On the error-sum function of Lüroth series

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Abstract

We introduce the error-sum function of Lüroth series. Some elementary properties of this function are studied. We also determine the Hausdorff dimension of the graph of this function.
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1. Introduction

For any \( x \in (0, 1] \), let \( d_1(x) \in \mathbb{N} \) and \( T(x) \in (0, 1] \) be defined by

\[
\frac{1}{d_1(x)} < x \leq \frac{1}{d_1(x) - 1}, \quad T(x) := d_1(x)(d_1(x) - 1) \left( x - \frac{1}{d_1(x)} \right).
\]

One can see that \( d_1(x) = [1/x] + 1 \), where \([ \ ]\) denotes the integer part. We define the sequence \( \{d_n(x), \ n \geq 1\} \) as follows:

\[
d_n(x) = d_1(T^{n-1}(x)), \quad n \geq 1,
\]

where \( T^n \) denotes the \( n \)th iterate of \( T \) (\( T^0 = \text{Id}_{(0,1]} \)).

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It is well known that from the algorithm (1), any \( x \in (0, 1] \) can be developed uniquely into an infinite series expansion of the form
\[
x = \sum_{n=1}^{\infty} \frac{1}{d_1(x)(d_1(x) - 1) \cdots d_{n-1}(x)(d_{n-1}(x) - 1)d_n(x)}.
\]
(3)

We call (3) the Lüroth series of \( x \) and denote it \( x = [d_1(x), \ldots, d_n(x), \ldots]_L \) for short.

For any \( x \in (0, 1] \) and \( n \geq 1 \), we obtain the Lüroth series convergents \( r_n(x) \) of \( x \) by truncating (3),
\[
r_n(x) = \sum_{i=1}^{n} \frac{1}{d_1(x)(d_1(x) - 1) \cdots d_{i-1}(x)(d_{i-1}(x) - 1)d_i(x)}.
\]

From algorithm (1), it is clear that
\[
x = r_n(x) + \frac{T^n x}{d_1(x)(d_1(x) - 1) \cdots d_n(x)(d_n(x) - 1)}.
\]
(4)

For any \( x \in (0, 1] \), let \( x = [d_1(x), \ldots, d_n(x), \ldots]_L \) be the Lüroth expansion of \( x \), then we have \( d_j(x) \geq 2 \) for any \( j \geq 1 \). On the other hand, any \( \{d_j, j \geq 1\} \) of integer sequence satisfying \( d_j \geq 2 \) for any \( j \geq 1 \) is a Lüroth admissible sequence, that is, there exists unique \( x \in (0, 1] \) such that \( d_j(x) = d_j \) for any \( j \geq 1 \), see [6].

The behavior of the sequence \( d_n(x) \) are of interest and the metric and ergodic properties of the sequence \( \{d_n(x), n \geq 1\} \) and \( T \) have been investigated by a number of authors, see [1–3,6,7,9,10].

For any \( x \in (0, 1] \), define
\[
S(x) = \sum_{n=1}^{\infty} (x - r_n(x)),
\]
(5)
and we call \( S(x) \) the error-sum function of Lüroth series. Since \( d_n(x) \geq 2 \) for any \( n \geq 1 \), \( S(x) \) is well defined. In this paper, we shall discuss some basic properties of \( S(x) \), also the Hausdorff dimension of the graph of \( S(x) \) is determined.

We remark that Ridley and Petruska [8] introduced the error-sum function of the regular continued fraction expansion, and discussed some elementary properties of that function.

2. Some basic properties of \( S(x) \)

In what follows, we shall often make use of the symbolic space.

