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Simple Lie algebras of small characteristic IV. Solvable and classical roots

Alexander Premet [a], Helmut Strade [b,*]

[a] *Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK*
[b] *Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany*

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Abstract

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ and T a torus of maximal dimension in the p-envelope of L in $\operatorname{Der} L$. In this paper we describe the T-semisimple quotients of the 2-sections of L relative to T and prove that if all 1-sections of L relative to T are compositionally classical or solvable then L is either classical or a Block algebra or a filtered Lie algebra of type S.

1. Introduction and preliminaries

This is the fourth paper in a series devoted to classifying all finite-dimensional simple Lie algebras over an algebraically closed field F of characteristic $p > 3$. As the previous one it will rely on the terminology and notation introduced in the first two papers of the series. Unless otherwise stated, all Lie algebras in this paper are assumed to be finite-dimensional over F. The classification of simple Lie algebras of absolute toral rank 2 obtained in [10] enables us now to deal with the general case implementing the programme successfully completed by the second author for $p > 7$.

Let $\mathfrak{g}_p$ be a p-envelope of a Lie algebra $\mathfrak{g}$ and

$$MT(\mathfrak{g}_p) := \max\{\dim \mathfrak{t} \mid \mathfrak{t} \text{ is a torus in } \mathfrak{g}_p\}.$$

* Corresponding author.
E-mail addresses: sashap@ma.man.ac.uk (A. Premet), strade@math.uni-hamburg.de (H. Strade).

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If $\mathfrak{g}$ is centerless then the *absolute toral rank* of $\mathfrak{g}$, denoted $TR(\mathfrak{g})$ is nothing but $MT(\mathcal{G})$ where $\mathcal{G}$ is the p-envelope of $\mathfrak{g}$ in $\operatorname{Der}\mathfrak{g}$ (if $C(\mathfrak{g}) \neq (0)$ the definition is slightly more complicated).

Given a subspace W in $\mathfrak{g}$, we denote by $\mathfrak{c}_{\mathfrak{g}}(W)$ the centralizer of W in $\mathfrak{g}$. Given a torus $\mathfrak{t}$ in $\mathfrak{g}_p$ and a restricted $\mathfrak{g}_p$-module V, we denote by $\Gamma^w(V,\mathfrak{t})$ the set of *all* weights of V relative to $\mathfrak{t}$. The set $\Gamma^w(V,\mathfrak{t}) \setminus \{0\}$ is denoted by $\Gamma(V,\mathfrak{t})$. If $V = \mathfrak{g}$ then $\Gamma = \Gamma(\mathfrak{g},\mathfrak{t})$ is nothing but the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{t}$. If $\mathfrak{t}$ is a torus of maximal dimension in $\mathfrak{g}_p$ then the centralizer $\mathfrak{c}_{\mathfrak{g}_p}(\mathfrak{t})$ is a Cartan subalgebra of $\mathfrak{g}_p$. The Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}_p$ of the form $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}_p}(\mathfrak{t}')$, where $\mathfrak{t}'$ is a torus of maximal dimension in $\mathfrak{g}_p$, are called *regular*. All regular Cartan subalgebras in $\mathfrak{g}_p$ have the same dimension [5], enjoy various nice properties (see our discussion below), and play an important role in the classification theory.

Now suppose that the torus $\mathfrak{t} \subset \mathfrak{g}_p$ is such that $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$ is nilpotent. Then so is $\mathfrak{h}_p$, the p-envelope of $\mathfrak{h}$ in $\mathfrak{g}_p$. Let $\tilde{\mathfrak{t}}$ denote the unique maximal torus in $\mathfrak{t} + \mathfrak{h}_p$ Then $\mathfrak{c}_{\mathfrak{g}}(\tilde{\mathfrak{t}}) = \mathfrak{h}$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma \in \Gamma(\mathfrak{g},\tilde{\mathfrak{t}})} \mathfrak{g}_\gamma$$

is the root space decomposition of $\mathfrak{g}$ relative to $\tilde{\mathfrak{t}}$. The subalgebra $\mathfrak{h}$ is said to act *triangulably* on a $\mathfrak{g}$-module V if all composition factors of V viewed as an $\mathfrak{h}$-module are 1-dimensional. If $\mathfrak{h}$ acts triangulably on $\mathfrak{g}$, one often says that $\mathfrak{h}$ is *triangulable*.

Let $x \in \mathfrak{h}$. If $r \in \mathbb{N}$ is large enough then $x^{p^r} \in \tilde{\mathfrak{t}}$. Thus any $\gamma \in \Gamma(\mathfrak{g},\tilde{\mathfrak{t}})$ can be viewed as an F-valued function on $\mathfrak{h}$. More precisely, we have that

$$\gamma(x) = \sqrt[p^r]{\gamma\left(x^{p^r}\right)} \quad (\forall x \in \mathfrak{h}).$$

If $\mathfrak{h}$ is triangulable then, of course, any root function is linear on $\mathfrak{h}$. We stress, however, that the triangulability of $\mathfrak{h}$ is *not* pre-supposed in this paper, and some of the results we obtain will be used in our next paper devoted to the case where roots functions are nonlinear.

From now on L will always denote a *simple* Lie algebra over F, and L_p will stand for the p-envelope of L in $\operatorname{Der} L$. Recall that L_p is a semisimple Lie algebra and any semisimple p-envelope of L is isomorphic to L_p as restricted Lie algebras (see [22], for example). Given a torus T of maximal dimension in L_p, we set $H := \mathfrak{c}_L(T)$ and $\widetilde{H} := \mathfrak{c}_{L_p}(T)$. We have already mentioned that $\widetilde{H}$ is a Cartan subalgebra of L_p. However, H need not be a Cartan subalgebra of L; we will see later that it does happen in some very interesting cases that $H = (0)$.

In [25], Wilson proved that for $p > 7$ all Cartan subalgebras of L are triangulable. This theorem is so important for the classification theory that it was later generalized (by both of us) in different directions.

In [15], the second author proved that for $p > 7$ the Cartan subalgebra $\widetilde{H}$ of L_p is triangulable. Using the terminology just introduced this result simply says that for $p > 7$ all *regular* Cartan subalgebras of L_p are triangulable. It should be stressed, however, that not all Cartan subalgebras of L_p are regular, in general, and there are many examples of *simple* Lie algebras whose semisimple p-envelopes contain non-triangulable Cartan subalgebras.

In [7], the first author modified Wilson's original proof to cover the cases $p = 5$ and $p = 7$. Again it should be stressed that some simple Lie algebras in characteristic 5 do possess non-triangulable Cartan subalgebras. So Wilson's theorem does not generalize directly in this case. Roughly speaking, the result in [7] says that in characteristic 5 the failure of H to be triangulable can be detected at the level of 2-sections. In characteristic 7, Wilson's result is valid in its original form. In Section 3 of this paper, we prove the following generalization.

Theorem A. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ and let T be a torus of maximal dimension in L_p, the p-envelope of L in* $\operatorname{Der} L$. *Let $H = \mathfrak{c}_L(T)$ and $\widetilde{H} = \mathfrak{c}_{L_p}(T)$. Then the following hold*:

(1) *If $p > 5$ then $\widetilde{H}$ is triangulable.*
(2) *If $p = 5$ and H is triangulable then $\widetilde{H}$ is triangulable, too.*

In Section 4 of this paper, we investigate the 2-sections of L relative to T. Let $\alpha, \beta \in \Gamma(L,T)$ be two roots such that the 2-section $L(\alpha,\beta) := H \oplus \sum_{i,j\in\mathbb{F}_p} L_{i\alpha+j\beta}$ is nonsolvable, and let $\operatorname{rad}_T L(\alpha,\beta)$ denote the maximal T-invariant solvable ideal of $L(\alpha,\beta)$. Put $L[\alpha,\beta] := L(\alpha,\beta)/\operatorname{rad}_T L(\alpha,\beta)$ and let $\widetilde{S} = \widetilde{S}[\alpha,\beta]$ denote the T-socle of $L[\alpha,\beta]$, the sum of all minimal T-invariant ideals of $L[\alpha,\beta]$. Then $\widetilde{S} = \bigoplus_{i=1}^r \widetilde{S}_i$ where each $\widetilde{S}_i$ is a minimal T-invariant ideal of $L[\alpha,\beta]$. Let $L(\alpha,\beta)_p$ denote the p-envelope of $L(\alpha,\beta)$ in L_p. It is easily seen that $T + L(\alpha,\beta)_p \subset L_p$ acts on $L[\alpha,\beta]$ as derivations and preserves $\widetilde{S}$. We thus have a natural restricted homomorphism $T + L(\alpha,\beta)_p \to \operatorname{Der}\widetilde{S}$ which we call $\Psi_{\alpha,\beta}$. We identify $L[\alpha,\beta]$ with $\Psi_{\alpha,\beta}(L(\alpha,\beta))$ and denote the torus $\Psi_{\alpha,\beta}(T)$ by $\overline{T}$.

By Block's theorem, $\widetilde{S}_i \cong S_i \otimes A(m_i;\underline{1})$ where S_i is a simple Lie algebra and $m_i \in \mathbb{N}_0$. It is shown in Section 4 that $r \leqslant 2$ and the equality $r = 2$ implies that each S_i is one of $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$. Moreover, if $r = 1$ then $\widetilde{S} = S \otimes A(m;\underline{1})$ where S is a simple Lie algebra with $TR(S) \leqslant 2$. According to [10, Theorem 1.1], S is either classical or of Cartan type or isomorphic to the restricted Melikian algebra $\mathfrak{g}(1,1)$ (in which case $p = 5$).

Our next result generalizes and strengthens [2, Theorem 9.1.1], an important intermediate result of the Block–Wilson classification.

Theorem B.

(i) *If $r = 2$ then there are $\mu_1, \mu_2 \in \Gamma(L,T)$ such that*

$$L[\mu_1]^{(1)} \oplus L[\mu_2]^{(1)} \subset L[\alpha,\beta] \subset L[\mu_1] \oplus L[\mu_2].$$

(ii) *If $r = 1$ and $TR(\widetilde{S}) = 2$ then $\widetilde{S}$ is simple and the following hold*:
  (1) *If $\widetilde{S}$ is restricted then $L[\alpha,\beta] = \widetilde{S}$.*
  (2) *If $\widetilde{S}$ is non-restricted then $\widetilde{S} \subset L[\alpha,\beta] \subset \widetilde{S} + \overline{T} = \widetilde{S}_p$ unless $\widetilde{S} \cong H(2;(2,1))^{(2)}$ in which case $H(2;(2,1))^{(2)} \subset L[\alpha,\beta] \subset H(2;(2,1))_p$.*

(iii) *If $r = 1$ and $TR(\widetilde{S}) = 1$ then one of the following occurs*:
  (1) *$L[\alpha,\beta] = L[\mu]$ for some μ. Moreover, $\widetilde{S} = L[\mu]^{(1)}$ and $\dim\overline{T} = 1$.*

(2) $\widetilde{S} = H(2;\underline{1})^{(2)}$ *and* $L[\alpha,\beta] = H(2;\underline{1})^{(2)} \oplus FD$ *where either* $D = 0$ *or* $D = D_H(x_1^{p-1}x_2^{p-1})$ *or* $p = 5$ *and* $D = x_1^4\partial_2$. *Moreover,* $\dim \overline{T} = 2$.

