains a maximal
element $\Sigma^*$ of $\mathcal{V}(\Psi; q)$, and that $\Sigma^* \neq 1$. Since $q \in \tau_1$, $\Sigma^* \subseteq \mathcal{O}_{p^j}(\Sigma)$. But $\mathcal{O}_{p^j}(\Sigma) = 1$, since $F(\Sigma)$ is a $p_1$-group, and we reach a contradiction.

Hence we may suppose that neither $F(\Sigma)$ nor $F(\Sigma_1)$ is of prime power order. Set $m_i = |\mathcal{O}_{p^j}(\Sigma)|$, $m_i' = |\mathcal{O}_{p^j}(\Sigma_1)|$, $i = 1, 2, \ldots, n$. Let $\Psi_i, \Psi_i'$ be $S_{\tau_i}$-subgroups of $S(\Sigma_i)$, $S(\Sigma_1)$ respectively, $i = 1, 2, \ldots, n$, and set $\Psi_i = N_S(\Psi_i)$, $\Psi_i' = N_S(\Psi_i')$, $i = 1, 2, \ldots, n$. Then by $\mathfrak{G}$, in $S(\Sigma_1)$, $S(\Sigma_1)$, $\Psi_i, \Psi_i'$ are $S_{\tau_i}$-subgroups of $S(\Sigma_i), S(\Sigma_1)$ respectively, and $\mathcal{O}_{p^j}(\Sigma)$ are $S_{\tau_i}$-subgroups of $\mathfrak{G}$. Hence if $\mathcal{O}_{p^j}(\Sigma_i)$ and $\mathcal{O}_{p^j}(\Sigma_1)$, it follows by induction that there exists, for each $i$, an element $G_i$ in $\mathfrak{G}$ such that $G_i^{G_i} = \mathcal{O}_{p^j}(\Sigma_i)$. Let $\Psi_i = \mathcal{O}_{p^j}(\Sigma_i)$, $i = 1, 2, \ldots, n$.

Suppose $\mathcal{O}_{p^j}(\Sigma_i) \cap \mathcal{O}_{p^j}(\Sigma) = 1$ for some $i, j$, $i \neq j$. Now if $\Sigma$ is an $S_\tau$-subgroup of $\mathfrak{G}$ for $q$ in $\tau$, then $\mathfrak{G} \cap \mathcal{O}_{p^j}(\Sigma) = \Sigma_1 \neq 1$ since $q \in \tau_1$. Furthermore, $C_\mathfrak{G}(\Sigma_1)$ is solvable by Lemma 3.3. Since $C_\mathfrak{G}(\Sigma) \subseteq C_\mathfrak{G}(\Sigma_1)$, we see that $C_\mathfrak{G}(\Sigma)$ is solvable. Hence $C_\mathfrak{G}(F(\Sigma))$ is solvable by Lemma 3.2. Since $C_{\mathfrak{G}}(F(\Sigma)) \subseteq F(\Sigma)$, it follows that $F(\Sigma)$ contains an $S_{\tau_i}$-subgroup of $C_\mathfrak{G}(F(\Sigma))$. Thus $\mathcal{O}_{p^j}(\Sigma_1)$, which is contained in $\mathfrak{G}$, is faithfully represented on $F(\Sigma)$. Let $\Psi_i$ be an $S_{\tau_i}$-subgroup of $\Psi_i$. Since $\Psi_i^{G_i}$ and $\mathcal{O}_{p^j}(\Sigma_1)$ are both contained in $N(\Sigma_1)$, there exists an element $X_i$ in $N(\Sigma_1)$ such that $X_i^{G_i} \mathcal{O}_{p^j}(\Sigma_1) = \mathcal{O}_{p^j}(\Sigma_1)$, hence without loss, we may assume that $\Psi_i^{G_i}$ and consequently also $\mathcal{O}_{p^j}(\Sigma_1)$, contains $\Psi_i$. Thus $\mathcal{O}_{p^j}(\Sigma_1)$ and $\mathcal{O}_{p^j}(\Sigma)$ are each normal subgroups of $\Psi_i$. Since they are disjoint, $\mathcal{O}_{p^j}(\Sigma_1)$ centralizes $\mathcal{O}_{p^j}(\Sigma)$, and hence $\mathcal{O}_{p^j}(\Sigma_1)$ clearly lies in $\mathcal{O}_{p^j}(\Sigma)$, so that $\mathcal{O}_{p^j}(\Sigma_1) \subseteq \mathcal{O}_{p^j}(\Sigma)$. But $\mathcal{O}_{p^j}(\Sigma)$ contains $\mathcal{O}_{p^j}(\Sigma)$ for all $k \neq i$, and hence $\mathcal{O}_{p^j}(\Sigma_1)$ centralizes $\mathcal{O}_{p^j}(\Sigma)$ for all $k \neq i$. Since $\mathcal{O}_{p^j}(\Sigma)$ is faithfully represented on $F(\Sigma)$, we conclude that $\mathcal{O}_{p^j}(\Sigma_1)$ is faithfully represented on $\mathcal{O}_{p^j}(\Sigma)$.

It follows therefore from Lemma 5.2 of F.T. that $m_j \leq m_i$. Similarly $m_i \leq m_i'$. If for all $i, j$, $i \neq j$, $i = i$, it were true that $\mathcal{O}_{p^j}(\Sigma_1) \cap \mathcal{O}_{p^j}(\Sigma) = 1$, it would follow that $m_j \leq m_i \leq m_j'$ for all $i, j$, and hence that $m_j = m_i = 1$ for all $i, j$. But this is not possible, since $\Sigma$ is a solvable normal $\tau$-subgroup of $\mathfrak{G}$ and $\mathfrak{G} \neq 1$.

Hence we may assume without loss that $\mathcal{O}_{p^j}(\Sigma_1) \cap \mathcal{O}_{p^j}(\Sigma) \neq 1$. As in the preceding paragraph, we may also assume that $G_2$ is chosen so that $\mathfrak{G} G_2$, and hence also $\mathfrak{G}$, contains an $S_{\tau_2}$-subgroup $\Psi_2$ of $\mathfrak{G}$. Let $\mathfrak{D}_2 = \mathfrak{G}_2 G_2$, so that $\mathfrak{D}_2$ is an $S_{\tau_2}$-subgroup of $S(\mathfrak{G})$ and $\mathfrak{D}_2 \subseteq \mathfrak{G}$. Since $\mathfrak{G}$ and $\mathfrak{G}_2$ contain $S_{\tau_2}$-subgroups of $\mathfrak{G}$ for each $q$ in $\tau$, it follows from Lemma 8.1 that $\mathcal{O}_{p^j}(\mathfrak{G})$ and $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ both contain maximal elements of $\mathcal{V}(\Psi_2; q)$ for each $q$ in $\tau_1$.

Now set $\Psi_2 = N(\mathfrak{D}_2)$. Since $\mathfrak{D}_2$ centralizes each of the groups $\mathcal{O}_{p^j}(\mathfrak{D}_2)$, $\mathcal{O}_{p^j}(\mathfrak{D}_2)$, and since $\mathfrak{D}_2 \subseteq \mathfrak{D}_2$, we have $\mathcal{O}_{p^j}(\mathfrak{D}_2) \subseteq \mathcal{O}_{p^j}(\mathfrak{D}_2), \mathfrak{D}_2 \subseteq \mathfrak{D}_2$. Furthermore, since $\Psi_2$ normalizes $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ and $\mathcal{O}_{p^j}(\mathfrak{D}_2)$, Lemma 4.1 implies that $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ and $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ both lie in $\mathcal{O}_{p^j}(\mathfrak{D}_2)$. Since $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ contains a maximal element of $\mathcal{V}(\Psi_2; q)$ for each $q$ in $\tau_1$, $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ is necessarily an $S_{\tau_2}$-subgroup of $\mathcal{O}_{p^j}(\mathfrak{D}_2)$; and the same is true of $\mathcal{O}_{p^j}(\mathfrak{D}_2)$. But then $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ and $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ are conjugate by $C_{\tau_2}$ in $\mathcal{O}_{p^j}(\mathfrak{D}_2)$. We conclude from this that $\mathcal{O}_{p^j}(\mathfrak{D}_2)$ and $\mathcal{O}_{p^j}(\mathfrak{D}_2)$
are conjugate in $\mathfrak{S}$. Hence without loss of generality, we may assume that $O_{\mu_1}(\Xi) = O_{\mu_1}(\Xi)$.

We can now easily complete the proof of the lemma. If $p \in \tau_1$, then $O_{\mu}(\Xi) = O_{\mu}(O_{\mu_1}(\Xi))$ and $O_{\mu}(\Xi) = O_{\mu}(O_{\mu_1}(\Xi))$. Since $O_{\mu_1}(\Xi) = O_{\mu_1}(\Xi)$ by assumption, it follows that $O_{\mu}(\Xi) = O_{\mu}(O_{\mu_1}(\Xi))$ for every $p \in \tau_1$. Set $\mathfrak{R} = N(O_{\mu_1}(\Xi))$, so that $\mathfrak{S}$ and $\mathfrak{S}_1$ both lie in $\mathfrak{R}$. Note that since $F(\Xi)$ is not a $p$-group, $O_{\mu_1}(\Xi) \neq 1$, and hence $\mathfrak{R} \subseteq \mathfrak{S}$. Since some conjugate of $\mathfrak{S}_1$ by an element of $\mathfrak{R}$ contains an $S_{\mu_1}$-subgroup of $\mathfrak{S}$, we may assume that $\mathfrak{S}$ and $\mathfrak{S}_1$ contain the same $S_{\mu_1}$-subgroup $\mathfrak{S}_1$ of $\mathfrak{S}$. Since $\mathfrak{S}$ contains an $S_{\mu_1}$-subgroup of $\mathfrak{S}$ for each $q \in \tau_1$, it follows once again from Lemma 4.1 that $O_{\mu_1}(\Xi) \leq O_{\mu_1}(\Xi)$ for each $q \in \tau_1$, and hence that $O_{\mu_1}(\Xi) \leq O_{\mu_1}(\Xi)$. But since $\mathfrak{S} \subseteq \mathfrak{R}$, $\mathfrak{S}_1 \cap O_{\mu_1}(\Xi) \leq O_{\mu_1}(\Xi)$. Thus $\mathfrak{S}_1 \cap O_{\mu_1}(\Xi) = O_{\mu_1}(\Xi)$. Similarly, we have $\mathfrak{S}_1 \cap O_{\mu_1}(\Xi) = O_{\mu_1}(\Xi)$, and consequently $O_{\mu}(\Xi) = O_{\mu_1}(\Xi)$. We conclude that $O_{\mu}(\Xi) = O_{\mu_1}(\Xi)$ for all $p \in \tau$. In particular, it follows that $F(\Xi) = F(\Xi_1)$.

It remains to show that $O_{\mu_1}(\Xi)$ and $O_{\mu_1}(\Xi_1)$ are conjugate for any $j = 1, 2, \ldots, n$. Let $\Psi_j$ be an $S_{\mu_1}$-subgroup of $\mathfrak{S}_1$. It follows once again from Lemmas 4.1 and 8.1 that $O_{\mu_1}(\Xi)$ and $O_{\mu_1}(\Xi_1)$ are both contained in $O_{\mu_1}(\Xi)$ and that each is an $S_{\mu_1}$-subgroup of $O_{\mu_1}(\Xi)$. Thus $O_{\mu_1}(\Xi)$ and $O_{\mu_1}(\Xi_1)$ are conjugate in $O_{\mu_1}(\Xi)$, and the lemma is proved.

