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Journal of Computational and Applied Mathematics 170 (2004) 79–102
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Stability criteria for certain second-order delay differential equations with mixed coefficients

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Received 3 February 2003; received in revised form 25 October 2003

Abstract

In this paper, we study asymptotic stability of the zero solution of the second-order linear delay differential equation:

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau),$$

where p_1, p_2, q_1, and q_2 are constants with $p_1 p_2 \geqslant 0$ and $q_1 q_2 < 0$. Here $\tau > 0$ is a constant. In proving our results we make use of Pontryagin's theory of quasi-polynomials.

MSC: 45E99; 34D99

Keywords: Asymptotic stability; Stability criteria; Sufficient conditions; Delay; Characteristic functions; Stability regions

1. Introduction

The aim of this paper is to study asymptotic stability of the zero solution of the delay differential equation

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau), \tag{1.1}$$

where $\tau > 0$ is a constant and p_1, p_2, q_1, and q_2 are constants. In particular, we assume that $p_1 p_2 \geqslant 0$ and $q_1 q_2 < 0$. In [7] we showed that if $p_1 p_2 \geqslant 0$ and $q_1 > 0$, $q_2 > 0$, then the zero solution of (1.1) is not asymptotically stable. Equations of type (1.1) with $q_1 q_2 < 0$ appear in many applications. This problem is of interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots. See [21,17]. These are illustrations of inverted pendulum problems.

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doi:10.1016/j.cam.2003.12.043

A typical example is the balancing of a stick (see [23]). The delay τ reflects human reflexes, and the linearization of the mathematical model leads to an equation of form (1.1) with $p_1 = 0$, $p_2 < 0$, $q_1 > 0$, and $q_2 < 0$. An other application is in machine tool analysis. An important source of instability in the cutting process is the so-called regenerative effect [22]. It is a "past-effect" in which the cutting force depends on the actual and delayed values of the relative displacement of the tool and the workpiece. A simple linearized mathematical model of this phenomenon leads to an equation of form (1.1) with $p_1 < 0$, $p_2 = 0$, $q_1 < 0$, and $q_2 > 0$. Equations of form of (1.1) can be used as test equations for numerical methods. The authors are not aware of a comprehensive study of this important equation. In general the study of asymptotic stability of linear delay differential equations is divided in two different approaches; one approach deals with finding stability criteria or regions of stability of the zero solution [9,10,26,5,12,3,7,24]. The other approach is finding sufficient conditions for asymptotic stability of the zero solution [4,11,18]. For other results on asymptotic stability of second-order delay differential equations, see [25,20,15,14,1,16,8]. However, there is no complete study of stability criteria of (1.1).

In this paper we obtain practical (either easily checked or algorithmically checked) stability criteria for the zero solution of (1.1) when $p_1 p_2 \geqslant 0$ and $q_1 q_2 < 0$. Note that with $\tau = 0$ the zero solution of (1.1) is asymptotically stable if and only if

$$P = p_1 + p_2 < 0 \quad \text{and} \quad Q = q_1 + q_2 < 0. \tag{1.2}$$

We will demonstrate some cases when stability occurs with $\tau > 0$ and condition (1.2) is not valid. In other words we expose some rare cases where the delay can stabilize Eq. (1.1).

This paper is organized as follows. In Section 2, we present the tools used in our asymptotic stability analysis, and we provide some special cases. In Section 3 we give our main results. In Section 4 we present some examples.

2. Background

In this section, we identify the characteristic function of (1.1) in order to study the asymptotic stability of the zero solution. We also quote the main results of Pontryagin related to the asymptotic stability [2, pp. 442–444] and the applications of Pontryagin's results [19, Sections 13.7–13.9].

The characteristic function of (1.1) is given by

$$\hat{H}(s) = s^2 - p_1 s - p_2 s \mathrm{e}^{-s\tau} - q_1 - q_2 \mathrm{e}^{-s\tau}. \tag{2.1}$$

Multiplying (2.1) by $\mathrm{e}^{s\tau}$ yields

$$\mathrm{e}^{s\tau}\hat{H}(s) = \mathrm{e}^{s\tau} s^2 - p_1 s \mathrm{e}^{s\tau} - p_2 s - q_1 \mathrm{e}^{s\tau} - q_2. \tag{2.2}$$

Letting $s = z/\tau$, we examine the zeros of

$$H(z) = \tau^2 \mathrm{e}^z \hat{H}\left(\frac{z}{\tau}\right) = z^2 \mathrm{e}^z - Az\mathrm{e}^z - B\mathrm{e}^z - Cz - D, \tag{2.3}$$

where

$$A = \tau p_1, \quad B = \tau^2 q_1, \quad C = \tau p_2 \quad \text{and} \quad D = \tau^2 q_2. \tag{2.4}$$

Theorem 2.1. *In order that all solutions of* (1.1) *approach zero as* $t \to \infty$ *it is necessary and sufficient that all zeros of* (2.1), *or equivalently* (2.3), *have negative real parts.*

See [19]. The function (2.3) is a special function, usually called an exponential polynomial or a quasi-polynomial. The problem of analyzing the distribution of the zeros in the complex plane of such functions has received a great deal of attention.

Definition 2.1. Let $h(z,w)$ be a polynomial in the two variables z and w (with complex coefficients),

$$h(z,w) = \sum_{m,n} a_{mn} z^m w^n \quad (m,n \text{ nonnegative integers}).$$

We call the term $a_{rs} z^r w^s$ the principal term of $h(z,w)$ if $a_{rs} \neq 0$, and for each term $a_{mn} z^m w^n$ with $a_{mn} \neq 0$, we have $r \geqslant m$ and $s \geqslant n$.

Note that $H(z) = h(z, \mathrm{e}^z)$ where

$$h(z,w) = wz^2 - (Az + B)w - (Cz + D). \tag{2.5}$$

It is clear from Definition 2.1 that $h(z,w)$ of (2.5) has principal term $z^2 w$. We now cite two theorems of Pontryagin, see [2, pp. 442–444, 19].

Theorem 2.2. *Let* $H(z) = h(z, \mathrm{e}^z)$, *where* $h(z,w)$ *is a polynomial with a principal term. The function* $H(\mathrm{i}y)$ *is now separated into real and imaginary parts*; *that is, we set* $H(\mathrm{i}y) = F(y) + \mathrm{i}G(y)$. *If all the zeros of the function* $H(z)$ *lie in the open left half plane*, *then the zeros of the functions* $F(y)$ *and* $G(y)$ *are real*, *are interlacing*, *and*

$$\Delta(y) = G'(y)F(y) - G(y)F'(y) > 0 \tag{2.6}$$

for all real y. *Moreover*, *in order that all the zeros of the function* $H(z)$ *lie in the open left half plane*, *it is sufficient that one of the following conditions be satisfied*:

(a) *All the zeros of the functions* $F(y)$ *and* $G(y)$ *are real and interlace*, *and the inequality* (2.6) *is satisfied for at least one value of* y.
(b) *All the zeros of the function* $F(y)$ *are real and for each of these zeros* $y = y_0$ *condition* (2.6) *is satisfied*; *that is*, $F'(y_0)G(y_0) < 0$.
(c) *All the zeros of the function* $G(y)$ *are real and for each of these zeros the inequality* (2.6) *is satisfied*; *that is*, $G'(y_0)F(y_0) > 0$.

In our case,

$$H(\mathrm{i}y) = (\mathrm{i}y)^2 \mathrm{e}^{\mathrm{i}y} - (A\mathrm{i}y + B)\mathrm{e}^{\mathrm{i}y} - (C\mathrm{i}y + D) \tag{2.7}$$

or

$$\begin{aligned} H(\mathrm{i}y) &= -y^2 \cos y + Ay \sin y - B\cos y - D + \mathrm{i}(-y^2 \sin y - Ay\cos y - B\sin y - Cy) \\ &= F(y) + \mathrm{i}G(y), \end{aligned} \tag{2.8}$$

where

$$F(y) = -y^2 \cos y + Ay \sin y - B \cos y - D \tag{2.9}$$

and

$$G(y) = -y^2 \sin y - Ay \cos y - B \sin y - Cy. \tag{2.10}$$

In order to study the location of the zeros of $H(z)$ one has to study the zeros of F and G. To do so, we need the following result which is useful in determining whether all roots of F and G are real. Let $f(z,u,v)$ be a polynomial in z, u, and v, which we write in the form

$$f(z,u,v) = \sum_{m,n} z^m \phi_m^{(n)}(u,v), \tag{2.11}$$

where $\phi_m^{(n)}(u,v)$ is a polynomial of degree n, homogeneous in u and v, let $z^r \phi_r^{(s)}(u,v)$ be the principal term of $f(z,u,v)$, and let $\phi^{*(s)}(u,v)$ denote the coefficient of z^r in $f(z,u,v)$, so that

$$\phi^{*(s)}(u,v) = \sum_{n \leqslant s} \phi_r^{(n)}(u,v).$$

Also we let

$$\Phi^{*(s)}(z) = \phi^{*(s)}(\cos z, \sin z).$$

Theorem 2.3. *Let $f(z,u,v)$ be a polynomial with principal term $z^r \phi_r^{(s)}(u,v)$. If ε is such that $\Phi^{*(s)}(\varepsilon + \mathrm{i}y) \neq 0$ for all real y, then in the strip $-2\pi k + \varepsilon \leqslant x \leqslant 2\pi k + \varepsilon$, $z = x + \mathrm{i}y$, the function $F(z) = f(z, \cos z, \sin z)$ will have, for all sufficiently large values of k, exactly $4sk + r$ zeros. Thus, in order for the function $F(z)$ to have only real roots, it is necessary and sufficient that in the interval $-2\pi k + \varepsilon \leqslant x \leqslant 2\pi k + \varepsilon$, it has exactly $4sk + r$ real roots for all sufficiently large k.*

Note that the functions $F(y)$ and $G(y)$ in (2.9) and (2.10) have principal terms $-y^2 \cos y$ and $-y^2 \sin y$, respectively. We will use Theorems 2.2 and 2.3 to study the asymptotic stability of (1.1). In the next section we will present the main results of this paper.

3. Main results

In this section we present the main results of this paper. First we start with the case when $p_1 > 0$, $p_2 > 0$, $q_1 > 0$, and $q_2 < 0$ which was encompassed in a result of an earlier paper [7].

Theorem 3.1. *If $p_1 > 0$, $p_2 > 0$, $q_1 > 0$ and $q_2 < 0$, then the zero solution of* (1.1) *is not asymptotically stable.*

Proof. See [6, Corollary 3.1].

From Theorem 2.2, we have the following necessary condition. □

Lemma 3.1. *If the zero solution of* (1.1) *is asymptotically stable, then* $\Delta(0) = (B+D)(A+B+C) > 0$.

We first consider special cases where at least one of the coefficients p_1, p_2 of (1.1) is zero. Recall that in all cases we assume that $p_1 p_2 \geqslant 0$ and $q_1 q_2 < 0$ (or, equivalently, $AC \geqslant 0$ and $BD < 0$). We start with the case $p_1 = p_2 = 0$ (or, equivalently, $A = C = 0$).

