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Journal of Functional Analysis 259 (2010) 2238–2252

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Turbulence and Araki–Woods factors

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Received 8 February 2010; accepted 23 June 2010

Available online 17 July 2010

Communicated by S. Vaes

Abstract

Using Baire category techniques we prove that Araki–Woods factors are not classifiable by countable structures. As a result, we obtain a far reaching strengthening as well as a new proof of the well-known theorem of Woods that the isomorphism problem for ITPFI factors is not smooth. We derive as a consequence that the odometer actions of $\mathbb{Z}$ that preserve the measure class of a finite non-atomic product measure are not classifiable up to orbit equivalence by countable structures.

Keywords: Von Neumann algebras; Classification of factors; Descriptive set theory; Borel reducibility; Turbulence

1. Introduction

The present paper continues a line of research into the structure of the isomorphism relation for separable von Neumann algebras using techniques from descriptive set theory, which was initiated in [21] and [20].

The central notion from descriptive set theory relevant to this paper is that of *Borel reducibility*. Recall that if E and F are equivalence relations on Polish spaces X and Y, respectively, we say that E is *Borel reducible* to F if there is a Borel function $f : X \to Y$ such that

$$(\forall x,\ x' \in X)\ xEx' \iff f(x)Ff\left(x'\right),$$

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doi:10.1016/j.jfa.2010.06.018

and if this is the case we write $E \leqslant_B F$. Borel reducibility is a notion of relative complexity of equivalence relations and the isomorphism problems they pose, and the statement $E \leqslant_B F$ is interpreted as saying that the points of X are classifiable up to E-equivalence by a Borel assignment of complete invariants that are F-equivalence classes. The requirement that f be Borel is a natural restriction to ensure that the invariants are assigned in a reasonably definable way. Without a definability condition on the function f reducibility would amount only to a consideration of the cardinality of the quotient spaces X/E and Y/F.

In [21] it was shown that the isomorphism relation in all the natural classes of separable von Neumann factors, II_1, II_∞, III_λ, $(0 \leqslant \lambda \leqslant 1)$ do not admit a classification by countable structures. That is, if $\mathcal{L}$ is a countable language and $\mathrm{Mod}(\mathcal{L})$ is the natural Polish space of countable $\mathcal{L}$-structures (see [10, §2.3]), then there is no Borel reduction of the isomorphism relation of von Neumann factors of any fixed type to the isomorphism relation $\simeq^{\mathrm{Mod}(\mathcal{L})}$ in $\mathrm{Mod}(\mathcal{L})$. This in particular implies that there is no Borel assignment of countable groups, graphs, fields or orderings as complete invariants for the isomorphism problem for factors.

Recently, Kerr, Li and Pichot in [16] obtained several non-classification results along the same lines exhibited here but for the automorphism groups of finite factors. For instance they showed that the conjugacy relation for the trace-preserving free weakly mixing actions of discrete groups on a II_1 factor is not classifiable by countable structures.

These types of results are much stronger than the classical smooth/non-smooth dichotomy, since they give specific information about the complexity of the kind of invariant that can be used in a complete classification. They are also stronger than the traditional smooth/non-smooth dichotomy for equivalence relation, since, for instance, isomorphism of countable groups is *not smooth*, yet in many cases countable groups are reasonable invariants.

The earliest non-smoothness result for the isomorphism relation of von Neumann algebras is Woods' Theorem [24], which asserts that the isomorphisms relation for ITPFI_2 factors is not smooth. Recall that a von Neumann algebra M is called an Araki–Woods factor or an ITPFI factor (short for *infinite tensor product of factors of type* I) if it is of the form

$$M = \bigotimes_{k=1}^{\infty} \bigl(M_{n_k}(\mathbb{C}), \phi_k\bigr)$$

where $M_{n_k}(\mathbb{C})$ denotes the algebra of $n_k \times n_k$ matrices and the ϕ_k are faithful normal states. In the case when $n_k = 2$ for all k, the factor M is called ITPFI_2. In this paper we will show:

Theorem 1.1. *The isomorphism relation for* ITPFI_2 *factors is not classifiable by countable structures.*

This solves a problem posed in [20], and provides a strengthening and a new proof of Woods' Theorem. It also provides a new and more direct proof that the isomorphism relation for injective type III_0 factors is not classifiable by countable structures, a result proven in [21] using Krieger's Theorem regarding the duality between flows and injective factors, [18]. Results of Krieger [17] and Connes and Woods [6] show that not all injective factors are ITPFI factors. Thus a natural question to ask was whether ITPFI factors are "simpler" objects to classify from the point of view of Borel reducibility. The results of this article show that even for this elementary class of von Neumann factors, the classification problem is too complicated to be distinguished by countable structures invariants, which might be surprising given the simplicity of their construction.

The ITPFI factors constructed in the proof of Theorem 1.1 correspond to group-measure space factors constructed from the measure-class preserving odometer actions of $\mathbb{Z}$ on $\{0,1\}^{\mathbb{N}}$, when $\{0,1\}^{\mathbb{N}}$ is equipped with a finite product measure. Therefore we obtain the following interesting corollary:

Theorem 1.2. *The odometer actions of $\mathbb{Z}$ on $\{0,1\}^{\mathbb{N}}$ preserving the measure class of a finite non-atomic ergodic product measure are not classifiable, up to orbit equivalence, by countable structures.*

This stands in contrast to Dye's Theorem for probability measure preserving actions, and may be compared with the theorem of Ioana, Kechris, Tsankov and Epstein in [12] on the non-classifiability up to orbit equivalence of probability measure preserving ergodic actions of a countable non-amenable group.

