New subclasses of bi-univalent functions

B.A. Frasin\textsuperscript{a}, M.K. Aouf\textsuperscript{b,∗}

\textsuperscript{a} Department of Mathematics, Al-al-Bayt University, P.O. Box: 130095 Mafraq, Jordan
\textsuperscript{b} Department of Mathematics, Mansoura University, Mansoura 35516, Egypt

**Abstract**

In this paper, we introduce two new subclasses of the function class $\Sigma$ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

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1. Introduction and definitions

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by $\delta$ we shall denote the class of all functions in $A$ which are univalent in $U$.

Ding et al. \cite{1} introduced the following class $Q_\lambda(\beta)$ of analytic functions defined as follows:

$$Q_\lambda(\beta) = \left\{ f \in A : \Re \left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \right) > \beta, \beta < 1, \lambda \geq 0 \right\}.$$

It is easy to see that $Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1, 0 \leq \beta < 1, Q_\lambda(\beta) \subset Q_1(\beta) = \{ f \in A : \Re f'(z) > \beta, 0 \leq \beta < 1 \}$ and hence $Q_1(\beta)$ is an univalent class (see \cite{2–4}).

It is well known that every function $f \in \delta$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \hspace{1cm} (z \in U)$$

and

$$f(f^{-1}(w)) = w \hspace{1cm} (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots$$

∗ Corresponding author.

E-mail addresses: bafarin@yahoo.com (B.A. Frasin), mkaouf127@yahoo.com (M.K. Aouf).

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let $\Sigma$ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a brief history and interesting examples in the class $\Sigma$, see [5].

Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses $\Delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$, respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in \mathcal{A}$ is in the class $\Delta^*_\alpha[\alpha]$ of strongly bi-starlike functions of order $\alpha (0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ z \in \mathcal{U})$$

and

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ w \in \mathcal{U}),$$

where $g$ is the extension of $f^{-1}$ to $\mathcal{U}$. The classes $\Delta^*_\alpha[\alpha]$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$, corresponding (respectively) to the function classes $\Delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\Delta^*_\alpha[\alpha]$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6,7]).

The object of the present paper is to introduce two new subclasses of the function class $\Sigma$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class $\Sigma$ employing the techniques used earlier by Srivastava et al. [5].

In order to derive our main results, we have to recall here the following lemma [9].

**Lemma 1.1.** If $h \in \mathcal{P}$ then $|c_k| \leq 2$ for each $k$, where $\mathcal{P}$ is the family of all functions $h$ analytic in $\mathcal{U}$ for which $\Re h(z) > 0$ $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ for $z \in \mathcal{U}$.

### 2. Coefficient bounds for the function class $\mathcal{B}_\Sigma(\alpha, \lambda)$

**Definition 2.1.** A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ \lambda \geq 1, \ z \in \mathcal{U})$$

(2.1)

and

$$\left| \arg \left( (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ \lambda \geq 1, \ w \in \mathcal{U}),$$

(2.2)

where the function $g$ is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$  

(2.3)

We note that for $\lambda = 1$, the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ reduces to the class $\mathcal{H}^*_{\alpha[\alpha]}$ introduced and studied by Srivastava et al. [5].

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$.

**Theorem 2.2.** Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}$$

(2.4)

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1},$$

(2.5)

**Proof.** It follows from (2.1) and (2.2) that

$$(1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) = [p(z)]^\alpha$$

(2.6)

and

$$(1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) = [q(w)]^\alpha$$

(2.7)
where \( p(z) \) and \( q(w) \) in \( \mathcal{P} \) and have the forms
\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\] (2.8)
and
\[
q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots.
\] (2.9)

Now, equating the coefficients in (2.6) and (2.7), we get
\[
(\lambda + 1) a_2 = \alpha p_1,
\] (2.10)
\[
(2\lambda + 1) a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2,
\] (2.11)
\[
-(\lambda + 1) a_2 = \alpha q_1
\] (2.12)
and
\[
(2\lambda + 1)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2.
\] (2.13)

From (2.10) and (2.12), we get
\[
p_1 = -q_1
\] (2.14)
and
\[
2(\lambda + 1)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2).
\] (2.15)

Now from (2.11), (2.13) and (2.15), we obtain
\[
2(2\lambda + 1)a_2^2 = \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} (p_1^2 + q_1^2)
\]
\[
= \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \frac{2(\lambda + 1)^2 a_2^2}{\alpha^2}.
\]
Therefore, we have
\[
a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}.
\]

Applying Lemma 1.1 for the coefficients \( p_2 \) and \( q_2 \), we immediately have
\[
|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}.
\]

This gives the bound on \( |a_2| \) as asserted in (2.4).