Lemma 3.2. *Assume that* $A = C = 0$. *If* $B > 0$ *and* $D < 0$, *then the zero solution of* (1.1) *is not asymptotically stable.*

Proof. For $A = 0$ and $C = 0$, (2.10) yields

$$G(y) = -(y^2 + B)\sin y$$

and thus G has nonreal zeros, $\pm i\sqrt{B}$. By Theorems 2.2 and 2.3 the zero solution of (1.1) is not asymptotically stable. □

Theorem 3.2. *Assume that* $A = C = 0$ *and* $D > 0$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if* $B < 0$, *and there exists* $k \in Z^+$ *such that*

$$2k\pi < \sqrt{-B} < (2k+1)\pi \tag{3.1}$$

and

$$D < \min(-(2k)^2\pi^2 - B, (2k+1)^2\pi^2 + B). \tag{3.2}$$

Proof. For $A = 0$ and $C = 0$, (2.9) and (2.10) yield

$$G(y) = (-y^2 - B)\sin y,$$

$$G'(y) = -2y\sin y + (-y^2 - B)\cos y$$

and

$$F(y) = -y^2\cos y - B\cos y - D.$$

We first prove necessity. If $B = 0$, then $\Delta(0) = 0$ and by Lemma 3.1 the zero solution of (1.1) is not asymptotically stable. If $B > 0$, the proof of Lemma 3.2 yields that the zero solution of (1.1) is not asymptotically stable. Thus it is necessary that $B < 0$. The zeros of G are $y = \pm\sqrt{-B}$ and $y = n\pi (n \in Z)$. If y is a zero of G, then

$$\Delta(y) = F(y)G'(y) = [-y^2\cos y - B\cos y - D][-2y\sin y + (-y^2 - B)\cos y]$$

and, in particular,

$$\Delta(-\sqrt{-B}) = \Delta(\sqrt{-B}) = 2D\sqrt{-B}\sin\sqrt{-B}.$$

Since $D > 0$, $\Delta(-\sqrt{-B}) = \Delta(\sqrt{-B}) > 0$ if and only if $\sin\sqrt{-B} > 0$, or equivalently, (3.1) holds for some nonnegative integer k. At the points $y = n\pi$ $(n \in Z)$ we have

$$\begin{aligned}\Delta(n\pi) &= [(-n^2\pi^2 - B)(-1)^n - D][(-n^2\pi^2 - B)(-1)^n] \\ &= (n^2\pi^2 + B)^2 + D(n^2\pi^2 + B)(-1)^n.\end{aligned}$$

Thus $\Delta(n\pi) > 0$ if and only if

$$(n^2\pi^2 + B)^2 > D(n^2\pi^2 + B)(-1)^{n+1}. \tag{3.3}$$

We distinguish two cases for n.

Case 1: Let $n > 2k$. Thus $n\pi > 2k\pi$ and $0 < 2k\pi < \sqrt{-B} < n\pi$, so that $(2k)^2\pi^2 < -B < n^2\pi^2$, and therefore $n^2\pi^2 + B > 0$. For n even, the right-hand side of (3.3) is negative, and inequality (3.3) is satisfied. For n odd, (3.3) is equivalent to

$$n^2\pi^2 + B > D. \tag{3.4}$$

Observe that (3.4) holds for all odd $n > 2k$ if and only if it holds for $n = 2k + 1$.

Case 2: Let $0 \leqslant n \leqslant 2k$. Thus $0 \leqslant n\pi \leqslant 2k\pi < \sqrt{-B}$, and $n^2\pi^2 + B < 0$. For n odd, (3.3) is satisfied. For n even (3.3) is equivalent to

$$-n^2\pi^2 - B > D. \tag{3.5}$$

Observe that (3.5) holds for all even n with $0 < n \leqslant 2k$ if and only if it holds for $n = 2k$. Condition (3.2) results in combining the last two cases. For n negative, no additional analysis is needed since Δ is even. For sufficiency, these arguments reverse yielding that G has all real zeros and $\Delta > 0$ at each of these zeros. Asymptotic stability of the zero solution of (1.1) follows from Theorems 2.1 and 2.2. This proof is now complete. □

Now we consider the cases, $A = 0$ or $C = 0$ (or, equivalently, $p_1 = 0$ or $p_2 = 0$ and $q_1q_2 < 0$). Our next special case is $A = 0$.

Lemma 3.3. *Assume that $A = 0$, $BD < 0$, $C \neq 0$. Necessary for the zero solution of* (1.1) *to be asymptotically stable is that*

1. $0 < B < \min(-C, -D)$ *and G has two zeros in $(2j\pi, (2j+1)\pi)$ $(j = 0, 1, \ldots, m+1)$ where m is the nonnegative integer such that $2m\pi < \sqrt{B} < (2m+2)\pi$, or*
2. $B < \min(-C, -D) < 0$, $C > 0$, *G has one zero in $(0, \pi)$ if $m = 0$, G has two zeros in $(2m\pi, (2m+1)\pi)$ if $m > 0$, and G has two zeros in $((2m+1)\pi, (2m+2)\pi)$ m is the nonnegative integer such that $2m\pi < \sqrt{-B} < (2m+1)\pi$, or*
3. $B < \min(-C, -D) < 0$, $C < 0$, *G has two zeros in $((2m-1)\pi, 2m\pi)$ if $m > 0$, and G has two zeros in $(2m\pi, (2m+1)\pi)$ where m is the nonnegative integer such that $(2m-1)\pi < \sqrt{-B} < (2m+1)\pi$.*

Proof. Since $A = 0$, (2.9) and (2.10) yield

$$G(y) = -(y^2 + B)\sin y - Cy, \tag{3.6a}$$

$$F(y) = -(y^2 + B)\cos y - D. \tag{3.6b}$$

Since $C \neq 0$, the zeros of G are $y = 0$ and the roots of the equation

$$\csc y = -\frac{1}{C}\left(y + \frac{B}{y}\right). \tag{3.7}$$

In each case below, sketches of left- and right-hand sides of (3.7) are useful. In addition, if r is a zero of G, then (3.6a) and (3.6b) yield that

$$F(r) = Cr\cot r - D. \tag{3.8}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}r}(Cr\cot r - D) = C\,\frac{(\sin 2r - 2r)}{2\sin^2 r}$$

is of a constant sign on each interval $(n\pi,(n+1)\pi)$ (n any integer), the function $Cr\cot r - D$ is monotone on this interval and thus can have at most two points of sign changes there. For the zeros of F and G to interlace, it is thus necessary that G have at most two zeros in each interval $(n\pi,(n+1)\pi)$ (n any integer). Necessary for $\Delta(y) \neq 0$ for all y is that all zeros of G be simple zeros. Now from Lemma 3.1,

$$\Delta(0) = (B+D)(B+C) > 0 \tag{3.9}$$

is necessary for the asymptotic stability of the zero solution of (1.1).

Assume that $B > 0$. Suppose that $C > 0$. Inside each interval $(n\pi,(n+1)\pi)$, the right-hand side of (3.7) is greater than $\csc y$ near both of the endpoints or less than $\csc y$ near both of the end points so that (3.7) has an even number of roots there. Also for k sufficiently large and $0 < \varepsilon < \pi/2$, (3.7) has no roots in $(2k\pi, 2k\pi+\varepsilon)$ and one in $(-2k\pi,-2k\pi+\varepsilon)$. It follows that G has $4M$ zeros in $(-2k\pi+\varepsilon, 2k\pi+\varepsilon)$ where the $2M$ is the number of zeros of G in $(0,2k\pi)$. Since $4M$ cannot equal $4k+2$, G has nonreal zeros contrary to the asymptotic stability of the zero solution of (1.1). Thus it is necessary that $C < 0$. (This argument will be repeated in this paper, and we call it the "standard zero counting argument".)

With $C < 0$, (3.9) leads us to two cases: (i) $B > -C$ and $B > -D$ or (ii) $B < -C$ and $B < -D$. If $B > -C$, or equivalently $-B/C > 1$, then $\csc y < -(1/C)(y + B/y)$ for all positive y sufficiently near 0. It would then follow that (3.7) has an odd number of roots in $(0,\pi)$. This and the fact that in this case (3.7) has one root in $(2k\pi, 2k\pi+\varepsilon)$ and none in $(-2k\pi,-2k\pi+\varepsilon)$ where k is sufficiently large would lead to a contradiction as above. The only remaining case is (ii), and it follows that $B < \min(-C,-D)$ is necessary for the zero solution of (1.1) to be asymptotically stable. In this case, (3.7) has an even number of roots in $(2j\pi,(2j+1)\pi)$ and no roots in $((2j+1)\pi,(2j+2)\pi)$ ($j=0,1,\ldots$). The standard counting method yields that necessary for the zero solution of (1.1) to be asymptotically stable is that G has precisely two zeros in $(2n\pi,(2n+1)\pi)$ ($n=0,1,\ldots$).

Now $-(1/C)(y+B/y)$ is increasing on $(\sqrt{B},\infty)$. Choose the integer m so that $2m\pi < \sqrt{B} < (2m+1)\pi$. It follows that if G has two zeros in $(2j\pi,(2j+1)\pi)$ ($j=0,1,2,\ldots,m+1$), then G has all the required zeros. The standard counting method reveals that G has now real zeros.

Assume that $B < 0$.

Suppose further that $C > 0$. Again (3.9) leaves cases (i) $B+C > 0$ and $B+D > 0$ or (ii) $B+C < 0$ and $B+D < 0$. The standard zero counting argument allows us to eliminate (i). Thus we have that $B < \min(-C,-D)$. It follows that $-B/C > 1$, and (3.7) has an odd number of zeros in $(0,\pi)$. As well, on $(0,\infty)$, $-(1/C)(y+B/y)$ is decreasing with y-intercept $\sqrt{-B}$. If $2m\pi < \sqrt{-B} \leqslant (2m+2)\pi$, the observation above and the standard counting argument yields that G must have one zero in $(0,\pi)$, and two zeros in $((2j+1)\pi,(2j+2)\pi)$ ($j=m,m+1,\ldots$). Since $-(1/C)(y+B/y)$ is decreasing in $(0,\infty)$, it follows that if G has one zero in $(0,\pi)$ when $m=0$, two zeros in $(2m\pi,(2m+1)\pi)$ when

$m > 0$ and two zeros in $((2m+1)\pi, (2m+2)\pi)$, then G has all the other required zeros. (That G has no more zeros follows from the standard counting argument.)

Suppose now that $C < 0$. By (3.9), $B + D < 0$ and so $B < -D = \min(-C, -D)$. In this case $-(1/C)(y + B/y)$ is increasing on $(0, \infty)$ with y-intercept $\sqrt{-B}$. Let $(2m-1)\pi < \sqrt{-B} < (2m+1)$ $(m = 0, 1, \ldots)$. The standard zero counting argument yields that G must have two zeros in $((2j+1)\pi,$ $(2j+2)\pi)$ $(j = 0, 1, 2, \ldots, m)$ and two zeros in $(2j\pi, (2j+1)\pi)$ $(j = m, m+1, \ldots)$ in order that G have all real zeros.