(3) $\widetilde{S} = S \otimes A(1;\underline{1})$ *where* S *is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$. *Moreover,* $L[\alpha,\beta] \subset (\operatorname{Der} S) \otimes A(1;\underline{1})$ *and* $\overline{T} = (Fh_0 \otimes 1) \oplus (F\operatorname{Id} \otimes (1+x)\partial)$ *where* h_0 *is a nonzero toral element in* S.

(4) $\widetilde{S} = S \otimes A(m;\underline{1})$ *where* S *is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$ *and* $m > 0$. *There exists a classical root* μ *such that*

$$L[\alpha,\beta] = S \otimes A(m;\underline{1}) + L[\alpha,\beta](\mu);$$

$$L[\mu,\nu] \cong \mathfrak{g}(1,1) \quad \textit{for some } \nu \in \Gamma(L,T).$$

(5) $\widetilde{S} = S \otimes A(1;\underline{1})$ *where* S *is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$, *and* $L[\alpha,\beta]$ *is a subalgebra in* $(\operatorname{Der} S) \otimes A(1;\underline{1}) + \operatorname{Id} \otimes W(1;\underline{1})$ *such that*

$$L[\alpha,\beta] = S \otimes A(1;\underline{1}) + \big(L[\alpha,\beta]\big)(\mu),$$

where μ *is a Witt root.*

(6) $\widetilde{S} = S \otimes A(2;\underline{1})$ *where* S *is one of* $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$, *and* $L[\alpha,\beta]$ *is a subalgebra in* $(\operatorname{Der} S) \otimes A(2;\underline{1}) + \operatorname{Id} \otimes W(2;\underline{1})$ *such that*

$$L[\alpha,\beta] = S \otimes A(2;\underline{1}) + \big(L[\alpha,\beta]\big)(\mu),$$

where μ *is a Hamiltonian root.*

Section 5 extends the results of [17] to the case where $p = 5$. Section 6 deals with the simple Lie algebras L whose all 1-sections relative to T are solvable. This is a very difficult, isolated case and the results we established so far (in [8–10]) do not really help here. Our arguments in Section 6 rely on several subsidiary results established in [20] (and valid for $p > 3$). However, our approach differs from that in [20] which allows us to shorten the proof even in the case where p is large. Our main result is identical to the one obtained in [20], the only difference being that it now holds for $p > 3$.

Theorem C. *Let* L *be a finite-dimensional simple Lie algebra over an algebraically closed field* F *of characteristic* $p > 3$ *and suppose that the* p*-envelope of* L *in* $\operatorname{Der} L$ *contains a torus* T *of maximal dimension such that for every root* $\alpha \in \Gamma(L,T)$ *the* 1*-section* $L(\alpha)$ *is solvable. Then the set* $A := \Gamma(L,T) \cup \{0\}$ *is an* $\mathbb{F}_p$*-subspace in* T^* *and either* $L \cong S(m;\underline{n};\Phi(\tau))^{(1)}$ *for some* $m \geqslant 3$ *and* $\underline{n} \in \mathbb{N}^m$ *or* L *is isomorphic to a Block algebra* $L(A,0,f)$ *for some* $\mathbb{F}_p$*-bilinear mapping* $f : A \times A \to F$. *In all cases, each* $L(\alpha)$ *is abelian and* $\mathfrak{c}_L(T) = (0)$.

Note that each Block algebra $L(A,0,f)$ is known to be of type H. The Cartan type Lie algebras $S(m;\underline{n};\Phi(\tau))^{(1)}$ with $m \geqslant 3$ and $\underline{n} \in \mathbb{N}^m$ can be described as follows [20]: let M be an m-dimensional vector space over F, and let A be an additive subgroup in M^* of order p^N such that $\bigcap_{\alpha\in A}\ker\alpha = (0)$. For $\alpha \in A$, we set $M_\alpha := \{\alpha\} \times \ker\alpha$, an isomorphic copy of $\ker\alpha$. Give

$$V(M, A) := \bigoplus_{\alpha\in A\setminus\{0\}} M_\alpha$$

an algebra structure by setting $[(\alpha, u), (\beta, v)] = (\alpha + \beta, \alpha(v)u - \beta(u)v)$ for all nonzero $\alpha, \beta \in A$ and all $u \in \ker\alpha$ and $v \in \ker\beta$. It is known that $V(M, A)$ is a simple Lie algebra isomorphic to one of $S(m; \underline{n}; \Phi(\tau))^{(1)}$ with $N = \sum n_i$. Conversely, each $S(m; \underline{n}; \Phi(\tau))^{(1)}$ is isomorphic to one of $V(M, A)$'s for a suitable choice of $A \subset M^*$.

Recall that a root $\delta \in \Gamma(L, T)$ is called *solvable* (respectively *classical*) if $L(\delta)$ is solvable (respectively $L(\delta)/\operatorname{rad} L(\delta) \cong \mathfrak{sl}(2)$). Section 7 deals with the case where all roots in $\Gamma(L, T)$ are either classical or solvable, and at least one classical root occurs. Our argument here relies on Theorem B and several subsidiary results obtained in [16] (and valid for $p > 3$). It is slightly shorter than the original argument in [16]. As expected, the result we obtain is identical to the one proved by the second author for $p > 7$.

Theorem D. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ and assume that the p-envelope of L in $\operatorname{Der} L$ contains a torus T of maximal dimension such that all roots in $\Gamma(L, T)$ are either solvable or classical. Assume further that at least one root in $\Gamma(L, T)$ is classical. Then L is a classical Lie algebra, that is there exists a simple algebraic group G of adjoint type over F such that $L \cong (\operatorname{Lie} G)^{(1)}$. In particular, L is restricted.*

We mention for completeness that if the group G is not of type A_{kp-1} then $\operatorname{Lie} G$ is simple (recall that $p > 3$). In this case, $L \cong \operatorname{Lie} G$ (and one can also replace the adjoint group G by its simply connected cover). If G is of type A_{kp-1} then $G \cong \mathrm{PGL}_{kp}(F)$ and $L \cong (\mathfrak{pgl}_{kp}(F))^{(1)} = \mathfrak{psl}_{kp}(F)$.

We would like to finish the introduction by announcing that our next paper will investigate the simple Lie algebras L with the property that $H = \mathfrak{c}_L(T)$ is non-triangulable for at least one torus T of maximal dimension in $L_p \subset \operatorname{Der} L$. It will be proved in our next paper that L is then isomorphic to one of the Melikian algebras $\mathfrak{g}(m, n)$ where $(m, n) \in \mathbb{N}^2$.

Given a Cartan type Lie algebra M, not necessarily simple, we denote by $M_{(k)}$ the kth component of the standard filtration of M.

2. 1-sections in Hamiltonian algebras

This section is of preliminary nature and aims at gathering some missing information on root space decomposition in non-restricted Hamiltonian algebras of absolute toral rank 2. The results we obtain here will be used in Sections 3 and 4. They refine [2, Lemmas 10.1.3, 11.1.3] and [18, Sections VI, VIII].

Given a subalgebra A of a Lie algebra M we denote by $\operatorname{nil} A$ the largest ideal of A acting nilpotently on M.

Proposition 2.1. *Let $\widetilde{S} = H(2;(2,1))$ and $S = H(2;(2,1))^{(2)}$. Let $\widetilde{S}_p$ and S_p be the p-envelopes of $\widetilde{S}$ and S in* Der S, *respectively. Let $\mathfrak{g}$ be a Lie subalgebra of $\widetilde{S}_p$ containing S, $\mathfrak{t}$ be a 2-dimensional torus in $\widetilde{S}_p$ satisfying $[\mathfrak{t},\mathfrak{g}] \subset \mathfrak{g}$, and $\mathfrak{h} = \mathfrak{c}_S(\mathfrak{t})$. Then the following are true*:

(1) $S_p = \mathfrak{t} + S$.
(2) *If $\alpha \in \Gamma(\mathfrak{g},\mathfrak{t})$ is such that $\alpha(\mathfrak{h}) \neq 0$ then $\widetilde{S}(\alpha) \cong H(2;\underline{1})$ and* $\operatorname{rad}\mathfrak{g}(\alpha) = (0)$.

Proof. Recall that $\widetilde{S}_p = \widetilde{S} \oplus FD_1^p$ is isomorphic to a restricted subalgebra in the p-envelope of $W(2;(2,1))$ in $\operatorname{Der} A(2;(2,1))$ and $S_{(0)}$ is a restricted subalgebra in $\widetilde{S}_p$. Moreover, $\operatorname{Der} H(2;(2,1))^{(2)} = F(x_1D_1 + x_2D_2) \oplus \widetilde{S}_p$ is isomorphic to a restricted subalgebra in $\operatorname{Der} A(2;(2,1))$ (see [2, Proposition 2.1.8(vii)], for example). Since $\operatorname{Der} A(2;(2,1)) \cong W(3;\underline{1})$ possesses a 3-dimensional toral Cartan subalgebra, we have $MT(\widetilde{S}_p) = MT(\operatorname{Der} H(2;(2,1))^{(2)}) - 1 \leqslant 2$ (by [19, Lemma 1.6(2)] and the main result of [5]). On the other hand, $MT(\widetilde{S}_p) \geqslant 2$ (by [18, Section VI] for example). Therefore, $\mathfrak{t}$ is a torus of maximal dimension in $\widetilde{S}_p$.

(a) Since $\mathfrak{t} \subset \widetilde{S}_p$ and $\widetilde{S}_p/S_p$ is p-nilpotent we have $\mathfrak{t} \subset S_p = S \oplus FD_1^p$ (by Jacobson's formula, the subalgebra on the right is restricted). As $\dim \mathfrak{t} = 2$ this implies that $\mathfrak{t} \cap S \neq (0)$. Suppose $\mathfrak{t} \subset S$. Since $S/S_{(0)}$ is a 2-dimensional module over $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$, each nonzero element in $\mathfrak{t} \cap S_{(0)}$ acts invertibly on $S/S_{(0)}$. So $\mathfrak{t} \cap S_{(0)} \neq (0)$ would imply $\mathfrak{t} \subset S_{(0)}$. But then $\mathfrak{t}$ would inject into $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ which is impossible. Thus under our present assumption on $\mathfrak{t}$ we must have that $\mathfrak{t} \cap S_{(0)} = (0)$. This forces $S = \mathfrak{t} + S_{(0)}$. But then $\mathfrak{t}$ must contain a toral element of the form $aD_1 + x$ with $a \in F^*$ and $x \in S_{(0)}$. Since $S_{(0)}$ is restricted we then have $D_1^p \in \mathfrak{t} + S = S$ which is not true. Hence $\mathfrak{t} \not\subset S$ and, consequently, $S_p = \mathfrak{t} + S$.