Next, in order to find the bound on \( |a_3| \), by subtracting (2.13) from (2.11), we get
\[
2(2\lambda + 1)a_3 - 2(2\lambda + 1)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left( \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \right).
\] (2.16)

It follows from (2.14)-(2.16) that
\[
2(2\lambda + 1)a_3 = \frac{\alpha^2 (2\lambda + 1)(p_1^2 + q_1^2)}{(\lambda + 1)^2} + \alpha (p_2 - q_2)
\]
or, equivalently,
\[
a_3 = \frac{\alpha^2 (p_1^2 + q_1^2)}{2(\lambda + 1)^2} + \frac{\alpha (p_2 - q_2)}{2(2\lambda + 1)}.
\]

Applying Lemma 1.1 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[
|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.
\]

This completes the proof of Theorem 2.2. \( \square \)

Putting \( \lambda = 1 \) in Theorem 2.2, we have

\[
|a_3| \leq \frac{2\alpha^2}{2} + \frac{2\alpha}{3}.
\]
Corollary 2.3 ([5]). Let \( f(z) \) given by (1.1) be in the class \( H^\alpha_\Sigma \) \((0 < \alpha \leq 1)\). Then
\[
|a_2| \leq \alpha \frac{2}{\sqrt{2 + \alpha}}
\]
and
\[
|a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.
\]

3. Coefficient bounds for the function class \( B^\Sigma_\beta(\lambda) \)

Definition 3.1. A function \( f(z) \) given by (1.1) is said to be in the class \( B^\Sigma_\beta(\lambda) \) if the following conditions are satisfied:
\[
f \in \Sigma \text{ and } \Re\left( (1-\lambda) \frac{f(z)}{z} + \lambda f'(z) \right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, z \in \mathbb{U})
\]
and
\[
\Re\left( (1-\lambda) \frac{g(w)}{w} + \lambda g'(w) \right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, w \in \mathbb{U}),
\]
where the function \( g \) is defined by (2.3).

We note that for \( \lambda = 1 \), the class \( B^\Sigma_\beta(\lambda) \) reduces to the class \( H^\Sigma_\beta \) introduced and studied by Srivastava et al. [5].

Theorem 3.2. Let \( f(z) \) given by (1.1) be in the class \( B^\Sigma_\beta(\lambda) \), \( 0 \leq \beta < 1 \) and \( \lambda \geq 1 \). Then
\[
|a_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda + 1}}
\]
and
\[
|a_3| \leq \frac{4(1-\beta)^2}{(\lambda + 1)^2} + \frac{2(1-\beta)}{2\lambda + 1}.
\]

Proof. It follows from (3.1) and (3.2) that there exist \( p \) and \( q \in P \) such that
\[
(1-\lambda) \frac{f(z)}{z} + \lambda f'(z) = \beta + (1-\beta)p(z)
\]
and
\[
(1-\lambda) \frac{g(w)}{w} + \lambda g'(w) = \beta + (1-\beta)q(w)
\]
where \( p(z) \) and \( q(w) \) have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields
\[
(\lambda + 1)a_2 = (1-\beta)p_1,
\]
\[
(2\lambda + 1)a_3 = (1-\beta)p_2,
\]
\[
-(\lambda + 1)a_2 = (1-\beta)q_1
\]
and
\[
(2\lambda + 1)(2a_2^2 - a_3) = (1-\beta)q_2.
\]
From (3.7) and (3.9), we get
\[
p_1 = -q_1
\]
and
\[
2(\lambda + 1)^2a_2^2 = (1-\beta)^2(p_1^2 + q_1^2).
\]
Also, from (3.8) and (3.10), we find that
\[
2(2\lambda + 1)a_2^2 = (1-\beta)(p_2 + q_2).
\]
Thus, we have

$$|a_2^2| \leq \frac{(1 - \beta)}{2(2\lambda + 1)}(|p_2| + |q_2|) = \frac{2(1 - \beta)}{2\lambda + 1}$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2(2\lambda + 1)a_3 - 2(2\lambda + 1)a_2^2 = (1 - \beta)(p_2 - q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1 - \beta)(p_2 - q_2)}{2(2\lambda + 1)}.$$  

Upon substituting the value of $a_2^2$ from (3.12), we obtain

$$a_3 = \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(\lambda + 1)^2} + \frac{(1 - \beta)(p_2 - q_2)}{2(2\lambda + 1)}.$$  

Applying Lemma 1.1 for the coefficients $p_1$, $p_2$, $q_1$ and $q_2$, we readily get

$$|a_3| \leq \frac{4(1 - \beta)^2}{(\lambda + 1)^2} + \frac{2(1 - \beta)}{2\lambda + 1}$$

which is the bound on $|a_3|$ as asserted in (3.4).  

Putting $\lambda = 1$ in Theorem 3.2, we have the following corollary.

**Corollary 3.3** ([5]). Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$, $(0 \leq \beta < 1)$. Then

$$|a_2| \leq \frac{2(1 - \beta)}{3}$$  

and

$$|a_3| \leq \frac{(1 - \beta)(5 - 3\beta)}{3}.$$  

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**References**