2. A turbulence lemma

In this section we establish a general lemma which shows that a wide class of natural actions are turbulent, in the sense of [10]. Recall that if G is a Polish group acting continuously on a Polish space X, then the action is said to be turbulent if the following holds[1]:

For all $x, y \in X$, all open $U \subseteq X$ with $x \in U$ and all open $V \subseteq G$ containing the identity, there is $y_0 \in U$ in the G-orbit of y, such that for all neighbourhoods U_0 of y_0 there is a finite sequence $x_i \in U$ $(0 \leqslant i \leqslant n)$ with $x_0 = x$ and a sequence $g_i \in V$ $(0 \leqslant i < n)$, such that

$$x_{i+1} = g_i \cdot x_i$$

and $x_n \in U_0$.

Recall moreover that a *Fréchet space* is a completely metrizable locally convex vector space (over $\mathbb{R}$ or $\mathbb{C}$).

Lemma 2.1. *Let F be a separable Fréchet space and let $G \subseteq F$ be a dense subgroup of the additive group $(F,+)$. Suppose $(G,+)$ has a Polish group topology such that the inclusion map $i : G \to F$ is continuous, and satisfies*

(∗) *for all $g \in G$ and open $V \subseteq G$ with $0 \in V$ there is $n \in \mathbb{N}$ such that $\frac{1}{n}g \in V$*

(*e.g. when G itself is a Fréchet space*). *Then either $G = F$ or the action of G on F by addition,*

$$g \cdot x = g + x,$$

is turbulent and has meagre dense classes.

In particular, if $(G, \|\cdot\|_G)$ and $(F, \|\cdot\|_F)$ are separable Banach spaces such that G is a dense subspace of F and the inclusion map $i : G \to F$ is bounded, then either $G = F$ or the action of $(G,+)$ on F by addition is turbulent and has meagre dense classes.

[1] Strictly speaking, Hjorth required a turbulent action to have dense, meagre orbits, but we keep those requirements separate.

Proof. Note that G is an analytic subset of F since $i : G \to F$ is continuous, and so it has the Baire Property in F (see [13, 8.21 and 21.6]). So if $G \neq F$ then G must be meagre in F, since otherwise by Pettis' Theorem [13, 9.9] G must contain a neighbourhood of the identity, and so $G = F$. Since G is dense in F it follows that all the G-orbits are meagre and dense. So it suffices to show that the action of G is turbulent. For this, let $x \in F$ and let $U \subseteq F$ be a convex open neighbourhood of F. Let $y \in U$, let U_0 be an open neighbourhood of y such that $y \in U_0 \subseteq U$, and let $V \subseteq G$ be a neighbourhood of 0 in G. Since G is dense in F we may find $g \in G$ such that $x + g \in U_0$. By assumption there is $n \in \mathbb{N}$ such that $\frac{g}{n} \in V$, and since U is convex we have

$$x, x + \frac{1}{n}g, x + \frac{2}{n}g, \ldots, x + \frac{n-1}{n}g, x + g \in U,$$

which shows that G acts turbulently. □

Remark 2.2. A version of Lemma 2.1 was already noted by Kechris in [14]. Many turbulence results found in the literature are special instances of the above lemma. For instance, let $(c_0, \|\cdot\|_\infty)$ denote the real Banach space of real valued sequences that converge to zero, equipped with the sup-norm. The elementary example [10, 3.23] that c_0 acts turbulently on $\mathbb{R}^{\mathbb{N}}$ by addition fits into this framework. Moreover, condition $(*)$ may be replaced by the weaker condition

$(**)$ for all $g \in G$ and $W, V \subseteq G$ open neighbourhoods such that $g \in W$ and $0 \in V$, there is $z \in V$ and $n \in \mathbb{N}$ such that $nz \in W$.

in which case [10, Proposition 3.25] also follows from the above.

The results in [22] also fall into this category. Indeed, let X be a locally compact not compact Polish space. The space $C(X, \mathbb{R}) = \{f : X \to \mathbb{R} \text{ continuous}\}$ is a separable Fréchet space with the topology given by uniform convergence in compact sets. $C_0(X, \mathbb{R}) = \{f \in C(X, \mathbb{R}):\ \lim_{x\to\infty} f(x) = 0\}$ is a dense subspace of $C(X, \mathbb{R})$ and it is Polish in the topology given by uniform convergence. It follows from the previous lemma that the natural action of $C_0(X, \mathbb{R})$ on $C(X, \mathbb{R})$ is turbulent, has meagre classes and every class is dense. The exponential map then gives [22, Theorem 1.1]. In a similar fashion one could also recover [22, Theorem 1.2].

S_∞-ergodicity and turbulence. Let S_∞ denote the group of all permutations of $\mathbb{N}$. The importance of the notion of turbulence comes from its relation to the notion of *S_∞-ergodicity*. Recall that an equivalence relation E on a Polish space X is said to be *generically S_∞-ergodic* if whenever S_∞ acts continuously on a Polish space Y giving rise to the orbit equivalence relation $E^Y_{S_\infty}$, and $f : X \to Y$ is a Baire measurable function such that

$$(\forall x, x')\ xEx' \implies f(x) E^Y_{S_\infty} f(x')$$

(i.e. f is a homomorphism of equivalence relations), then there is a single S_∞-orbit $[y]_{E^Y_{S_\infty}}$ such that

$$\{x \in X:\ f(x) \in [y]_{E^Y_{S_\infty}}\}$$

is comeagre. Note that if $E \subseteq E'$, where E' is also an equivalence relation, and E is generically S_∞-ergodic, then so is E'.