(b) According to [18, Theorem VI.2(2)], there is a torus $\mathfrak{t}'$ in S_p such that $|\Gamma(S,\mathfrak{t}')| = p^2 - 1$ and $\dim S_\gamma = p$ for all $\gamma \in \Gamma(S,\mathfrak{t}')$. Combining this with [9, Corollary 2.11], we obtain that the same is true for $\mathfrak{t}$, that is

$$\bigl|\Gamma(S,\mathfrak{t})\bigr| = p^2 - 1 \quad \text{and} \quad \dim S_\gamma = p \quad \forall \gamma \in \Gamma(S,\mathfrak{t}).$$

Since $\mathfrak{t}$ acts nilpotently on $\widetilde{S}_p/S_p$, we also have that $\widetilde{S}_\gamma = S_\gamma$ for all $\gamma \in \Gamma(\widetilde{S},\mathfrak{t})$. The standard filtration of S (respectively $\widetilde{S}$) induces a filtration in its subalgebra $S(\alpha)$ (respectively $\widetilde{S}(\alpha)$). The corresponding graded Lie algebras $\operatorname{gr} S(\alpha)$ and $\operatorname{gr} \widetilde{S}(\alpha)$) are naturally identified with graded Lie subalgebras in S and $\widetilde{S}$, respectively.

There is a toral element $t \in \mathfrak{t}$ such that $S(\alpha) = \mathfrak{c}_S(t)$ (and likewise for $\widetilde{S}$ and $\mathfrak{g}$). We first suppose that $t \notin S$. Then $t = aD_1^p + u$ where $a \in F^*$ and $u \in S$. It is easily seen (and first observed in [2, p. 232]) that $\operatorname{gr} S(\alpha)$ is contained in $\mathfrak{c}_S(D_1^p) \cong H(2;\underline{1})^{(2)}$ while $\operatorname{gr} \widetilde{S}(\alpha)$ lies in $\mathfrak{c}_{\widetilde{S}}(D_1^p) \cong H(2;\underline{1})$. Since

$$\dim \operatorname{gr} S(\alpha) = \dim S(\alpha) = \dim S - p\bigl(p^2 - p\bigr) = p^3 - 2 - \bigl(p^3 - p^2\bigr) = p^2 - 2,$$

we deduce that $\operatorname{gr} S(\alpha) \cong H(2, \underline{1})^{(2)}$ as Lie algebras. Likewise $\dim \operatorname{gr} \widetilde{S}(\alpha) = p^2 + 1$ yielding $\operatorname{gr} \widetilde{S}(\alpha) \cong H(2; \underline{1})$. Since for any ideal $I \subset S(\alpha)$ the subspace $\operatorname{gr} I$ is an ideal in $\operatorname{gr} S(\alpha)$ the Lie algebra $S(\alpha)$ must be simple. A similar reasoning applied to $\widetilde{S}(\alpha)$ shows that any nonzero ideal in $\widetilde{S}(\alpha)$ has dimension $\geqslant p^2 - 2$ (this is due to the fact that any nonzero ideal in $H(2; \underline{1})$ contains $H(2; \underline{1})^{(2)}$). Since $S(\alpha)$ is a 1-section in a simple Lie algebra of absolute toral rank 2, we now obtain that $S(\alpha) \cong H(2; \underline{1})^{(2)}$ (see [9, p. 193]).

The adjoint action of $\widetilde{S}(\alpha)$ on its ideal $S(\alpha)$ gives rise to a Lie algebra homomorphism $\phi : \widetilde{S}(\alpha) \to \operatorname{Der} H(2; \underline{1})^{(2)}$. As $S(\alpha)$ is simple ϕ must be injective (otherwise our earlier discussion would imply $S(\alpha) \subset \ker \phi$ which is impossible). Thus $\widetilde{S}(\alpha)$ is isomorphic to a Lie subalgebra of dimension $p^2 + 1$ in $\operatorname{Der} H(2; \underline{1})^{(2)}$. As mentioned at the beginning of the proof, $\mathfrak{t}$ is a torus of maximal dimension in $\widetilde{S}_p$. By [19, Theorem 1.9(2)], this implies that $TR(\widetilde{S}(\alpha)) \leqslant 1$. On the other hand, it is well known that $\operatorname{Der} H(2; \underline{1})^{(2)}$ has dimension $p^2 + 2$ and contains a 2-dimensional torus. Since $\widetilde{S}(\alpha)/S(\alpha)$ is nilpotent it is immediate from the description of $\operatorname{Der} H(2; \underline{1})^{(2)}$ given in [2, Theorem 2.1.8(vii)] that the 3-dimensional image of $\phi(\widetilde{S}(\alpha))$ in the restricted quotient $\operatorname{Der} H(2; \underline{1})^{(2)}/H(2; \underline{1})^{(2)}$ consists of p-nilpotent elements. So it must coincide with the image of $H(2; \underline{1})$. This enables us to conclude that $\widetilde{S}(\alpha) \cong H(2; \underline{1})$. But then $\mathfrak{g}(\alpha) \subset \widetilde{S}(\alpha)$ can be identified with a subalgebra of $\operatorname{Der} H(2; \underline{1})^{(2)}$ containing $H(2; \underline{1})^{(2)}$. Since any such subalgebra is semisimple, we obtain the second statement of the proposition (under our present assumption on t).

(c) Next we suppose that $t \in S \setminus S_{(0)}$. Then $t = aD_1 + bD_2 + w$ where $a, b \in F$ and $w \in S_{(0)}$. Since $(aD_1 + bD_2)^p = a^p D_1^p$ and $w^p \in S$, the equality $t^p = t$ combined with Jacobson's formula gives $a = 0$ and $b \neq 0$. We now look at the graded Lie algebra $\operatorname{gr} S(\alpha) = \operatorname{gr} \mathfrak{c}_S(t)$ which is naturally identified with a graded subalgebra of $\mathfrak{c}_S(D_2)$ (cf. [2, p. 232]). This observation enabled Block and Wilson to deduce that $\mathfrak{c}_S(t)$ is solvable. However, in this proof we need more information on $\mathfrak{c}_S(t)$. We claim that $\mathfrak{c}_S(t)$ is nilpotent and acts triangulably on S. To see this we first recall that, as in the former case, $\dim \mathfrak{c}_S(t) = p^2 - 2$. We define

$$u_i := \bigl((\operatorname{ad} t)^{p-1} - \operatorname{Id}\bigr)\bigl(D_H\bigl(x_1^{(i)} x_2^{(p-1)}\bigr)\bigr), \quad 1 \leqslant i \leqslant p^2 - 2,$$

all of which lie in $\mathfrak{c}_S(t)$. Since $b \neq 0$, the element $\operatorname{gr} u_i \in H(2; (2,1))^{(2)}$ is a nonzero multiple of $D_H(x_1^{(i)})$. It follows that the $\operatorname{gr} u_i$'s are linearly independent. But then so are the u_i's. Then they form a basis of $\mathfrak{c}_S(t)$. Since $t \in S \setminus S_{(0)}$ and $u_i \in S_{(1)}$ for $i \geqslant 3$, we get $\mathfrak{c}_S(t) = Ft \oplus Fu_2 \oplus \mathfrak{c}_S(t) \cap S_{(1)}$. Then $\mathfrak{c}_S(t)$ is nilpotent and $[\mathfrak{c}_S(t), \mathfrak{c}_S(t)] \subset S_{(1)}$, hence the claim. As a consequence, $\mathfrak{h} = \mathfrak{c}_S(\mathfrak{t}) = Ft \oplus \operatorname{nil} \mathfrak{h}$. But then α vanishes on $\mathfrak{h}$ contrary to our assumption on $\mathfrak{h}$.

(d) Finally, suppose $t \in S_{(0)}$. Then $0 \neq \operatorname{gr} t \in S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ acts invertibly on $S/S_{(0)}$. It follows that $\mathfrak{c}_S(t) = Ft \oplus S_{(1)}(t)$ forcing $\mathfrak{h} = Ft \oplus \operatorname{nil} \mathfrak{h}$. Then again α vanishes on $\mathfrak{h}$ contrary to our assumption. Thus our assumption on $\mathfrak{h}$ implies that $t \notin S$. Then we are in case (b) and the proof of the proposition is now complete. □

As in the earlier papers, we denote by $A(\!(m)\!)$ the divided power algebra in m variables, a complete, linearly compact, local algebra over F (see, e.g., [2, (1.1)] where the notation is a bit different). For $k \geqslant 0$, the kth part of the standard (decreasing) filtration in $A(\!(m)\!)$ is denoted by $A(\!(m)\!)_{(k)}$. Recall that the exponential mapping

$$\exp : A(\!(m)\!)_{(1)} \longrightarrow 1 + A(\!(m)\!)_{(1)}, \quad f \mapsto \exp(f) := \sum_{i \geqslant 0} f^{(i)},$$

is bijective. The inverse mapping $1 + A(\!(m)\!)_{(1)} \to A(\!(m)\!)_{(1)}$ is given by

$$1 + g \longmapsto \ln(1+g) := \sum_{i=1}^{p} (-1)^{i-1}(i-1)!g^{(i)}.$$

Of course, $\exp(f+g) = \exp(f)\exp(g)$ and $\ln(1+f)(1+g) = \ln(1+f) + \ln(1+g)$ for all $f, g \in A(\!(m)\!)_{(1)}$.

We are now going to investigate the 1-sections in the Albert–Zassenhaus algebra $\mathfrak{g} = H(2; \underline{1}; \Delta)$. Recall that $\mathfrak{g}$ consists of all $D \in W(2; \underline{1})$ with $D\omega_\Delta = 0$ where

$$\omega_\Delta = \exp\bigl(x_1^{(p)}\bigr) dx_1 \wedge dx_2.$$

The Lie algebra $\mathfrak{g}$ is simple and has dimension p^2 (see [2, (2.1)], for example). The standard filtration in $\mathfrak{g}$ is induced by that of $W(2; \underline{1})$.

Proposition 2.2. *Let $\mathfrak{g} = H(2; \underline{1}; \Delta)$ and let $\mathfrak{t}$ be a 2-dimensional torus in the semisimple p-envelope $\mathfrak{g}_p$ of $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$, $\alpha \in \Gamma(\mathfrak{g}, \mathfrak{t})$, and suppose that $\alpha(\mathfrak{h}) \neq 0$. Then the union*

$$\bigcup_{i \in \mathbb{F}_p^*} \bigl(\operatorname{rad} \mathfrak{g}(\alpha)_{i\alpha} \cup \bigl[\mathfrak{g}(\alpha)_{i\alpha}, \operatorname{rad} \mathfrak{g}(\alpha)_{-i\alpha}\bigr]\bigr)$$

consists of p-nilpotent elements of $\mathfrak{g}_p$.

Proof. It is well known (see, e.g., [18, Chapter VIII]) that $\mathfrak{g}_p = \operatorname{Der} \mathfrak{g} \cong H(2; 1; \Delta) \oplus Fx_1D_1 \subset W(2; \underline{1})$. Moreover, $\mathfrak{g}_p$ contains the 2-dimensional torus $Fx_1D_1 \oplus Fx_2D_2$ which will be denoted by $\mathfrak{t}_1$. This description implies that $\mathfrak{g}_p$ acts on the line $F\omega_\Delta$. According to [18, p. 459], $|\Gamma(\mathfrak{g}, \mathfrak{t}_1)| = p^2 - 1$ and $\dim \mathfrak{g}_\gamma = 1$ for any $\gamma \in \Gamma(\mathfrak{g}, \mathfrak{t}_1)$. Therefore, $\mathfrak{t}$ has $p^2 - 1$ roots on $H(2; \underline{1}; \Delta)$ and all root spaces for $\mathfrak{t}$ are 1-dimensional (see [9, Corollary 2.10]).