Since $-(1/C)(y + B/y)$ is increasing on $(0, \infty)$, it follows that if G has two zeros in $((2m-1)\pi, 2m\pi)$ (an empty condition if $m = 0$) and two zeros in $(2m\pi, (2m+1)\pi)$, then G has all the other required zeros above. (That G could have no more than those follow from the standard zero counting argument.) $\square$

Remark 3.1. In each of the cases in Lemma 3.3, G has all real zeros, and their rough locations are determined. In case 1, for $j = 0, 1, \ldots,$ G has precisely two zeros $r_{2j+1} < r_{2j+2}$ in the interval $(2j\pi, (2j+1)\pi)$. The other zeros of G are $y = 0$ and $-r_n$ $(n = 1, 2, \ldots)$. In case 2, G has one zero r_1 in $(0, \pi)$, for $j = 1, 2, \ldots, m$ G has two zeros $r_{2j} < r_{2j+1}$ in $(2j\pi, (2j+1)\pi)$, and for $j = m, m+1, \ldots, G$ has two zeros $r_{2j+2} < r_{2j+3}$ in $((2j+1)\pi, (2j+2)\pi)$. As in case 1, the other zeros of G are $y = 0$ and $-r_n$ $(n = 1, 2, \ldots)$. In case 3, for $j = 1, \ldots,$ G has two zeros $r_{2j-1} < r_{2j}$ in $((2j-1)\pi, 2j\pi)$, and for $j = m, m+1, \ldots$ G has two zeros $r_{2j+1} < r_{2j+2}$ in $(2j\pi, (2j+1)\pi)$. Again the other zeros of G are $y = 0$ and $-r_n$ $(n = 1, 2, \ldots)$. In each of the three cases, $F(0) = -(B + D) > 0$. In order that the zeros of F and G interlace it is necessary that $(-1)^n F(r_n) > 0$ $(n = 1, 2, \ldots)$.

Theorem 3.3. *Assume $A = 0$, $BD < 0$, and $C \neq 0$. Necessary and sufficient for the zero solution of* (1.1) *to be asymptotically stable is that the conditions in Lemma* 3.3 *hold and* $(-1)^n F(r_n) > 0$ $(n = 1, 2, \ldots)$.

Proof. Necessity follows from Lemma 3.3 and Remark 3.1 as the interlacing of zeros of F and G and the fact that $F(0) > 0$ yield these sign changes. We prove sufficiency. We start with the case $0 < B < \min(-C, -D)$. Since $F(0) = -(B + D) > 0$, F has at least one zero between 0 and r_1. Also by hypothesis F has at least one zero between r_j and r_{j+1} for $j = 1, 2, \ldots, 2m+3$ and $F(r_{2m+3}) < 0$. The function $-(1/C)(y + B/y)$ is decreasing, and it follows that G has a zero r_{2m+4} (equivalently, a root of $\csc y = -(1/C)(y + B/y)$) in the interval $((2m+3)\pi, (2m+3)\pi + \pi/2)$ and a zero r_{2m+5} in the interval $((2m+3)\pi + \pi/2, (2m+4)\pi)$. At r_{2m+4} the cosine function is negative and $y^2 + B$ is positive so that $F(r_{2m+4}) = -\cos r_{2m+4}(r_{2m+4}^2 + B) - D > 0$. Now $r_{2m+5} > r_{2m+3} + 2\pi$ so that $\cos r_{2m+5} > \cos r_{2m+3} > 0$ and $r_{2m+5}^2 + B > r_{2m+3}^2 + B > 0$. It follows that $F(r_{2m+5}) < F(r_{2m+3}) < 0$. Continuation of this argument reveals that G has a zero r_{2j+4} in $((2j+3)\pi, (2j+3)\pi + \pi/2)$ and $F(r_{2j+4}) > 0$ and a zero r_{2j+5} in $((2j+3)\pi + \pi/2, (2j+4)\pi)$ with $F(r_{2j+5}) < 0$ for $j = m+1, m+2, \ldots$. Recall that G has all real zeros, and they are 0, the r_j, and their negatives. As well, F has zeros interlacing with the zeros of G, and standard zero counting argument reveals that F has all real zeros and exactly one zero between consecutive zeros of G. Asymptotic stability now follows.

For the case $C < 0$, the sufficiency condition is similar.

In Theorem 3.3 the number of conditions to be examined is infinite. In the following we will provide an algorithm to reduce the infinitely many conditions to a finitely many, and thus we obtain practical implementation of the above theorem.

We define the residue modulo 2π of a number x to be the real number $[x]$ for which $0 \leqslant [x] < 2\pi$ and $x - [x]$ is an integer multiple of 2π. □

Algorithmic Stability Test I. *Assume that $A = 0$, $BD < 0$, $C \neq 0$. Moreover assume the necessary conditions of Lemma 3.3 are satisfied.*

In Case 1:

1. *If $F(r_1) < 0$ and $F(r_{2l+1}) < 0$, for $l = 1, 2, \ldots, n$ where $n > m$ and $[r_{2n+1}] < \pi/2$, then $F(r_{2l+1}) < 0$, for all l.*
2. *If $F(r_{2l}) > 0$, for $l = 1, 2, \ldots, n$, where $n > m$ and $[r_{2n}] > \pi/2$, then $F(r_{2l}) > 0$ for all l.*
   Items (1) and (2) imply that the zero solution of (1.1) is asymptotically stable.

In Case 2:

3. *If $F(r_1) < 0$ and $F(r_{2l+1}) < 0$, for $l = 1, 2, \ldots, n$ where $n > m$ and $[r_{2n+1}] > \frac{3}{2}\pi$ then $F(r_{2l+1}) < 0$, for all l.*
4. *If $F(r_{2l}) > 0$, for $l = 1, 2, \ldots, n$, where $n > m$ and $\pi < [r_{2n}] < \frac{3}{2}\pi$, then $F(r_{2l}) > 0$ for all l.*
   Items (3) and (4) imply that the zero solution of (1.1) is asymptotically stable.

In Case 3:

5. *If $F(r_1) < 0$ and $F(r_{2l+1}) < 0$, for $l = 1, 2, \ldots, n$ where $n > m$ and $[r_{2n+1}] < \pi/2$ then $F(r_{2l+1}) < 0$, for all l.*
6. *If $F(r_{2l}) > 0$, for $l = 1, 2, \ldots, n$, where $n > m$ and $[r_{2n}] > \pi/2$, then $F(r_{2l}) > 0$ for all l.*
   Items (5) and (6) imply that the zero solution of (1.1) is asymptotically stable. Note that we automatically have $n > m$.

Proof. The proof is based on the monotonicity of $[r_{2l}]$ and $[r_{2l+1}]$ when $l > m$. In case 2 for example $[r_{2l}] \downarrow \pi$ and $[r_{2l+1}] \uparrow 2\pi$ while in case 3 $[r_{2l+1}] \downarrow 0$ and $[r_{2l}] \uparrow \pi$. In case 1 $[r_{2l+1}] \downarrow 0$ and $[r_{2l}] \uparrow \pi$. In case 1, if $F(r_{2l+1}) < 0$ when $[r_{2l+1}] < \pi/2$ and $[r_{2l+3}] < [r_{2l+1}]$, then $\cos r_{2l+1} = \cos[r_{2l+1}] > \cos[r_{2l+3}] = \cos r_{2l+3} > 0$ and $r_{2l+3}^2 + B > r_{2l+1}^2 + B > 0$ so that

$$F(r_{2l+3}) = (r_{2l+3}^2 + B)\cos r_{2l+3} - D < -(r_{2l+1}^2 + B) - D = f(r_{2l+1}) < 0.$$

The rest of the proof is easily obtained from Eq. (2.9) and Theorem 3.3. □

Now we consider the case $C = 0$.

Lemma 3.4. *Suppose that $C = 0$, $A \neq 0$, $BD < 0$. Necessary conditions for the zero solution of (1.1) to be asymptotically stable are $A < 0$ and $B + D < 0$.*

Proof. With $C = 0$, (2.9) and (2.10) yield

$$F(y) = -(y^2 + B)\cos y + Ay\sin y - D \tag{3.10a}$$

and

$$G(y) = -(y^2 + B)\sin y - Ay\cos y. \tag{3.10b}$$

The zeros of G are $y = 0$ and the roots of

$$\cot y = -\frac{1}{A}\left(y + \frac{B}{y}\right). \tag{3.11}$$

As before, sketches of the left- and right-hand sides of (3.11) are useful. Now if r is a root of (3.11), then (3.10a) and (3.10b) yield

$$F(r) = \frac{Ar}{\sin r} - D. \tag{3.12}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Ar}{\sin r - D}\right) = \frac{A\cos r(\tan r - r)}{\sin^2 r}. \tag{3.13}$$

We first establish that $A<0$ is a necessary condition. To this end, assume that $A>0$. By Lemma 3.1, $A+B\neq 0$, and we consider two cases: (i) $A+B>0$ and (ii) $A+B<0$. In case (i), $-B/A<1$, and thus (3.11) has an odd number of roots in $(j\pi,(j+1)\pi)$ for $j=0,1,\ldots$. Since $A>0$, (3.11) has no roots in $(2k\pi, 2k\pi+\varepsilon)$ and one root in $(-2k\pi,-2k\pi+\varepsilon)$ for k sufficiently large. Thus for k sufficiently large, G has $4m$ zeros in $(-2k\pi+\varepsilon, 2k\pi+\varepsilon)$, and so G has nonreal zeros, a contradiction to the asymptotic stability of the zero solution of (1.1).

In case (ii), $-B/A>1$ and so $B<0$. By hypothesis, $D>0$, and by Lemma 3.1, $B+D>0$. Now (3.11) has an even number of roots in $(0,\pi)$ and odd number of roots in $(j\pi,(j+1)\pi)$ for $j=1,2,\ldots$. If (3.11) has two or more roots in $(0,\pi)$, then since $F(0)=-(B+D)<0$, interlacing of the zeros of G and F would force F and thus the right-hand side of (3.13) to have a change from positive to negative in $(0,\pi)$. Thus, (3.13) would assume a negative value in $(0,\pi)$ which is false. If (3.11) has no roots in $(0,\pi)$ and three roots in $(\pi,2\pi)$, then interlacing would force F and thus the right-hand side would assume values that are positive, negative, and positive at three values $r_1<r_2<r_3$, respectively, in $(\pi,2\pi)$. It would then follow that (3.13) changes sign from negative to positive in $(\pi,2\pi)$. However, (3.13) only changes sign from positive to negative there. For subsequent intervals between consecutive multiples of π, the analysis is similar. We thus have that G has no zeros in $(0,\pi)$ and precisely one zero in $(j\pi,(j+1)\pi)$. The standard zero counting method yields that G has $4k-2$ zeros in $(-2k\pi+\varepsilon, 2k\pi+\varepsilon)$ for k sufficiently large contrary to the asymptotic stability of the zero solution of (1.1). Thus we have $A<0$.