The fundamental theorem in Hjorth's theory of turbulence is that if a Polish group G acts continuously and turbulently on a Polish space X, then the associated orbit equivalence relation E_G^X is generically S_∞-ergodic, see [10, Theorem 3.18]. Since the isomorphism relation $\simeq^{\mathrm{Mod}(\mathcal{L})}$ in the Polish space $\mathrm{Mod}(\mathcal{L})$ of countable $\mathcal{L}$-structures is induced by a continuous S_∞-action, turbulence provides an obstruction to Borel reducibility of E_G^X to $\simeq^{\mathrm{Mod}(\mathcal{L})}$ if the G-orbits are meagre. The isomorphism relations for countable groups, graphs, fields, orderings, etc., are special instances of $\simeq^{\mathrm{Mod}(\mathcal{L})}$ for appropriate choices of the language $\mathcal{L}$, and so turbulence can be used to prove the impossibility of obtaining a complete classification by a reasonable (i.e. Borel or Baire measurable) assignment of such countable objects as invariants.

3. Proof of the main theorem

We will now define a family $(M_x)_{x\in c_0}$ of ITPFI factors parameterized by elements of c_0. The family is chosen with great care so that it will be possible to prove for each $x \in c_0$, the set

$$\{y \in c_0\colon\ T(M_y) = T(M_x)\}$$

is meagre, and so in particular, isomorphism of the M_x is meagre in the parameter $x \in c_0$. The motivation behind the definition can be traced back to the results in [9]. For $j \in \mathbb{N}$ define $N_j = 2^{j!}$, and for each $x \in c_0$ let

$$l_j^x = \ln(2)\, j!\, e^{x(j)/j!}.$$

Let ϕ_j^x be the state on $M_2(\mathbb{C})$ given by

$$\phi_j^x(a) = \frac{1}{1+e^{-l_j^x}} \operatorname{Tr}\left(a \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-l_j^x} \end{bmatrix}\right).$$

Then we define M_x to be the ITPFI_2 factor

$$M_x = \bigotimes_{j=1}^{\infty} \big(M_2(\mathbb{C}), \phi_j^x\big)^{\otimes N_j}.$$

In other words, M_x is the ITPFI_2 factor with eigenvalue list $(\lambda_n^x, 1-\lambda_n^x)_{n\in\mathbb{N}}$ where λ_n^x is given by

$$\lambda_n^x = \frac{1}{1+e^{-l_j^x}}$$

whenever $\sum_{i=1}^{j-1} N_i < n \leqslant \sum_{i=1}^{j} N_i$ for some $j \in \mathbb{N}$. Since $l_j^x \to \infty$ and $\sum_j N_j e^{-l_j^x} = \infty$, all the factors M_x are of type III, [3, III.4.6.6].

Theorem 1.1 will be proved by showing that the family of factors $(M_x)_{x\in c_0}$ is not classifiable up to isomorphism by countable structures. An outline of the proof is as follows: First we will show that the equivalence relation

$$x \sim_{\mathrm{iso}} x' \iff M_x \text{ is isomorphic to } M_{x'}$$

has meagre classes, thus showing that the family (M_x) contains uncountably many non-isomorphic factors. Then we will show that there is a subgroup $G \subseteq c_0$ of the additive group $(c_0, +)$ that satisfies the hypothesis of Lemma 2.1, and with the additional property that

$$M_{g+x} \simeq M_x$$

for all $g \in G$ and $x \in c_0$, where $\simeq$ denotes isomorphism of von Neumann algebras. From this fact it will be easy to deduce that the equivalence relation $\sim_{\mathrm{iso}}$ is not classifiable by countable structures. Finally we will show that the map $x \mapsto M_x$ is Borel (in a precise way) and thus provides a Borel reduction of $\sim_{\mathrm{iso}}$ to $\simeq$.

The main tool used to distinguish uncountably many non-isomorphic elements of the family $(M_x)_{x\in c_0}$ is Connes' invariant $T(M)$. Recall that if M is a von Neumann algebra with a faithful semifinite normal weight φ, the Tomita–Takesaki theory associates to it a one parameter group of automorphisms of M, the so-called *modular automorphism group*. If σ_t^φ denotes the modular automorphism group of (M, φ), the T-set of M is the additive subgroup of $\mathbb{R}$ defined by

$$T(M) = \bigl\{t \in \mathbb{R}\colon\ \sigma_t^\varphi \text{ is an inner automorphism}\bigr\}.$$

Even though σ_t^φ depends on φ, Connes' non-commutative Radon–Nikodym Theorem guarantees that $T(M)$ is independent of the choice of the faithful semifinite normal weight φ. The T-set is arguably the most important invariant employed to distinguish injective type III_0 factors and it can be found already in Araki and Woods's seminal article [1]. A thorough treatment of these important concepts that are at the heart of the structural theory of factors of type III can be found in [5, 5.3–5.5], [3, III.3, III.4] and [23]. For the purpose of this article we will only need the following lemma.