(a) Let $\mathfrak{t}_0 = F(1 + x_1)D_1 \oplus F(1 + x_2)D_2$ and $\mathfrak{t}_2 = F(1 + x_1)D_1 \oplus Fx_2D_2$. By Demushkin's theorem, there is $\phi \in \operatorname{Aut} W(2; \underline{1})$ such that $\phi(\mathfrak{t}) = \mathfrak{t}_s$ for some $s \in \{0, 1, 2\}$. Furthermore, ϕ is induced by a continuous automorphism of the divided power algebra $A(\!(2)\!)$ preserving $A(2; \underline{1}) \subset A(\!(2)\!)$. Let

$$J(\phi) = D_1\bigl(\phi(x_1)\bigr)D_2\bigl(\phi(x_2)\bigr) - D_2\bigl(\phi(x_1)\bigr)D_1\bigl(\phi(x_2)\bigr),$$

an invertible element in $A(2;\underline{1})$. Since $\phi(dx_1 \wedge dx_2) = J(\phi)dx_1 \wedge dx_2$ we have that $\phi(\omega_\Delta) = a\omega_\Delta$ where $a = \exp(\phi(x_1)^{(p)})J(\phi)$. For $i = 1, 2$ set $a_i := a^{-1}D_i(a)$. There exist $u \in A((2))_{(1)}$ and $\mu \in F^*$ such that $a = \mu\exp(u)$. Then $a_i = D_i(u)$ where $i = 1, 2$. Since D_i respects the divided power maps, we also have that

$$a_i = \phi(x_1)^{(p-1)}D_i\big(\phi(x_1)\big) + J(\phi)^{-1}D_i\big(J(\phi)\big).$$

As a consequence, $a_1, a_2 \in A(2;\underline{1})$. Since ω_Δ is a weight vector for $\mathfrak{t}$ and $dx_1 \wedge dx_2$ is a weight vector for each of the tori $\mathfrak{t}_0$, $\mathfrak{t}_1$, $\mathfrak{t}_2$, the divided power series a is a weight vector for $\mathfrak{t}_s$. But then so are the truncated polynomials a_1 and a_2. Furthermore, a_i has the same weight as D_i. Since $a \notin A(2;\underline{1})$ it follows from [2, Lemma 2.1.3] that $\phi(\mathfrak{g}) = \{\mathcal{D}_a(f) \mid f \in A(2;\underline{1})\}$ where

$$\mathcal{D}_a(f) := (D_2 + a_2)(f)D_1 - (D_1 + a_1)(f)D_2.$$

(b) There is a toral element $t \in \mathfrak{t}$ such that $\mathfrak{g}(\alpha) = \mathfrak{c}_\mathfrak{g}(t)$. Set

$$\mathfrak{c}' := \mathfrak{c}_{\phi(\mathfrak{g})}\big(\phi(t)\big) = \phi\big(\mathfrak{g}(\alpha)\big).$$

If $t \notin W(2;\underline{1})_{(0)}$, it can be assumed that $\phi(t) = (1 + x_1)D_1$ (see [9, Theorem 2.3]).

Suppose $s = 0$. Since all $\mathfrak{t}_0$-weight spaces in $A(2;\underline{1})$ are 1-dimensional, there are $\lambda_1, \lambda_2 \in F$ such that $D_i(u) = \lambda_i(1 + x_i)^{p-1}$ for $i = 1, 2$. This system of differential equations has a unique solution in $A((2))_{(1)}$, namely,

$$u = \lambda_1 \ln(1 + x_1) + \lambda_2 \ln(1 + x_2).$$

If $\lambda_1, \lambda_2 \in \mathbb{F}_p$ then $u = \ln(1 + x_1)^{\lambda_1}(1 + x_2)^{\lambda_2}$ yielding

$$a = \mu\exp(u) = \mu(1 + x_1)^{\lambda_1}(1 + x_2)^{\lambda_2} \in A(2;\underline{1}),$$

a contradiction. Thus, either $\lambda_1 \notin \mathbb{F}_p$ or $\lambda_2 \notin \mathbb{F}_p$. It follows from our remarks earlier in the proof that $\mathfrak{c}'$ has basis $\{\mathcal{D}_a((1 + x_2)^i(1 + x_1)) \mid i \in \mathbb{F}_p\}$ where

$$\mathcal{D}_a\big((1 + x_2)^i(1 + x_1)\big) = (i + \lambda_2)(1 + x_2)^{i-1}(1 + x_1)D_1 - (1 + \lambda_1)(1 + x_2)^i D_2.$$

If $\lambda_1 \neq -1$, the natural projection $\mathfrak{c}' \to A(2;\underline{1})D_2$ induces an isomorphism $\mathfrak{g}(\alpha) \cong W(1;\underline{1})$. Then $\operatorname{rad}\mathfrak{g}(\alpha) = (0)$. If $\lambda_1 = -1$ then $\lambda_2 \notin \mathbb{F}_p$, hence $\mathfrak{c}'$ is spanned by the elements $(1 + x_2)^i(1 + x_1)D_1$ with $i \in \mathbb{F}_p$. Therefore, $\mathfrak{c}_{\phi(\mathfrak{g})}(\phi(\mathfrak{t})) = F(1 + x_1)D_1$ so that $\mathfrak{h} = Ft$ and α vanishes on $\mathfrak{h}$.

(c) Suppose $s = 2$ and $t \notin W(2;\underline{1})_{(0)}$. Then $a_1 = \lambda_1(1 + x_1)^{p-1}$ and $a_2 = \lambda_2 x_1^{(p-1)}$ for some $\lambda_1, \lambda_2 \in F$ and $\mathfrak{c}'$ has basis $\{\mathcal{D}_a(x_2^{(i)}(1 + x_1)) \mid 0 \leqslant i \leqslant p - 1\}$ where

$$\mathcal{D}_a\big(x_2^{(i)}(1 + x_1)\big) = \big((1 - \delta_{i,0})x_2^{(i-1)} + \lambda_2\delta_{i,0}x_2^{(p+i-1)}\big)(1 + x_1)D_1 - (1 + \lambda_1)x_2^{(i)}D_2.$$

If $\lambda_1 \neq -1$ then, as before, $\mathfrak{g}(\alpha) \cong W(1;\underline{1})$ and $\operatorname{rad}\mathfrak{g}(\alpha) = (0)$. If $\lambda_1 = -1$ then $\lambda_2 \neq 0$ (otherwise $u = \ln(1+x_1)^{p-1}$ and $a = \mu(1+x_1)^{p-1} \in A(2;\underline{1})$, a contradiction). Then again $\mathfrak{c}_{\phi(\mathfrak{g})}(\phi(\mathfrak{t})) = F(1+x_1)D_1$ and $\alpha(\mathfrak{h}) = 0$.

(d) Suppose $t \in W(2;\underline{1})_{(0)}$ and $s = 2$. In this case $\phi(t) = rx_2D_2$ for some $r \in \mathbb{F}_p^*$ while a_1 and a_2 are as in part (c). Then $\mathfrak{c}'$ has basis $\{\mathcal{D}_a((1+x_1)^i x_2) \mid i \in \mathbb{F}_p\}$ and

$$\mathcal{D}_a\big((1+x_1)^i x_2\big) = (1+x_1)^i D_1 - (i+\lambda_1)(1+x_1)^{i-1}x_2D_2.$$

So $\mathfrak{g}(\alpha)$ is isomorphic to $W(1;\underline{1})$ and $\operatorname{rad}\mathfrak{g}(\alpha) = (0)$.

(e) Suppose $s = 1$. Then there are $\lambda_1, \lambda_2 \in F$ such that $a_i = \lambda_i x_i^{(p-1)}$ for $i = 1, 2$. First, we consider the case where $\phi(t)$ acts noninvertibly on the subspace spanned by D_1 and D_2. Then $\phi(t) = rx_kD_k$ where $r \in \mathbb{F}_p^*$ for $k = 1, 2$. We assume that $k = 1$, the case $k = 2$ being similar. Since $\mathfrak{c}'$ is spanned by $\{\mathcal{D}_a(x_2^{(i)}x_1) \mid 0 \leqslant i \leqslant p-1\}$ and

$$\mathcal{D}_a\big(x_2^{(i)}x_1\big) = \big((1-\delta_{i,0})x_2^{(i-1)} + \lambda_2\delta_{i,0}x_2^{(p+i-1)}\big)x_1D_1 - x_2^{(i)}D_2,$$

we have $\mathfrak{g}(\alpha) \cong W(1;\underline{1})$. So $\operatorname{rad}\mathfrak{g}(\alpha) = (0)$ in this case.

Next, we suppose that $\phi(t)$ is a nonzero multiple of $x_1D_1 + x_2D_2$. Then $\mathfrak{c}'$ is spanned by all $\mathcal{D}_a(x_1^{(i)}x_2^{(j)})$ with $0 \leqslant i, j \leqslant p-1$ and $i + j - 2 \equiv 0 \pmod p$. It is easily checked that $\mathfrak{s} := \operatorname{span}\{\mathcal{D}_a(x_1^2), \mathcal{D}_a(x_1x_2), \mathcal{D}_a(x_2^{(2)})\}$ is a 3-dimensional simple Lie subalgebra in $W(2;\underline{1})_{(0)}$. From this it follows that $\mathfrak{c}' = \mathfrak{s} \oplus \operatorname{rad}\mathfrak{c}'$ where $\operatorname{rad}\mathfrak{c}' = \mathfrak{c}' \cap W(2;\underline{1})_{(1)}$. So $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2) \oplus \operatorname{rad}\mathfrak{g}(\alpha)$ and $\operatorname{rad}\mathfrak{g}(\alpha)$ consists of p-nilpotent elements of $\mathfrak{g}_p$.

Finally, suppose $\phi(t)$ acts invertibly on the span of D_1, D_2, x_1D_2, and x_2D_1. Then $\mathfrak{c}' \subset \mathfrak{t}_1 \oplus W(2;\underline{1})_{(1)}$ which implies that $\mathfrak{g}_p(\alpha) = (\mathfrak{t} \cap \mathfrak{g}_p(\alpha)) \oplus \operatorname{nil}\mathfrak{g}(\alpha)$. But then $\mathfrak{g}_{i\alpha} \subset \operatorname{nil}\mathfrak{g}(\alpha)$ for all $i \in \mathbb{F}_p^*$.

We have considered all cases and the proof of the proposition is now complete. □

3. Triangularity

Let M and A be Lie algebras and suppose that A acts on M as derivations. We say that A *acts triangulably* on M if $A^{(1)}$ acts on M as nilpotent linear transformations. If A is a subalgebra in M and $\operatorname{ad} A$ acts triangulably on M we often say that A is *triangulable*. Given a T-invariant Lie subalgebra $Q \subset L_p$ we say that T is *standard* with respect to Q if the centralizer $\mathfrak{c}_Q(T)$ acts triangulably on Q.

The starting point for the second author's classification has been the observation that certain important subalgebras of L_p are triangulable. In this section we will generalize these results to our present case $p > 3$. We first generalize [14, Theorem 3.5].

