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New subclasses of bi-univalent functions

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ABSTRACT

In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Ding et al. [1] introduced the following class $Q_\lambda(\beta)$ of analytic functions defined as follows:

$$Q_\lambda(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \right) > \beta, \beta < 1, \lambda \geq 0 \right\}.$$

It is easy to see that $Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1, 0 \leq \beta < 1, Q_\lambda(\beta) \subset Q_1(\beta) = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \beta, 0 \leq \beta < 1\}$ and hence $Q_\lambda(\beta)$ is univalent class (see [2–4]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a brief history and interesting examples in the class Σ, see [5].

Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$, respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha (0 < \alpha \le 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ w \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6,7]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [5].

In order to derive our main results, we have to recall here the following lemma [9].

Lemma 1.1. *If $h \in \mathcal{P}$ then $|c_k| \le 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{U}$ for which* Re$h(z) > 0$ *$h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$ for $z \in \mathcal{U}$.*

2. Coefficient bounds for the function class $\mathcal{B}_\Sigma(\alpha, \lambda)$

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ \lambda \ge 1, z \in \mathcal{U}) \tag{2.1}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, \lambda \ge 1, w \in \mathcal{U}), \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2.3}$$

We note that for $\lambda = 1$, the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ reduces to the class $\mathcal{H}^\alpha_\Sigma$ introduced and studied by Srivastava et al. [5].

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$.

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$, $0 < \alpha \le 1$ and $\lambda \ge 1$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}} \tag{2.4}$$

and

$$|a_3| \le \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = [p(z)]^\alpha \tag{2.6}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = [q(w)]^\alpha \tag{2.7}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(\lambda + 1)a_2 = \alpha p_1, \tag{2.10}$$

$$(2\lambda + 1)a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \tag{2.11}$$

$$-(\lambda + 1)a_2 = \alpha q_1 \tag{2.12}$$

and

$$(2\lambda + 1)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(\lambda + 1)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\begin{aligned} 2(2\lambda + 1)a_2^2 &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) \\ &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \frac{2(\lambda + 1)^2 a_2^2}{\alpha^2}. \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$2(2\lambda + 1)a_3 - 2(2\lambda + 1)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2\right). \tag{2.16}$$

It follows from (2.14)–(2.16) that

$$2(2\lambda + 1)a_3 = \frac{\alpha^2 (2\lambda + 1)(p_1^2 + q_1^2)}{(\lambda + 1)^2} + \alpha(p_2 - q_2)$$

or, equivalently,

$$a_3 = \frac{\alpha^2 (p_1^2 + q_1^2)}{2(\lambda + 1)^2} + \frac{\alpha(p_2 - q_2)}{2(2\lambda + 1)}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.$$

This completes the proof of Theorem 2.2. □

Putting $\lambda = 1$ in Theorem 2.2, we have

Corollary 2.3 (*[5]*). *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leq 1)$. *Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{2+\alpha}} \tag{2.17}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.18}$$

3. Coefficient bounds for the function class $\mathcal{B}_{\Sigma}(\beta, \lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \mathrm{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, z \in \mathcal{U}) \tag{3.1}$$

and

$$\mathrm{Re}\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, w \in \mathcal{U}), \tag{3.2}$$

where the function g is defined by (2.3).

We note that for $\lambda = 1$, the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$ reduces to the class $\mathcal{H}_{\Sigma}(\beta)$ introduced and studied by Srivastava et al. [5].

Theorem 3.2. *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{B}_{\Sigma}(\beta, \lambda)$, $0 \leq \beta < 1$ *and* $\lambda \geq 1$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda+1}} \tag{3.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(\lambda+1)a_2 = (1-\beta)p_1, \tag{3.7}$$
$$(2\lambda+1)a_3 = (1-\beta)p_2, \tag{3.8}$$
$$-(\lambda+1)a_2 = (1-\beta)q_1 \tag{3.9}$$

and

$$(2\lambda+1)(2a_2^2 - a_3) = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(\lambda+1)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$2(2\lambda+1)a_2^2 = (1-\beta)(p_2 + q_2).$$

Thus, we have

$$|a_2^2| \leq \frac{(1-\beta)}{2(2\lambda+1)}(|p_2|+|q_2|) = \frac{2(1-\beta)}{2\lambda+1}$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2(2\lambda+1)a_3 - 2(2\lambda+1)a_2^2 = (1-\beta)(p_2-q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\beta)(p_2-q_2)}{2(2\lambda+1)}.$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(\lambda+1)^2} + \frac{(1-\beta)(p_2-q_2)}{2(2\lambda+1)}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1}$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Putting $\lambda = 1$ in Theorem 3.2, we have the following corollary.

Corollary 3.3 ([5]). *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$, $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \tag{3.13}$$

and

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{3.14}$$

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