We are now in a position to prove Theorem A. In view of Theorem 3, we may assume that $\tau$ consists of at least 3 primes. Arguing by induction, we may assume that $\mathfrak{S}$ contains a proper $S_{\mu}$-subgroup for every proper subset $\sigma$ of $\tau$. If $\tau = \{p_1, p_2, \ldots, p_n\}$, set $\tau_i = \tau - p_i$ and $\tau_{ij} = \tau - p_i - p_j$, $i \neq j$. Let $\mathfrak{S}_i$ be an $S_{\mu_i}$-subgroup of $\mathfrak{S}$, and let $m_{ij} = |O_{\mu_i}(\mathfrak{S}_j)|$, $i \neq j$. By Lemma 8.3, $m_{ij}$ depends only on $i$ and $j$ and not upon the choice of the $S_{\mu_i}$-subgroup $\mathfrak{S}_i$.

Let $\Psi_i$ be an $S_{\mu_i}$-subgroup of $\mathfrak{S}$, and let $\Xi_j^\tau_i$, $\Xi_k^\tau_i$ be $S_{\tau_{ij}}$- and $S_{\tau_{ik}}$-subgroups of $\mathfrak{S}$ containing $\Psi_i$, $i \neq j \neq k \neq i$. Let $\mathfrak{D}_j$ be an $S_{\tau_{ik}}$-subgroup of $\mathfrak{S}(\Xi_j^\tau_i)$ normalized by $\Psi_j$. It follows from $C_{\tau_{ij}}^\mathfrak{S}$ in $\mathfrak{S}(\Xi_j^\tau_i)$ that $\Xi_{jk} = N_{\Xi_j^\tau_i}(\mathfrak{D}_j)$ is an $S_{\tau_{jk}}$-subgroup of $\mathfrak{S}$ and that $\Xi_{jk}$ contains $\Psi_i$. Similarly, $\Xi_{ij}^\tau_i$ contains an $S_{\tau_{ij}}$-subgroup $\Xi_{jk}$ containing $\Psi_i$. Now by Lemma 8.3, $O_{\mu_i}(\Xi_{jk})^G = O_{\mu_i}(\Xi_{jk})$ for some element $G$ in $\mathfrak{S}$. Since $\Psi_j^G$ and $\Psi_i$ are each $S_{\mu_i}$-subgroups of $N(O_{\mu_1}(\Xi_{jk}))$, we may assume that $G$ is chosen so that $\Psi_j^G = \Psi_j$. Hence replacing $\Xi_j^\tau_i$ by a conjugate, if necessary, we may assume that $O_{\mu_1}(\Xi_{jk}) = O_{\mu_1}(\Xi_{jk})$.

Let $\Psi_{ij} = O_{\mu_1}(\Xi_{ij})$ and $\Psi_{ik} = O_{\mu_1}(\Xi_{ik})$. We shall prove that the assumption $\Psi_{ij} \cap \Psi_{ik} = 1$ implies that $m_{ij} \leq m_{ik}$. For simplicity we assume that $i = 1$, $j = 2$, $k = 3$. Let $\mathfrak{D}_{ij}$ be an $S_{\tau_{ij}}$-subgroup of $\mathfrak{S}(\Xi_j^\tau_i)$, $j = 2, 3$. By definition of $\Xi_j^\tau_i$, $\mathfrak{D}_{ij} \subset \Xi_j^\tau_i$. Let $F(\Xi_j)$. Now if $\mathfrak{D}$ is an $S_{\mu}$-subgroup of
$O_{q',q}(D_3)$ for any $q$ in $\tau_3$, then $C_{\tau_3}(Q)$ is solvable by Lemma 3.3. Hence $C_{\tau_3}(\tau_3)$ is solvable, and consequently $C_{\tau_3}(\tau_3)$ is solvable by Lemma 3.2. Thus $C_{\tau_3}(\tau_3) \cap \Psi_1 \leq \Psi_3$. But $C_{\tau_3}(\tau_3) \subseteq \tau_3$, hence $C_{\tau_3}(\tau_3) \cap \Psi_1 \leq \tau_3$.

Since $\Psi_1 \cap \tau_3 = 1$ by assumption, and $\Psi_3$ is an $S_{\tau_3}$-subgroup of $\tau_3$, $\Psi_1$ is represented faithfully on $\tau_3$. Let $\Psi$ be the $S_{\tau_3}$-subgroup of $\tau_3$. Since $\Psi$ normalizes $\tau_3$, $\tau_3 \leq O_{p_1}(\tau_3)$ by Lemma 4.1. But $O_{p_1}(\tau_3) = O_{p_1}(\tau_3)$ by assumption, and hence $\tau_3 \leq \tau_3$. Since $\Psi_1 \leq \tau_3$, $\tau_3$ is a $S_{\tau_3}$-group. However, $\tau_3 < \tau_3$, and consequently $\tau_3$ is a $S_{\tau_3}$-group. We conclude that $\tau_3$ centralizes $\tau_3$. Furthermore, $\Psi_1$ and $\tau_3$ are disjoint normal subgroups of $\Psi_1$, whence $\tau_3$ centralizes $\tau_3$. Since $\tau_3$ is the $S_{\tau_3}$-subgroup of $\tau_3$, and since $\Psi_1$ is faithfully represented on $\tau_3$, we see that $\Psi_1$ must be faithfully represented on $\tau_3$, the $S_{\tau_3}$-subgroup of $\tau_3$. Now Lemma 5.2 of F.T. yields the desired conclusion $m_{12} \leq m_{23}$.

If $\Psi_{ij} \cap \Psi_{ik} = 1$ for all $i$, $j$, $k$, the preceding argument shows that $m_{ij} \leq m_{jk}$ for all $i$, $j$, $k$, $i \neq j \neq k \neq i$. Permuting $i$, $j$, $k$ cyclically, we have $m_{ij} \leq m_{jk} \leq m_{ki} \leq m_{ij}$. Since the integers $m_{ij}$, $m_{jk}$, $m_{ki}$ are pairwise relatively prime, it follows that $m_{ij} = 1$ for all $i$, $j$, $i \neq j$. But this is impossible, since $\tau_3$ is a solvable normal $S_{\tau_3}$-subgroup of $\tau_3$ and $\tau_3 \neq 1$. Hence some $\Psi_{ij} \cap \Psi_{ik} = 1$. For definiteness, assume $\Psi_{ij} \cap \Psi_{ik} = 1$, and set $\tau_{123} = \tau_{12} \cap \tau_{13}$. Then $\tau = \langle \Psi_1, O_{\alpha}(\tau_2), O_{\alpha}(\tau_3) \rangle$ is a proper subgroup of $\tau_3$ normalizing $\tau_{123}$. Since $O_{\alpha}(\tau_2)$ and $O_{\alpha}(\tau_3)$ are normalized by $\Psi_1$, they lie in $O_{\alpha}(\tau)$ by Lemma 4.1. Thus $\tau$ possesses a normal $\rho_1$ complement. Let $\Psi$ be a $\rho_1$-invariant $S_{\alpha}$-subgroup of $O_{\alpha}(\Psi)$ containing $O_{\alpha}(\tau_3)$. Since any two such subgroups are conjugate by an element of $O_{\alpha}(\tau) \cap C(\Psi_1)$, $O_{\alpha}(\tau_3) \simeq \Psi$ for some element $\Psi$ in $O_{\alpha}(\tau) \cap C(\Psi_1)$. Furthermore, by Lemma 8.1 applied to the groups $\tau_3$ and $\tau_3$, $O_{\alpha}(\tau_3)$ contains a maximal element of $\tau(\tau_3)$, and $O_{\alpha}(\tau_3)$ contains a maximal element of $\tau(\tau_3)$, $i = 2, 3, \ldots, n$.

