Signed Diagonal Flips and the Four Color Theorem

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We introduce a signed version of the diagonal flip operation. We then formulate the conjecture that any two triangulations of a given polygon may be transformed into one another by a signable sequence of diagonal flips. Finally, we show that this conjecture, if true, would imply the four color theorem.

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I. INTRODUCTION

Let \( P \) be a fixed convex \( n \)-gon in the plane, \( n \geq 3 \). By a triangulation of \( P \), we mean a plane graph \( T \) with the \( n \) vertices and \( n \) edges of \( P \), and with \( n - 3 \) additional edges, called diagonals, which subdivide the inner face of \( P \) into triangles. The number of triangulations of \( P \) is the Catalan number \( c_{n-2} = \frac{1}{2n-3} \binom{2n-3}{n-2} \).

**Figure 1.** The five triangulations of a pentagon.

A triangulation \( T \) of \( P \) may be transformed into another one by a diagonal flip. The diagonal flip, or flip for short, of a diagonal \( e \) in \( T \) is the following operation: in the quadrilateral \( Q = v_1v_2v_3v_4 \) formed by the two faces of \( T \) adjacent to \( e = v_1v_3 \), remove \( e \) and replace it by the opposite diagonal \( e' = v_2v_4 \). The result is a new triangulation \( T' \) of \( P \).

**Figure 2.** A diagonal flip.

It is well known and easy to see that given any two triangulations \( T_1, T_2 \) of \( P \), there is a sequence of diagonal flips transforming \( T_1 \) into \( T_2 \). The least possible length of any such sequence is the flip distance \( d(T_1, T_2) \) of \( T_1, T_2 \). The problem of determining the maximal flip distance \( d(n) \) of all pairs of triangulations of \( P \) has been studied in [9, 12], where it is shown that \( d(n) \leq 2n - 10 \) for \( n \geq 13 \), and that equality holds for \( n \) sufficiently large. It is conjectured that \( d(n) = 2n - 10 \) already for \( n \geq 13 \).

Here we introduce an additional structure, by attaching signs to the inner faces of a triangulation of \( P \), and by defining a signed version of the diagonal flip operation.
A signed triangulation of $P$ is a pair $(T, \epsilon)$, where $T$ is a triangulation of $P$, and $\epsilon : \mathcal{F}(T) \rightarrow \{+1, -1\}$ is any sign function on the set $\mathcal{F}(T)$ of inner faces of $T$.

Let $(T, \epsilon)$ be a signed triangulation of $P$, and $e$ a diagonal in $T$. The signed diagonal flip of $e$ is only defined if the two inner faces $F_1, F_2$ of $T$ adjacent to $e$ have the same sign. If $\epsilon(F_1) = \epsilon(F_2) = \mu$, the signed diagonal flip of $e$ in $(T, \epsilon)$ results in the signed triangulation $(T', \epsilon')$ of $P$, defined as follows. $T'$ is obtained by flipping $e$ in $T$ as above, and $\epsilon'(F'_1) = \epsilon'(F'_2) = -\mu$ if $F'_1, F'_2$ are the two faces adjacent to the new diagonal $e'$ in $T'$, whereas for the other inner faces $F$ of $T'$, which are also inner faces of $T$, there is no sign change, i.e., $\epsilon'(F) = \epsilon(F)$.

By a sequence of signed diagonal flips, we mean a sequence of signed triangulations of $P$ such that each term is obtained from the preceding one by a signed diagonal flip.

Every sequence of signed diagonal flips $(R_1, \epsilon_1), \ldots, (R_k, \epsilon_k)$ induces a sequence of diagonal flips, namely $R_1, \ldots, R_k$, by forgetting the sign functions.

We will say that a sequence of diagonal flips $R_1, \ldots, R_k$ is signable if it is induced in the above sense by a sequence of signed diagonal flips.

We have already observed that any triangulation of $P$ may be transformed into any other one by a suitable sequence of diagonal flips. Is it always possible, however, to do so by a signable flip sequence?

**Conjecture.** Let $n \geq 3$, and let $T_1, T_2$ be two triangulations of the $n$-gon $P$. Then there is a sequence of diagonal flips from $T_1$ to $T_2$ which is signable.

We will refer to this as the signed flips conjecture. If true, it would have a very interesting consequence.

**Theorem.** The truth of the signed flips conjecture implies the four color theorem.

The proof of this implication, given in Section 3, depends on the relationship between proper four-vertex-colorings and Heawood signings for the triangulations of the sphere. This is briefly recalled in Section 2.