Theorem 3.1. *Let $\mathfrak{t}_0 \subset L_p$ be a torus in L_p such that $\mathfrak{c}_L(\mathfrak{t}_0)$ is nilpotent and acts triangulably on L. Let $\alpha_1, \ldots, \alpha_s \in \Gamma(L, \mathfrak{t}_0)$ and assume that $L(\alpha_1, \ldots, \alpha_s)$ is nilpotent. Then $L(\alpha_1, \ldots, \alpha_s)$ acts triangulably on L.*

Proof. Put $\mathfrak{t} := \mathfrak{t}_0 \cap \bigcap_{i=1}^{s} \ker \alpha_i$, $\mathfrak{h} := \mathfrak{c}_L(\mathfrak{t}) = L(\alpha_1, \ldots, \alpha_s)$, and let $\mathfrak{h}_p$ denote the p-envelope of $\mathfrak{h}$ in $L_p \subset \operatorname{Der} L$. Let $\tilde{\mathfrak{t}}$ be the unique maximal torus in the restricted nilpotent subalgebra $\tilde{\mathfrak{h}} := \mathfrak{t} + \mathfrak{h}_p$. Put $\widetilde{L} := \tilde{\mathfrak{h}} + L$ and let

$$\widetilde{L} = \tilde{\mathfrak{h}} \oplus \sum_{\gamma \in \Gamma} \widetilde{L}_\gamma, \quad \Gamma \subset \tilde{\mathfrak{t}}^* \setminus \{0\},$$

be the root space decomposition of $\widetilde{L}$ with respect to $\tilde{\mathfrak{t}}$.

(a) Suppose $\mathfrak{h}$ acts nontriangulably on L. Then [7, Theorem 1] shows that $p = 5$ and there exist $\alpha, \beta \in \Gamma$ linearly independent over $\mathbb{F}_5$ in $\tilde{\mathfrak{t}}^*$ and a maximal ideal $\widetilde{R}(\alpha, \beta)$ of the 2-section $\widetilde{L}(\alpha, \beta)$ such that

$$\widetilde{L}(\alpha, \beta)/\widetilde{R}(\alpha, \beta) \cong \mathfrak{g}(1, 1)$$

is the 125-dimensional restricted Melikian algebra. Moreover, the proof of this theorem shows that the image of $\tilde{\mathfrak{h}}$ in $\mathfrak{g}(1, 1)$ is a nontriangulable Cartan subalgebra in $\mathfrak{g}(1, 1)$. The p-envelope $\widetilde{L}(\alpha, \beta)_p$ of $\widetilde{L}(\alpha, \beta)$ in L_p preserves $\widetilde{R}(\alpha, \beta)$ hence acts on $\widetilde{L}(\alpha, \beta)/\widetilde{R}(\alpha, \beta)$ as derivations. Since $\mathfrak{g}(1, 1) \cong \operatorname{Der} \mathfrak{g}(1, 1)$ (see [19, Theorem 3.37] for example) this gives rise to an epimorphism of restricted Lie algebras

$$\phi_1 : \widetilde{L}(\alpha, \beta)_p \longrightarrow \mathfrak{g}(1, 1).$$

Note that $\phi_1(\tilde{\mathfrak{t}})$ is a 2-dimensional nonstandard torus in $\mathfrak{g}(1, 1)$.

(b) Suppose $\widetilde{R}(\alpha, \beta) \cap \tilde{\mathfrak{h}}$ contains an element h acting nonnilpotently on $\widetilde{L}(\alpha, \beta)$ and let $h_s \in \tilde{\mathfrak{t}}$ denote the semisimple part of h. There exists a nonzero $\nu \in \mathbb{F}_5\alpha + \mathbb{F}_5\beta$ such that $\nu(h_s) \neq 0$. But then $\widetilde{L}_\nu \subset \widetilde{R}(\alpha, \beta)$, so that ν is not a $\phi_1(\tilde{\mathfrak{t}})$-root of $\mathfrak{g}(1, 1)$. However, $\mathfrak{g}(1, 1)$ has $p^2 - 1$ roots relative to each of its 2-dimensional tori (by [7, Lemma 4.1] and [9, Corollary 2.10]). This contradiction shows that $\widetilde{R}(\alpha, \beta) \cap \tilde{\mathfrak{h}}$ acts nilpotently on $\widetilde{L}(\alpha, \beta)$.

Let I be any ideal of $\widetilde{L}(\alpha, \beta)$ not contained in $\widetilde{R}(\alpha, \beta)$. The maximality of $\widetilde{R}(\alpha, \beta)$ implies that $\widetilde{L}(\alpha, \beta) = I + \widetilde{R}(\alpha, \beta)$. Both I and $\widetilde{R}(\alpha, \beta)$ are ideals in $\widetilde{L}(\alpha, \beta))$ hence $\tilde{\mathfrak{t}}$-stable. Then $\tilde{\mathfrak{h}} = I \cap \tilde{\mathfrak{h}} + \widetilde{R}(\alpha, \beta) \cap \tilde{\mathfrak{h}}$. Thus any $t \in \tilde{\mathfrak{t}}$ can be written as $t = h_1 + h_2$ with $h_1 \in I \cap \tilde{\mathfrak{h}}$ and $h_2 \in \widetilde{R}(\alpha, \beta) \cap \tilde{\mathfrak{h}}$. By our discussion above, h_2 acts nilpotently on $\widetilde{L}(\alpha, \beta)$. Also $0 = [t, h_2] = [h_1, h_2]$. Hence, for r big enough, $t^{p^r} = h_1^{p^r} + h_2^{p^r} \in h_1^{p^r} + \tilde{\mathfrak{t}} \cap \ker \alpha \cap \ker \beta$. But then $\widetilde{L}_\gamma \subset I$ for any $\gamma \in (\mathbb{F}_5\alpha + \mathbb{F}_5\beta) \setminus \{0\}$. In other words,

$$\widetilde{L}(\alpha, \beta) = I + \tilde{\mathfrak{h}} \cap \widetilde{R}(\alpha, \beta).$$

(c) Note that $\mathfrak{h}$, $\mathfrak{h}_p$, and the center $C(\mathfrak{h}_p)$ are $\mathfrak{t}_0$-invariant. Therefore,

$$C(\mathfrak{h}_p)^p = \left(\sum_{\delta \in \mathfrak{t}_0^*} C(\mathfrak{h}_p)_\delta \right)^p \subset \sum_{\delta \in \mathfrak{t}_0^*} C(\mathfrak{h}_p)_{p\delta}$$

is centralized by $\mathfrak{t}_0$. Then $[\mathfrak{t}_0, \tilde{\mathfrak{t}}] = [\mathfrak{t}_0, \tilde{\mathfrak{t}}^p] \subset [\mathfrak{t}_0, C(\mathfrak{h}_p)^p] = (0)$, so that $\mathfrak{t}_0$ respects the root space decomposition of $\widetilde{L}$ relative to $\tilde{\mathfrak{t}}$. In other words, $[\mathfrak{t}_0, \tilde{\mathfrak{h}}] \subset \tilde{\mathfrak{h}}$ and $[\mathfrak{t}_0, L_\gamma] \subset L_\gamma$ for all $\gamma \in \Gamma(\widetilde{L}, \tilde{\mathfrak{t}})$. In particular, $\widetilde{L}(\alpha, \beta)$ is $\mathfrak{t}_0$-invariant.

Let $\mathcal{J}$ be the sum of all $\mathfrak{t}_0$-invariant ideals of $\widetilde{L}(\alpha, \beta)_p$ contained in $\ker\phi_1$ and

$$\phi_2 : \widetilde{L}(\alpha, \beta)_p \longrightarrow \widetilde{L}(\alpha, \beta)_p/\mathcal{J}$$

be the canonical homomorphism. Since $\mathcal{J}$ is $\mathfrak{t}_0$-invariant, the torus $\mathfrak{t}_0$ acts on its image $\phi_2(\widetilde{L}(\alpha, \beta)_p)$. Let $(0) \neq \mathcal{I} \subset \phi_2(\widetilde{L}(\alpha, \beta)_p)$ be a minimal $\mathfrak{t}_0$-invariant ideal, $I := \phi_2^{-1}(\mathcal{I})$, and $I' := I \cap \widetilde{L}(\alpha, \beta)$. By the minimality of $\mathcal{I}$, there are two possibilities: either $\mathcal{I} \subset \phi_2(\widetilde{L}(\alpha, \beta))$ or $\mathcal{I} \cap \phi_2(\widetilde{L}(\alpha, \beta)) = (0)$. Suppose the second possibility occurs. Then $[I, \widetilde{L}(\alpha, \beta)_p] \subset I' \subset \ker\phi_2 = \mathcal{J} \subset \ker\phi_1$, hence $\phi_1(I) \subset C(\mathfrak{g}(1,1)) = (0)$. But then $I \subset \mathcal{J}$, by the definition of $\mathcal{J}$, and $\mathcal{I} = (0)$, a contradiction. So $\mathcal{I} \subset \phi_2(\widetilde{L}(\alpha, \beta))$. Moreover, $I' \not\subset \ker\phi_1$. By part (b), $\widetilde{L}(\alpha, \beta) = I' + \tilde{\mathfrak{h}} \cap \widetilde{R}(\alpha, \beta)$. Also, $\phi_2(I') = \mathcal{I}$, by the minimality of $\mathcal{I}$. Since $\tilde{\mathfrak{h}}$ is nilpotent, this shows that $\mathcal{I} = \mathcal{I}^{(\infty)} = \phi_2(\widetilde{L}(\alpha, \beta))^{(\infty)}$ is the unique minimal $\mathfrak{t}_0$-invariant ideal of $\phi_2(\widetilde{L}(\alpha, \beta))$. In particular, $\mathcal{I}$ is nonsolvable. By Block's theorem, there are $m \in \mathbb{N}_0$ and a simple Lie algebra S such that $\mathcal{I} \cong S \otimes A(m; \underline{1})$ as Lie algebras. Since $\mathcal{J} \subset \ker\phi_1$ there exists a Lie algebra epimorphism

$$\phi_3 : \widetilde{L}(\alpha, \beta)_p/\mathcal{J} \twoheadrightarrow \widetilde{L}(\alpha, \beta)_p/\ker\phi_1 \cong \mathfrak{g}(1,1)$$

such that $\phi_1 = \phi_3 \circ \phi_2$. Note that $\phi_2(\tilde{\mathfrak{h}} \cap \widetilde{R}(\alpha, \beta)) \subset \ker\phi_3$ and the image $\phi_3(\mathcal{I}) \cong \mathfrak{g}(1,1)$ is simple. Clearly, $S \otimes A(m; \underline{1})_{(1)}$ is the unique maximal ideal of $S \otimes A(m; \underline{1})$. Since this ideal is nilpotent so is

$$\phi_2\big(\widetilde{R}(\alpha, \beta)\big) = \phi_2\big(\widetilde{L}(\alpha, \beta)\big) \cap \ker\phi_3 = \mathcal{I} \cap \ker\phi_3 + \phi_2\big(\tilde{\mathfrak{h}} \cap \widetilde{R}(\alpha, \beta)\big),$$

while $S \cong \mathfrak{g}(1,1)$.