Suppose first that $\Psi_1 = 1$. We shall prove that $\Psi = N(\tau)$ contains an $S_{\tau_3}$-subgroup of $\tau$. Let $\Psi_1$, $\Psi_2$, $\ldots$, $\Psi_n$ be a Sylow system of $\Psi$ normalized by $\Psi_1$, and set $\Psi_i = n(\tau)$, $i = 1, 2, \ldots, n$. Let $\Psi_i \in \tau(\tau)$ and hence $\tau_i \leq \tau_i \leq \tau_i$. Also set $\Psi_i = n(\tau)$. Let $\Psi_i \in \tau(\tau)$ and hence $\tau_i \leq \tau_i \leq \tau_i$. Thus $\tau_i = O_{n_i}(\tau_i) \tau_i(N(\Psi_i))$ by Sylow's theorem. Furthermore, $\tau_i \subseteq \tau_i \subseteq \tau_i$, and consequently an $S_{\tau_3}$-subgroup of $\tau_i = n(\Psi_i)$ by Lemma 4.1. Since $\tau_i = n(\Psi_i)$ contains an $S_{\tau_3}$-subgroup of $\tau_i = n(\Psi_i)$ and since $\tau_i$ is a solvable normal $S_{\tau_3}$-subgroup of $\tau_i$, it follows that $|N(n(\Psi_i)) : C(n(\Psi_i))|_n = p^{i-1}$. We conclude that $|N(n(\Psi_i)) : C(n(\Psi_i))| = p^{i-1}$. Now set $\Psi_1 = N(\tau)$, $\Psi \leq O_{\alpha}(\tau)$ by Lemma 5.5, and hence $\Psi_1 = O_{n_1}(\Psi_1) N(\Psi)$ by Sylow's theorem. Thus if $\Psi$ is a coset of $O_{\alpha}(\tau)$ in $\Psi$, $\Psi$ contains an element $N$ of $\Psi(\Psi)$. Since $\Psi$ is a maximal element of
$N(\mathfrak{g}_i; p_i)$, it follows from Theorem 1 and Corollary 4.4 that $N_{i-1} = C_{\mathfrak{g}_i} C_i$, $i = 2, 3, \ldots, n$, where each $C_i \in C(\mathfrak{g}_i)$. Set $\mathfrak{A} = \langle \mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n \rangle$. Then $\mathfrak{A}$ is a proper subgroup of $\mathfrak{G}$ since $\mathfrak{A}$ centralizes $\mathfrak{D}_2 \cap Z(\mathfrak{g}_1) \neq 1$. Since $C(\mathfrak{g}_i) = \mathfrak{D} \times \mathfrak{D}$, where $\mathfrak{D}$ is a $p_i$-group, and since $\mathfrak{g}_i$ is $\mathfrak{D}$-invariant, we may assume that each $C_i \in \mathfrak{D}$. But then $\mathfrak{A}$ and each $C_i$ lie in $O_{p_i}(\mathfrak{A})$ by Lemma 4.1. Thus $\mathfrak{A} = O_{p_i}(\mathfrak{A})$, and consequently $O_{p_i}(\mathfrak{A})$ contains both $\mathfrak{M}$ and $\mathfrak{M}^{N-1}$. Now $\mathfrak{M}$ and $\mathfrak{M}^{N-1}$ are each $S_{\mathfrak{g}_i}$-subgroups of $O_{p_i}(\mathfrak{A})$ and each is $\mathfrak{D}$-invariant, and hence $\mathfrak{M}^{N-1} = \mathfrak{M}^C$ for some element $C$ in $\mathfrak{D}$. Thus $CN \in N(\mathfrak{M})$. Since $Z(\mathfrak{g}_i) \subseteq \mathfrak{D}$, $C \in O_{p_i}(\mathfrak{g}_i)$ by Lemma 4.1, and we conclude that $\mathfrak{g}_i = O_{p_i}(\mathfrak{g}_i)$, where $\mathfrak{D} = \mathfrak{g}_i \cap N(\mathfrak{M})$. Since $\mathfrak{D} \subseteq O_{p_i}(\mathfrak{g}_i)$ by Lemma 5.5, it follows now from Sylow's theorem that $\mathfrak{g}_i = O_{p_i}(\mathfrak{g}_i) N_2(\mathfrak{g}_i)$. This implies that $\mathfrak{g}_i = O_{p_i}(\mathfrak{g}_i) \cap N_2(\mathfrak{g}_i)$ as well as $\mathfrak{D} = O_{p_i}(\mathfrak{g}_i) \cap N(\mathfrak{M})$. But $O_{p_i}(\mathfrak{g}_i) \cap N_2(\mathfrak{g}_i) \subseteq C_2(\mathfrak{g}_i)$ and $O_{p_i}(\mathfrak{g}_i) \cap N(\mathfrak{M}) \subseteq C(\mathfrak{g}_i)$, and consequently $N_2(\mathfrak{g}_i)/C_2(\mathfrak{g}_i)$ is isomorphic to $N(\mathfrak{M})/C(\mathfrak{g}_i)$. Thus $N(\mathfrak{M})/C_2(\mathfrak{g}_i)$ contains a subgroup $\mathfrak{g}_i \subseteq N_2(\mathfrak{g}_i)$, $i = 2, 3, \ldots, n$. Since $N_2(\mathfrak{g}_i) \cap \mathfrak{M}$ centralizes $\mathfrak{g}_i$, we see that $\mathfrak{g}_i = \mathfrak{g}_i \cap \mathfrak{M}$ for each $i = 2, 3, \ldots, n$. But then $N(\mathfrak{M}) \nmid p_i$ for each $i = 2, 3, \ldots, n$. Since $N(\mathfrak{M})$ contains an $S_{\mathfrak{g}_i}$-subgroup of $\mathfrak{G}$ for all $i = 1, 2, \ldots, \mathfrak{n}$, $\mathfrak{M}$ satisfies $E'$. Hence if $\mathfrak{D}^*$ denotes an $S_\tau$-subgroup of $S(N(\mathfrak{M}))$ and if $\mathfrak{D}^* = N(\mathfrak{M})(\mathfrak{D}^*)$, it follows readily that $\mathfrak{D}^*$ is an $S_\tau$-subgroup of $\mathfrak{G}$. Thus $\mathfrak{G}$ satisfies $E'$. Suppose finally that $\mathfrak{M} = 1$, in which case $\mathfrak{D}_2$ and $\mathfrak{D}_3$ are each $p_i$-groups. By Lemma 3.4, $\mathfrak{D}_2 \subseteq \mathfrak{D}_j$, $j = 2, 3$ and therefore $S_{\mathfrak{g}_2}^*$, $S_{\mathfrak{g}_3}^*$ both lie in $\mathfrak{g}_1$ by Lemma 5.4. It follows at once that $\mathfrak{g}_1$ contains an $S_{\mathfrak{g}_i}$-subgroup of $\mathfrak{G}$, and again $\mathfrak{G}$ satisfies $E'$. Theorem B is almost an immediate corollary of Theorem A. For if $\mathfrak{G}$ is a simple group which is $\tau$-tame for some set of primes $\tau$ in $\mathfrak{D}_3$, $\mathfrak{G}$ possesses an $S_\tau$-subgroup $\mathfrak{S}$ by Theorem A. Let $\mathfrak{D}$ be an $S_\tau$-subgroup of $S(\mathfrak{g}_i)$, so that $\mathfrak{D} \subseteq \mathfrak{S}$ and $\mathfrak{D}(\mathfrak{D}) = \tau$ by definition of $\mathfrak{S}$. Now $F(\mathfrak{D})$ is noncyclic, since $F(\mathfrak{D})$ centralizes its own centralizer in $\mathfrak{D}$. And $\mathfrak{D}$ contains an element of $F(\mathfrak{D})/C_2(\mathfrak{g}_i)$ for each $p$ in $\tau$. Choose $p$ in $\tau$ so that an $S_p$-subgroup $\mathfrak{V}_p$ of $F(\mathfrak{D})$ is noncyclic, and let $\mathfrak{B}$ be an $S_p$-subgroup of $\mathfrak{S}$. Since $\mathfrak{V}_p \cap \mathfrak{B}$ is noncyclic, $\mathfrak{V}_p$ contains a normal subgroup $\mathfrak{B}$ of $\mathfrak{B}$ of type $(p, p)$. Now by Lemma 8.1, $O_{p_i}(\mathfrak{D})$ contains a maximal element $\mathfrak{V}$ of $\mathfrak{V}(\mathfrak{B}; r)$ for each $r$ in $\tau$. By Lemma 8.9 of F.T., there exists an element $\mathfrak{V}$ of $F(\mathfrak{D})$ containing $\mathfrak{V}$. Then by Corollary 4.4, $\mathfrak{V}$ is a maximal element of $\mathfrak{V}(\mathfrak{B}; r)$. Since $\mathfrak{B} \subseteq F(\mathfrak{D})$, $\mathfrak{B}$ centralizes $\mathfrak{V}$. But any two maximal elements of $\mathfrak{V}(\mathfrak{B}; r)$ are conjugate by an
element of $C(\mathfrak{U})$ by Theorem 1, and hence $\mathfrak{B}$ centralizes every maximal element of $\mathcal{U}(\mathfrak{U}; r)$. Thus $\mathfrak{B}$ centralizes every element of $\mathcal{U}(\mathfrak{B}; r)$. Since this is true for each $r$ in $\pi - p$, Theorem B is proved.

We turn now to the proof of Theorem C. Let $\mathfrak{B}$ be an $S_{\mu}$-subgroup of $G$ and $\mathfrak{B}$ a normal subgroup of $\mathfrak{B}$ of type $(p, p)$ which satisfies the hypothesis of Theorem C. Let $\sigma$ denote the set of primes $r$ such that $\mathfrak{B}$ centralizes every element of $\mathcal{U}(\mathfrak{B}; r)$. Then by assumption there exists at most one prime $q$ such that $\mathcal{U}(\mathfrak{B}; q)$ is nontrivial and $q \notin \sigma$.

Let $\mathfrak{A}$ be an element of $\mathcal{U}(\mathfrak{B}; q)$ containing $\mathfrak{B}$. Then $\mathcal{U}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, where $\mathfrak{D}$ is a $p'$-group. Set $\mathfrak{A} = N(\mathfrak{B})$, and let $\mathfrak{B} = O_{p'}(\mathfrak{A})$. Then $\mathfrak{B}$ centralizes $\mathfrak{D}$ and also centralizes a maximal element $\mathfrak{R}$ of $\mathcal{U}(\mathfrak{B}; r)$ for each $r$ in $\sigma$. Lemma 4.1 implies that $\mathfrak{D} \subseteq \mathfrak{Y}$ and that $\mathfrak{R} \subseteq \mathfrak{Y}$ for each $r$ in $\sigma$. Thus $\mathfrak{Y}$ contains a maximal element of $\mathcal{U}(\mathfrak{B}; r)$ for each $r$ in $\sigma$. Let $\mathfrak{X}$ be an $\mathfrak{A}$-invariant $S_{\mu}$-subgroup of $\mathfrak{Y}$. Then by the hypothesis of Theorem C, $|\mathfrak{Y} : \mathfrak{X}| = 1$ or $q^r$ for some prime $q$.

Suppose now that the exceptional prime $q$ exists, and let $\mathfrak{Q}_0$ be a maximal element in $\mathcal{U}(\mathfrak{B}; q)$. Then to prove Theorem C, we need the following lemma:

**Lemma 9.1.** $\langle \mathfrak{Y}, \mathfrak{Q}_0 \rangle$ is a $p'$-group.

*Proof.* Let $\mathfrak{Z}$ be the set of $\mathfrak{A}$-invariant $q$-subgroups $\mathfrak{Q}_1$ of $\mathfrak{Q}_0$ such that $\langle \mathfrak{Y}, \mathfrak{Q}_1, \mathfrak{A} \rangle \subseteq \mathfrak{Y}$. Then clearly $1 \in \mathfrak{Z}$. Let $\mathfrak{Q}_1 \in \mathfrak{Z}$ and set $\mathfrak{Z}_1 = \langle \mathfrak{Y}, \mathfrak{Q}_1 \rangle$ and $\mathfrak{Y}_1 = \mathfrak{Z}_1 \mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{Y}$ by assumption, both $\mathfrak{Y}$ and $\mathfrak{Q}_1$ lie in $O_{p'}(\mathfrak{Y})$ by Lemma 4.1. Hence $\mathfrak{Y}_1 = O_{p'}(\mathfrak{Y})$. Since $\mathfrak{A}$ normalizes $\mathfrak{Z}_1$, it follows from Corollary 4.1 that $\mathcal{U}(\mathfrak{Y}; r)$ is nontrivial for every $r$ in $\pi(\mathfrak{Z})$. Since $\mathfrak{X}$ contains a maximal element of $\mathcal{U}(\mathfrak{Y}; r)$ for every $r$ in $\sigma$, it follows that $|\mathfrak{Z}_1 : \mathfrak{X}| = q^r$.

Now let $\mathfrak{Z}_0$ be an $\mathfrak{A}$-invariant $S_{\mu}$-subgroup of $\mathfrak{Z}_1$. Then by Theorem 1, $\mathfrak{Z}_0 \subseteq \mathfrak{Q}_0$ for some element $D$ in $\mathfrak{D}$. But $\mathfrak{D} \subseteq \mathfrak{Y} \subseteq \mathfrak{Z}_1$, and hence $\mathfrak{Z}_0 \subseteq \mathfrak{Z}_1$. Thus $\mathfrak{Z}_0 \subseteq \mathfrak{Z}_1$ is an $S_{\mu}$-subgroup of $\mathfrak{Z}_1$. Since $|\mathfrak{Z}_1 : \mathfrak{X}| = q^r$, we conclude that $\mathfrak{Z}_1 = \mathfrak{X}(\mathfrak{Z}_0 \cap \mathfrak{Z}_1)$ and that $\mathfrak{Z}_0 \cap \mathfrak{Z}_1$ is an $\mathfrak{A}$-subgroup of $\mathfrak{Z}_1$. Hence $\mathfrak{Z}_0 \subseteq \mathfrak{Z}_1$. Since $\mathfrak{Q}_1 \subseteq \mathfrak{Z}_0 \subseteq \mathfrak{Z}_1$, it follows that $\langle \mathfrak{Q}_1 \cap \mathfrak{Q}_1 \cap \mathfrak{Z}_1 = \mathfrak{Y} \rangle$ is permutable with $\mathfrak{X}$. Thus $\mathfrak{X} \mathfrak{Z}_0$ is a $p'$-group containing $\mathfrak{Z}_0$. Since $\mathfrak{A}$ normalizes $\mathfrak{X} \mathfrak{Z}_0$, $\langle \mathfrak{Y}, \mathfrak{Z}_0, \mathfrak{A} \rangle \subseteq \mathfrak{Y}$, and hence $\mathfrak{Z}_0 \subseteq \mathfrak{Z}_1$. This means that $\mathfrak{Z}_1$ possesses a unique maximal element.

Observe now that for each $B$ in $\mathfrak{Y}$, $\mathcal{C}(B)$ contains $\mathfrak{Y}$ and $\mathfrak{C}_{\mathcal{U}_0}(B)$, whence $\langle \mathfrak{Y}, \mathfrak{C}_{\mathcal{U}_0}(B) \rangle \subseteq \mathfrak{Y}$. Furthermore, $\mathfrak{C}_{\mathcal{U}_0}(B)$ is invariant under $\mathfrak{A}$. Clearly $p \in \pi_3(\langle \mathfrak{Y}, \mathfrak{C}_{\mathcal{U}_0}(B) \rangle, \mathfrak{A})$, and hence $\mathfrak{C}_{\mathcal{U}_0}(B) \subseteq \mathfrak{Z}_1$ for each $B$ in $\mathfrak{Y}$. Since $\mathfrak{Z}_0$ is generated by its subgroups $\mathfrak{C}_{\mathcal{U}_0}(B)$ with $B$ in $\mathfrak{Y}$, we conclude that $\mathfrak{Z}_0 = \mathfrak{Z}_0$, completing the proof of the lemma.