Lemma 3.1. *(See [4, Corollaire 1.3.9].) If M is an* ITPFI_2 *factor with eigenvalue list* $(\lambda_n, 1-\lambda_n)$ *then the T-set is given by the formula*

$$T(M) = \left\{t \in \mathbb{R}\colon\ \sum_{n=1}^{\infty}\bigl(1 - \bigl|\lambda_n^{1+it} + (1-\lambda_n)^{1+it}\bigr|\bigr) < \infty\right\}.$$

The following slightly abusive notation is convenient in this paper: For a real $s \in \mathbb{R}$, write $s \pmod{2\pi}$ for the unique element of

$$\{s + 2\pi p\colon\ p \in \mathbb{Z}\} \cap (-\pi, \pi].$$

For $x \in c_0$ and $t \in \mathbb{R}$ define

$$\delta_j^x(t) = t l_j^x \pmod{2\pi}.$$

When the value of t is clear from the context we will usually write δ_j^x for $\delta_j^x(t)$.

The next lemma is stated only for the family $(M_x)_{x\in c_0}$, but is a special case of a well-known consequence of Lemma 3.1 which has been observed in many places in the literature (see e.g. [9] and [1]). We include its proof for the sake of completeness.

Lemma 3.2. *For each $t \in \mathbb{R}$, $t \in T(M_x)$ iff*

$$\sum_{j=1}^{\infty} N_j e^{-l_j^x} \big(\delta_j^x(t)\big)^2 < \infty.$$

Proof. By Lemma 3.1,

$$\begin{aligned}
t \in T(M_x) \quad &\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j \left(1 - \left| \left(\frac{1}{1+e^{-l_j^x}}\right)^{1+it} + \left(\frac{e^{-l_j^x}}{1+e^{-l_j^x}}\right)^{1+it} \right| \right) < \infty \\
&\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j \left(1 - \frac{1}{1+e^{-l_j^x}} \big|1 + e^{-l_j^x} e^{-il_j^x t}\big| \right) < \infty \\
&\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j \left(1 - \frac{1}{1+e^{-l_j^x}} \big|1 + e^{-l_j^x} e^{-i\delta_j^x(t)}\big| \right) < \infty \\
&\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j e^{-l_j^x} \big(1 - \cos\big(\delta_j^x(t)\big)\big) < \infty \\
&\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j e^{-l_j^x} \Big(\big(\delta_j^x(t)\big)^2 + O\big(\big(\delta_j^x(t)\big)^4\big)\Big) < \infty \\
&\Longleftrightarrow \quad \sum_{j=1}^{\infty} N_j e^{-l_j^x} \big(\delta_j^x(t)\big)^2 < \infty. \qquad \square
\end{aligned}$$

Remark 3.3. The previous lemma sheds light on the motivation behind the definition of the family $(M_x)_{x \in c_0}$. Indeed, since $N_j e^{-l_j^x} = 2^{j!(1-e^{x(j)/j!})}$ goes to 1 when $j \to \infty$, then to control the sum $\sum_{j=1}^{\infty} N_j e^{-l_j^x} (\delta_j^x(t))^2$, and thus the T-set, it will be enough to control the size of $\delta_j^x(t)$. This fact is what we will exploit in the next two lemmas.

Lemma 3.4. *For each $x \in c_0$, $T(M_x) \neq \{0\}$.*

Proof. Define

$$t = \frac{1}{\ln 2} \sum_{j=1}^{\infty} \frac{a(j)}{j! e^{x(j)/j!}}$$

where $a(j) \in (0, 3\pi]$ is defined recursively by letting $a(1) = 1$ and in general for $j > 1$,

$$a(j) = \left[-\sum_{k=1}^{j-1} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} \right] \pmod{2\pi} + 2\pi.$$

Then $0 < t < \infty$ and we have

$$\begin{aligned} l_j^x t &= \sum_{k=1}^{\infty} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} \\ &= \sum_{k=1}^{j-1} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} + a(j) + \sum_{k=j+1}^{\infty} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} \end{aligned}$$

and so

$$\delta_j^x(t) = \sum_{k=j+1}^{\infty} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} \pmod{2\pi}.$$

If j is large enough so that for all $k \geqslant j$ we have $1/2 \leqslant e^{x(k)/k!} \leqslant 2$ then

$$\begin{aligned} 0 \leqslant \sum_{k=j+1}^{\infty} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} &\leqslant 12\pi \sum_{k=j+1}^{\infty} \frac{j!}{k!} \\ &\leqslant 12\pi \sum_{k=1}^{\infty} \frac{1}{(j+1)^k} \\ &= \frac{12\pi}{j}. \end{aligned}$$

Hence for j sufficiently large it holds that

$$\delta_j^x(t) = \sum_{k=j+1}^{\infty} j! e^{x(j)/j!} \frac{a(k)}{k! e^{x(k)/k!}} \sim \frac{1}{j}$$

and so by (†) and Lemma 3.2 we have $t \in T(M_x)$. □

Lemma 3.5. *For each $t \in \mathbb{R} \setminus \{0\}$ the set $\{x \in c_0\colon\ t \notin T(M_x)\}$ is a dense G_δ subset of c_0.*

Proof. Since $T(M_x)$ is a subgroup of $(\mathbb{R}, +)$, we may assume that $t > 0$. For each $K \in \mathbb{N}$ let

$$A_K = \left\{ x \in c_0\colon\ (\exists L \in \mathbb{N}) \sum_{j=1}^{L} N_j e^{-l_j^x} \left(\delta_j^x\right)^2 > K \right\}.$$