(d) Since the ideal $\mathcal{I}$ is $\mathfrak{t}_0$-invariant, ϕ_2 gives rise to a natural homomorphism of restricted Lie algebras $\widetilde{L}(\alpha, \beta)_p + \mathfrak{t}_0 \to \operatorname{Der}\mathcal{I}$. Since $\mathcal{I} \cong S \otimes A(m; \underline{1})$ the latter induces a restricted homomorphism

$$\Phi : \widetilde{L}(\alpha, \beta)_p + \mathfrak{t}_0 \longrightarrow (\operatorname{Der} S) \otimes A(m; \underline{1}) + \operatorname{Id} \otimes W(m; \underline{1})$$

such that $S \otimes A(m; \underline{1}) \subset \Phi(\widetilde{L}(\alpha, \beta)_p + \mathfrak{t}_0)$ and $\pi_2(\Phi(\widetilde{L}(\alpha, \beta)_p + \mathfrak{t}_0)$ is a transitive subalgebra in $W(m; \underline{1})$ (recall that we denote by π_2 the canonical projection

$$(\operatorname{Der} S) \otimes A(m; \underline{1}) + \operatorname{Id} \otimes W(m; \underline{1}) \longrightarrow W(m; \underline{1}),$$

see [9] for more detail). Since $S \cong \operatorname{Der} S$ there exists a restricted transitive Lie subalgebra $\mathcal{D} \subset W(m; \underline{1})$ such that

$$\Phi\big(\widetilde{L}(\alpha, \beta)_p + \mathfrak{t}_0\big) = S \otimes A(m; \underline{1}) + \operatorname{Id} \otimes \mathcal{D}.$$

It follows from the maximality of $\widetilde{R}(\alpha,\beta)$ and our discussion in part (c) that

$$\Phi\big(\widetilde{R}(\alpha,\beta)\big)=S\otimes A(m;\underline{1})_{(1)}+\mathrm{Id}\otimes\mathcal{D}_0,$$

where $\mathcal{D}_0$ is a subalgebra of $\mathcal{D}$. As $\Phi(\widetilde{R}(\alpha,\beta))$ is a Lie algebra it must be that $\mathcal{D}_0\subset W(m;\underline{1})_{(0)}$. Let $\widetilde{\mathcal{D}}_0$ denote the p-envelope of $\mathcal{D}_0$ in $W(m;\underline{1})_{(0)}$. As $S\cong\mathfrak{g}(1,1)$ is a restricted Lie algebra and $\widetilde{L}(\alpha,\beta)\subset I'+\widetilde{R}(\alpha,\beta)$, we have

$$\Phi\big(\widetilde{L}(\alpha,\beta)_p\big)\subset S\otimes A(m;\underline{1})+\mathrm{Id}\otimes\widetilde{\mathcal{D}}_0.$$

As $\mathcal{D}$ is transitive this shows that so must be $\Phi(\mathfrak{t}_0)$. Thanks to [9, Theorem 2.6] it can be assumed that there exist toral elements $t_1,\dots,t_m\in\mathfrak{t}_0$ and a subtorus $\mathfrak{t}_0'\subset\mathfrak{t}_0$ such that $\mathfrak{t}_0=\mathfrak{t}_0'\oplus\bigoplus_{i=1}^m Ft_i$ and

$$\Phi(t_i)=\mathrm{Id}\otimes(1+x_i)\partial_i\quad\forall i\leqslant m,$$
$$\Phi(x)=\lambda_1(x)\otimes 1+\mathrm{Id}\otimes\lambda_2(x)\quad\forall x\in\mathfrak{t}_0',$$

where λ_1 and λ_2 are restricted homomorphisms from $\mathfrak{t}_0'$ into S and $W(m;\underline{1})_{(0)}$, respectively. As $[\lambda_2(\mathfrak{t}_0'),(1+x_i)\partial_i]\in(\pi_2\circ\Phi)(\mathfrak{t}_0^{(1)})=(0)$ for all $i\leqslant m$ and $\lambda_2(\mathfrak{t}_0')$ lies in $W(m;\underline{1})_{(0)}$ it must be that $\lambda_2=0$. So

$$\Phi(\mathfrak{t}_0)=\left(\sum_{i=1}^m F\mathrm{Id}\otimes(1+x_i)\partial_i\right)\oplus\big(\lambda_1\big(\mathfrak{t}_0'\big)\otimes F\big).$$

As $\widetilde{L}(\alpha,\beta)_p$ is $\mathfrak{t}_0$-invariant and $\widetilde{\mathcal{D}}_0=(\pi_2\circ\Phi)(\widetilde{L}(\alpha,\beta)_p)\subset W(m;\underline{1})_{(0)}$ the transitivity of $\Phi(\mathfrak{t}_0)$ yields $\widetilde{\mathcal{D}}_0=(0)$. But then $\Phi(\widetilde{L}(\alpha,\beta)_p)=\mathfrak{g}(1,1)\otimes A(m;\underline{1})$, a perfect Lie algebra. Consequently,

$$\Phi\big(\widetilde{L}(\alpha,\beta)_p\big)=\Phi\big(\big(\widetilde{L}(\alpha,\beta)_p\big)^{(\infty)}\big)=\Phi\big(L(\alpha,\beta)\big).$$

(e) Recall that $\mathfrak{t}\subset\mathfrak{t}_0\cap\widetilde{L}(\alpha,\beta)$. Then $\Phi(\mathfrak{t})\subset\mathfrak{g}(1,1)\otimes A(m;1)$ forcing $\mathfrak{t}\subset\mathfrak{t}_0'$, so that $\lambda_1(\mathfrak{t})\subset\lambda_1(\mathfrak{t}_0')$. Both $\lambda_1(\mathfrak{t})$ and $\lambda_1(\mathfrak{t}_0')$ are tori in $\mathfrak{g}(1,1)$. Since $MT(\mathfrak{g}(1,1))=2$ (see [7, Lemma 4.4(ii)]), one has

$$0\leqslant\dim\lambda_1(\mathfrak{t})\leqslant\dim\lambda_1\big(\mathfrak{t}_0'\big)\leqslant 2.$$

Since $\mathfrak{h}=\mathfrak{c}_L(\mathfrak{t})$ is nilpotent so is $\mathfrak{c}_{\mathfrak{g}(1,1)}(\lambda_1(\mathfrak{t}))$. Therefore, $\dim\lambda_1(\mathfrak{t})\neq 0$. If $\dim\lambda_1(\mathfrak{t})=1$ then [7, Theorem 1] implies that $\mathfrak{c}_{\mathfrak{g}(1,1)}(\lambda_1(\mathfrak{t}))$ acts triangulably on $\mathfrak{g}(1,1)$.

Suppose $\dim\lambda_1(\mathfrak{t})=2$. Then $\lambda_1(\mathfrak{t})=\lambda_1(\mathfrak{t}_0')$ and

$$\Phi\big(\mathfrak{c}_L(\mathfrak{t}_0)\big)=\Phi\big(\mathfrak{c}_{L(\alpha,\beta)}(\mathfrak{t}_0)\big)=\mathfrak{c}_{\mathfrak{g}(1,1)}\big(\lambda_1\big(\mathfrak{t}_0'\big)\big)\otimes F=\mathfrak{c}_{\mathfrak{g}(1,1)}\big(\lambda_1(\mathfrak{t})\big)\otimes F.$$

Since $\mathfrak{c}_L(\mathfrak{t}_0)$ is triangulable by assumption we obtain that $\mathfrak{c}_{\mathfrak{g}(1,1)}(\lambda_1(\mathfrak{t}))$ acts triangulably on $\mathfrak{g}(1,1)$ in all cases.

The evaluation map $f \mapsto f(0)$ from $A(m;\underline{1})$ onto F induces a natural homomorphism of restricted Lie algebras $\epsilon : \mathfrak{g}(1,1) \otimes A(m;\underline{1}) \twoheadrightarrow \mathfrak{g}(1,1)$. Chasing through the maps shows that ϵ sends $\Phi(\tilde{\mathfrak{h}}) \subset \mathfrak{g}(1,1) \otimes A(m;\underline{1})$ onto a restricted subalgebra in $\mathfrak{g}(1,1)$ isomorphic to $\mathfrak{c}_{\mathfrak{g}(1,1)}(\phi_1(\tilde{\mathfrak{t}}))$. By part (a), the latter acts nontriangulably on $\mathfrak{g}(1,1)$. Hence $\Phi(\tilde{\mathfrak{h}})$ acts nontriangulably on $\mathfrak{g}(1,1) \otimes A(m;\underline{1})$. Since $\tilde{\mathfrak{h}}^{(1)} = \mathfrak{h}^{(1)}$ so does $\Phi(\mathfrak{h})$, too. On the other hand, it is easy to see that ϵ sends $\Phi(\mathfrak{h}) \subset \Phi(\tilde{\mathfrak{h}})$ onto $\mathfrak{c}_{\mathfrak{g}(1,1)}(\lambda_1(\mathfrak{t}))$, a triangulable subalgebra in $\mathfrak{g}(1,1)$. This entails that $\Phi(\mathfrak{h})$ acts triangulably on $\mathfrak{g}(1,1) \otimes A(m;\underline{1})$. Thus the assumption we made in (a) leads to a contradiction. Therefore, $\mathfrak{h}$ acts triangulably on L as desired. □

Recall that for a subalgebra A of a Lie algebra M the *toral rank of A in M*, denoted $TR(A, M)$, is defined as

$$TR(A,M) := MT\big(\mathcal{A}/\big(\mathcal{A} \cap C(\mathcal{M})\big)\big),$$

where $\mathcal{M}$ is any p-envelope of M and $\mathcal{A}$ is the restricted subalgebra in $\mathcal{M}$ generated by A (this is known to be independent of the choice of $\mathcal{M}$, see [19, Theorem 1.3]).

Theorem 3.2. *Let $\mathfrak{g}$ be a perfect Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$ with $TR(\mathfrak{h},\mathfrak{g}) = 1$. Then the following hold*:

(1) $\mathfrak{h}$ *acts triangulably on* $\mathfrak{g}$;
(2) $\operatorname{rad}\mathfrak{g}$ *is the unique maximal ideal in* $\mathfrak{g}$;
(3) $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ *is one of* $\mathfrak{sl}(2)$, $W(1;\underline{n})$, $H(2;\underline{n};\Psi)^{(2)}$.

Proof. Let $\mathfrak{g}_p$ be a p-envelope of $\mathfrak{g}$, $\mathfrak{h}_p$ the p-envelope of $\mathfrak{h}$ in $\mathfrak{g}_p$, and $\mathfrak{t}$ the unique maximal torus in $\mathfrak{h}_p$. Then $\dim \mathfrak{t}/\mathfrak{t} \cap C(\mathfrak{g}_p) = TR(\mathfrak{h},\mathfrak{g}) = 1$. There is a nonzero toral element t in $\mathfrak{t}$ such that $\mathfrak{t} = Ft \oplus \mathfrak{t} \cap C(\mathfrak{g}_p)$. All eigenvalues of $\operatorname{ad} t$ lie in $\mathbb{F}_p$. Let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{i\in\mathbb{F}_p^*} \mathfrak{g}_i$ be the eigenspace decomposition of $\mathfrak{g}$ relative to $\operatorname{ad} t$.