The proof of Theorem C is now easily completed. Set $\mathfrak{Z} = \langle \mathfrak{Y}, \mathfrak{Q}_0 \rangle = \mathfrak{X} \mathfrak{Z}_0$ if $q$ exists, otherwise set $\mathfrak{Z} = \mathfrak{Y}$. Since $p \in \pi_3$ and $\mathfrak{Z}$ contains a maximal element of $\mathcal{U}(\mathfrak{Q}; r)$ for every $r$, we must have $\mathfrak{Z} \neq 1$. Hence $\mathfrak{W} = N(\mathfrak{Z})$ is a
proper subgroup of \( G \). Since \( \mathcal{P} \) normalizes \( \mathcal{Q} \) and \( 1 \subseteq \mathcal{M} \). By assumption (V), \( p \in \pi_p(\mathcal{M}) \). We shall prove that \( \mathcal{M} \) is the unique subgroup of \( G \) which is maximal subject to the conditions \( \mathcal{P} \subseteq \mathcal{M} \) and \( p \in \pi_p(\mathcal{M}) \).

Suppose \( \mathcal{M} \subseteq \mathcal{M}_1 \), with \( p \in \pi_p(\mathcal{M}_1) \). Then \( \mathcal{M} \subseteq \mathcal{O}_p(\mathcal{M}_1) \) by Lemma 4.1. Since \( \mathcal{Z} \) contains a maximal element of \( \mathcal{N}(\mathcal{Q}; r) \) for every \( r \), we conclude that \( \mathcal{Z} = \mathcal{O}_p(\mathcal{M}_1) \). Hence \( \mathcal{M}_1 \subseteq \mathcal{N}(\mathcal{Z}) = \mathcal{M} \), and consequently \( \mathcal{M} = \mathcal{M}_1 \) is maximal subject to \( \mathcal{P} \subseteq \mathcal{M} \) and \( p \in \pi_p(\mathcal{M}) \). Now set \( \mathcal{V} = \mathcal{V}(c, \iota(\mathcal{M}); \mathcal{P}) \).

To prove the uniqueness of \( \mathcal{M} \), we first show that \( \mathcal{N}(\mathcal{Q} \mathcal{M}) \subseteq \mathcal{M} \). If \( \mathcal{R} \) is a \( \mathcal{P} \)-invariant \( S_r \)-subgroup of \( \mathcal{Z} \) for any \( r \in \pi(\mathcal{Z}) \), then \( \mathcal{R} \) is a maximal element of \( \mathcal{N}(\mathcal{P}; r) \) and of \( \mathcal{N}(\mathcal{Q}; r) \). If \( N \in \mathcal{N}(\mathcal{Q}) \), then \( \mathcal{R}^{N^{-1}} \subseteq \mathcal{P} \) and \( \mathcal{R} \) normalizes \( \mathcal{R}^{N} \), whence by Theorem 1, \( \mathcal{R}^{N} = \mathcal{R}^{P} \) for some \( D \) in \( \mathcal{D} \). Since \( \mathcal{D} \subseteq \mathcal{Z} \), \( \mathcal{R}^{N} \subseteq \mathcal{Z} \), and we conclude that \( \mathcal{R}^{N} = \mathcal{Z} \). Thus \( \mathcal{M} \) contains \( \mathcal{N}(\mathcal{Q}) \).

Now let \( \mathcal{V} \) be an arbitrary subgroup of \( \mathcal{M} \) containing \( \mathcal{Q} \) for which \( p \in \pi_p(\mathcal{V}) \). By Lemma 5.5, \( \mathcal{V} = \mathcal{O}_p(\mathcal{V}) \mathcal{N}_p(\mathcal{V}) \). Since \( \mathcal{N}(\mathcal{Q}) \subseteq \mathcal{M} \), the uniqueness of \( \mathcal{M} \) will follow if we can show that \( \mathcal{O}_p(\mathcal{V}) \subseteq \mathcal{M} \). If \( \mathcal{R} \) is an \( S_r \)-subgroup of \( \mathcal{O}_p(\mathcal{V}) \) invariant under \( \mathcal{Q} \), then by Theorem 1, \( \mathcal{R}^{P} \) lies in an \( S_r \)-subgroup of \( \mathcal{Z} \) for some \( D \) in \( \mathcal{D} \). Since \( \mathcal{D} \subseteq \mathcal{Z} \), \( \mathcal{R} \subseteq \mathcal{Z} \), whence \( \mathcal{O}_p(\mathcal{V}) \subseteq \mathcal{Z} \subseteq \mathcal{M} \). This completes the proof of Theorem C.

X. Proof of Theorem D

For the application to groups with dihedral Sylow 2-subgroups, the following four lemmas are stated in slightly greater generality than is needed for the proof of Theorem D.

**Lemma 10.1.** Let \( \mathcal{R} \) be a group such that \( p \in \pi_p(\mathcal{R}) \) and \( \mathcal{O}_p(\mathcal{R}) = 1 \). Let \( \mathcal{Z} \) be a nontrivial subgroup of \( \mathcal{Z}(\mathcal{O}_p(\mathcal{R})) \) such that (i) \( \mathcal{Z} \leq \mathcal{R} \), (ii) \( \mathcal{O}_p(\mathcal{Z}) = 1 \), and (iii) \( \mathcal{R} \) is \( p \)-restricted with respect to \( \mathcal{Z} \). Let \( \mathcal{P} \) be an \( S_p \)-subgroup of \( \mathcal{R} \) and assume that \( \mathcal{P} \notin \mathcal{C}_p(\mathcal{Z}) \). Then \( \mathcal{P} \) possesses an elementary normal subgroup \( \mathcal{E} \) of type \( (p, p, p) \) such that \( \mathcal{P} : \mathcal{C}_p(\mathcal{E}) = p \) and such that any element of \( \mathcal{P} \) is represented indecomposably on \( \mathcal{E} \).

**Proof.** Since \( \mathcal{R} \) is \( p \)-restricted with respect to \( \mathcal{Z} \) and \( \mathcal{P} \notin \mathcal{C}_p(\mathcal{Z}) \), it follows from the definition that \( \mathcal{P} \) contains a normal subgroup \( \mathcal{P}_1 \) such that \( \mathcal{P}_1 : \mathcal{P}_1 \) is cyclic and \( \gamma^* \mathcal{O}_p(\mathcal{Z}_1) = \mathcal{P}_1 \). Hence we can find a subgroup \( \mathcal{Z}_2 \) of \( \mathcal{O}_p(\mathcal{Z}_1) \) of order at least \( p^3 \) such that \( X \) acts indecomposably on \( \mathcal{Z}_2 \). The unique \( \mathcal{P} \)-invariant subgroup of \( \mathcal{Z}_2 \) of order \( p^3 \) can then be taken as \( \mathcal{E} \).

In the following three lemmas, \( \mathcal{G} \) is assumed to be a simple group which is \( p \)-tame for some prime \( p \) in \( \pi_4 \).
**Lemma 10.2.** Let $\mathfrak{B}$ be an $S_\nu$-subgroup of $\mathfrak{C}$ and suppose that $\mathfrak{B}$ possesses an elementary normal subgroup $\mathfrak{E}$ of type $(p, p, p)$ such that $\gamma : C_{\mathfrak{E}}(\mathfrak{E}) = p$ and such that any element of $\mathfrak{B} - C_{\mathfrak{E}}(\mathfrak{E})$ is represented indecomposably on $\mathfrak{E}$. Then $\gamma C(\mathfrak{E}) \mathfrak{E}^2 = 1$ for every element $E$ in $\mathfrak{E}^2$.

**Proof.** Let $\mathfrak{B}_0 = C_{\mathfrak{E}}(\mathfrak{E})$ and $\mathfrak{A} = \Omega_1(\mathfrak{B}(\mathfrak{B}_0))$ so that $\mathfrak{E} \subset \mathfrak{A}$. Let $\mathfrak{A} \subset \mathfrak{A}^2$ and set $\mathfrak{C} = C(\mathfrak{A})$. Let $\mathfrak{B}^* = an S_\nu$-subgroup of $\mathfrak{C}$ containing $\mathfrak{B}_0$. Since $\mathfrak{C}$ is contained in an element of $J_4$, $\mathfrak{B}_0$ contains an element of $J_4$, whence $O_{\nu}(\mathfrak{C}) = 1$ by Corollary 4.1. Suppose first that $\mathfrak{B}^* = \mathfrak{B}_0$. Since $\mathfrak{B} : \mathfrak{B}_0 = p$, Lemma 3.3 implies that $C(O_\nu(\mathfrak{C}))$ is solvable, and hence $\mathfrak{A} \subset O_\nu(\mathfrak{C})$. The identity $\gamma C(\mathfrak{E}) \mathfrak{E}^2 = 1$ follows at once.

Suppose next that $\mathfrak{B}^* : \mathfrak{B}_0 = p$. Then $\delta(\mathfrak{B}, \mathfrak{B}^*) \subset N(\mathfrak{B}_0)$ so that $\langle \mathfrak{B}, \mathfrak{B}^* \rangle$ normalizes $\mathfrak{A}$ and $\mathfrak{B}, \mathfrak{B}^*$ are conjugate in $N(\mathfrak{B}_0)$. It follows from Lemma 10.1 that every element of $\mathfrak{B}^*$ has minimal polynomial of degree at least 3 on $\mathfrak{A}$. Set $\mathfrak{S}_0 = O_\nu(\mathfrak{C})$. Then $\mathfrak{S}_0 : \mathfrak{S} \cap \mathfrak{B} = -1$ or $p$. Thus $\mathfrak{S} \mathfrak{A} = \mathfrak{B}_0$ and $\mathfrak{S} \mathfrak{A} \mathfrak{E} = 1$. Since $\mathfrak{B}^*$ is an $S_\nu$-subgroup of $\mathfrak{C}$, $\mathfrak{E}$ is $p$-stable and consequently $\mathfrak{A} \subset \mathfrak{C}$. If $\mathfrak{S} \subset \mathfrak{B}_0$, $\gamma C(\mathfrak{A}) = 1$ follows. If $\mathfrak{S} \not\subset \mathfrak{B}_0$, then $D(\mathfrak{S}) \subset \mathfrak{B}_0$, so $\mathfrak{A} \subset C_{\mathfrak{B}}(D(\mathfrak{S})) = \mathfrak{S}$. If $\mathfrak{S} \subset \mathfrak{B}_0$, $\gamma C(\mathfrak{A}) = 1$ follows once again. Suppose then that $\mathfrak{S}$ contains an element of $\mathfrak{S} - (\mathfrak{S} \cap \mathfrak{B}_0)$. Then the argument at the beginning of the paragraph implies that $c(\mathfrak{S}) \geq 3$. But if $X, Y \in \mathfrak{S}$, then $[X, Y] \in \mathfrak{S} \cap \mathfrak{B}$. Since $\mathfrak{S} \subset D(\mathfrak{S})$, we have $[X, Y, Z] = 1$ for all $X, Y, Z$ in $\mathfrak{S}$. This contradiction shows that $\gamma C(\mathfrak{A}) \mathfrak{S}^2 = 1$ for all $\mathfrak{S}$ in $\mathfrak{A}^2$. In particular, $\gamma C(\mathfrak{E}) \mathfrak{E}^2 = 1$ for all $E$ in $\mathfrak{E}^2$ since $\mathfrak{E} \subset \mathfrak{A}$.