Suppose further that $A+B>0$. Then $-B/A>1$. It now follows that (3.11) has an even number of roots in $(0,\pi)$, and odd number of roots in $(j\pi,(j+1)\pi)$ for $j=1,2,\ldots$, one root in $(2k\pi,2k\pi+\varepsilon)$ and no roots in $(-2k\pi,-2k\pi+\varepsilon)$ for k sufficiently large. The standard counting argument yields a contradiction. As well. $A+B\neq 0$ by Lemma 3.1. It follows that $A+B<0$, and by Lemma 3.1 again, $B+D<0$. □

Remark 3.2. If the necessary conditions of Lemma 3.4 hold, then G has all real zeros using the standard counting method. In both cases of the Lemma 3.4 the function G has one zero r_{j+1} in each interval $(j\pi,(j+1)\pi)$, $j=0,1,2,\ldots$. In all cases, the standard counting method yields that these constitute all positive zeros of G. The other zeros of G are 0 and the opposites of these zeros.

Theorem 3.4. *Suppose that* $C=0$, $A\neq 0$, $BD<0$. *The zero solution of* (1.1) *is asymptotically stable if and only if the necessary conditions of Lemma* 3.4 *hold and*

1. *if* $B<0$,

$$F(r_{2k})>0, \quad k=1,2,\ldots\ . \tag{3.14}$$

2. *if* $B > 0$,

$$F(r_{2k+1}) < 0, \quad k = 0, 1, 2, \dots, \tag{3.15}$$

where $r_1, r_2, \dots$ *are the positive zeros of* G *given in Remark* 3.2.

Proof. The proof of necessity is based on Lemma 3.4 and $F(0) = -(B + D) > 0$. For sufficiency, observe that when $B < 0$, $D > 0$ the right-hand side of (3.12) is negative on $(2j\pi, (2j + 1)\pi)$ for $j = 0, 1, \dots$. When $B > 0$, $D < 0$ the right-hand side of (3.12) is positive on $((2j + 1)\pi, 2j\pi)$ for $j = 0, 1, 2, \dots$. For k sufficiency large $F(2k\pi) = -(2k\pi)^2 - B - D < 0$ and $F(r_{2k}) > 0$ so F must have at least one zero between r_{2k} and $2k\pi$. Therefore F has at least $2k + 1$ zeros in $(0, 2k\pi)$ and thus at least $4k + 2$ zeros in $(-2k\pi, 2k\pi)$. (Recall F is an even function and G is an odd function.) By Theorem 2.3, F has exactly $4k + 2$ zeros in $(-2k\pi, 2k\pi)$ or exactly one zero between consecutive zeros of G. Thus F has all real zeros and they interlace with those of G. By Theorems 2.2 and 2.3 the zero solution of (1.1) is asymptotically stable. □

Again we provide algorithms to reduce the infinitely many conditions given in (3.10) and (3.11) to finitely many yielding a practical implementation of Theorem 3.4.

Algorithmic Stability Test II. *Suppose that* $C = 0$, $A \neq 0$, $BD < 0$. *Moreover assume that the necessary conditions of Lemma* 3.4 *are satisfied*:

1. *Suppose* $B < 0$. *If* $F(r_{2l}) > 0$, *for* $l = 0, 1, 2, \dots, n$, *where* $\frac{3}{2}\pi > [r_{2n}] > \pi$, *and* $-(r_{2n}^2 + B)\cos r_{2n} - D > 0$, *then* $F(r_{2l}) > 0$ *for all* l.
2. *Suppose* $B > 0$. *If* $F(r_{2l+1}) < 0$, *for* $l = 1, 2, \dots, m$, *where* $-(r_{2m+1}^2 + B)\cos r_{2m+1} - D < 0$, *then* $F(r_{2l+1}) < 0$ *for all* l.

Proof. The proof is based on the observation that for case 1, $[r_{2l}] \downarrow \pi$ and $-(r_{2l}^2 + B)\cos r_{2l} + Ar_{2l}\sin r_{2l} - D > -(r_{2l}^2 + B)\cos r_{2l} - D$ when $\pi < [r_{2l}] < \frac{3}{2}\pi$, and for case 2, $[r_{2l+1}] \downarrow 0$ and $-(r_{2l+1}^2 + B)\cos r_{2l+1} + Ar_{2l+1}\sin r_{2l+1} - D < -(r_{2l+1}^2 + B)\cos r_{2l+1} - D$. Note that in the first case $r_{2n}^2 + B > 0$. □

Now we consider Eq. (1.1) with all coefficients nonzero.

Lemma 3.5. *Suppose that* $A > 0$, $C > 0$, $B < 0$, *and* $D > 0$. *Let* m *be the nonnegative integer for which* $\sqrt{-B} \in (2m\pi, (2m + 2)\pi]$. *Then necessary conditions for the zero solution of* (1.1) *to be asymptotically stable are* $-B > A + C$ *and* $-B > D$. *Furthermore*,

1. *if* $C > A$, *then necessary for the zero solution of* (1.1) *to be asymptotically stable is that*
(i) G *has one zero in* $(0, \pi)$ *and two distinct zeros in* $(\pi, 2\pi)$ *if* $m = 0$ *and*
(ii) G *has one zero in* $(0, \pi)$ *and two distinct zeros in* $(2m\pi, (2m + 1)\pi)$ *and two distinct zeros in* $((2m + 1)\pi, (2m + 2)\pi)$ *if* $m \geqslant 1$;
2. *if* $C < A$, *then necessary for the zero solution of* (1.1) *to be asymptotically stable is that*
(i) G *has two zeros in* $(0, \pi)$, *if* $m = 0$ *and*
(ii) G *has no zeros in* $(0, \pi)$ *and three zeros in* $(2k\pi, (2k + 1)\pi)$ *for some* $k = 1, 2, \dots, m$ *if* $m \geqslant 1$;
3. *if* $C = A$, *then necessary for the zero solution of* (1.1) *to be asymptotically stable is that*
(i) G *has one zero in* $(0, \pi)$ *and one zero in* $(\pi, 2\pi)$ *if* $m = 0$ *and*

(ii) *G has no zeros in* $(0,\pi)\cup(\pi,2\pi)$ *and one zero in* $(2j\pi,(2j+1)\pi)$ *for,* $j=1,2,\ldots,m$ *and two zeros in* $(2m\pi,(2m+1)\pi)$ *if* $m\geqslant 1$.

Proof. From (2.9) and (2.10)

$$F(y)=-(y^2+B)\cos y+Ay\sin y-D$$

and

$$G(y)=-(y^2+B)\sin y-Ay\cos y-Cy.$$

By Lemma 3.1 necessary for the zero solution of (1.1) to be asymptotically stable is that $\Delta(0)=(B+D)(A+B+C)>0$ which is equivalent to (a) $B+D<0$ and $A+B+C<0$, or (b) $B+D>0$ and $A+B+C>0$. We start with the case (a).

When $C\neq A$, it is evident that $G(y)=0$ if and only if $y=0$ or $w(y)=\zeta(y)$ where $w(y)=A\cot y+C\csc y$ and $\zeta(y)=-(y+B/y)$.

The function w has rather different forms for the three cases $C>A$, $C<A$, and $C=A$. We establish the necessary and the additional necessary conditions separately for these three cases.

Case 1: Suppose $C>A$. In this case, $w(y)$ resembles the cosecant function in that w is an odd function, has a period 2π, is positive valued and concave upward in $(0,\pi)$, and has limit ∞ at 0 and π when the limit is taken from inside the interval $(0,\pi)$. With $B<0$, ζ has limit ∞ at 0 taken from inside $(0,\infty)$, is decreasing and concave upward on $(0,\infty)$, and has asymptote $\xi=-y$. Note also that $\zeta(y)>0$ for $0<y<\sqrt{-B}$, and $\zeta(y)<0$ for $y>\sqrt{-B}$.

Suppose that (b) $-B<A+C$ and $B>-D$. Since $-B<A+C$, $\zeta(y)<w(y)$ for y positive and sufficiently near 0. It follows then that $w(y)=\zeta(y)$ has an even number of roots in each interval between successive multiples of π. In addition, for $0<\varepsilon<\pi$ and k sufficiently large, $(2k\pi-\varepsilon,2k\pi)$ (and hence $(-2k\pi,-2k\pi+\varepsilon)$) contains one root of $w(y)=\zeta(y)$ while $(2k\pi,2k\pi+\varepsilon)$ does not. Since 0 is a zero of G, we see that for k sufficiently large, $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ contains 4M zeros of G where 2M is the number of zeros of G in $(0,2k\pi)$. This count cannot be equal to $4k+2$, and it follow that G has nonreal zeros.

Thus the zero solution of (1.1) is not asymptotically stable. Thus we have that (a) $-B>A+C$ and $B+D<0$. It follows that $w(y)=\zeta(y)$ has an odd number of roots in $(0,\pi)$ and an even number of roots in all other intervals between successive multiples of π in $(0,\infty)$.

If $w(y)=\zeta(y)$ has three roots in $(0,\pi)$, then F must have at least three zeros in $(0,\pi)$ in order that the zeros of F and G interlace. Now $F(0)=-(B+D)>0$. If $0<r_1<r_2<r_3<\pi$ are consecutive zeros of G in $(0,\pi)$, then interlacing forces $F(r_1)<0$, $F(r_2)>0$, and $F(r_3)<0$. If r is a nonzero zero of G, it can be shown that

$$F(r)=\frac{Ar}{\sin r}\left(1+\frac{C}{A}\cos r\right)-D. \tag{3.16}$$

The derivative of the right-hand side of (3.16) is

$$\frac{A}{\sin^2 y}\left[\sin y\left(1+\frac{C}{A}\cos y\right)-y\left(\cos y+\frac{C}{A}\right)\right]. \tag{3.17}$$

In the present case, $C>A>0$. We show that

$$\sin y\left(1+\frac{C}{A}\cos y\right)<y\left(\cos y+\frac{C}{A}\right) \tag{3.18}$$

on $(0,\pi)$. It is evident that (3.18) holds when $1+(C/A)\cos y \leqslant 0$. when $1+(C/A)\cos y > 0$, $\cos y < 1 < C/A$ implies $(1-\cos y)(C/A-1) > 0$, and thus

$$1+\frac{C}{A}\cos y < \cos y+\frac{C}{A}. \tag{3.19}$$

Since $0 < \sin y < y$ and $1+(C/A)\cos y > 0$, (3.18) follows from (3.19). We thus have (3.18) is negative on $(0,\pi)$. This contradicts the fact that the right-hand side of (3.16) is negative at r_1 and positive at r_2.

Hence, it is necessary that G has only one zero in $(0,\pi)$.

If $\sqrt{-B}\in(2m\pi,(2m+2)\pi]$, G has one zero in $(0,\pi)$ and at most two zeros in $(2j\pi,(2j+1)\pi)$ ($j=1,2,\ldots,m$). If G had more than two zeros in one of these intervals, the argument above yields a contradiction.

Now we have at most two zeros in $(2i\pi,(2i+1)\pi)$ for $i=1,2,\ldots,m$. On the other hand G has at most two zeros in each interval $((2j+1)\pi,(2j+2)\pi)$ for $j=m,m+1,\ldots$ because of the opposite concavities of w and ζ. If in any of the intervals $(2j\pi,(2j+1)\pi)$ ($j=1,2,\ldots,m$) or $((2j+1)\pi,(2j+2)\pi)$ ($j=m,m+1,\ldots$), G does not have two zeros, the standard counting method reveals that G has at most $4k-2$ zeros in $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ for k sufficiently large, and asymptotic stability fails. Case 1 is complete.