The set A_K is open since for each $j \in \mathbb{N}$ the function $x \mapsto (\delta_j^x)^2$ is continuous. By Lemma 3.2 we have

$$\left\{x \in c_0\colon\ t \notin T(M_x)\right\} = \bigcap_{K \in \mathbb{N}} A_K$$

so it suffices to show that $\bigcap_{K \in \mathbb{N}} A_K$ is dense. Let $y \in c_0$ and $\varepsilon > 0$. Pick $j_0 \in \mathbb{N}$ such that for all $j > j_0$,

$$\left|y(j)\right| < \frac{\varepsilon}{2}$$

and

$$\frac{1}{t\ln(2)\sqrt{j_0}} < \frac{\varepsilon}{2}.$$

For $j \leqslant j_0$ define $x(j) = y(j)$. For $j > j_0$ define $x(j) = j!\ln(1 + \frac{a(j)}{t\ln(2)j!\sqrt{j}})$, where $a(j)$ is defined according to the following rule:

$$a(j) = \begin{cases} 0 & \text{if } |t\ln(2)j! \pmod{2\pi}| \geqslant \frac{1}{2\sqrt{j}}, \\ 1 & \text{if } |t\ln(2)j! \pmod{2\pi}| < \frac{1}{2\sqrt{j}}. \end{cases}$$

It is clear that if $j > j_0$ then

$$0 \leqslant x(j) \leqslant j!\frac{a(j)}{t\ln(2)j!\sqrt{j}} \leqslant \frac{1}{t\ln(2)\sqrt{j}} < \frac{\varepsilon}{2},$$

so $x \in c_0$ and $\|x - y\|_\infty < \varepsilon$. On the other hand we have that

$$tl_j^x = t\ln(2)j!e^{x(j)/j!} = t\ln(2)j!\left(1 + \frac{a(j)}{t\ln(2)j!\sqrt{j}}\right) = t\ln(2)j! + \frac{a(j)}{\sqrt{j}}.$$

By the choice of $a(j)$ we have $|\delta_j^x| = |tl_j^x \pmod{2\pi}| \geqslant \frac{1}{2\sqrt{j}}$. It follows that

$$\sum_{j=1}^{\infty} N_j e^{-l_j^x}\left(\delta_j^x\right)^2 \geqslant \sum_{j=j_0+1}^{\infty} 2^{j!(1-e^{x(j)/j!})}\left(\frac{1}{2\sqrt{j}}\right)^2 = \infty,$$

which shows that $x \in \bigcap_{K\in\mathbb{N}} A_K$. $\square$

Recall that the equivalence relation $\sim_{\text{iso}}$ in c_0 is defined by $x \sim_{\text{iso}} x' \iff M_x \simeq M_{x'}$. For $x \in c_0$ let $[x]_{\sim_{\text{iso}}} = \{y \in c_0\colon\ y \sim_{\text{iso}} x\}$.

Lemma 3.6. *For each $x \in c_0$, $[x]_{\sim_{\text{iso}}}$ is meagre.*

Proof. By Lemma 3.4 there exists $t_0 \in T(M_x) \setminus \{0\}$. Then

$$[x]_{\sim_{\text{iso}}} \subseteq \left\{y \in c_0\colon\ t_0 \in T(M_y)\right\}$$

and so $[x]_{\sim_{\text{iso}}}$ is meagre by Lemma 3.5. $\square$

Remark 3.7. It will be shown below that, in a precise way, $x \mapsto M_x$ is Borel. By (the proof of) [8, Theorem 2.2] the isomorphism relation $\simeq$ is analytic (see also [21, Corollary 15].) It follows that $\sim_{\text{iso}}$ is analytic, and so by the Kuratowski–Ulam Theorem [13, 8.41] we get from Lemma 3.6 that $\sim_{\text{iso}}$ is meagre as a subset of $c_0 \times c_0$.

We will need the following fact, a proof of which may be found in [1, Lemma 2.13], see also [4, Lemme 1.3.8].

Proposition 3.8. *If M_1 and M_2 are* ITPFI_2 *factors with eigenvalue lists* $(\lambda_{n,1}, 1-\lambda_{n,1})_{n\in\mathbb{N}}$ *and* $(\lambda_{n,2}, 1-\lambda_{n,2})_{n\in\mathbb{N}}$, *respectively, and*

$$\sum_{n=1}^{\infty}\bigl((\lambda_{n,1})^{\frac{1}{2}}-(\lambda_{n,2})^{\frac{1}{2}}\bigr)^2+\bigl((1-\lambda_{n,1})^{\frac{1}{2}}-(1-\lambda_{n,2})^{\frac{1}{2}}\bigr)^2<\infty,$$

then M_1 and M_2 are unitarily isomorphic (*and so they are isomorphic*).

Remark 3.9. Denote by $c_{00}\subseteq c_0$ the set of all eventually zero sequences. Then c_{00} acts continuously on c_0 by addition. By Proposition 3.8 it follows easily that if $g\in c_{00}$ then

$$M_{g+x}\simeq M_x$$

for all $x\in c_0$. Thus the action of c_{00} on c_0 preserves $\sim_{\mathrm{iso}}$. Since $\sim_{\mathrm{iso}}$ is meagre in $c_0\times c_0$ and clearly c_{00}-orbits are dense, we can now apply [2, Theorem 3.4.5], by which it follows that $E_0\leqslant_B\sim_{\mathrm{iso}}$. Here E_0 denotes the equivalence relation in $\{0,1\}^{\mathbb{N}}$ defined by

$$x E_0 y \iff (\exists N)\ (\forall n\geqslant N)\ x(n)=y(n).$$

Below we will show that the assignment $x\mapsto M_x$ is Borel, and so it follows that E_0 is Borel reducible to isomorphism of ITPFI_2 factors. Since E_0 is not smooth this provides a new proof of the following:

Theorem. *(See Woods [24].) E_0 is Borel reducible to isomorphism of* ITPFI_2 *factors. In particular the isomorphism relation for* ITPFI_2 *factors is not smooth.*

Arguably the proof exhibited here is simpler than the argument given in [24], partly because we avoid to construct an explicit Borel reduction from E_0 to isomorphism of ITPFI_2 factors that made Woods' original proof quite involved. Observe that since c_{00} doesn't admit a Polish group structure, Hjorth's theory of turbulence does not apply to its actions. In what follows we will overcome this difficulty by defining a group G that can play the role of c_{00}, but which is also Polish. Specifically, consider the set

$$G=\left\{a\in c_0\colon \sum_{j=1}^{\infty}2^{j!}a(j)^2<\infty\right\}.$$

The set G becomes a separable real Hilbert space when equipped with the inner product given by $\langle a,b\rangle=\sum_{j=1}^{\infty}2^{j!}a(j)b(j)$. Since $c_{00}\subset G$, it follows that G is dense in c_0. Moreover, since $G\neq c_0$, by Lemma 2.1 the action of G on c_0 by addition is turbulent, has meagre classes and all the classes are dense. The following lemma shows that the G-action on c_0 preserves the $\sim_{\mathrm{iso}}$ classes:

Lemma 3.10. *If $a\in G$, then M_x is unitarily equivalent to M_{a+x}. In particular, $x\sim_{\mathrm{iso}}(a+x)$.*

Proof. By Proposition 3.8 it is enough to check that the sum

$$\sum_{j=1}^{\infty} N_j\left\{\left[\left(\frac{1}{1+e^{-l_j^x}}\right)^{\frac{1}{2}}-\left(\frac{1}{1+e^{-l_j^{a+x}}}\right)^{\frac{1}{2}}\right]^2+\left[\left(\frac{e^{-l_j^x}}{1+e^{-l_j^x}}\right)^{\frac{1}{2}}-\left(\frac{e^{-l_j^{a+x}}}{1+e^{-l_j^{a+x}}}\right)^{\frac{1}{2}}\right]^2\right\}$$

is finite. Since the derivatives of the functions $f(s) = (1+e^{-s})^{-\frac{1}{2}}$ and $h(s) = (\frac{e^{-s}}{1+e^{-s}})^{\frac{1}{2}}$ are bounded by 1 whenever $s > 0$, the previous sum is bounded by

$$\begin{aligned}\sum_{j=1}^{\infty} 2N_j\left[l_j^x - l_j^{a+x}\right]^2 &= 2\ln^2(2)\sum_{j=1}^{\infty} 2^{j!}(j!)^2 e^{2x(j)/j!}\left[1-e^{a(j)/j!}\right]^2\\ &< K\sum_{j=1}^{\infty} 2^{j!}(j!)^2\left[\left(a(j)/j!\right)^2 + O\left(a(j)/j!\right)^3\right]\\ &< \tilde{K}\sum_{j=1}^{\infty} 2^{j!}a(j)^2\end{aligned}$$

for appropriate constants K and $\tilde{K}$, and this is finite whenever $a \in G$. □

Theorem 3.11. *The equivalence relation $\sim_{\mathrm{iso}}$ is generically S_∞-ergodic, and $\sim_{\mathrm{iso}}$ is not classifiable by countable structures.*

Proof. Let G be as above, and let $E_G^{c_0}$ denote the orbit equivalence relation induced by the action of G on c_0. Then by Lemma 3.10 we have $E_G^{c_0} \subseteq \sim_{\mathrm{iso}}$. Since G acts turbulently, it follows by [10, Theorem 3.18] that $E_G^{c_0}$ is generically S_∞-ergodic, and so as noted in the discussion of S_∞-ergodicity in §2, $\sim_{\mathrm{iso}}$ is generically S_∞-ergodic.

Suppose, seeking a contradiction, that S_∞ acts continuously on the Polish space Y and

$$\sim_{\mathrm{iso}} \leqslant_B E_{S_\infty}^Y.$$

If $f : c_0 \to Y$ were a Borel reduction witnessing this then f would map a comeagre set in c_0 to the same S_∞-class. But this would contradict that all $\sim_{\mathrm{iso}}$ classes are meagre by Lemma 3.6, and so f can't be a reduction. Hence $\sim_{\mathrm{iso}}$ is not classifiable by countable structures. □

Remark 3.12. It follows from Theorem 3.11 that the set

$$\left\{x \in c_0\colon\ T(M_x) \text{ is uncountable}\right\}$$

is comeagre in c_0, since otherwise the assignment $x \mapsto T(M_x)$ would give an $\sim_{\mathrm{iso}}$-invariant assignment of countable subsets of $\mathbb{R}$ on a comeagre set, and so by [10, Lemma 3.14] the function $x \mapsto T(M_x)$ would be constant on a comeagre set. But this contradicts Lemma 3.4 and 3.5.

It follows from the above and Lemma 3.4 that

$$\{x \in c_0\colon\ M_x \text{ is of type } \mathrm{III}_0\}$$

is comeagre, as it should be, since ITPFI factors of type III_λ, $0 < \lambda \leqslant 1$, are classified by a single real number λ. It actually follows from [9, Proposition 1.3] that for all $x \in c_0$, $T(M_x)$ is uncountable, thus M_x is of type III_0, but using an entirely different line of argument.