**Lemma 10.3.** Let $\mathfrak{S}$ be a $p$-stable subgroup of $\mathfrak{C}$ such that $p \in \pi(\mathfrak{S})$ and $O_\nu(\mathfrak{S}) = 1$. Let $\mathfrak{Z}_1$ be a nonidentity subgroup of $Z(O_\nu(\mathfrak{S}))$, let $\mathfrak{L}$ be the largest normal subgroup of $\mathfrak{S}$ which centralizes $\mathfrak{Z}_1$, and set $\mathfrak{Z} = C(\mathfrak{L}) \cap Z(O_\nu(\mathfrak{S}))$. Assume that $O_\nu(\mathfrak{S}/\mathfrak{L}) = 1$ and that $\mathfrak{S}$ is $p$-restricted with respect to $\mathfrak{L}$. Suppose further that $\mathfrak{S}$ possesses an elementary subgroup $\mathfrak{B}$ of order at least $p^3$ such that $\gamma C(\mathfrak{B}) \mathfrak{B}^2 = 1$ for every element $B$ in $\mathfrak{B}^2$. Then if $\mathfrak{B}_1$ is an $S_\nu$-subgroup of $\mathfrak{S}$, we have $\mathfrak{S} \subset Z(O_\nu(\mathfrak{S}))$, where $\mathfrak{L}_1 = V(ccl_0(\mathfrak{S}); \mathfrak{B}_1)$.

In particular, $O_\nu(\mathfrak{S}/\mathfrak{L}) = 1$ and $\mathfrak{L} = C_{\mathfrak{S}}(3)$ whenever $\mathfrak{Z}_1 \subset Z(\mathfrak{S}_1)$.

**Proof.** Since $\mathfrak{L}$ and $Z(O_\nu(\mathfrak{S}))$ are normal in $\mathfrak{S}$, it follows that $\mathfrak{Z} \subset \mathfrak{S}$. Since $\mathfrak{Z}_1 \subset \mathfrak{S}$, $C_{\mathfrak{S}}(3)$ is a normal subgroup of $\mathfrak{S}$ centralizing $\mathfrak{Z}_1$, and hence $C_{\mathfrak{S}}(3) \subset \mathfrak{L}$ by our maximal choice of $\mathfrak{L}$. But clearly $\mathfrak{L} \subset C_{\mathfrak{S}}(3)$, and consequently $\mathfrak{L} = C_{\mathfrak{S}}(3)$. Thus $O_\nu(\mathfrak{S}/C_{\mathfrak{S}}(3)) = 1$, and we see that it is meaningful to speak of $\mathfrak{S}$ being $p$-restricted with respect to $\mathfrak{Z}$. We first establish the final statement of the lemma; so assume $\mathfrak{Z}_1 \subset Z(\mathfrak{S}_1)$. If $\mathfrak{L}_1$ denotes the inverse image of $O_\nu(\mathfrak{S}/\mathfrak{L})$ in $\mathfrak{S}$, we have $\mathfrak{L}_1 = \mathfrak{L}_1(\mathfrak{S}_1 \cap \mathfrak{L}_1)$ and $\mathfrak{L}_1 \subset \mathfrak{S}$. But then $\mathfrak{L}$ and $\mathfrak{S}_1 \cap \mathfrak{L}$ each centralize $\mathfrak{Z}_1$, so that $\mathfrak{L} = \mathfrak{L}_1$ by our maximal choice of $\mathfrak{L}$. Hence $O_\nu(\mathfrak{S}/\mathfrak{L}) = 1$, and thus the final assertion of the lemma holds.
Now let $\mathfrak{P}_2 = \mathfrak{P}_1 \cap \mathfrak{P}$. The lemma will follow at once from Sylow’s theorem if $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$. Hence we may suppose that some conjugate $\mathfrak{H}$ of $\mathfrak{B}$ lies in $\mathfrak{P}_1$, but not in $\mathfrak{P}_2$. Suppose $\mathfrak{H} \cap \mathfrak{P}_2 = 1$. Since $\mathfrak{H}$ is conjugate to $\mathfrak{B}$, it follows from our hypotheses that $\gamma^2 \mathfrak{C}(\mathfrak{H} \cap \mathfrak{P}_2) = 1$. But $\mathfrak{H} \subseteq \mathfrak{C}(\mathfrak{H} \cap \mathfrak{P}_2)$, and hence $\gamma^2 \mathfrak{C}(\mathfrak{H}) = 1$. Since $\mathfrak{H}$ is $p$-stable, this implies that $\mathfrak{H} \subseteq \mathfrak{P}_2$, a contradiction. Thus $\mathfrak{H} \cap \mathfrak{P}_2 = 1$. Since $\mathfrak{H}$ is $p$-restricted with respect to $\mathfrak{H}$, it follows that $\mathfrak{H}$ contains a subgroup $\mathfrak{H}_0$ such that $|\mathfrak{H}/\mathfrak{H}_0| = p$ and $\gamma^2 \mathfrak{H}_0 = 1$, where $\mathfrak{H}_0 = \mathfrak{C}_\mathfrak{H}(\mathfrak{H}_0)$. But $\gamma^2 \mathfrak{C}(\mathfrak{H}_0) = 1$, and again we reach a contradiction.

**Lemma 10.4.** Let $\mathfrak{H}$ be a subgroup of $\mathfrak{B}$ which satisfies the following conditions: $O_{p'}(\mathfrak{H}) = 1$, $p \in \pi_\mathfrak{H}(\mathfrak{B})$, an $S_p$-subgroup of $\mathfrak{H}$ is an $S_p$-subgroup of $N(O_p(\mathfrak{B}))$, and $\mathfrak{H}$ is a $p$-restricted and a $p$-reductive group. Let $\mathfrak{P}$ be an $S_p$-subgroup of $\mathfrak{H}$ and assume further that $\mathfrak{P}$ possesses an elementary subgroup $\mathfrak{E}$ of order $p^3$ such that $\gamma^2 \mathfrak{C}(\mathfrak{E}) = 1$ for all $\mathfrak{E}$ in $\mathfrak{E}^\mathfrak{H}$ and $\mathfrak{E} \cap \mathfrak{B} = 1$. Let $\gamma(\mathfrak{P}^\mathfrak{B})$ be the subgroup of $\mathfrak{B}$ generated by its subgroups $\mathfrak{P}$ which are of index $p$ in $\mathfrak{B}$. Then if $(\mathfrak{U}, \mathfrak{W})$ is any one of the pairs $(\mathfrak{Z}(\mathfrak{P}), \mathfrak{W})$, $(\mathfrak{Z}(\mathfrak{E}), \mathfrak{W})$, or $(\mathfrak{Z}(\mathfrak{B}^\mathfrak{H}), \mathfrak{W})$, we have $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{N}(\mathfrak{X})) (\mathfrak{H} \cap \mathfrak{N}(\mathfrak{Y}))$.

**Proof.** Since $O_{p'}(\mathfrak{H}) = 1$ and $\mathfrak{P}$ is an $S_p$-subgroup of $N(O_p(\mathfrak{B}))$, it follows that $O_p(\mathfrak{H})$ is an $S_p$-subgroup of $O_p'(N(O_p(\mathfrak{B})))$. Since $\mathfrak{B}$ is $p$-constrained by assumption, this implies that $\mathfrak{C}(O_p(\mathfrak{H}))$ is solvable, and consequently $C_{\mathfrak{H}}(O_p(\mathfrak{B})) \subseteq O_p(\mathfrak{H})$ by Lemma 1.2.3 of [5]. Thus $Z(\mathfrak{P}) \subseteq O_p(\mathfrak{H})$. Furthermore, since $\mathfrak{B}$ is $p$-stable, it follows from the definition that $\mathfrak{H}$ is a $p$-stable group.

Suppose first that $\mathfrak{X} = \mathfrak{Z}(\mathfrak{P})$ and let $\mathfrak{L}$ be the largest normal subgroup of $\mathfrak{H}$ which centralizes $\mathfrak{Z}(\mathfrak{P})$. If $\gamma = \mathfrak{W}$, we can therefore apply the preceding lemma with $\mathfrak{Z}(\mathfrak{P})$ in the role of $\mathfrak{X}$ and $\mathfrak{E}$ in the role of $\mathfrak{P}$ to conclude that $\mathfrak{H} = L N(\mathfrak{B})$, thus proving the first case of the lemma. If $\gamma = \mathfrak{W}^\mathfrak{H}$, we let $\mathfrak{E}_1, \mathfrak{E}_2, \cdots, n$ be the distinct subgroups of $\mathfrak{E}$ of order $p^2$, and let $\mathfrak{W}_i = V(ccl(\mathfrak{E}_i); \mathfrak{P})$. Applying the preceding lemma once again, we conclude that $\mathfrak{H} = L N(\mathfrak{B})$, $i = 1, 2, \cdots, n$. Since $O_p(\mathfrak{H}/\mathfrak{L}) = 1$, this forces each $\mathfrak{W}_i$ to lie in $\mathfrak{B} \cap \mathfrak{L}$. But by definition of $\mathfrak{W}^\mathfrak{H}$, we have $\mathfrak{W}^\mathfrak{H} = L N(\mathfrak{B})$. Thus the lemma also holds when $\mathfrak{X} = \mathfrak{Z}(\mathfrak{P})$ and $\gamma = \mathfrak{W}^\mathfrak{H}$.

Suppose next that $\mathfrak{X} = \mathfrak{Z}(\mathfrak{W})$ in which case $\mathfrak{B} = \mathfrak{B},$ and assume that the lemma does not hold for $\mathfrak{H}$. Since $\mathfrak{Z}(\mathfrak{W}^\mathfrak{H})$ is an abelian normal subgroup of $\mathfrak{P}$, we have $\gamma^2 \mathfrak{P} \mathfrak{Z}(\mathfrak{W})^2 = 1$. As $\mathfrak{P}$ is $p$-stable, this implies that $\mathfrak{Z}(\mathfrak{W}^\mathfrak{H}) \subseteq O_p(\mathfrak{H})$. Furthermore, $\mathfrak{W}$ is generated by all conjugates of $\mathfrak{E}$ in $\mathfrak{B}$ which lie in $\mathfrak{P}$ and $\mathfrak{W} = V(ccl(\mathfrak{E}, \mathfrak{P})$. Thus $\mathfrak{W}$ satisfies the required conditions (i) and (ii) in the definition of $p$-reduction. Finally if we set $\mathfrak{L} = C_{\mathfrak{P}}(\mathfrak{Z}(\mathfrak{W}^\mathfrak{H}))$ in the present case, then $\mathfrak{H} \neq L N(\mathfrak{B})$; otherwise the lemma would hold for $\mathfrak{H}$. 


We see then that \( R \) satisfies all the conditions for \( p \)-reduction with \( \mathbb{Z}(W^*) \) in the role of \( \psi \), and \( R \) in the role of \( \mathcal{R} \). Since \( R \) is \( p \)-reductive by hypothesis, we conclude that \( R \) contains a subgroup \( \mathcal{R} \) which satisfies the following conditions:

(a) \( O_{\mu}(R) \subseteq \mathcal{R} \).

(b) If \( \mathfrak{L}_0 \) denotes the largest normal subgroup of \( \mathcal{R} \) which centralizes \( \mathbb{Z}(W^*) \) and if \( \mathcal{L}_1 \) is the inverse image of \( O_{\mu}(\mathfrak{L}_0) \) in \( \mathcal{R} \), then \( | R \cap \mathcal{L}_1 | = p \) for some conjugate \( \mathcal{R} \) of \( \mathcal{C} \) contained in \( \mathcal{R} \cap \mathcal{R} \).