In analogy with the unsigned case, we define $d_s(T_1, T_2)$ to be the length of a shortest signable flip sequence from $T_1$ to $T_2$, if such sequences exist, or $\infty$ otherwise. We also define $d_s(n)$ as the maximum of $d_s(T_1, T_2)$ for all triangulations $T_1, T_2$ of the $n$-gon $P$. The above conjecture says that $d_s(n)$ is finite for every $n \geq 3$.

The conjecture is true for $n \leq 7$. For $n = 7$, we have $d_s(7) = 6$, whereas $d(7) = 5$. As an example, set

\[ T_1 = \begin{array}{c}
\end{array} \quad \text{and} \quad T_2 = \begin{array}{c}
\end{array} \]
the flip distance of \( T_1, T_2 \) is easily seen to be equal to 5 (see Figure 4), but none of the two flip sequences of length 5 from \( T_1 \) to \( T_2 \) is signable. However, there is a signable flip sequence of length 6 from \( T_1 \) to \( T_2 \) (see Figure 5).

More generally, a signable flip sequence between two triangulations \( T_1, T_2 \) of a polygon can be shown to exist in the following two instances. We leave this as an exercise to the reader.

The first one is the case where \( T_2 \) is obtained by flipping each diagonal of \( T_1 \) at most once. This will happen for example when \( T_2 \) is a star, i.e., when all its diagonals have a common endvertex, and \( T_1 \) is arbitrary.

More interestingly perhaps, in the second instance the vertex set \( V(P) \) of \( P \) is assumed to have a nontrivial partition \( V(P) = A \biguplus B \) such that all diagonals of \( T_1 \) and \( T_2 \) join a vertex in \( A \) to a vertex in \( B \).

2. **Heawood Signings**

Let \( G \) be a triangulation of a closed surface \( S \). Here we mean that \( G \) is a finite graph, loop-free but possibly with multiple edges, embedded in the surface \( S \) and subdividing it into triangular faces (i.e., all faces have exactly three incident edges).

As for polygons, we define a *signed triangulation* of \( S \) to be a pair \( (G, \epsilon) \), where \( G \) is a triangulation of \( S \) and \( \epsilon \) is a sign function on the faces of \( G \). Signed diagonal flips are defined for signed triangulations of \( S \) exactly as above. By convention, the diagonal flip of an edge \( e \) is allowed only if the vertices \( v_1, v_2, v_3, v_4 \) incident to the two faces \( F_1, F_2 \) adjacent to \( e \) are distinct. In other words, a diagonal flip is not allowed to create a loop.

We denote by \( \mathcal{F}(G) \) the set of faces of \( G \), and by \( \mathcal{F}_v \) the subset of faces incident to some vertex \( v \). If \( (G, \epsilon) \) is a signed triangulation of \( S \), and if \( v \) is a vertex of \( G \), we denote by \( s_\epsilon(v) \) the sum of the signs of the faces \( F \) incident to \( v \), i.e.,

\[
s_\epsilon(v) = \sum_{F \in \mathcal{F}_v} \epsilon(F).
\]

The signing \( \epsilon \) is a *Heawood signing* if at every vertex \( v \) of \( G \), one has

\[
s_\epsilon(v) \equiv 0 \mod 3.
\]

**Figure 4.** An unsignable flip sequence of length 5 from \( T_1 \) to \( T_2 \).

**Figure 5.** A signed flip sequence of length 6.
This notion is dual to the notion of ‘Heawood vertex characters’ for cubic graphs on $S$, where signs are attached to the vertices of the graph, with the condition that the sum of the signs of the vertices around every region is divisible by 3.

The main point about signed diagonal flips is that they preserve the Heawood property.

**Lemma.** Let $(G, \epsilon)$ and $(G', \epsilon')$ be signed triangulations of a closed surface $S$, differing by one signed diagonal flip. If $\epsilon$ is a Heawood signing of $G$, then $\epsilon'$ is a Heawood signing of $G'$.

**Proof.** The triangulations $G, G'$ of $S$ coincide everywhere except inside a quadrilateral $Q$, which has one diagonal $e$ in $G$ and the opposite diagonal $e'$ in $G'$.