For any \( n \geq 1 \), let
\[
D_n = \{ (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{N}^n : \sigma_k \geq 2 \text{ for all } 1 \leq k \leq n \}.
\]

Define
\[
D = \bigcup_{n=0}^{\infty} D_n \quad (D_0 := \emptyset).
\]

For any \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in D_n \), write
\[
A_\sigma = \frac{1}{\sigma_1} + \frac{1}{\sigma_1(\sigma_1 - 1)\sigma_2} + \cdots + \frac{1}{\sigma_1(\sigma_1 - 1) \cdots \sigma_{n-1}(\sigma_{n-1} - 1)\sigma_n},
\]
(6)
\[
B_\sigma = A_\sigma + \frac{1}{\sigma_1(\sigma_1 - 1) \cdots (\sigma_{n-1} - 1)\sigma_n(\sigma_n - 1)}.
\]
(7)
We use $J_\sigma$ to denote the following subset of $(0, 1]$:

$$J_\sigma = \{ x \in (0, 1]: d_1(x) = \sigma_1, d_2(x) = \sigma_2, \ldots, d_n(x) = \sigma_n \}. \quad (8)$$

From [6] Theorem 4.14, we have $J_\sigma = (A_\sigma, B_\sigma]$. We define

$$I = \{ A_\sigma, \sigma \in D_n, n \geq 1 \}. \quad (9)$$

It is easy to see that for every $\sigma$ we have either $B_\sigma = 1$ or $B_\sigma = A'_\sigma$ with a suitable $\sigma'$. Therefore $\{B_\sigma\} = \{A_\sigma\} \cup \{1\}$.

The following lemma is easy to check.

**Lemma 2.1.** $x \in I$ if and only if $d_n(x) = 2$ ultimately.

**Lemma 2.2.** (i) For any $x \in (0, 1]$, $0 < S(x) \leq x$.

(ii) For any $n \geq 1$,

$$S(x) = \sum_{i=1}^{n} (x - r_i(x)) + \frac{S(T^n x)}{d_1(d_1 - 1) \cdots d_n(d_n - 1)}. \quad (10)$$

**Proof.** (i) From (4),

$$S(x) = \left( x - \frac{1}{d_1} \right) + \left( x - \left( \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)d_2} \right) \right) + \cdots$$

$$= \frac{T x}{d_1(d_1 - 1)} + \frac{T^2 x}{d_1(d_1 - 1)d_2(d_2 - 1)} + \cdots$$

$$\leq \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)d_2} + \cdots = x.$$  

(ii) From (4), we have

$$x - r_n(x) = \frac{T^n x}{d_1(d_1 - 1) \cdots d_n(d_n - 1)}.$$ 

Thus

$$S(x) = \sum_{i=1}^{\infty} (x - r_i(x)) = \sum_{i=1}^{n} (x - r_i(x)) + \sum_{i=n+1}^{\infty} (x - r_i(x))$$

$$= \sum_{i=1}^{n} (x - r_i(x)) + \sum_{i=n+1}^{\infty} \frac{T^i x}{d_1(d_1 - 1) \cdots d_i(d_i - 1)}$$

$$= \sum_{i=1}^{n} (x - r_i(x)) + \frac{1}{d_1(d_1 - 1) \cdots d_n(d_n - 1)}$$

$$\times \sum_{j=1}^{\infty} \frac{T^{n+j} x}{d_1(T^n x)(d_1(T^n x) - 1) \cdots d_j(T^n x)(d_j(T^n x) - 1)}$$

$$= \sum_{i=1}^{n} (x - r_i(x)) + \frac{1}{d_1(d_1 - 1) \cdots d_n(d_n - 1)} \cdot \sum_{j=1}^{\infty} [T^n x - r_j(T^n x)]$$

$$= \sum_{i=1}^{n} (x - r_i(x)) + \frac{S(T^n x)}{d_1(d_1 - 1) \cdots d_n(d_n - 1)}. \quad \square$$
From (i) of Lemma 2.2, we have the following corollary immediately.

**Corollary 2.3.** \( S(x) \) is bounded.

**Theorem 2.4.** The function \( S(x) \) is left continuous everywhere on \((0, 1]\), continuous at each points of \((0, 1) \setminus I\), and has a right jump discontinuity at each point of \( I \).