(c) \( \mathcal{L}_1 = \mathcal{R} \cap \mathcal{L}_1 \) is an \( \mathcal{R} \)-subgroup of \( \mathcal{R} \).

(d) \( \mathcal{R} \) is \( p \)-stable group.

Since \( O_{\mu}(\mathcal{R}) \subseteq \mathcal{L}_1 \), we have \( \mathbb{Z}(W^*) \subseteq \mathcal{L}_1 \). Furthermore, \( | \mathcal{R} \cap \mathcal{L}_1 | = p^2 \) by (b) and (c), and hence \( \mathcal{R} \cap \mathcal{L}_1 \subseteq \mathfrak{L}^* \). Let \( \mathfrak{L}^* \) be the normal closure of \( \mathcal{R} \cap \mathcal{L}_1 \) in \( \mathcal{R} \). Then \( \mathfrak{L}^* \subseteq \mathfrak{L}_1 \), and consequently \( \mathfrak{L}^* \) centralizes \( \mathfrak{L}^* \). Therefore if \( \mathfrak{C}^* = C_{\mathfrak{L}_1}(\mathfrak{L}^*) \), we have \( \mathbb{Z}(\mathfrak{L}^*) \subseteq \mathfrak{C}^* \) and \( \mathfrak{C}^* \subseteq \mathfrak{L}_1 \). Set \( \mathfrak{L}^* = C_{\mathfrak{L}_1}(\mathfrak{C}^*) \). Since \( \mathcal{R} = \mathfrak{L}_0 \mathcal{L}_1 \) by Sylow's theorem, it follows that \( \mathfrak{L}_0 \mathfrak{L}^* \subseteq \mathcal{R} \). But \( \mathfrak{L}_0 \) and \( \mathfrak{L}^* \) each centralize \( \mathfrak{L}^* \), whence \( \mathfrak{L}^* \subseteq \mathfrak{L}_0 \) by our maximal choice of \( \mathfrak{L}_0 \). Thus \( \mathfrak{L}^* \mathfrak{L}_1 \subseteq \mathfrak{L}_0 \mathfrak{L}_1 \mathfrak{L}_1 = \mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1 \). On the other hand, \( O_{\mu}(\mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1) = 1 \) since \( O_{\mu}(\mathfrak{L}_1 \mathfrak{L}_1) = 1 \). As \( \mathfrak{L}_1 \) is an \( \mathcal{R} \)-subgroup of \( \mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1 \), we conclude that \( O_{\mu}(\mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1) = 1 \). Furthermore, since \( \mathcal{R} = \mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1 \) and \( O_{\mu}(\mathfrak{L}_1 \mathfrak{L}_1 \mathfrak{L}_1) = 1 \), we have \( \mathfrak{L}_1 = O_{\mu}(\mathfrak{L}_1) \).

Finally if \( \mathfrak{C} \) is an \( \mathfrak{R} \)-subgroup of \( \mathfrak{L} \) containing an \( \mathfrak{R} \)-subgroup of \( \mathfrak{R} \), we conclude from Lemma 3.4 that \( O_{\mu}(\mathfrak{R}) \) contains every element of \( \mathfrak{R} \). Since \( \mathfrak{C} \subseteq \mathfrak{L}_1 \), it follows therefore from Corollary 4.3 that \( O_{\mu}(\mathfrak{C}) = 1 \). Since \( \mathfrak{C}_1 = O_{\mu}(\mathfrak{C}) \), we see that \( \mathfrak{C}_1 \) is an \( \mathfrak{R} \)-subgroup of \( O_{\mu}(\mathfrak{R}) \). But \( \mathfrak{C}_1 \) is a \( \mathfrak{R} \)-stable group by (d) and \( \mathfrak{R} \subseteq \mathfrak{C} \mathfrak{R} \mathfrak{R} \mathfrak{R} = C_{\mathfrak{L}_1}(\mathfrak{C}^*) O_{\mu}(\mathfrak{C}) \). Since we also have \( O_{\mu}(\mathfrak{C} \mathfrak{R} \mathfrak{R} \mathfrak{R} \mathfrak{C}) = 1 \), we conclude therefore from the definition of \( \mathfrak{C} \)-stability that \( \mathfrak{R} \mathfrak{C} \mathfrak{C} \mathfrak{R} \mathfrak{C} \neq 1 \). But \( \mathfrak{C}^* \) centralizes \( \mathfrak{R} \subseteq \mathfrak{C} \), and consequently it follows from our hypotheses that \( \mathfrak{C} \mathfrak{C} \mathfrak{C} \mathfrak{C} \mathfrak{R} \mathfrak{C} \mathfrak{C} \mathfrak{C} \mathfrak{C} = 1 \). This contradiction completes the proof of the lemma.

**Remark.** We note that the assumption that \( \mathfrak{C} \) is an \( \mathfrak{R} \)-subgroup of \( \mathfrak{L} \) is not fully used in the above argument which follows the existence of the subgroup \( \mathfrak{C} \) satisfying conditions (a)-(d). In fact, if \( \mathfrak{C} \) is any \( \mathfrak{R} \)-subgroup of \( \mathfrak{L} \) containing \( \mathfrak{C} \), and if \( \mathfrak{W}, \mathfrak{W}^* \) are assumed to be defined relative to \( \mathfrak{C} \), the argument shows that \( \mathfrak{L} \) cannot possess a subgroup \( \mathfrak{C} \) satisfying conditions (a)-(d). At one point in obtaining a similar result for groups with dihedral Sylow 2-subgroups, it is necessary to take \( \mathfrak{C} \) to be a maximal element of \( \mathfrak{C}_{\mathfrak{L}}(\mathfrak{X} ; p) \) instead of as an \( \mathfrak{R} \)-subgroup of \( \mathfrak{L} \), where \( \mathfrak{X} \) is a four subgroup of \( \mathfrak{L} \).

On the basis of these results we can now easily establish Theorem D. Assume then that \( \mathfrak{C} \) is a simple group which is strongly \( p \)-tame for some prime \( p \in \pi_4 \). Let \( \mathfrak{C} \) be a fixed \( \mathfrak{R} \)-subgroup of \( \mathfrak{C} \) and set \( \mathfrak{C}_1 = \mathfrak{N}(\mathfrak{C}) \).
Suppose \( \mathcal{N}_1 \) is not the unique subgroup of \( \mathcal{G} \) containing \( \mathcal{P} \) which is maximal subject to \( p \in \pi_\mathcal{G}(\mathcal{N}_1) \). Then \( \mathcal{G} \) possesses a subgroup \( \mathcal{R} \) such that \( \mathcal{P} \leq \mathcal{R} \), \( p \in \pi_\mathcal{G}(\mathcal{R}) \), and \( \mathcal{R} \not\subseteq \mathcal{N}_1 \). Since \( \mathcal{P} \leq \mathcal{R} \) and \( p \in \pi_\mathcal{G}(\mathcal{R}) \), it follows that \( O_p(\mathcal{R}) = 1 \), that \( \mathcal{R} \) is \( p \)-stable and \( p \)-restricted, and that \( Z(\mathcal{P}) \leq O_p(\mathcal{R}) \). Let \( \mathcal{Q} \) be the largest normal subgroup of \( \mathcal{R} \) which centralizes \( Z(\mathcal{P}) \) and set

\[ J = C(\mathcal{P}) \cap Z(O_p(\mathcal{R})). \]

Then \( J \leq \mathcal{R} \), \( Z(\mathcal{P}) \leq J \), and we have \( \mathcal{Q} = C(\mathcal{P})(J) \) and \( O_p(\mathcal{R}/J) = 1 \) by the final assertion of Lemma 10.3. If \( \mathcal{P} \leq \mathcal{Q} \), then \( \mathcal{R} = \mathcal{N}_R(\mathcal{P}) \) by Sylow’s theorem. Since \( \mathcal{G} \) and \( N_R(\mathcal{P}) \) are contained in \( \mathcal{N}_1 = N(\mathcal{Z}(\mathcal{P})) \), it follows that \( \mathcal{R} \not\leq \mathcal{N}_1 \), a contradiction. Thus \( \mathcal{P} \not\subseteq \mathcal{Q} \). We can therefore apply Lemmas 10.1 and 10.2 to conclude that \( \mathcal{P} \) possesses an elementary subgroup \( \mathcal{E} \) of order \( p^3 \) such that \( M(C(E)) \leq 1 \) for each \( E \in \mathcal{E} \).

As in Lemma 10.4, set \( \mathcal{M} = V(\mathcal{G}) \cap \mathcal{G}(\mathcal{P}) \) and let \( \mathcal{M}^{*} \) be the subgroup of \( \mathcal{P} \) generated by its subgroups \( \mathcal{P} \) which are of index \( p \) in \( \mathcal{G} \) for suitable \( \mathcal{G} \) in \( \mathcal{G} \).

Set \( \mathcal{N}_1 = N(\mathcal{P}) \) and \( \mathcal{N}_2 = N(\mathcal{M}^{*}) \). Since \( \mathcal{G} \) is strongly \( p \)-tame and since \( \mathcal{N}_1 = N(\mathcal{Z}(\mathcal{P})) \), \( \mathcal{N}_2 \), and \( \mathcal{N}_3 \) contain the \( S_p \)-subgroup \( \mathcal{P} \) of \( \mathcal{G} \), it follows that each \( \mathcal{N}_i \), \( i = 1, 2, 3 \) is \( p \)-restricted and \( p \)-reductive. Hence by Lemma 10.4 applied with \( \mathcal{N}_1 \) as \( \mathcal{R} \), \( \mathcal{N}_1 \) as \( \mathcal{M} \), and \( \mathcal{P} \) as \( \mathcal{Z}(\mathcal{M}^{*}) \), we see that \( \mathcal{N}_1 \leq \mathcal{N}_2 \mathcal{N}_3 \) and \( \mathcal{N}_1 \leq \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \). Next apply the same lemma with \( \mathcal{N}_2 \) as \( \mathcal{R} \), \( \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_1 \) as \( \mathcal{M} \), and \( \mathcal{P} \) as \( \mathcal{Z}(\mathcal{M}^{*}) \). Since \( N(\mathcal{M}^{*}) \leq N(\mathcal{Z}(\mathcal{M}^{*})) \), we conclude that \( \mathcal{N}_3 \leq \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \) and \( \mathcal{N}_2 \leq \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \).

Finally apply the lemma with \( \mathcal{N}_3 \) as \( \mathcal{R} \), \( \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_1 \) as \( \mathcal{M} \), and \( \mathcal{P} \) as \( \mathcal{Z}(\mathcal{M}^{*}) \) to obtain \( \mathcal{N}_3 \leq \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \) and \( \mathcal{N}_2 \leq \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \). Thus \( \mathcal{N}(\sigma(1)) \leq \mathcal{N}(\sigma(2)) \mathcal{N}(\sigma(3)) \) for every permutation \( \sigma \) of \( \{1, 2, 3\} \). But then Lemma 8.6 of F.T. implies that \( \mathcal{N}_1 \), \( \mathcal{N}_2 \), and \( \mathcal{N}_3 \) are pairwise permutable. If \( \mathcal{N}_1 \mathcal{N}_2 = \mathcal{G} \), then \( \mathcal{N}_1 \) contains \( O_p(\mathcal{N}_2)^G \) for every \( G \) in \( \mathcal{G} \).

Since \( O_p(\mathcal{N}_2) \neq 1 \), \( \mathcal{G} \) is not simple, a contradiction. Thus \( \mathcal{N}_1 \mathcal{N}_2 \) is a proper subgroup of \( \mathcal{G} \).