Let $v_1, v_2, v_3, v_4$ be the four vertices of $Q$, numbered in cyclic order. Up to renumbering, we may assume that $e$ joins $v_1, v_3$ in $G$ and that $e'$ joins $v_2, v_4$ in $G'$. Let $F_1, F_2$ be the faces of $G$ adjacent to $e$, and $F'_1, F'_2$ the faces of $G'$ adjacent to $e'$. Since $(G', \epsilon')$ is obtained from $(G, \epsilon)$ by a signed diagonal flip, we have

$$\epsilon(F_1) = \epsilon(F_2) = \mu$$

for some $\mu = \pm 1$, and correspondingly

$$\epsilon'(F'_1) = \epsilon'(F'_2) = -\mu.$$  

In order to compare the sign sums $s_\epsilon$ and $s_{\epsilon'}$, we will compute the difference

$$\Delta(v) = s_\epsilon(v) - s_{\epsilon'}(v)$$

at every vertex $v$, and show that this difference is always divisible by 3.

If $v$ is distinct from $v_1, v_2, v_3, v_4$ then clearly $\Delta(v) = 0$, since the faces at $v$ and their signs are not affected by the signed diagonal flip of $e$.

If now $v$ is one of $v_1, v_2, v_3, v_4$, we claim that $\Delta(v) = 3\mu$. To start with, the faces at $v$ outside the quadrilateral $Q$, and their signs, are again the same in $(G, \epsilon)$ as in $(G', \epsilon')$ and thus contribute 0 to $\Delta(v)$.

Consider now the respective contributions to $\Delta(v)$ of the signs of the faces inside $Q$, namely of $\epsilon(F_1) = \epsilon(F_2) = \mu$ and $\epsilon'(F'_1) = \epsilon'(F'_2) = -\mu$.

Looking at Figure 6, we find that $\Delta(v) = 2\mu - (-\mu)$ if $v = v_1$ or $v_3$. Similarly, $\Delta(v) = \mu - (-2\mu)$ if $v = v_2$ or $v_4$. In either case, $\Delta(v) = 3\mu$ as claimed.

Since $\Delta(v) \equiv 0 \mod 3$ for every vertex $v$, and since $\epsilon$ is a Heawood signing of $G$ by hypothesis, it follows that

$$s_{\epsilon'}(v) \equiv s_\epsilon(v) \equiv 0 \mod 3$$

at every vertex $v$, whence $\epsilon'$ is a Heawood signing of $G'$. This completes the proof of the Lemma. \qed
Heawood discovered (see [4]) that, for a triangulation $G$ of the sphere $S^2$, every proper four-vertex-coloring yields a Heawood signing, and conversely.

Assume first that $G$ is a triangulation of a closed oriented surface $S$, and that $\gamma : V(G) \to F_4$ is a proper four-coloring of $G$ with values in the finite field $F_4 = \{0, 1, \alpha, \alpha^2\}$ with four elements.

If $vw$ is an edge of $G$, define $\partial \gamma(vw) = \gamma(v) + \gamma(w)$, which is an element of $F_4^2 = \{1, \alpha, \alpha^2\}$.

As is easily seen, the function $\partial \gamma$ is then a Grünbaum coloring of $G$, i.e., a coloring of the edges with three colors such that at each face, the three bounding edges get distinct colors.

Now, for each face $F$ of $G$, set $\epsilon(F) = 1$ if the colors $\{1, \alpha, \alpha^2\}$ appear in a clockwise order on the edges of $F$, and $\epsilon(F) = -1$ otherwise.

Using the relation $\alpha^3 = 1$ in $F_4$, it is not difficult to check that the signing $\epsilon$ just defined is in fact a Heawood signing.

On the sphere $S^2$, this construction can be reversed, as it can be shown that every Heawood signing of a triangulation $G$ is induced as above by a proper four-vertex-coloring. One of the crucial facts involved is that the cycle space of a plane graph is spanned by the cycles bounding its faces. (See e.g., Chapters 8 and 9 in [10] for more details. See also [13].)

On the torus however, as on orientable surfaces of higher genus, it is no longer true that every Heawood signing of a triangulation $G$ comes from a Grünbaum coloring, nor is it true that every Grünbaum coloring comes from a proper four-vertex-coloring.

As an example, consider the embedding of the complete graph $K_7$ in the torus. Since the degree of every vertex is 6, the constant function $+1$ on the faces of $K_7$ is a Heawood signing, which is induced by a Grünbaum coloring of the edges as easily seen. This Grünbaum coloring is of course not induced by a proper four-vertex-coloring!