**Proof.** The function \( T \) and each of its iterates are left continuous everywhere on \((0, 1]\). This implies that the digits functions \( d_n(x) \) are also everywhere left continuous on \((0, 1]\). Therefore, the same is true for the functions \( r_n(x) \), and thus each term of the definition of \( S(x) \) is left continuous on \((0, 1]\). Since the sum is uniformly convergent, the same is true for \( S(x) \).

It is easy to see that if \( x \in (0, 1) \setminus I \), then \( T^n(x) \neq 1 \) for every \( n \) and thus \( T^n \) is continuous at \( x \) for every \( n \). The same is true for the digit functions \( d_n(x) \). Thus the terms of the definition of \( S(x) \) are continuous at \( x \) and then, by the uniform convergence, \( S(x) \) is continuous at \( x \).

Let \( x_1 \in I \). Suppose \( x_1 = A_\sigma \), where \( \sigma \in D_n \), then \( d_l(x) = \sigma_i \) for every \( x \in [A_\sigma, B_\sigma] \) and \( i < n \). It is easy to see that \( T^n \) is a strictly increasing linear function on \((A_\sigma, B_\sigma] \) with \( \lim_{x \to x_1^+} T^n(x) = 0 \). Therefore, it follows from (10) above that

\[
\lim_{x \to x_1^+} S(x) = \sum_{i=1}^{n-1} (x_1 - r_i(x_1)) < S(x_1). 
\]

(Note that by (4) \( \lim_{x \to x_1^+} (x - r_n(x)) = 0 \).) This proves that \( S(x) \) has a right jump discontinuity at \( x_1 \).

The value

\[
S(x_1) - \lim_{x \to x_1^+} S(x) = \sum_{i=n}^{\infty} (x_1 - r_i(x_1))
\]

can be computed as follows. The Lüroth series expansion of \( x_1 \) is \([\sigma_1, \ldots, \sigma_{n-1}, \sigma_n + 1, 2, 2, \ldots]_L\). Put

\[
U = \frac{1}{\sigma_1} + \frac{1}{\sigma_1(\sigma_1 - 1)\sigma_2} + \cdots + \frac{1}{\sigma_1(\sigma_1 - 1)\cdots\sigma_{n-2}(\sigma_{n-2} - 1)\sigma_{n-1}}
\]

and \( C = \sigma_1(\sigma_1 - 1)\cdots\sigma_{n-2}(\sigma_{n-2} - 1)\sigma_{n-1} \), then \( x_1 = U + \frac{1}{C\sigma_n} \), and

\[
ri(x_1) = U + \frac{1}{C(\sigma_n + 1)} + \frac{1}{C(\sigma_n + 1)\sigma_n \cdot 2} + \cdots + \frac{1}{C(\sigma_n + 1)\sigma_n \cdot 2^{-n}}
\]

for every \( i \geq n \). Thus

\[
x_1 - r_i(x_1) = \frac{1}{C(\sigma_n + 1)\sigma_n \cdot 2^{-n}} \quad (i \geq n)
\]

and

\[
\sum_{i=n}^{\infty} (x_1 - r_i(x_1)) = \sum_{i=n}^{\infty} \frac{1}{C(\sigma_n + 1)\sigma_n \cdot 2^{-n}} = \frac{2}{C(\sigma_n + 1)\sigma_n}. \quad \Box
\]

**Lemma 2.5.** For any \( n \geq 1 \) and \( \sigma \in D_n \), write \( x_1 = A_\sigma \), \( x_2 = B_\sigma \). Then for any \( x \in J_\sigma \),

\[
S^*(x_1) < S(x) \leq S(x_2),
\]

where \( S^*(x_1) = S(x_1) - \frac{2}{\sigma_1(\sigma_1-1)\cdots\sigma_n(\sigma_n+1)} \).
Proof. By (11),
\[ S(x) > \sum_{i=1}^{n-1} (x - r_i(x)) = \sum_{i=1}^{n-1} (x - r_i(x_1)) > \sum_{i=1}^{n-1} (x_1 - r_i(x_1)) = S^*(x_1). \]
Lemma 2.2(ii) leads to the second inequality. \(\square\)