Case 2: Suppose that $C<A$. In this case, $w(y)$ resembles the cotangent function in that w is decreasing on $(0,\pi)$ and has limits ∞ and $-\infty$ at 0 and π, respectively, when the limits are taken from the inside $(0,\pi)$.

Suppose that (b) $-B<A+C$. The number of zeros of G in $(0,\pi)$ as well as in all other intervals between consecutive multiples of π is odd. In the intervals $(-2k\pi,-2k\pi+\varepsilon)$ and $(2k\pi,2k\pi+\varepsilon)$ for k sufficiently large and $0<\varepsilon<\pi$, G has one and no zeros, respectively. As in case 1, G has $4M$ rather that $4k+2$ zeros in $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ for k sufficiently large. Thus G has nonreal zeros, and by Theorems 2.2 and 2.3 the zero solution of (1.1) is not asymptotically stable. Thus it is necessary that $-B>A+C$ for the zero solution of (1.1) to be asymptotically stable.

It now follows that G has an even number of zeros in $(0,\pi)$ and an odd number of zeros in every other interval between consecutive multiples of π in $(0,\infty)$. This yields a minimal count of zeros of G in these intervals: none in $(0,\pi)$ and one in each interval $(j\pi,(j+1)\pi)$ ($j=1,2,\ldots$). This minimal count (accounting for $y=0$ as a zero of G and these being one and no zeros of G in $(-2k\pi,-2k\pi+\varepsilon)$ and $(2k\pi,2k\pi+\varepsilon)$, respectively) leads to $4k-2$ zeros in $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ for k sufficiently large. Thus a further necessary condition for the zero solution of (1.1) to be asymptotically stable is that $(0,\pi)$ contains two zeros of G or that $(j\pi,(j+1)\pi)$ contains three zeros of G for some $j=1,2,\ldots$. To complete the proof of case 2, we rule out the cases when j is odd or greater than $2m$.

Recall that if r is a nonzero zero of G, then

$$F(r)=\frac{Ar}{\sin r}\left(1+\frac{C}{A}\cos r\right)-D. \tag{3.20}$$

The facts that $0<C<A$ and $D>0$ show that the right-hand side of (3.20) is negative in $(j\pi,(j+1)\pi)$ when j is odd. As such, if G had three zeros in $(j\pi,(j+1)\pi)$, then the zeros of F could not interlace with these zeros.

Suppose that G has three zeros in $(2k\pi,(2k+1)\pi)$ when $k>m$. On this interval $\zeta<0$, and thus $w(r)<0$ for each zero r of G in $(2k\pi,(2k+1)\pi)$. Thus these zeros of G lie in the interval $(s,(2k+1)\pi)$ when $w(s)=0$ and $s>2k\pi+\pi/2$.

Observe that the derivative of the right-hand side of (3.20) is

$$A\csc r + C\cot r - r\csc r w(r) = \frac{A}{\sin r}\left(1+\frac{C}{A}\cos r\right) - r\csc r w(r). \tag{3.21}$$

Now $1+(C/A)\cos r>0$, $w(r)<0$, and $A/\sin r>0$ so that (3.21) is positive. Thus three zeros of G in $(s,(2j+1)\pi)$ cannot interlace with the zeros of F. Case 2 is complete.

Case 3: Suppose that $C=A$. In this case $y=0$ is a zero of G as are all the odd multiples of π and the roots of $\zeta(y)=w(y)$. In this case, w is decreasing on $(0,\pi)$ and has limits ∞ and 0 at 0 and π, respectively, taken from inside the interval $(0,\pi)$. Also for $\pi<y<2\pi$, $w(y)=-w(2\pi-y)$.

Observe that if $\sqrt{-B}$ were an odd multiple of π, then the determinant Δ would be zero at this point. So necessary for the zero solution of (1.1) to be asymptotically stable is that $\sqrt{-B}$ is not an odd multiple of π.

Suppose that (b) $-B<A+C$. In this case G has an odd number of zeros in $(0,\pi)\cup(\pi,2\pi)$. Also, for $n=1,2,\ldots,(2n\pi,(2n+1)\pi)\cup((2n+1)\pi,(2n+2)\pi))$ contains an odd number of zeros of G. As in the previous case $y=0$ is a zero of G as are all odd multiples of π. In the intervals $(-2k\pi,-2k\pi+\varepsilon)$ and $(2k\pi,2k\pi+\varepsilon)$ for k sufficiently large G has one and no zeros, respectively. Using the standard counting method the number of zeros of G in $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ is $4M$, and thus G has nonreal zeros. By Theorems 2.2 and 2.3, the zero solution of (1.1) is not asymptotically stable.

Thus we have that $-B>A+C$ and $B+D<0$. It follows that $\zeta(y)=w(y)$ has an even number of roots in $(0,\pi)\cup(\pi,2\pi)$ and an odd number of roots in $(2n\pi,(2n+1)\pi)\cup((2n+1)\pi,(2n+2)\pi)$ $(n=1,2,\ldots)$. The standard zero counting method yields that in order for G have all real zeros it is necessary that $\zeta(y)=w(y)$ has two roots in $(0,\pi)\cup(\pi,2\pi)$ or three roots in $(2n\pi,(2n+1)\pi)\cup((2n+1)\pi,(2n+2)\pi)$ for some $n=1,2,\ldots$. From (3.16), for every root r of $\zeta(r)=w(r)$, we have that

$$F(r)=\frac{Ar}{\sin r}(1+\cos r)-D. \tag{3.22}$$

In each interval $((2n+1)\pi,(2n+2)\pi)$, the right-hand side of (3.22) is negative, and thus for the zeros of F and G to interlace, G can have at most one zero in $((2n+1)\pi,(2n+2)\pi)$. By (3.17) the derivative of the right-hand side of (3.22) is

$$w(r)-r\csc r w(r)=\frac{w(r)}{\sin r}(\sin r - r). \tag{3.23}$$

Now (3.23) is negative throughout each interval $(2n\pi,(2n+1)\pi)$ $(n=0,1,2,\ldots)$. For the zeros of F and G to interlace, it is then necessary that G have at most two zeros in $(2n\pi,(2n+1)\pi)$ $(n=1,2,\ldots)$. Furthermore, $F(0)=-(B+D)>0$. If G had two zeros in $(0,\pi)$, say $r_1<r_2<\pi$, then interlacing would force $F(r_1)<0<F(r_2)$ and it would then follow that (3.23) is positive somewhere in $(0,\pi)$. To summarize, if G has more than the minimal count of zeros in $(2n\pi,(2n+2)\pi)$, then G must have zeros in $(2n\pi,(2n+1)\pi)$ and in $((2n+1)\pi,(2n+2)\pi)$. That is, ζ must change sign in $(2n\pi,(2n+2)\pi)$ so that $m=n$. The proof is now complete. □

Remark 3.3. If the necessary conditions of Lemma 3.5 hold, then G has all real zeros.

Suppose $C > A$. Then G has one zero r_1 in $(0,\pi)$, two zeros $r_{2j} < r_{2j+1}$ in $(2j\pi,(2j+1)\pi)$ for $j=1,2,\ldots,m$, and two zeros $r_{2j+2} < r_{2j+3}$ in $((2j+1)\pi,(2j+2)\pi)$ for $j=m,m+1,\ldots$ (the second designation is empty when $m=0$).

Suppose $C < A$. If $\sqrt{-B}\in(0,2\pi)$, then G has two $r_1 < r_2$ zeros in $(0,\pi)$ and one zero r_{j+2} in $(j\pi,(j+1)\pi)$ for $j=1,2,\ldots$. If $\sqrt{-B}\in(2m\pi,(2m+2)\pi)$ for some $m\geqslant 1$, then there is a k with $1\leqslant k\leqslant m$ such that G has no zeros in $(0,\pi)$, one zero r_j in $(j\pi,(j+1)\pi)$ for $j=1,\ldots,2k$, three zeros $r_{2k+1} < r_{2k+2} < r_{2k+3}$ in $(2k\pi,(2k+1)\pi)$, and one zero r_{j+2} in $(j\pi,(j+1)\pi)$ for $j=2k+2,2k+3,\ldots$. Here $k\leqslant m$ is described in Lemma 3.5.

Suppose $C=A$. If $\sqrt{-B}\in(0,2\pi)$, G has one zero r_1 in $(0,\pi)$ and one zero r_2 in $(\pi,2\pi)$ and one zero r_{2j} in $((2j-1)\pi,2j\pi)$ $(j=2,3,\ldots)$. The zeros of G include $r_{2j+1}=(2j+1)\pi$ for $j=1,2,\ldots$. If $\sqrt{-B}\in(2m\pi,(2m+2)\pi)$ where $m\geqslant 1$, then G has no zeros in $(0,\pi)\cup(\pi,2\pi)$ and one zero r_{2j} in $(2j\pi,(2j+1)\pi)$ for $j=1,2,\ldots,m-1$, and two zeros $r_{2m} < r_{2m+1}$ in $(2m\pi,(2m+1)\pi)$ and one zero r_{2j+3} in $((2j+1)\pi,(2j+2)\pi)$ for $j\geqslant m$. In this case, the odd multiple of π are also zeros of G, and we label them as $r_{2j+1}=(2j+1)\pi$ $(j=0,1,\ldots,m-1)$ and $r_{2j+2}=(2j+1)\pi$ $(j=m,m+1,\ldots)$.

In all cases, the standard counting method yields that these constitute all positive zeros of G. The other zeros of G are 0 and the opposites of these zeros.

Theorem 3.5. *Suppose that* $A>0$, $C>0$, $B<0$, $D>0$. *If* $C>A$ *or* $C<A$, *the zero solution of* (1.1) *is asymptotically stable if and only if the necessary conditions of Lemma* 3.5 *hold and*

$$(-1)^k F(r_k) > 0,\quad k=1,2,\ldots, \tag{3.24}$$

where $r_1,r_2,\ldots$ *are the positive zeros of* G *given in Remark* 3.3.

If $C=A$, *the zero solution of* (1.1) *is asymptotically stable if and only if the necessary conditions of Lemma* 3.5 *hold and*

1. *if* $\sqrt{-B}\in(0,2\pi)$ *then* $F(r_1)<0$, *and* $\pi^2+B-D>0$, *and*
2. *if* $\sqrt{-B}\in(2m\pi,(2m+2)\pi)$, *then* $((2m-1)\pi)^2+B-D<0$ *and* $F(r_{2j})>0$, $j=1,2,\ldots,m$, $F(r_{2m+1})<0$, *and* $((2m+1)\pi)^2+B-D>0$.

Proof. When $C\neq A$, the proof of necessity and sufficiency is essentially the same as the proof of Theorem 3.3.