(a) Let I be any ideal in $\mathfrak{g}$. Clearly, $I = I \cap \mathfrak{h} \oplus \sum_{i\in\mathbb{F}_p^*} I \cap \mathfrak{g}_i$ is an $\mathbb{F}_p$-grading of the Lie algebra I. If $I \cap \mathfrak{h}$ acts nilpotently on I then I is solvable (see [19, Proposition 1.14]). But then $I \subset \operatorname{rad}\mathfrak{g}$. If $I \cap \mathfrak{h}$ acts nonnilpotently on I then there is $h \in I \cap \mathfrak{h}$ whose semisimple part h_s (in $\mathfrak{g}_p$) is not contained in $C(\mathfrak{g}_p)$. Since $h_s \in \mathfrak{t}$, we have that $h_s = at + z$ for some $a \in \mathbb{F}^*$ and $z \in C(\mathfrak{g}_p)$. As I is an ideal, this gives $\sum_{i\in\mathbb{F}_p^*} \mathfrak{g}_i \subset I$, so that $\mathfrak{g} = \mathfrak{h} + I$. As $\mathfrak{g}$ is perfect and $\mathfrak{h}$ is nilpotent we get $\mathfrak{g} = I$. This proves (2).

(b) Let $\bar{\mathfrak{h}}$ denote the image of $\mathfrak{h}$ in $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$, and $\bar{t}$ be the image of t in $\operatorname{Der}(\mathfrak{g}/\operatorname{rad}\mathfrak{g})$. By part (a), $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{c}_{\mathfrak{g}/\operatorname{rad}\mathfrak{g}}(\bar{t}) = \bar{\mathfrak{h}}$. Besides, the maximal torus of the p-envelope of $\bar{\mathfrak{h}}$ in $\operatorname{Der}(\mathfrak{g}/\operatorname{rad}\mathfrak{g})$ is nothing but $F\bar{t}$. So [7, Theorem 1] applies and shows that $\bar{\mathfrak{h}}^{(1)}$ acts nilpotently on $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. This, in turn, shows that

$$\big[\big(\mathfrak{h}_p^{(1)}\big) \cap \mathfrak{t}, \mathfrak{g}\big] \subset \operatorname{rad}\mathfrak{g}.$$

If $(\mathfrak{h}^{(1)}{}_p) \cap \mathfrak{t} \not\subset C(\mathfrak{g}_p)$ then $\mathfrak{g} = \mathfrak{h} + \operatorname{rad}\mathfrak{g}$, a contradiction. Thus $(\mathfrak{h}^{(1)}{}_p) \cap \mathfrak{t} \subset C(\mathfrak{g}_p)$ which means that all elements of $\mathfrak{h}^{(1)}$ act nilpotently on $\mathfrak{g}$. This proves (1).

(c) We have already established that $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is a simple Lie algebra and $\bar{\mathfrak{h}}$ is a Cartan subalgebra of toral rank 1 in $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. Now [7, Theorem 2] yields (3). □

One often obtains important information of L by studying 2-sections of L relative to a torus $\mathfrak{t} \subset L_p$. This reduces the investigation to simple Lie algebras of smaller absolute toral rank.

Proposition 3.3 (cf. [2, Lemma 10.21]). *Let $\mathfrak{t}_0 \subset L_p$ be a torus such that $\mathfrak{h} = \mathfrak{c}_L(\mathfrak{t}_0)$ is nilpotent, $\mathfrak{t}$ be the unique maximal torus in $\mathfrak{t}_0 + \mathfrak{h}_p \subset L_p$, and suppose one of the following two conditions holds for some $\alpha, \beta \in \Gamma(L, \mathfrak{t})$:*

(a) *there are $h_1 \in [L_\alpha, L_{-\alpha}]$ and $h_2 \in [L_\beta, L_{-\beta}]$ such that $\alpha(h_1) = 0$, $\beta(h_1) \neq 0$, and $\alpha(h_2) \neq 0$;*
(b) *$\mathfrak{t}$ is a maximal torus of L_p and there are $u \in L_\alpha$ and $h_2 \in [L_\beta, L_{-\beta}]$ such that $\beta(u^p) \neq 0$ and $\alpha(h_2) \neq 0$.*

Set $\mathfrak{g} := \sum_{\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)\setminus\{0\}} (L_\gamma + [L_\gamma, L_{-\gamma}])$. Then the following hold:

(1) *Every ideal of $\mathfrak{g}$ is $\mathfrak{t}$-invariant.*
(2) *If I is a maximal ideal of $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathfrak{g}/I$ is the canonical homomorphism then $h_1, h_2 \notin I$ (respectively $u, h_2 \notin I$) and $\pi(\mathfrak{g})$ is simple with $TR(\pi(\mathfrak{g})) \geqslant 2$. Moreover, if* (a) *holds for L then $\pi(\mathfrak{h} \cap \mathfrak{g})$ is a Cartan subalgebra in $\pi(\mathfrak{g})$ with*

$$\dim \pi(\mathfrak{h} \cap \mathfrak{g})/\big(\pi(\mathfrak{h} \cap \mathfrak{g}) \cap \operatorname{nil} \pi(\mathfrak{h} \cap \mathfrak{g})_p\big) = 2 \quad \textit{and} \quad TR\big(\pi(\mathfrak{h} \cap \mathfrak{g}), \pi(\mathfrak{g})\big) = 2,$$

where the pth powers are taken in $\pi(\mathfrak{g})_p \subset \operatorname{Der}\mathfrak{g}$.
(3) *Suppose further that $\mathfrak{t}$ is a maximal torus in L_p. Then $\operatorname{rad}\mathfrak{g}$ is nilpotent and, moreover, the unique maximal ideal in $\mathfrak{g}$. If $\mathfrak{h}'$ is a subalgebra in $\mathfrak{c}_{L_p}(\mathfrak{t})$ such that all elements in the union $\bigcup_{h \in \mathfrak{h}'} (\operatorname{ad} h)^{p-1}(\mathfrak{h} \cap \mathfrak{g})$ act nilpotently on $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ then $\operatorname{rad}\mathfrak{g}$ is $\mathfrak{h}'$-invariant.*

Proof. (a) Let $t_1, t_2 \in \mathfrak{t}$ denote the semisimple parts of $h_1, h_2 \in \mathfrak{h}$ in case (a) and of u^p, h_2 in case (b). In case (a), $\alpha(h_1) = 0$ by our assumption, while in case (b) the maximality of $\mathfrak{t}$ implies that $\alpha(u^p) = 0$. Thus

$$\alpha(t_1) = 0, \qquad \beta(t_1) \neq 0, \qquad \alpha(t_2) \neq 0.$$

Consequently, $t = Ft_1 \oplus Ft_2 \oplus (\mathfrak{t} \cap \ker\alpha \cap \ker\beta)$. Since $\mathfrak{t} \cap \ker\alpha \cap \ker\beta$ annihilates $\mathfrak{g}$ and $t_1, t_2 \in \mathfrak{g}_p$ every ideal of $\mathfrak{g}$ is $\mathfrak{t}$-invariant.

(b) Given an ideal I of $\mathfrak{g}$, we denote by I_p be the p-envelope of I in L_p. Suppose $I_p \cap \mathfrak{t} \not\subset \ker\alpha \cap \ker\beta$. Then there is $t \in I_p \cap \mathfrak{t}$ with $\alpha(t) \neq 0$ or $\beta(t) \neq 0$. Suppose $\alpha(t) \neq 0$ the case $\beta(t) \neq 0$ being similar. Since $L_{i\alpha} = [\mathfrak{t}, L_{i\alpha}] \subset I$ for all $i \in \mathbb{F}_p^*$, we have that $h_1 \in I$

in case (a) and $u \in I$ in case (b). Then $t_1 \in I_p \cap \mathfrak{t}$ in both cases showing that $L_\gamma \subset I$ for all $\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\}$. This gives $I = \mathfrak{g}$. Thus if $I \neq \mathfrak{g}$ then $I_p \cap \mathfrak{t}$ centralizes $\mathfrak{g}$.

(c) Let I be a maximal ideal of $\mathfrak{g}$. Since $t_1, t_2 \in \mathfrak{g}_p$ one has $\mathfrak{g}^{(1)} = \mathfrak{g}$. So $\mathfrak{g}/I$ is simple. Since $I_p \cap \mathfrak{t} \subset \ker\alpha \cap \ker\beta$, by part (b), it is clear that $h_1, h_2 \notin I$ in case (a) and $u, h_2 \notin I$ in case (b). This implies that $L_\alpha \not\subset I$ and $L_\beta \not\subset I$. So $\pi(h_1)$ and $\pi(h_2)$ (respectively $\pi(u)$ and $\pi(h_2)$) generate a torus in $\pi(\mathfrak{g})_p$ which distinguishes $\pi(\mathfrak{g}_\alpha) \neq (0)$ and $\pi(\mathfrak{g}_\beta) \neq (0)$. From this it is immediate that $TR(\pi(\mathfrak{g})) \geqslant 2$. If (a) holds then $\pi(\mathfrak{h} \cap \mathfrak{g})$ is self-normalizing, hence a Cartan subalgebra in $\pi(\mathfrak{g})$. Moreover, $\pi(h_1)$ and $\pi(h_2)$ are linearly independent modulo $\operatorname{nil}\pi(\mathfrak{h} \cap \mathfrak{g})_p$, so that

$$2 \leqslant TR\big(\pi(\mathfrak{h} \cap \mathfrak{g}), \pi(\mathfrak{g})\big) \leqslant TR\big(\pi(\mathfrak{t}), \pi(\mathfrak{g})_p\big) = 2.$$

(d) Now suppose that $\mathfrak{t}$ is a maximal torus in L_p. Let $I \subset \mathfrak{g}$ be a proper ideal of $\mathfrak{g}$ and $x \in I_\gamma = \mathfrak{g}_\gamma \cap I$ where $\gamma \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$. As $\mathfrak{t}$ is maximal, $x^{p^r} \in \mathfrak{t}$ for $r \gg 0$. We have shown in (b) that $I_p \cap \mathfrak{t}$ centralizes $\mathfrak{g}$. It follows that $\bigcup_{\gamma \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta} \operatorname{ad}_\mathfrak{g} I_\gamma$ is a weakly closed set consisting of nilpotent endomorphisms. So the Engel–Jacobson theorem yields that I acts nilpotently on $\mathfrak{g}$. Therefore, $I \subset \operatorname{rad}\mathfrak{g}$. Moreover, $\mathfrak{g} \neq \operatorname{rad}\mathfrak{g}$, for $\mathfrak{g}^{(1)} = \mathfrak{g}$. Then $\operatorname{rad}\mathfrak{g}$ is nilpotent.