Now let \( \mathcal{P} \) be any subgroup of \( \mathcal{G} \) containing \( \mathcal{P} \) such that \( p \in \pi_\mathcal{G}(\mathcal{P}) \). Since \( \mathcal{G} \) is strongly \( p \)-tame, \( \mathcal{P} \leq \mathcal{N}_1 \mathcal{N}_2 \) by Lemma 10.4, and hence \( \mathcal{M} = \mathcal{N}_1 \mathcal{N}_2 \) is the unique subgroup of \( \mathcal{G} \) containing \( \mathcal{P} \) which is maximal subject to the condition \( p \in \pi_\mathcal{G}(\mathcal{N}_1) \), which proves Theorem D.

XI. PROOF OF THEOREM E

Let \( \mathcal{G} \) be weakly \( p \)-tame for some odd prime \( p \), let \( \mathcal{P} \) be a fixed \( S_p \)-subgroup of \( \mathcal{G} \), and let \( \mathcal{M} \) be any subgroup of \( \mathcal{G} \) which is maximal subject to the conditions \( \mathcal{M} \leq \mathcal{P} \) and \( p \in \pi_\mathcal{G}(\mathcal{M}) \). Then we define

\[ \mathcal{A}(\mathcal{P}) = \{ \mathcal{P}_0 \mid \mathcal{P}_0 \leq \mathcal{P}, \mathcal{P}_0 \text{ contains } \mathcal{M} \text{ for some } \mathcal{M} \in \mathcal{A}(\mathcal{G}), \mathcal{P}_0 \text{ and suitable } M \in \mathcal{M} \}. \]

\[ \mathcal{A}(\mathcal{M})(\mathcal{P}) = \{ \mathcal{P}_0 \mid \mathcal{P}_0 \leq \mathcal{P}, \mathcal{P}_0 \text{ contains a subgroup } \mathcal{P}_1 \text{ of type } (p, p) \text{ such that } C_{\mathcal{P}_1}(P) \in \mathcal{A}(\mathcal{M})(\mathcal{P}^M) \text{ for each } P \in \mathcal{P}_1 \text{ and suitable } M \in \mathcal{M}, i = 2, 3, 4. \]

(Here \( \mathcal{A}(\mathcal{M})(\mathcal{P}^M) \) denotes the set of subgroups \( \mathcal{P}_1^M \) with \( \mathcal{P}_0 \) in \( \mathcal{A}(\mathcal{G}) \).

We first prove...
LEMMA 11.1. Theorem E holds if \( M \) is the unique subgroup of \( G \) which is maximal subject to containing an element of \( \mathcal{A}_1^*(\Psi) \) and \( p \in \pi_1(M) \).

Proof. Let \( j \) be the least value of 2, 3, 4 for which there exists an element \( \Psi_0 \) of \( \mathcal{A}_1^*(\Psi) \) such that \( \Psi_0 \subseteq M_1 \), \( p \in \pi_1(M_1) \), and \( M_1 \not\subseteq M \); and choose \( \Psi_0 \) in \( \mathcal{A}_1^*(\Psi) \) so that \( |\Psi_0| \) is maximal. Let \( \Psi^* \) be an \( S_j \)-subgroup of \( M_1 \) containing \( \Psi_0 \). The assumptions of the lemma imply that \( \Psi_0 \) does not contain an element of \( \mathcal{A}_1^*(\Psi) \), and hence \( \Psi_0 \subseteq \Psi^* \). Thus \( N_{\Psi}(\Psi_0) \supseteq \Psi_0 \), and it follows that \( N(\Psi_0) \subseteq M \). In particular, \( N_{\Psi^*}(\Psi_0) \subseteq M \); whence \( N(\Psi_0) \subseteq \mathcal{A}_1^*(\Psi) \). Our maximal choice of \( \Psi_0 \) forces \( N_{\Psi^*}(\Psi_0) = \Psi_0 \), and consequently \( \Psi_0 = \Psi^* \) is an \( S_j \)-subgroup of \( M_1 \).

Let \( \Psi_1 \) be a subgroup of \( \Psi_0 \) of type \((p, p')\) with the property that \( C_{\Psi_1}(P) \) contains an element \( c_{\Psi_1}(P) \) for each \( P \) in \( \Psi_1 \) and suitable \( M \) in \( M \). Since \( O_{\Psi}(M_1) \) is generated by its subgroups \( O_{\Psi}(M_1) \subseteq C_{\Psi}(\Psi) \), \( P \in \Psi_1^* \), it follows that \( O_{\Psi}(M_1) \subseteq M \). Set \( \Psi_0 = \Psi_0 \cap O_{\Psi}(M_1) \). Since \( M_1 = O_{\Psi}(M_1) \), \( M_1 \) by Sylow's theorem, where \( M_1 = \Psi_1 \cap \Psi_1 \), we see that \( M_1 \not\subseteq M \). But now the maximality of \( \Psi_0 \) implies that \( \Psi_0 \) is an \( S_\alpha \)-subgroup of \( M_1 \), so that by Lemma 3.4, \( \Psi_0 \) contains every element of \( \mathcal{A}_1^*(\Psi) \), contrary to the fact that \( \Psi_0 \) contains no element of \( \mathcal{A}_1^*(\Psi) \).

Thus \( M \) is the unique subgroup of \( G \) which is maximal subject to containing an element of \( \mathcal{A}_1^*(\Psi) \), \( \alpha = 1, 2, 3, 4 \) and \( p \in \pi_1(M) \). The final assertion of Theorem E is immediate and is proved in Lemma 24.2 of F.T.

LEMMA 11.2. Theorem E holds under the assumptions of Theorem C.

Proof. In view of the preceding lemma, we may assume that there exists an element \( \Psi_0 \) of \( \mathcal{A}_1^*(\Psi) \) which is contained in a subgroup \( M_1 \) of \( G \) such that \( p \in \pi_1(M_1) \) and \( M_1 \not\subseteq M \). If \( \Psi_0 \) is chosen so that \( |\Psi_0| \) is maximal, it follows as in Lemma 11.1 that \( \Psi_0 \) is an \( S_\alpha \)-subgroup of \( M_1 \), and that

\[
\Psi_0 = \Psi_0 \cap O_{\Psi}(M_1)
\]

contains every element of \( \mathcal{A}_1^*(\Psi) \). Let \( \Psi \in \mathcal{A}_{1,s}(\Psi) \). Since \( \Psi \subseteq \Psi_0 \), it follows from Lemma 3.4 that \( N(\Psi_0) = N_1 = N(\Psi) \), where \( \Psi = \mathcal{V}(c_{\Psi}(\Psi)) \). Since \( \Psi \subseteq \Psi_1 \), \( p \in \pi_1(M_1) \), \( N_1 \subseteq M \) by Theorem C. Hence \( N(\Psi_0) \subseteq M \). Since \( M_1 = O_{\Psi}(M_1) \), \( M_1 \subseteq M \). Since \( M \) is the unique subgroup of \( G \) which is maximal subject to \( \Psi \subseteq M \) and \( p \in \pi_1(M) \), it follows from Corollary 4.2 that \( M \) contains every element of \( \mathcal{V}(\Psi; q) \) for every \( q \). Since \( C_{\Psi}(M) \subseteq M_1 \subseteq M \), \( M \) contains every maximal element of \( \mathcal{V}(\Psi; q) \) by Theorem 1. We conclude that \( O_{\Psi}(M_1) \subseteq M \), a contradiction.

Hence throughout the balance of the section we may assume that \( p \in \pi_1 \).

The next two lemmas are proved under the following assumptions:
1. $\mathfrak{G}$ is weakly $p$-tame.

$(E)$ 2. $p \in \pi_4$.

3. Theorem $E$ is false.

Remark. Since these lemmas will be needed for future applications, it is important to point out that they are independent of Theorem D. Furthermore, we remark that Lemma 11.1 is also independent of Theorem D. Hence if $\mathfrak{M}$ is any subgroup of $\mathfrak{G}$ which is maximal subject to $\mathfrak{M} \subseteq \mathfrak{W}$ and $p \in \pi_4(\mathfrak{M})$, $(E3)$ and Lemma 11.1 imply that there exists an element $\mathfrak{P}_0$ in $\mathcal{A}^*_4(\mathfrak{M})$ which is contained in a subgroup $\mathfrak{W}_1$ of $\mathfrak{G}$ such that $p \in \pi_4(\mathfrak{W}_1)$, $\mathfrak{W}_1 \subseteq \mathfrak{M}$, $\mathfrak{P}_0$ is an $S_p$-subgroup of $\mathfrak{W}_1$, and $\mathfrak{P}_0$ is an $S_p$-subgroup of $N(\mathfrak{P}_0 \cap O_{p'}(\mathfrak{W}_1))$.

Lemma 11.3. There exists an element $\mathfrak{V}$ in $\mathcal{U}(\mathfrak{V})$ whose normal closure in $C(\Omega_1(\mathcal{Z}(\mathfrak{V})))$ is abelian.

Proof. Let $\mathfrak{W}$ be a subgroup of $\mathfrak{G}$ containing $C(\Omega_1(\mathcal{Z}(\mathfrak{V})))$ and maximal subject to $p \in \pi_4(\mathfrak{W})$. Since $p \in \pi_4$, $O_p(\mathfrak{W}) = 1$, and Lemma 3.4 implies that $\mathfrak{V} = O_p(\mathfrak{W})$ contains every element of $\mathcal{U}(\mathfrak{V})$. If $\mathfrak{V}$ possesses a noncyclic characteristic abelian subgroup $\mathfrak{C}$, then $\mathfrak{C}$ contains an element $\mathfrak{B}$ of $C(\mathfrak{V})$ and $\mathfrak{B}^\mathfrak{W}$ is abelian. Since $C(\Omega_1(\mathcal{Z}(\mathfrak{V}))) \subseteq \mathfrak{W}$, the lemma follows in this case. Hence we may assume that $\mathfrak{V}$ contains no such characteristic subgroup $\mathfrak{C}$, so that by (3.5) of F.T., $\mathfrak{V}$ is the central product of an extraspecial group and a cyclic group.

Let $\mathfrak{P}_0$ be an element of $\mathcal{A}^*_4(\mathfrak{M})$ which is contained in a subgroup $\mathfrak{W}_1$ of $\mathfrak{G}$ such that $p \in \pi_4(\mathfrak{W}_1)$, $\mathfrak{W}_1 \subseteq \mathfrak{M}$, $\mathfrak{P}_0$ is an $S_p$-subgroup of $\mathfrak{W}_1$, and $\mathfrak{P}_0$ is an $S_p$-subgroup of $N(\mathfrak{P}_0 \cap O_{p'}(\mathfrak{W}_1))$. By Lemma 3.4, $\mathfrak{I}$ contains every element of $\mathcal{U}(\mathfrak{W}, \mathfrak{V})$. Since $p \in \pi_4$, $O_p(\mathfrak{W}_1) = 1$, $\mathfrak{I} = O_p(\mathfrak{W}_1)$, and $\mathfrak{I}$ contains $\mathcal{Z}(\mathfrak{V})$. Let $\mathfrak{L} = C_{\mathfrak{W}_1}(\mathcal{Z}(\mathfrak{I}))$ and let $\mathfrak{L}_1$ be the inverse image of $O_p(\mathfrak{W}_1/\mathfrak{I})$ in $\mathfrak{M}_1$. Then $\mathfrak{L}_1 \subseteq C(\mathcal{Z}(\mathfrak{V})) \subseteq \mathfrak{W}$. If $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \mathfrak{L}_1$, then $\mathfrak{W}_1 = \mathfrak{P}_1 N_{\mathfrak{W}_1}(\mathfrak{P}_1)$ by Sylow's theorem. Hence if $\mathfrak{M}_1$ is chosen of minimal order subject to the above conditions, it follows that $N_{\mathfrak{W}_1}(\mathfrak{P}_1) = \mathfrak{M}_1$ and consequently that $\mathfrak{P}_1 = \mathfrak{I}$ and $\mathfrak{I} = \mathfrak{P}_0 \cap \mathfrak{L}$ is an $S_p$-subgroup of $\mathfrak{L}$. Thus $O_p(\mathfrak{W}_1/\mathfrak{L}) = 1$.