We now consider the case $C=A$. First we prove necessity. In (1), $F(0)=-(B+D)>0$ and from Lemma 3.5, $r_1\in(0,\pi)$. If $F(r_1)\geqslant 0$, then interlacing fails. Now $F(r_2)=F(\pi)=\pi^2+B-D>0$ in order that interlacing holds. In (2), $F(0)=-(B+D)>0$ and by Lemma 3.5 it is necessary that G has no zeros in $(0,\pi)\cup(\pi,2\pi)$. Thus it is necessary that $F(\pi)=\pi^2+B-D<0$, $F(r_2)>0$, and $F(3\pi)=(3\pi)^2+B-D<0$. We continue until we have two zeros in $(2m\pi,(2m+1)\pi)$. In order for interlacing to hold it is necessary that $F(r_{2m})>0$, $F(r_{2m+1})<0$, and $F((2m+1)\pi=((2m+1)\pi)^2+B-D>0$.

For sufficiency, we only need to show that F strictly alternates in sign at the zeros of G. We consider case (1). We have that $F(0)=-(B+D)>0$ (by Lemma 3.5) and $F(r_1)<0$ by hypothesis. Now $F(r_2)=F(\pi)=\pi^2+B-D>0$ by hypothesis, and for $j>1$, $F(r_{2j})=F((2j+1)\pi)=(2j+1)^2\pi^2+B-D>\pi^2+B-D>0$. For j odd and greater than 1, $r_{2j}\in(j\pi,(j+1)\pi)$ and $F(r_{2j})<0$ follow from (3.16). The proof of case (2) is similar.

In the following we will provide algorithms to reduce the infinitely many conditions given in (3.19) to a finitely many conditions, and thus we obtain practical implementations of the above theorem. Note that in the case $A = C$ the number of conditions is finite and an algorithmic stability test is not needed. □

Algorithmic Stability Test III. *Suppose that* $A > 0$, $C > 0$, $B < 0$, $D > 0$ *and* $C > A$. *Moreover assume the necessary conditions of Lemma* 3.5 *are satisfied*:

(i) *If* $F(r_{2l}) > 0$, *for* $l = 0, 1, 2, \dots, v$, *where* $v > m$, $[r_{2v}] < 3\pi/2$, $-r_{2v}\cos r_{2v} > A + (-B + D)/r_{2v}$, *then* $F(r_{2l}) > 0$ *for all* l.

(ii) *If* $F(r_{2l+1}) < 0$, *for* $l = 0, 1, 2, \dots, n$, n, *where* $n > m$ *and* $[r_{2n+1}] > 3\pi/2$, *then* $F(r_{2l+1}) < 0$ *for all* l.

Items (i) *and* (ii) *imply that* (3.24) *holds for all* l.

Proof. Observe that $[r_{2l}] \downarrow \pi$ and $[r_{2l+1}] \uparrow 2\pi$ when $l > m$. For (ii), if $l \geqslant n > m$ and $[r_{2n+1}] > 3\pi/2$, then $[r_{2l+1}] > 3\pi/2$, and thus $F(r_{2l+1}) < 0$. For (i)

$$\begin{aligned} F(r_{2v+2}) &= r_{2v+2}(-r_{2v+2}\cos r_{2v+2} + A\sin r_{2v+2}) - B\cos r_{2v+2} - D \\ &\geqslant -r_{2v+2}(\cos r_{2v} + A\sin r_{2v+2}) - B\cos r_{2v+2} - D \\ &> r_{2v+2}(A + (-B + D)/r_{2v} + A\sin r_{2v+2}) - B\cos r_{2v+2} - D > 0. \end{aligned}$$

Note also that

$$-r_{2v+2}\cos r_{2v+2} > A + (-B + D)/r_{2v+2}. \qquad \square$$

Algorithmic Stability Test IV. *Suppose that* $A > 0$, $C > 0$, $B < 0$, $D > 0$, *and* $A > C$. *Moreover assume the necessary conditions of Lemma* 3.5 *are satisfied*:

(i) *Suppose* $\sqrt{-B} \in (0, \pi)$ *so that* G *has two zeros* $r_1 < r_2$ *in* $(0, \pi)$ *and* $F(r_1) < 0$ *and* $F(r_2) > 0$. *If* $F(r_{2j}) > 0$, $j = 1, 2, \dots, n$, *when* $[r_{2n}] > \pi/2$, *and* $-\cos r_{2n}(r_{2n}^2 + B) - D > 0$, *then* (3.24) *is satisfied for all* l.

(ii) *Suppose* $\sqrt{-B} \in (2m\pi, (2m + 2)\pi)$, $m \geqslant 1$, *so that* G *has no zeros in* $(0, \pi)$ *and three zeros in* $((2k)\pi, (2k + 1)\pi)$, *for some* $1 \leqslant k \leqslant m$. *If* $F(r_{2k}) > 0$, $F(r_{2k+1}) < 0$, $F(r_{2k+2}) > 0$ *and if* $F(r_{2l}) > 0$, $l = 1, 2, \dots, n$ *that is*, (3.24) *holds for all* r_{2l+1} *smaller than* $(2m + 2)\pi$ *where* $r_{2n} > \sqrt{-B}$ *and* $[r_{2n}] > \pi/2$, $-\cos r_{2n}(r_{2n}^2 + B) - D > 0$, *then* (3.24) *holds for all* k.

Proof. First recall that

$$F(r) = \frac{Ar}{\sin y}\left(1 + \frac{C}{A}\cos r\right) - D < 0,$$

where r is a zero of G in any interval $((2i + 1)\pi, (2i + 2)\pi)$, $i = 0, 1, 2, \dots$. In addition $[r_{2l}] \downarrow \pi$ in both cases. The rest of the proof obtained easily from Eq. (2.9) for $F(y)$.

We also obtained the following nonasymptotic stability result. □

Theorem 3.6. *Assume that* $A<0$, $C<0$, $B>0$, $D<0$, $B+D>0$, *and* $B>-A-C$. *Then the zero solution of* (1.1) *is not asymptotically stable.*

Proof. We will consider three cases $-C>-A$, $-C=-A$ and $-C<-A$. In all cases we will show that G has nonreal zeroes, and by Theorems 2.2 and 2.3, the zero solution of (1.1) is not asymptotically stable.

(i) Assume that $-C>-A$. Eq. (2.9) yields

$$G(y)=-y^2\sin y-Ay\cos y-B\sin y-Cy. \tag{3.25}$$

Evidently that $G(y)=0$ if and only if $y=0$ or $w(y)=\zeta(y)$ where $w(y)=-A\cot y-C\csc y$ and $\zeta(y)=y+B/y$. In this case $w(y)$ resembles the cosecant function in the same sense as in the proof of Lemma 3.5. Since $B>-A-C$, $w(y)=\zeta(y)$ has an odd number of zeros in $(0,\pi)$ and an even number of zeros in each $(2l\pi,(2l+2)\pi)$. Also $y=0$ is a zero of G and for k sufficiently large G has a zero in $(2k\pi,2k\pi+\varepsilon)$ and no zeros in $(-2k\pi+\varepsilon,-2k\pi)$. Thus the total number of zeros in $(-2k\pi+\varepsilon,2k\pi+\varepsilon)$ is $4M$ where M is a positive integer. Since G has $4k+2$ zeros in the strip $-2k\pi+\varepsilon\leqslant Re\,z\leqslant 2k\pi+\varepsilon$ for k sufficiently large, G has nonreal zeros.

(ii) Assume that $-A>-C$. Again the zeros of G are $y=0$ and the roots of $w(y)=\zeta(y)$. In this case, $w(y)$ resembles the cotangent function in the same sense as in the proof of Lemma 3.5, and $w=\zeta$ has an even number of zeros in $(0,\pi)$ and odd number in every interval $(l\pi,(l+1)\pi)$ with $l\geqslant 1$. Also $y=0$ is a zero of G and for k sufficiently large G has a zero in $(2k\pi,2k\pi+\varepsilon)$ and G has no zeros in $(-2k\pi,-2\pi k+\varepsilon)$. If $w=\zeta$ has no zeros in $(0,\pi)$ and one zero in each $(l\pi,(l+1)\pi)$ interval then the total number of zeros in $(-2\pi k+\varepsilon,2\pi k+\varepsilon)$, is $4k$, and G has nonreal zeros. If $w=\zeta$ has two zeros in $(0,\pi)$, then G has $4k+4$ zeros in $(2k\pi+\varepsilon,2k\pi+\varepsilon)$ which is impossible. By the same argument if $w=\zeta$ has three zeros in any $(l\pi,(l+1)\pi)$ ($l\geqslant 1$), then G would have too many zeros.

(iii) Assume that $-A=-C$. In this case $y=0$ and the odd multiples of π are zeros of G. Also the roots of $w(y)=\zeta(y)$ are the zeros of G. In the interval $(0,2\pi)$, $w=\zeta$ has an even number of zeros, and $w=\zeta$ has an odd number of zeros in every $(2l\pi,(2i+2)\pi)$, $l\geqslant 1$. For k sufficiently large G has a zero in $(2k\pi,2k\pi+\varepsilon)$ and G has no zeros in $(-2k\pi,-2\pi k+\varepsilon)$. If $w=\zeta$ has no zeros in $(0,2\pi)$ and one zero in every interval $(2l\pi,(2l+2)\pi)$, $l\geqslant 1$, then the total number of zeros of G is $4k$. As before if $w=\zeta$ has two zeros in $(0,2\pi)$ or three zeros in any interval $(2l\pi,(2l+2)\pi)$, $l>1$, then G has at least $4k+4$ which is impossible. Thus G has nonreal zeros. By Theorem 2.3 the zero solution of (1.1) is not asymptotically stable.

Recall that if $A\neq C$, then $G=0$ if and only if $y=0$ and $w(y)=\zeta(y)$ where $w(y)=-C\csc y-A\cot y$ and $\zeta(y)=y+B/y$. Recall also that G is an odd function. □

Lemma 3.6. *Assume* $A<0$, $B>0$, $C<0$, $D<0$, *and* $-C=-A$. *Then necessary conditions for the zero solution of* (1.1) *to be asymptotically stable are* $B+D<0$ *and* $B<-A-C$. □

Proof. The proof is straightforward from Lemma 3.1 and the standard counting technique. We omit it.

Remark 3.4. The zeros of G are real, and they include the odd multiples of π, denoted $r_{2j}=(2j-1)\pi$, $j=1,2,3,\dots$. In each interval $(2j\pi,(2j+1)\pi)$ $j=0,1,2,\dots,G$ has one zero denoted r_{2j+1}. The other zeros of G are 0 and the opposite of these zeros.

Lemma 3.7. *Suppose that $A<0$, $B>0$, $C<0$, $D<0$, and $-C<-A$. Then necessary conditions for the zero solution of (1.1) to be asymptotically stable are $B+D<0$ and $B<-A-C$.*

Proof. The proof is straightforward from the standard counting technique, and we omit it. □

Remark 3.5. In each interval $((j-1)\pi, j\pi)$ $j=1,2,\dots,G$ has one zero denoted r_j. The other zeros of G are 0 and the opposite of these zeros.

Theorem 3.7. *Suppose that $A<0$, $B>0$, $C<0$, $D<0$, and $-C\leqslant -A$. The zero solution of (1.1) is asymptotically stable if and only if $B+D<0$, $B<-A-C$ and*

1. *if $-C=-A$*

$$F(r_{2j+1})<0, \tag{3.26}$$

where $j=1,2,\dots,r_{2j+1}$ are the positive zeros of G given in Remark 3.4, and

2. *if $-C<-A$*

$$F(r_{2j+1})<0, \tag{3.27}$$

where $j=0,1,2,\dots,r_{2j+1}$ are the positive zeros of G given in Remark 3.5.