Lemma 2.6. \(\sup_{x,y \in J_\sigma} |S(x) - S(y)| = (n + 1)\lambda(J_\sigma),\) where \(\lambda(J_\sigma)\) is the Lebesgue measure of \(J_\sigma.\)

Proof. From Lemma 2.5, we have for any \(\sigma \in D_n,\)
\[
\sup_{x,y \in J_\sigma} |S(x) - S(y)| = S(x_2) - S^*(x_1)
= \sum_{i=1}^{n-1} (x_2 - r_i(x_2)) + \frac{2}{C(\sigma_n - 1)\sigma_n}
- \left( \sum_{i=1}^{n-1} (x_1 - r_i(x_1)) + \frac{2}{C(\sigma_n + 1)\sigma_n} - \frac{2}{C(\sigma_n + 1)\sigma_n} \right)
= \frac{n + 1}{\sigma_1(\sigma_1 - 1) \cdots \sigma_n(\sigma_n - 1)} = (n + 1)\lambda(J_\sigma). \quad (12)
\]

Proposition 2.7. \(S(x)\) is not of bounded variation.

Proof. Let \(K > 0\) be arbitrary, and fix an integer \(n > K.\) The intervals \(J_\sigma (\sigma \in D_n)\) cover \((0, 1],\) and thus \(\sum_{\sigma \in D_n} \lambda(J_\sigma) = 1.\) Therefore we can select a finite subset \(\Lambda \subset D_n\) such that
\[
\sum_{\sigma \in \Lambda} (n + 1) : \lambda(J_\sigma) > n > K.
\]
By Lemma 2.6, there are points \(x_\sigma, y_\sigma \in J_\sigma (\sigma \in \Sigma)\) such that
\[
\sum_{\sigma \in \Lambda} |S(x_\sigma) - S(y_\sigma)| > K,
\]
which shows that the variation of \(S(x)\) is greater than \(K.\) \(\square\)

3. Hausdorff dimension of graph for \(S(x)\)

In this section, we determine the Hausdorff dimension of the graph for \(S(x).\) For the definition and property of Hausdorff dimension, we refer to [4,5].

Write
\[ Gr(S) = \{ (x, S(x)) \mid x \in [0, 1] \}. \]

Theorem 3.1. \(\dim_H Gr(S) = 1.\)

Proof. For any \(n \geq 1, \{ J_\sigma \times S(J_\sigma), \sigma \in D_n \} \) is a covering of \(Gr(S).\) From (11), \(J_\sigma \times S(J_\sigma)\) can be covered by \(n + 1\) squares with side of length \(\lambda(J_\sigma).\) For any \(\epsilon > 0,\)
\[ H^{1+\epsilon}(Gr(S)) \leq \liminf_{n \to \infty} \sum_{\sigma \in D_n} (n + 1)(\sqrt{2})^{1+\epsilon}(\lambda(J_{\sigma}))^{1+\epsilon} \]

\[ \leq \liminf_{n \to \infty} (n + 1)(\sqrt{2})^{1+\epsilon}2^{-n\epsilon} \sum_{\sigma \in D_n} (\lambda(J_{\sigma})) \]

\[ = \liminf_{n \to \infty} (n + 1)(\sqrt{2})^{1+\epsilon}2^{-n\epsilon} = 0. \]

Thus

\[ \dim_H Gr(S) \leq 1. \]

It is obvious that \( \dim_H Gr(S) \geq 1 \), so \( \dim_H Gr(S) = 1 \). \( \Box \)

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References