Now let $\mathfrak{A} \in \mathcal{U}(\mathcal{U}(\mathfrak{V}))$. Since $\mathfrak{A} \subseteq \mathfrak{I}$, $\mathcal{Z}(\mathfrak{I}) \subseteq \mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{G}$ by Lemma 3.4, it follows that $\mathcal{Z}(\mathfrak{I}) \subseteq \mathfrak{S}$. Since $cl(\mathfrak{S}) = 2$, we have $\gamma^2(\mathcal{Z}(\mathfrak{I})) \subseteq \mathfrak{S}$. Hence if $\mathfrak{S}_0 = \mathfrak{S} \cap \mathfrak{P}_0$, $\gamma^2(\mathfrak{S}_0) = 1$. Since $\mathfrak{W}_1$ is $p$-stable, and $O_p(\mathfrak{W}_1/\mathfrak{L}) = 1$, $\mathfrak{S}_0 \subseteq \mathfrak{L}$. But $\mathfrak{L}$ is an $S_p$-subgroup of $\mathfrak{L}$, whence $\mathfrak{S}_0 \subseteq \mathfrak{L}$. Since $\mathfrak{P}_0$ is an $S_p$-subgroup of $\mathcal{Z}(\mathfrak{I})$ and $\mathfrak{I}$ normalizes $\mathfrak{S}_0$, it follows that $\mathfrak{S}_0 \subseteq \mathfrak{I}$. But $C_{\mathfrak{I}}(\mathfrak{S}_0) \subseteq \mathfrak{I}$ by Lemma 3.3 and Lemma 1.2.3 of [5]. It follows that $\Omega_1(\mathcal{Z}(\mathfrak{I})) = \Omega_1(\mathcal{Z}(\mathfrak{V})) < \mathfrak{M}_1$, whence $\mathfrak{M}_1 \subseteq \mathfrak{W}$, a contradiction.

Let $\mathfrak{B}$ be an element of $\mathcal{U}(\mathfrak{V})$ which satisfies the conditions of Lemma 11.3.
Lemma 11.4. \( \gamma^2 C(B) \cong 1 \) for every \( B \) in \( \mathfrak{B}^\theta \).

Proof. Set \( \mathfrak{A} = C(B) \) and \( \mathfrak{A}_1 = \mathfrak{A} \cap \mathfrak{B} \), so that \( | \mathfrak{A} : \mathfrak{A}_1 | = 1 \) or \( p \). Let \( \mathfrak{A}^* \) be an \( S_p \)-subgroup of \( \mathfrak{A} \) containing \( \mathfrak{A}_1 \). Since \( \forall \subseteq \mathfrak{A} \) for some \( \forall \) in \( \mathcal{A}, N \mathfrak{A}_1(\mathfrak{B}) \), \( \forall \cong \mathfrak{A} \) and hence \( O_p(\mathfrak{A}) = 1 \). If \( \mathfrak{B}_2 = O_p(\mathfrak{A}) \), then \( C_{\mathfrak{A}_1}(\mathfrak{B}_2) \) is solvable by Lemma 3.3. If \( \mathfrak{A}^* \) is not an \( S_p \)-subgroup of \( \mathfrak{B} \), then \( \mathfrak{B}^* = \mathfrak{A}_1 \) and \( B \notin Z(\mathfrak{B}) \). Since \( C_{\mathfrak{A}_1}(\mathfrak{B}_2) \) is solvable, \( Z(\mathfrak{B}) \subseteq \mathfrak{B}_2 \). But in this case, \( \mathfrak{B} = \langle \Omega_2(\mathfrak{Z}(\mathfrak{B})) \rangle \), \( B \rangle \), and hence \( \mathfrak{B} \subseteq \mathfrak{B}_2 \). On the other hand, if \( \mathfrak{A}^* \) is an \( S_p \)-subgroup of \( \mathfrak{B} \), we see that \( \gamma^2 \mathfrak{B}^2 = 1 \). Since \( \mathfrak{B} \) is \( p \)-stable, \( \mathfrak{B} \subseteq \mathfrak{B}_2 \) in this case as well. If \( \mathfrak{B} \subseteq Z(\mathfrak{B}) \), then \( \gamma^2 \mathfrak{B}^2 = 1 \). In particular, this will hold if \( \mathfrak{B} \subseteq \mathfrak{Z}(\mathfrak{B}) \). Hence we may suppose that \( \mathfrak{Z}(\mathfrak{B}) \) is cyclic. If \( B \in Z(\mathfrak{B}) \), then \( \mathfrak{B} \subseteq C(\Omega_2(Z(\mathfrak{B}))) \), so that by the preceding lemma, \( \mathfrak{B}^R \) is abelian. This implies that \( \gamma^2 \mathfrak{B}^2 = 1 \).

Thus we may assume that \( \mathfrak{Z}(\mathfrak{B}) \) is cyclic, and \( B \notin Z(\mathfrak{B}) \). We may also assume that \( \mathfrak{B}_2 \notin \mathfrak{A}_1 \), otherwise \( \forall \subseteq Z(\mathfrak{B}) \) and \( \gamma^2 \mathfrak{B}^2 = 1 \) follows. Thus \( | \mathfrak{A}^* : \mathfrak{A}_1 | = p \) and \( \mathfrak{B}_1 = \mathfrak{B}^* \). It follows that \( D(\mathfrak{B}_2) \subseteq \mathfrak{A}_1 \) and hence that \( \mathfrak{B} \subseteq C_{\mathfrak{A}_1}(D(\mathfrak{B}_2)) = \mathfrak{B} \). If \( \mathfrak{B} \subseteq Z(\mathfrak{A}) \), \( \gamma^2 \mathfrak{B}^2 = 1 \) follows since \( Z(\mathfrak{A}) \cong \mathfrak{A}_1 \). Since \( \forall \subseteq Z(\mathfrak{A}) \) if \( \mathfrak{B} \subseteq \mathfrak{A}_1 \), we can assume that \( \forall \subseteq \mathfrak{A}_1 \) and that \( \mathfrak{B} \notin Z(\mathfrak{A}) \). Choose \( R \in \mathfrak{A} \cap C_{\mathfrak{A}_1}(\mathfrak{B}) \). Since \( \mathfrak{A}^* \) centralizes \( B \) and \( B \notin Z(\mathfrak{B}) \), it follows that \( R \) does not centralize \( \mathfrak{A} \subseteq \Omega_2(Z(\mathfrak{B})) \subseteq \mathfrak{B} \). Let \( \mathfrak{A} = \langle D, R \rangle \) and set \( F = [D, R] \) so that \( F \neq 1 \). Now \( \forall \subseteq Z(\mathfrak{B}) \cong \mathfrak{B}^* \), and consequently \( F \in Z(\mathfrak{B}) \). On the other hand, \( F \in D(\mathfrak{B}_2) \). Since \( R \in \mathfrak{A} \), \( R \) centralizes \( F \). But \( \langle \mathfrak{A}_1, R \rangle = \mathfrak{A}^* \), and therefore \( F \in Z(\mathfrak{B}^*) \). But \( F \) is of order \( p \) and \( \langle B \rangle = \Omega_2(Z(\mathfrak{B}^*)) \), since \( \mathfrak{B}^* \) is an \( S_p \)-subgroup of \( \mathfrak{B} \) and \( Z(\mathfrak{B}^*) \) is cyclic. It follows that \( \langle F \rangle \cong \langle B \rangle \), whence \( R \) normalizes \( \mathfrak{B} \) and with respect to the basis \( (D, F) \) of \( \mathfrak{B} \) has the matrix \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). On the other hand, \( \mathfrak{B} \) possesses an element \( X \) which normalizes \( \mathfrak{B} \) and with respect to the basis \( (D, F) \) has the matrix \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \).

We conclude that \( N(\mathfrak{B})/C(\mathfrak{B}) \) is isomorphic to \( SL(2, p) \). Set \( \mathfrak{V} = N(\mathfrak{B}) \) and \( \mathfrak{V}_1 = O_p(\mathfrak{V}) \). Now \( \mathfrak{B} \subseteq \mathfrak{V} \) and \( \mathfrak{V} \) contains a \( p \)-element \( Y \) which does not lie in \( C(\mathfrak{B}) \). \( \mathfrak{V}_1 = C(\mathfrak{B}) \). On the other hand, since \( \mathfrak{B} \) is of type \( (p, p) \), \( \gamma^2 \mathfrak{B}^2 Y^N = 1 \). Furthermore, \( O_p(\mathfrak{V}/C(\mathfrak{B})) = 1 \), and since \( \mathfrak{B} \) is \( p \)-stable, it follows that \( Y \in C(\mathfrak{B}) \mathfrak{V}_1 = C(\mathfrak{B}) \), a contradiction.

It is now an easy matter to complete the proof of Theorem E under the assumption that \( p \in \pi_4 \) and \( \mathfrak{B} \) is strongly \( p \)-tame. By Theorem D, there exists a unique subgroup \( \mathfrak{M} \) of \( \mathfrak{B} \) which is maximal subject to \( p \in \pi_4(\mathfrak{M}) \) and \( \mathfrak{B} \cong \mathfrak{M}^N \); and by Lemma 11.1, we need only show that \( \mathfrak{M} \) is the unique subgroup of \( \mathfrak{B} \) which is maximal subject to containing an element of \( \mathfrak{A}_1 \) and \( p \in \pi_4(\mathfrak{B}) \). We proceed as in Lemma 11.2, and define \( \mathfrak{M}_0 \) and \( \mathfrak{M}_1 \) as in that lemma. Since \( \mathfrak{M}_0 \cap O_p(\mathfrak{M}_1) \) contains an element \( \mathfrak{A} \) of \( \mathcal{A}_N \mathfrak{N}_3(\mathfrak{B}) \), \( O_p(\mathfrak{M}_1) = 1 \) and \( Z(\mathfrak{B}) \subseteq \mathfrak{A} \subseteq O_p(\mathfrak{M}_1) \). Since \( \mathfrak{M}_0 \) is an \( S_p \)-subgroup of \( \mathfrak{B} \),
N(O_p(M_1)), and \( \Phi \) is strongly \( p \)-tame, \( M_1 \) is a \( p \)-restricted group. We can therefore apply Lemma 10.3 to conclude that \( M_1 = \Phi N_{\Phi}(\Phi) \), where \( \Phi \) is the largest normal subgroup of \( M_1 \) which centralizes \( Z(\Phi) \) and \( \Phi = V(ccl(\Phi); \gamma) \), \( \Phi \) an element of \( H(\Phi) \) which satisfies the conditions of Lemma 11.3. Since \( M_1 \subseteq M_1 \), \( \Phi_0 \subseteq \Phi \), and hence \( N_{\Phi_0}(\Phi) \supseteq \Phi_0 \). It follows therefore from our maximal choice of \( \Phi_0 \) that \( N(\Phi) \subseteq M_1 \). Thus \( N_{M_1}(\Phi) \subseteq M_1 \). Furthermore, \( \Phi \subseteq C(Z(\Phi)) \subseteq M_1 \). It follows that \( M_1 \subseteq M_1 \), a contradiction.

References