Recall from [21] and [20] that if $\mathcal{H}$ is a separable complex Hilbert space, then $\mathrm{vN}(\mathcal{H})$ denotes the standard Borel space of von Neumann algebras acting on $\mathcal{H}$, equipped with the Effros Borel structure originally introduced in [7] and [8]. Let $\simeq^{\mathrm{vN}(\mathcal{H})}$ denote the isomorphism relation in $\mathrm{vN}(\mathcal{H})$.

Theorem 3.13. *The isomorphism relation for* ITPFI_2 *factors is not classifiable by countable structures.*

Proof. It suffices to show that there is a Borel function $f : c_0 \to \mathrm{vN}(\ell^2(\mathbb{N}))$ such that for all $x \in c_0$ we have $f(x) \simeq M_x$, since then by Theorem 3.11 it follows that $\sim_{\mathrm{iso}} \leqslant_B \simeq^{\mathrm{vN}(\ell^2(\mathbb{N}))}$. That such a function f exists follows from the next three lemmas. $\square$

Lemma 3.14. *Suppose X is a standard Borel space and $(H_x\colon x \in X)$ is a family of infinite dimensional separable Hilbert spaces, and that $(e_n^x)_{n\in\mathbb{N}}$ is an orthonormal basis of H_x for each $x \in X$. Suppose further that Y is a standard Borel space and $(T_y^x\colon x \in X,\ y \in Y)$ is a family of operators such that $T_y^x \in \mathcal{B}(H_x)$ for all $y \in Y$, $x \in X$ and that the functions*

$$X \times Y \to \mathbb{C} : (x, y) \mapsto \langle T_y^x e_n^x, e_m^x \rangle$$

are Borel for all n, m. Then there is a Borel function $\theta : X \times Y \to \mathcal{B}(\ell^2(\mathbb{N}))$ and a family $(\varphi_x\colon x \in X)$ such that

(1) *$\varphi_x \in \mathcal{B}(H_x, \ell^2(\mathbb{N}))$ satisfies $\varphi_x(e_n^x) = e_n$, where $(e_n)_{n\in\mathbb{N}}$ is the standard basis for $\ell^2(\mathbb{N})$.*
(2) *For all $x \in X$, $y \in Y$ and $\xi \in H_x$ we have $\theta(x, y)(\varphi_x(\xi)) = \varphi_x(T_y^x(\xi))$.*

Moreover, if M_x is the von Neumann algebra generated by the family $(T_y^x\colon y \in Y)$, and there are Borel functions

$$\psi_n : X \to Y$$

such that $(T_{\psi_n(x)}^x\colon n \in \mathbb{N})$ generates M_x for each $x \in X$, then there is a Borel function $\hat{\theta} : X \to \mathrm{vN}(\ell^2(\mathbb{N}))$ such that $\hat{\theta}(x) \simeq M_x$ for all $x \in X$.

Proof. The family $(\varphi_x\colon x \in X)$ is uniquely defined by (1), and θ is uniquely defined by

$$\theta(x, y) = T \iff (\forall n, m)\ \langle T e_n, e_m \rangle_{\ell^2(\mathbb{N})} = \langle T_y^x e_n^x, e_m^x \rangle_{H_x},$$

which also gives a Borel definition of the graph of θ, so θ is Borel by [13, 14.12] since $\mathcal{B}(\ell^2(\mathbb{N}))$ is a standard Borel space when given the Borel structure generated by the weak topology. If we let

$$f_n : X \to \mathrm{vN}\big(\ell^2(\mathbb{N})\big)\colon\ x \mapsto \theta\big(x, \psi_n(x)\big).$$

Then the "moreover" part follows from [7, Theorem 2] since M_x is isomorphic to

$$\{f_n(x)\colon n \in \mathbb{N}\}'' \in \mathrm{vN}(\ell^2(\mathbb{N})). \qquad \square$$

Lemma 3.15. *There is a Borel function* $f : (0,1)^{\mathbb{N}} \to \mathrm{vN}(\ell^2(\mathbb{N}))$ *such that for all* $x \in (0,1)^{\mathbb{N}}$, $f(x)$ *is isomorphic to*

$$N_x = \bigotimes_{n=1}^{\infty} \big(M_2(\mathbb{C}), \big(x(n), 1 - x(n)\big)\big),$$

the ITPFI *factor with eigenvalue list* $(x(n), 1 - x(n))_{n \in \mathbb{N}}$.

Proof. Let $M_2(\mathbb{C})$ act on itself by multiplication. Then let $\eta : (0,1)^{\mathbb{N}} \to (M_2(\mathbb{C})^4)^{\mathbb{N}}$ be a Borel function such that

$$\eta(x)(n)_1 = \begin{pmatrix} \sqrt{x(n)} & 0 \\ 0 & \sqrt{1 - x(n)} \end{pmatrix}$$

and $\{\eta(x)(n)_i\colon i \in \{1,2,3,4\}\}$ is an orthonormal basis for $M_2(\mathbb{C})$. For each $\vec{i} \in \{1,2,3,4\}^{\mathbb{N}}$ such that $\vec{i}(k) = 1$ eventually, let

$$e^x_{\vec{i}} = \eta(x)(1)_{\vec{i}(1)} \otimes \eta(x)(2)_{\vec{i}(2)} \otimes \cdots \otimes \eta(x)(n)_{\vec{i}(n)} \otimes \cdots.$$