Proof. Note that in this case for $-C=-A$, $F((2j+1)\pi)=(2j+1)^2\pi^2+B-D>0$. For the case $-C<-A$ at r_{2j} the value of $F(r_{2j})=Ar_{2j}/\sin r_{2j}(1+(C/A)\cos r_{2j})-D>0$, $j=1,2,\dots$. The remainder of the proof of this theorem is very similar to the proof of Theorem 3.5. We omit it. □

Algorithmic Stability Test V. *Suppose that $A<0$, $C<0$, $B>0$, $D<0$. Moreover assume the necessary conditions of Lemmas 3.6 and 3.7 are satisfied:*

(i) *Suppose $-C=-A$. If $F(r_{2j+1})<0$, $j=0,1,2,\dots,n$, when $[r_{2n}]<\pi/2$, and $-\cos r_{2n}(r_{2n}^2+B)-D<0$, then $F(r_{2l+1})<0$ for all l.*

(ii) *Suppose $-C<-A$. If $F(r_{2j+1})<0$, $j=0,1,2,\dots,n$, when $[r_{2n}]<\pi/2$, and $-\cos r_{2n}(r_{2n}^2+B)-D<0$, then $F(r_{2l+1})<0$ for all l.*

Proof. In both cases $[r_{2l+1}]\downarrow 0$. The rest of the proof obtained easily from Eq. (2.9). □

Lemma 3.8. *Suppose that $A<0$, $B>0$, $C<0$, $D<0$, and $-C>-A$. Then necessary conditions for the zero solution of (1.1) to be asymptotically stable are $B+D<0$, and $B<-A-C$, and G has two zeros in $(2j\pi,(2j+1)\pi)$ for $j=0,1,2,\dots,m+1$ where $\sqrt{B}\in(2m\pi,(2m+2)\pi)$.*

Proof. From Lemma 3.1, $\Delta(0)=(B+D)(A+B+C)>0$, and from Theorem 3.6, it is necessary that $A+B+C<0$ and $B+D<0$. The zeros of G are $y=0$ and the roots of $w(y)=\zeta(y)$ where $w(y)=-A\cot y-C\csc y$ and $\zeta(y)=y+B/y$. When $-C>-A$, $w(y)$ resembles the cosecant function as previously described. With $B>0$, ζ has limit ∞ at 0 taken from inside $(0,\infty)$, is decreasing on $(0,\sqrt{B})$ and is increasing on $(\sqrt{B},\infty)$ and concave upward on $(0,\infty)$, and has asymptote $\xi(y)=y$. Note also that $w(y)=\zeta(y)$ has even number of zeros on each interval $(2l\pi,(2l+1)\pi)$ for $l=0,1,\dots$. We claim that G cannot have four or more zeros in any of these intervals.

Consider the interval $(2l\pi,(2l+1)\pi)$. At every nonzero zero of G, (3.16) holds, and since $-C>-A>0$ (3.18) holds on $(2l\pi,(2l+1)\pi)$ and thus the derivative of the right-hand side of (3.16) is positive on $(2l\pi,(2l+1)\pi)$. In order that interlacing holds, G can thus have at most two zeros in $(2l\pi,(2l+1)\pi)$.

That G has precisely two distinct zeros in each $(2l\pi,(2l+1)\pi)$ follows from the standard zero counting method. □

Remark 3.6. If the necessary conditions of Lemma 3.6 hold, then G has all real zeros. Since ζ is increasing on $(\sqrt{B},\infty)$, G has two zeros in all of the intervals $(2l\pi,(2l+1)\pi)$. The standard counting method yields that these constitute all of the positive zeros of G. The other zeros of G are 0 and the opposite of these zeros.

Theorem 3.8. *Suppose that $A<0$, $B>0$, $C<0$, $D<0$, and $-C>-A$. The zero solution of (1.1) is asymptotically stable if and only if the necessary conditions of Lemma 3.8 hold and*

$$(-1)^k F(r_k)>0,\quad k=1,2,\ldots, \tag{3.28}$$

where $r_1,r_2,\ldots$ are the positive zeros of G given in Remark 3.6.

Proof. The proof of this theorem is similar to the proof of Theorem 3.5. We omit it. □

There are an infinite numbers of conditions in (3.28), and we reduce this to a finite number of conditions.

Algorithmic Stability Test VI. *Suppose that $A<0$, $C<0$, $B>0$, $D<0$ and $-C>-A$. Moreover assume the necessary conditions of Lemmas 3.8 are satisfied:*

(i) *If $F(r_{2j+1})<0$, $j=0,1,2,\ldots,u$ when $[r_{2u+1}]<\pi/2$, $u>m$ and $-(r_{2u+1}^2+B)\cos r_{2u+1}-D<0$ then $F(r_{2l+1})<0$ for all l.*
(ii) *If $F(r_{2j})>0$, $j=0,1,2,\ldots,u$ when $[r_{2u}]>\pi/2$, $u>m$, and $-\cos r_{2u}(r_{2u})+A\sin r_{2u}>0$, then $F(r_{2l})>0$ for all l.*

Proof. Observe that $[r_{2l+1}]\downarrow 0$ and $[r_{2j}]\uparrow\pi$. The rest of the proof obtained easily from Eq. (2.9) for $F(y)$. □

Lemma 3.9. *Assume $A<0$, $B<0$, $C<0$, $D>0$, and $-C>-A$. Then necessary conditions for the zero solution of (1.1) to be asymptotically stable are*

1. $B+D<0$,
2. *if $\sqrt{-B}\in(0,\pi)$, then G has two distinct zeros in $(0,\pi)$ and*
3. *if $\sqrt{-B}\in((2m-1)\pi,(2m+1)\pi)$ for m positive integer, when G has two distinct zeros in $((2j-1)\pi,2j\pi)$, $j=1,2,\ldots,m$ and two distinct zeros in $(2m\pi,(2m+1)\pi)$.*

Proof. From Lemma 3.1 $\Delta(0)=(B+D)(A+B+C)>0$. Since $A+B+C<0$, it is necessary that $B+D<0$. From (2.9) and (2.10)

$$F(y)=-(y^2+B)\cos y+Ay\sin y-D$$

and

$$G(y) = -(y^2 + B)\sin y - Ay\cos y - Cy.$$

The zeros of G are $y=0$ and the roots of $\zeta(y)=w(y)$ where $\zeta(y)=y+B/y$ and $w(y)=-A\cot y - C\csc y$. When $-C>-A$, $w(y)$ resembles the cosecant function as previously described. With $B<0$, ζ has limit $-\infty$ at 0 taken from inside $(0,\infty)$, is increasing and concave downward on $(0,\infty)$, and has asymptote $\xi(y)=y$. Note also that $\zeta(y)<0$ for $0<y<\sqrt{-B}$ and $\zeta(y)>0$ for $y>\sqrt{-B}$.

If $\sqrt{-B}\in(0,\pi)$, then $w(y)=\zeta(y)$ has two or zero roots in $(0,\pi)$ due to the opposite concavities of w and ζ there. If $w(y)=\zeta(y)$ has no roots there, the standard zero counting argument yields that G has $4k$ or fewer zeros in $(-2k\pi+\varepsilon, 2k\pi+\varepsilon)$ for k sufficiently large. Thus G has nonreal zeros. Thus (2) is necessary for the zero solution of (1.1) to be asymptotically stable.

Suppose $\sqrt{-B}\in((2m-1)\pi, 2m\pi]$ for m a positive integer. Consider the interval $((2m-1)\pi, 2m\pi)$. G has an even number of zeros in this interval since w has limit $-\infty$ as y approaches $(2m-1)\pi$ or $2m\pi$ from inside the interval $((2m-1)\pi, 2m\pi)$. If G has four zeros in $((2m-1)\pi, 2m\pi)$, let s be the point in $((2m-1)\pi, 2m\pi)$ where $1+(C/A)\cos y=0$. In the interval $(s, 2m\pi)$ the function w is decreasing while ζ is increasing, and therefore $w=\zeta$ has one zero there. Now in the interval $((2m-1)\pi, s)$, the function G must have three zeros. From (3.16) at any zero r of G, we have that

$$F(r) = \frac{Ar}{\sin r}\left(1+\frac{C}{A}\cos r\right) - D.$$

On the interval $((2m-1)\pi, s)$, $1+(C/A)\cos r<0$, $A\sin r>0$ and $-D<0$. Thus $F(r)<0$, and if G has three zeros in $((2m-1)\pi, s)$, interlacing fails. Similarly in each interval $((2j-1)\pi, 2j\pi)$ for $j=1,2,\ldots,m$, if G has four zeros there, interlacing fails. Thus G must have two zeros there. In addition, in each interval $(2k\pi,(2k+1)\pi)$, for $k=m,m+1,\ldots$ because of opposite concavities of w and ζ G can have two zeros only, and if G has two in $(2m\pi,(2m+1)\pi)$, then due to the monotonicity of ζ, G has two zeros in every $(2k\pi,(2k+1)\pi)$, for $k=m+1,m+2,\ldots$. If G has no zeros in $(2m\pi,(2m+1)\pi)$, then G has at most $4k$ zeros using the standard counting method. Thus conditions (3) is necessary. □

Remark 3.7. If the necessary conditions of Lemma 3.9 hold, then G has all real zeros, using the standard counting method.

If $\sqrt{-B}\in(0,\pi)$, then G has two zeros $r_1<r_2$ in $(0,\pi)$ and two zeros $r_{2j+1}<r_{2j+2}$ in $(2j\pi,(2j+1)\pi)$ for $j=1,2,\ldots$. If $\sqrt{-B}\in[(2m-1)\pi,(2m+1)\pi)$ for m a positive integer, then G has two zeros $r_{2j-1}<r_{2j}$, in $((2j-1)\pi, 2j\pi)$, $j=1,2,\ldots,m$ and two zeros $r_{2m+1}<r_{2m+2}$ in $(2m\pi,(2m+1)\pi)$ and two zeros $r_{2k+1}<r_{2k+2}$ in $(2k\pi,(2k+1)\pi)$ for $k=m+1,m+2,\ldots$.

In all cases, the standard counting method yields that these constitute all of the positive zeros of G. The other zeros of G are 0 and the opposite of these zeros.

Theorem 3.9. *Suppose that $A<0$, $B<0$, $C<0$, $D>0$, and $-C>-A$. The zero solution of* (1.1) *is asymptotically stable if and only if the necessary conditions of Lemma* 3.6 *hold and*

$$(-1)^k F(r_k) > 0,\ k=1,2,\ldots, \tag{3.29}$$

where $r_1, r_2, \ldots$ are the positive zeros of G given in Remark 3.7.

Proof. The proof of this theorem is similar to the proof of Theorem 3.5. We omit it. □

In the next test we will replace the infinite number of conditions (3.29) by finitely many conditions.