Another Heawood signing of $K_7$ can be obtained by assigning $+1$ and $-1$ to the faces in alternation around every vertex. This is clearly a Heawood signing again, but this one is not induced by a Grünbaum coloring as readily checked.

One may conjecture that every simple triangulation $G$ of a closed orientable surface admits a Heawood signing on its faces. This would follow, in the way described above, from Grünbaum’s conjecture (see [3], and [5, page 61]) stating that every such graph $G$ admits a Grünbaum coloring of its edges. See also [1, 2] for more details on the relationship between four-colorings, Grünbaum colorings and Heawood signings.

3. Proof of the theorem

Assuming the truth of the signed flips conjecture, we will show that every Hamiltonian triangulation of the sphere $S^2$ admits a Heawood signing of its faces, and hence a proper four-coloring of its vertices. This would be sufficient to prove the four color theorem, by a well known result of H. Whitney [14].

Let $T_1, T_2$ be two triangulations of the $n$-gon $P$. We denote by $G = T_1 \cup_P T_2$ the triangulation of $S^2$ obtained by gluing $T_1, T_2$ along their common boundary $P$, with $T_1$ triangulating the northern hemisphere, $T_2$ triangulating the southern one, and $P$ confined to the equator. The graph $G$ has vertex set $V(G) = V(P)$, the $n$ edges of $P$ on the equator, $n - 3$ diagonals in each hemisphere, and is Hamiltonian by construction. Conversely, every Hamiltonian triangulation of $S^2$ with $n$ vertices has an embedding of the above form.

We will construct a Heawood signing for $G = T_1 \cup_P T_2$ under the hypothesis that there is a signable sequence of flips from $T_1$ to $T_2$.

So let $T_1 = R_1, R_2, \ldots, R_k = T_2$ be a signable sequence of flips from $T_1$ to $T_2$. By hypothesis, there exist functions $\epsilon_i : \mathcal{F}(R_i) \to \{+1, -1\}$ such that for every $i \leq k - 1$, the signed triangulation $(R_{i+1}, \epsilon_{i+1})$ of $P$ is obtained from $(R_i, \epsilon_i)$ by a signed diagonal flip.
Let \( G_i \) denote the sphere triangulation \( R_i \cup P R_1 \), \( i = 1, \ldots, k \). The set \( \mathcal{F}(G_i) \) of faces of \( G_i \) is the disjoint union of the faces of \( R_i \) in the northern hemisphere, and of those of \( R_1 \) in the southern one.

Consider the sign function \( \nu_i = \epsilon_i \cup P (-\epsilon_1) \) on \( \mathcal{F}(G_i) \), defined by gluing the function \( \epsilon_i \) on the northern faces and the function \( -\epsilon_1 \) on the southern ones. In formula:

\[
\nu_i(F) = \begin{cases} 
\epsilon_i(F) & \text{if } F \text{ is a northern face} \\
-\epsilon_1(F) & \text{if } F \text{ is a southern face.}
\end{cases}
\]

We claim that the function \( \nu_i \) is a Heawood signing of \( G_i \) for all \( i = 1, \ldots, k \).

The claim is true for \( i = 1 \). Indeed, in \( G_1 = R_1 \cup P R_1 \), every northern face has a corresponding southern face having the same three vertices, and these corresponding faces have opposite signs under the function \( \nu_1 = \epsilon_1 \cup P (-\epsilon_1) \). It follows that \( s_{\nu_1}(v) = 0 \) at every vertex \( v \), hence in particular that \( \nu_1 \) is a Heawood signing of \( G_1 \).

Let \( 2 \leq i \leq k \). Since \( (G_i, \nu_i) \) is obtained from \( (G_{i-1}, \nu_{i-1}) \) by a signed diagonal flip, and since \( \nu_1 \) is a Heawood signing of \( G_1 \), it follows from the Lemma in Section 2 that \( \nu_i \) is a Heawood signing of \( G_i \) for all \( i \). Hence \( G = G_k = T_1 \cup P T_2 \) admits a Heawood signing and therefore is four-colorable, as claimed. This completes the proof of the Theorem.

**NOTE.** The signed flips conjecture and its connection with the four color theorem are due independently to the author and, in a dual form using binary trees and transplantations or rotations on them, to S. I. Kryuchkov in an unpublished preprint [8]. Kryuchkov’s discussion is based on L. H. Kauffman’s reformulation of the four color theorem involving the vector cross product [6]. See also [7].

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