Then $(e^x_{\vec{i}}\colon \vec{i}(k) = 1$ eventually$)$ is an orthonormal basis for the Hilbert space

$$H_x = \bigotimes_{n=1}^{\infty} \big(M_2(\mathbb{C}), \eta(x)(n)_1\big).$$

Let $M_2(\mathbb{C})^{<\mathbb{N}}$ denote the set of sequences in $\vec{a} \in M_2(\mathbb{C})^{\mathbb{N}}$ that $\vec{a}(k) = I$ eventually. Then for all $\vec{i}, \vec{j}$ the map $(0,1)^{\mathbb{N}} \times M_2(\mathbb{C})^{<\mathbb{N}} \to \mathbb{C}$ given by

$$(x, \vec{a}) \mapsto \big\langle \vec{a}(1)\eta(x)(1)_{\vec{i}(1)} \otimes \vec{a}(2)\eta(x)(2)_{\vec{i}(2)} \otimes \cdots, \eta(x)(1)_{\vec{j}(1)} \otimes \eta(x)(2)_{\vec{j}(2)} \otimes \cdots \big\rangle$$

is continuous. Since for each x fixed, $M_2(\mathbb{Q}[i])^{<\mathbb{N}}$ generates N_x as a von Neumann algebra, the lemma now follows from Lemma 3.14. $\square$

Lemma 3.16. *There is a Borel function* $f : c_0 \to \mathrm{vN}(\ell^2(\mathbb{N}))$ *such that* $f(x) \simeq M_x$.

Proof. Immediate by the previous lemma, since

$$x \mapsto \left(\frac{1}{1 + e^{-l^x_j}}\colon j \in \mathbb{N}\right)$$

is continuous. $\square$

Remarks 3.17. 1. Consider Cantor space $\{0,1\}^{\mathbb{N}}$ and the odometer action of $\mathbb{Z}$ on $X=\{0,1\}^{\mathbb{N}}$ (i.e. "adding one with carry"). Let $z\in(0,1)^{\mathbb{N}}$, and let μ^z be the product measure

$$\mu^z=\prod_{n=1}^{\infty}\bigl(z(n)\delta_0+\bigl(1-z(n)\bigr)\delta_1\bigr),$$

where δ_0 and δ_1 denote the Dirac measures on $\{0,1\}$ concentrating on 0 and 1, respectively. The measure class of μ^z is preserved by the odometer action, and if μ^z is ergodic for the odometer we let $N_z=L^{\infty}(X,\mu^z)\rtimes\mathbb{Z}$ be the Krieger factor obtained from the group-measure space construction. Then

$$N_z=L^{\infty}\bigl(X,\mu^z\bigr)\rtimes\mathbb{Z}\simeq\bigotimes_{n=1}^{\infty}\bigl(M_2(\mathbb{C}),\bigl(z(n),1-z(n)\bigr)\bigr),$$

see [3, III.3.2.18]. By Krieger's celebrated Theorem ([18, 8.4], see also [3, III.3.2.19]) the group measure space factors N_z and $N_{z'}$ are isomorphic precisely when the corresponding measure class-preserving odometer actions are orbit equivalent. If we now, for each $x\in c_0$, let $z_x(n)=\lambda_n^x$, where $(\lambda_n^x,1-\lambda_n^x)_{n\in\mathbb{N}}$ is the eigenvalue list of the factor M_x, then since all the factors M_x are type III, the measure μ^{z_x} is non-atomic and ergodic for the odometer. Thus we obtain the following consequence of Theorem 3.13:

Theorem 3.18. *The odometer actions of $\mathbb{Z}$ on $\{0,1\}^{\mathbb{N}}$ that preserve the measure class of some ergodic non-atomic μ^z as above, are not classifiable up to orbit equivalence by countable structures.*

2. The observation made in [21, Corollary 8] is equally pertinent to the main result of this paper: Since the proof relies only on Baire category techniques, Theorem 3.13 shows that it is not possible to construct in Zermelo–Fraenkel set theory without the Axiom of Choice a function that completely classifies ITPFI_2 factors up to isomorphism by assigning countable structures type invariants.

3. In [21] it was shown that isomorphism of separable factors is Borel reducible to an equivalence relation arising from a Borel action of the unitary group of $\ell^2(\mathbb{N})$ on a standard Borel space. This in particular then applies to ITPFI factors. Since by Theorem 1.1 we have $E_0<_B\simeq^{\mathrm{ITPFI}}$, it follows from [15] that $\simeq^{\mathrm{ITPFI}}$ is $\leqslant_B$-incomparable with the equivalence relation E_1 on $\mathbb{R}^{\mathbb{N}}$, given by

$$xE_1y\quad\Longleftrightarrow\quad(\exists n)\ (\forall m>n)\ x(n)=y(n).$$

This remark also applies to $\simeq^{\mathrm{II}_1}$, $\simeq^{\mathrm{II}_\infty}$ and $\simeq^{\mathrm{III}_\lambda}$, $0\leqslant\lambda\leqslant1$, using the results of [21].

4. In [20] we asked (Problem 4) if all possible K_σ subgroups of $\mathbb{R}$ appear as the T-set of some ITPFI factor. Stefaan Vaes has kindly pointed out to us that this is already known *not* to be the case: This follows from the results of [11, §2], see also [19, §2].

Acknowledgments

Research for this paper was mainly carried out at the Hausdorff Institute for Mathematics in Bonn during the 'Rigidity' programme in the fall semester of 2009. We wish to thank the

Hausdorff Institute for kind hospitality and support. A. Törnquist was also supported in part through Austrian Science Foundation FWF grant No. P19375-N18 and a Marie Curie grant No. 249167 from the European Union.

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