Algorithmic Stability Test VII. *Assume that* $A<0$, $B<0$, $C<0$, $D>0$, *and* $-C>-A$ *and that the necessary conditions of Lemma* 3.9 *hold*:

1. *Suppose that* $\sqrt{-B}\in(0,\pi)$.
2. *If* $F(r_{2j+1})<0$, $j=0,1,2,\dots,n$ *when* $[r_{2n+1}]<\pi/2$, *and* $-\cos r_{2n+1}(r_{2n+1}^2+B)-D<0$, *then* $F(r_{2l+1})<0$ *for all* l.
3. *If* $F(r_{2j})>0$, $j=0,1,2,\dots,n$ *when* $[r_{2n}]>\pi/2$, *and* $-\cos r_{2n}(r_{2n})+A\sin r_{2n}>0$, *then* $F(r_{2l})>0$ *for all* l.
4. *Suppose* $\sqrt{-B}\in((2m-1)\pi,(2m+1)\pi)$ *for* m *positive integer and suppose that* $(-1)^jF(r_j)<0$ *for* $j=1,2,\dots,2m+1$.
5. *If* $F(r_{2j+1})<0$, $j=m,m+1,\dots,k$, *and* $r_{2k+1}^2+B>0$, $[r_{2k+1}]<\pi/2$, *and* $-\cos r_{2k+1}(r_{2k+1}^2+B)-D<0$, *then* $F(r_{2l+1})<0$ *for all* l.
6. *If* $F(r_{2l})>0$, $l=m,m+1,\dots,n$ *when* $[r_{2n}]>\pi/2$ *and* $-\cos r_{2n}(r_{2n})+A\sin r_{2n}>0$, *then* $F(r_{2l})>0$ *for all* l *and conditions* (3.29) *are satisfied.*

Proof. The proof is easily obtained from Eq. (2.9) of $F(y)$ and the observation that $[r_{2l}]\uparrow\pi$ and $[r_{2l+1}]\downarrow 0$.

Lemma 3.10. *Suppose that* $A<0$, $B<0$, $C<0$, $D>0$, *and* $-C<-A$. *Then* G *has all real zeros.*

Proof. The proof is straightforward from the standard counting technique, and we omit it.

Remark 3.8. In each interval $((j-1)\pi, j\pi)$ $j=1,2,\dots,G$ has one zero denoted by r_j. The other zeros of G are 0 and the opposite of these zeros.

Theorem 3.10. *Suppose that* $A<0$, $B<0$, $C<0$, $D>0$ *and* $-C<-A$. *The zero solution of* (1.1) *is asymptotically stable if and only if* $B+D<0$ *and*

$$F(r_{2k})>0,\quad k=1,2,\dots, \tag{3.30}$$

where r_k, $k=1,2,\dots$, *are the positive zeros of* G *given in Remark* 3.8.

Proof. The proof of this theorem is very similar to the proof of Theorem 3.5. We omit it. □

In the next test we will replace the infinite number of conditions (3.30) by finitely many conditions.

Algorithmic Stability Test VIII. *Suppose that* $A<0$, $B<0$, $C<0$, $D>0$ *and* $-C<-A$. *If* $F(r_{2l})>0$, *for* $l=0,1,2,\dots,n$, $r_{2n}^2+B>0$, *where* $[r_{2n}]<\pi/2$ *and* $-\cos r_{2n}(r_{2n}^2+B)<D$, *then* $F(r_{2l})>0$ *for all* $l=0,1,\dots$.

Proof. The proof is similar to previous cases, and we omit it. □

Lemma 3.11. *Suppose that* $A<0$, $B<0$, $C<0$, $D>0$, *and* $-C=-A$ *and* $\sqrt{-B}\neq(2n-1)\pi$ *for all* $n=1,2,\dots$. *Then* G *has all real and distinct zeros.*

Proof. Note that if $-C=-A$ then all the odd multiples of π are zeros of G. The other zeros of G are zero and the roots of $w(y)=\zeta(y)$ where $\zeta(y)=y+B/y$ and $w(y)=-A\cot y-C\csc y$. If $\sqrt{-B}$ is not an n odd multiple of π, then $w=\zeta$ has one root in each set $(2k\pi,(2k+1)\pi)\cup((2k+1)\pi,(2k+2)\pi)$ for $k=0,1,2,\ldots$. The rest of the proof is straightforward using the standard counting technique. □

Remark 3.9. If $\sqrt{-B}\in(0,\pi)$, then in each interval $(2j\pi,(2j+1)\pi)$, $j=0,1,2,\ldots,w=\zeta$ has one root denoted by r_{2j+1}. In addition, all odd multiples of π are zeros of G. We also denote $r_{2j}=(2j-1)\pi$, $j=1,2,\ldots$. The other zeros of G are 0 and the opposite of these zeros. If $\sqrt{-B}\in((2m-1)\pi,(2m+1)\pi)$ for a fixed positive integer m, then in each interval $((2j-1)\pi,2j\pi), j=1,2,\ldots,m$, $w=\zeta$ has one root denoted r_{2j}, and in each interval $(2k\pi,(2k+1)\pi)$, $k=m,m+1,\ldots$ $w=\zeta$ has one root r_{2k+1} again the odd multiples of π are zeros of G. We denote $r_{2j-1}=(2j-1)\pi$, $j=1,2,\ldots,m$, and $r_{2k}=(2k+1)\pi$, $k=m,m+1,\ldots$. The other zeros of G are 0 and the opposite of these zeros.

Theorem 3.11. *Suppose that $A<0$, $B<0$, $C<0$, $D>0$, and $-C=-A$. Then the zero solution of (1.1) is asymptotically stable if and only if $B+D<0$, $\sqrt{-B}\neq(2n-1)\pi$ for all $n=1,2,\ldots,$ and*

1. *if $\sqrt{-B}\in(0,\pi)$, then $\pi^2+B-D>0$;*
2. *if $\sqrt{-B}\in[2m\pi,(2m+1)\pi)$ for m a positive integer, then $(2m-1)^2\pi^2+B-D<0$, $(2m+1)^2\pi^2+B-D>0$ and*

$$F(r_{2j})>0,\quad j=1,2,\ldots,m. \tag{3.32}$$

Proof. If $\sqrt{-B}$ is an odd multiple of π, then G would have a double root and so $\Delta=0$ there. If $\sqrt{-B}\in(0,\pi)$, then (3.16) yields that $F(r_{2j+1})<0$ for all $j=0,1,\ldots$. Also $F(r_{2j})=(2j-1)^2\pi^2+B-D$ will be positive for all $j=1,2,\ldots$ if $\pi^2+B-D>0$. The proof of (2) is analogous.

Note that Theorem 3.11 requires checking a finite number of conditions for the roots r_{2j-1}, $j=1,2,\ldots$ in both cases.

4. Examples

The main results of this paper give a complete characterization of the asymptotic stability of the zero solution of Eq. (1.1) when $p_1p_2\geqslant 0$ and $q_1q_2<0$ which has not previously been accomplished. To determine the asymptotic stability one needs to know the values of p_1, p_2, q_1, q_2, and τ. In all cases either the zero solution is not asymptotically stable or one has to determine if the given values are in the region of stability or to check finite number of conditions. In the cases where stability criteria was given via infinite number of conditions, we determine asymptotic stability by using one of the Asymptotically Stability Tests, which are practical and easy to use.

Example 4.1. Consider (1.1)

$$y''(t)=p_1y'(t)+p_2y'(t-\tau)+q_1y(t)+q_2y(t-\tau) \tag{4.1}$$

with

$$A=\tau p_1=0.3,\quad C=\tau p_2=0.6,\quad B=\tau^2q_1=-11,\quad D=\tau^2q_2=1.$$

In this example we apply Algorithmic Stability Test III. We found that $r_1 = 2.84$, $r_2 = 3.67$, and $r_3 = 6.07$. We evaluated $F(r_1) = -3.56$, $F(r_2) = 0.58$ $F(r_3) = -26.68$. Also $[r_3] > \frac{1}{2}\pi$, $-r_2 \cos r_2 + A \sin r_2 = 3.02 > 0$, and $r_2(-r_2 \cos r_2 + A \sin r_2) - D = 10.07 > 0$. By Algorithmic Stability Test III, the zero solution is asymptotically stable. With $\tau = 0$ the zero solution is not asymptotically stable. This is one of the cases where the delay has a stabilizing effect.

Example 4.2. Consider (1.1)

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau) \tag{4.2}$$

with

$$A = \tau p_1 = 0.6, \quad C = \tau p_2 = 0.3, \quad B = \tau^2 q_1 = -2, \quad D = \tau^2 q_2 = 1.$$

In this example, we apply Algorithmic Stability Test IV. We found that G has two zeros in $(0, \pi)$, $r_1 = 1.15$, and $r_2 = 3.02$ and $F(r_1) = -0.09$, $F(r_2) = 6.27$. We also found that $r_3 = 6.13$, $r_4 = 9.39$ and $[r_4] > \pi/2$. Moreover, $-\cos r_4(r_4^2 + B) - D = 85.17 > 0$. By Algorithmic Stability Test IV, the zero solution is asymptotically stable. With $\tau = 0$ the zero solution is not asymptotically stable.

Example 4.3. Consider (1.1)

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau) \tag{4.3}$$

with

$$A = \tau p_1 = 0, \quad C = \tau p_2 = 2, \quad B = \tau^2 q_1 = -3, \quad D\tau^2 q_2 = 1.$$

In this case, G has one zero in $(0, \pi)$ and two zeros in $(\pi, 2\pi)$. The zero in $(0, \pi)$ is $r_1 = 0.49$. We have $F(r_1)$=0.49 and since $F(r_1) > 0$. By Lemma 3.3 the zero solution of (1.1) is not asymptotically stable. Without delay the zero solution is also not asymptotically stable since $C > 0$.

Example 4.4. Consider (1.1)

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau) \tag{4.4}$$

with

$$A = \tau p_1 = 0, \quad C = \tau p_2 = 1, \quad B = \tau^2 q_1 = -3, \quad D = \tau^2 q_2 = 1.$$

In this case G also has one zero $r_1 = 1.29$, but in this example $F(r_1) = -0.63 < 0$. Here we continue to find more zeros of G. We found $r_2 = 3.52$ and $F(r_2) = 7.74$, $r_3 = 6.10$ and $F(r_3) = -34.71$, $r_4 = 9.53$ and $F(r_4) = 86.37$, $r_5 = 12.48$ and $F(r_5) = -153.35$. It is clear that $[r_5] > \frac{3}{2}\pi$ and $\pi < [r_6] < \frac{3}{2}\pi$ and by Algorithmic Stability Test I the zero solution of (1.1) is asymptotically stable. With $\tau = 0$ the zero solution is not asymptotically stable.

One can write the second-order delay differential equation as a system of first-order delay differential equations. Since in all the examples the zero solution is not asymptotically for $\tau = 0$ the results for systems given by Yuanhong [26] cannot be applied. Also the results given in [8] cannot easily be applied to our examples. Also the results in [13] cannot be applied (the spectrum is not negative). The results in [6] are not algorithmic type and are not applied to our examples. Since $p_2 \neq 0$ the results in [22] cannot be applied. As a system we also cannot applied the results in [3] because of the type of the system. Since $A + C > 0$ the results in [11] also cannot be applied.

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