(e) Let $\mathfrak{h}'$ be a Lie subalgebra in $\mathfrak{c}_{L_p}(\mathfrak{t})$. Clearly, $[\mathfrak{h}', L_\gamma] \subset L_\gamma$ for all γ. Then $[\mathfrak{h}', \mathfrak{g}] \subset \mathfrak{g}$ forcing $[\mathfrak{h}', \mathfrak{h} \cap \mathfrak{g}] \subset \mathfrak{h} \cap \mathfrak{g}$. Let R be the maximal $\mathfrak{h}'$-invariant solvable ideal in $\mathfrak{g}$, and let

$$\phi : \mathfrak{h}' + \mathfrak{g} \longrightarrow (\mathfrak{h}' + \mathfrak{g})/R$$

denote the canonical homomorphism. Let $\mathcal{J}$ be a nonzero $\mathfrak{h}'$-invariant ideal of $\phi(\mathfrak{g})$, and $I = \phi^{-1}(\mathcal{J})$. Then I is an $\mathfrak{h}'$-invariant ideal of $\mathfrak{g}$ satisfying $I \not\subset \operatorname{rad}\mathfrak{g}$. So part (d) of this proof shows that $I = \mathfrak{g}$ and, as a consequence, $\mathcal{J} = \phi(\mathfrak{g})$. This means that $\phi(\mathfrak{g})$ is $\mathfrak{h}'$-simple. By Block's theorem, there is a simple algebra S and $m \in \mathbb{N}_0$ such that

$$\phi(\mathfrak{g}) \cong S \otimes A(m; \underline{1}) \subset \phi(\mathfrak{h}' + \mathfrak{g}) \subset (\operatorname{Der} S) \otimes A(m; \underline{1}) + \operatorname{Id} \otimes W(m; \underline{1}).$$

By part (d) of this proof, $\mathfrak{g}/\operatorname{rad}\mathfrak{g} = \pi(\mathfrak{g}) \cong S$. The associative algebra $A(m; \underline{1})$ is isomorphic to the centroid of $\phi(\mathfrak{g}) \cong \pi(\mathfrak{g}) \otimes A(m; \underline{1})$, hence acts on $\phi(\mathfrak{g})$ via

$$(x \otimes f, a) \longmapsto (x \otimes f) \bullet a := x \otimes fa, \quad \forall x \in \pi(\mathfrak{g}),\ \forall f, a \in A(m; \underline{1}).$$

Decomposing $\phi(h_2) \bullet a \in \phi(\mathfrak{g})$ into root spaces relative to $\mathfrak{t}$ and applying $\operatorname{ad}\phi(h_1)$ and $\operatorname{ad}\phi(h_2)$ in case (a) (respectively $\operatorname{ad}\phi(u)$ and $\operatorname{ad}\phi(h_2)$ in case (b)), we observe that

$$\phi(h_2) \bullet A(m; \underline{1}) \subset \phi(\mathfrak{h} \cap \mathfrak{g}).$$

Suppose there is $h \in \mathfrak{h}'$ such that $(\pi_2 \circ \phi)(h) \notin W(m; \underline{1})_{(0)}$. Then $(\pi_2 \circ \phi)(h) = E + \sum_{i=1}^m a_i \partial_i$ where $E \in W(m; \underline{1})_{(0)}$ and $a_{i_0} \neq 0$ for some $i_0 \leqslant m$. Hence

$$\big(\operatorname{ad}\phi(h)\big)^{p-1}\big(\phi(h_2) \bullet x_{i_0}^{p-1}\big) \equiv (p-1)!a_0^{p-1}\pi(h_2) \otimes 1 \quad \big(\operatorname{mod} S \otimes A(m; \underline{1})_{(1)}\big).$$

Thus for each $y \in \phi^{-1}(\phi(h_2) \bullet x_{i_0}^{p-1}) \cap (\mathfrak{h} \cap \mathfrak{g})$ we have $\alpha(\phi((\operatorname{ad} h)^{p-1}(y))) \neq 0$. However, $\pi(\operatorname{ad} h)^{p-1}(y))$ acts nilpotently on $\pi(\mathfrak{g})$ by our assumption. So $S \otimes A(m; \underline{1})_{(1)}$ is a $\phi(\mathfrak{h}')$-invariant ideal of $\phi(\mathfrak{g})$. Since $\phi(\mathfrak{g})$ is $\phi(\mathfrak{h}')$-simple by our earlier remark, $m = 0$ necessarily holds. Then $\phi(\mathfrak{g}) = \pi(\mathfrak{g})$ is simple and, consequently, $\ker \phi = \operatorname{rad} \mathfrak{g}$ is $\mathfrak{h}'$-invariant. □

Proposition 3.4. *Let $\mathfrak{t}_0 \subset L_p$ be a torus such that $\mathfrak{h} := \mathfrak{c}_L(\mathfrak{t}_0)$ is nilpotent. Let $\mathfrak{t}$ denote the maximal torus in $\mathfrak{t}_0 + \mathfrak{h}_p \subset L_p$ and $\alpha \in \Gamma(L, \mathfrak{t})$ be such that $\alpha(\mathfrak{h}) \neq 0$. Then*

$$\alpha\bigl([L_\alpha, L_{-\alpha}]^2\bigr) = 0 \quad \textit{and} \quad [L_\alpha, L_{-\alpha}]^3 \subset \operatorname{nil} \mathfrak{h}_p.$$

Proof. (a) Suppose $\alpha([L_\alpha, L_{-\alpha}]^2) \neq 0$ and set

$$\mathfrak{g} := \sum_{i \in \mathbb{F}_p^*} \bigl(L_{i\alpha} + [L_{i\alpha}, L_{-i\alpha}]\bigr).$$

Our assumption implies that there is $h \in [L_\alpha, L_{-\alpha}]^2 \subset (\mathfrak{h} \cap \mathfrak{g})^{(1)}$ such that $\alpha(h) \neq 0$. Then $\mathfrak{g}^{(1)} = \mathfrak{g},\ TR(\mathfrak{h} \cap \mathfrak{g}, \mathfrak{g}) \geqslant 1,$ and $\mathfrak{h} \cap \mathfrak{g}$ is self-normalizing in $\mathfrak{g}$. In particular, $\mathfrak{h} \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. On the other hand, $TR(\mathfrak{h} \cap \mathfrak{g}, \mathfrak{g}) \leqslant \dim \mathfrak{t}/\ker \alpha = 1$. But then Theorem 3.2 applies showing that $(\mathfrak{h} \cap \mathfrak{g})^{(1)}$ acts nilpotently on $\mathfrak{g}$. So our present assumption leads to a contradiction which proves that $\alpha([L_\alpha, L_{-\alpha}]^2) = 0$.

(b) Suppose $[L_\alpha, L_{-\alpha}]^3 \not\subset \operatorname{nil} \mathfrak{h}_p$. Since $[L_\alpha, L_{-\alpha}]^3$ is an ideal of $\mathfrak{h}_p$ and $\operatorname{nil} \mathfrak{h}_p$ is the sum of all p-nilpotent ideals in $\mathfrak{h}_p$, there is $h \in [L_\alpha, L_{-\alpha}]^3$ whose semisimple part h_s is nonzero. Then there is $\kappa \in \Gamma(L, \mathfrak{t})$ with $\kappa(h_s) \neq 0$. It follows that the set

$$\Omega := \bigl\{\kappa \in \Gamma(L, \mathfrak{t}) \bigm| \kappa\bigl([L_\alpha, L_{-\alpha}]^3\bigr) \neq 0\bigr\}$$

is not empty. Since L is simple, we have, by Schue's lemma, that

$$\mathfrak{h} = \sum_{\kappa \in \Omega} [L_\kappa, L_{-\kappa}].$$

If the union $\bigcup_{\kappa \in \Omega} \operatorname{ad}_{L_\alpha} [L_\kappa, L_{-\kappa}]$ consisted entirely of nilpotent endomorphisms then the Engel–Jacobson theorem would imply that $\alpha(\mathfrak{h}) = 0$. Since this is not the case, by assumption, there is $\kappa \in \Omega$ such that $[L_\kappa, L_{-\kappa}]$ acts nonnilpotently on L_α. This means that $\alpha([L_\kappa, L_{-\kappa}]) \neq 0$. We deduce that

$$\alpha\bigl([L_\alpha, L_{-\alpha}]^3\bigr) = 0, \qquad \kappa\bigl([L_\alpha, L_{-\alpha}]^3\bigr) \neq 0, \qquad \alpha\bigl([L_\kappa, L_{-\kappa}]\bigr) \neq 0,$$

thereby verifying the conditions of Proposition 3.3 (case (a)).

(c) Now define

$$\mathfrak{g} := \sum_{\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\kappa)\setminus\{0\}} \big(L_\gamma + [L_\gamma, L_{-\gamma}]\big)$$

and let $I \neq \mathfrak{g}$ be a maximal ideal of $\mathfrak{g}$. In accordance with Proposition 3.3, put $\pi(\mathfrak{g}) = \mathfrak{g}/I$ and let $\bar{\mathfrak{h}}_0 := (\mathfrak{g} \cap \mathfrak{h} + I)/I \subset \pi(\mathfrak{g})$. Proposition 3.3 yields that $\pi(\mathfrak{g})$ is simple, that $\bar{\mathfrak{h}}_0$ is a Cartan subalgebra of toral rank 2 in $\pi(\mathfrak{g})$ and, as a consequence, that $[\pi(\mathfrak{g}_\alpha), \pi(\mathfrak{g}_{-\alpha})]^3$ acts nonnilpotently on $\pi(\mathfrak{g})$. Then [7, Theorem 1] shows that $\pi(\mathfrak{g})$ is isomorphic to the 125-dimensional Melikian algebra $\mathfrak{g}(1,1)$ and $\bar{\mathfrak{h}}_0$ is a nontriangulable Cartan subalgebra in $\pi(\mathfrak{g})$. As $\mathfrak{g}(1,1)$ is restricted there is a nonzero toral element $t_\alpha \in \bar{\mathfrak{h}}_0$ such that $\alpha(t_\alpha) = 0$. According to [7, Lemma 4.3], all nontriangulable Cartan subalgebras in $\mathfrak{g}(1,1)$ are conjugate under $\operatorname{Aut}\mathfrak{g}(1,1)$. Combining this result with [13, Theorem 2.1], it is easy to observe that there exists $\sigma \in \operatorname{Aut}\mathfrak{g}(1,1)$ such that $\sigma(\bar{\mathfrak{h}}_0) = \mathfrak{c}_{\mathfrak{g}(1,1)}(F(1+x_1)\partial_1 + F(1+x_2)\partial_2)$ and $\sigma(t_\alpha) = (1+x_1)\partial_1$. The description in [7, p. 697] yields $\dim \bar{\mathfrak{h}}_0 = 5$, $\dim C(\bar{\mathfrak{h}}_0) = 2$, and

$$\bar{\mathfrak{h}}_0^3 \subset C(\bar{\mathfrak{h}}_0) \subset \sum_{i \in \mathbb{F}_p^*} \big[\pi(\mathfrak{g}_{i\alpha}), \pi(\mathfrak{g}_{-i\alpha})\big] \subsetneq \bar{\mathfrak{h}}_0.$$

It follows that $C(\bar{\mathfrak{h}}_0)$ has codimension $\leqslant 2$ in $[\pi(\mathfrak{g}_\alpha), \pi(\mathfrak{g}_{-\alpha})]$. Then $[\pi(\mathfrak{g}_\alpha), \pi(\mathfrak{g}_{-\alpha})]^2 \subset C(\bar{\mathfrak{h}}_0)$ forcing $[\pi(\mathfrak{g}_\alpha), \pi(\mathfrak{g}_{-\alpha})]^3 = (0)$. However, this is impossible as the latter space acts nonnilpotently on $\pi(\mathfrak{g})$. This contradiction proves the proposition. □

We are now ready to determine 1-sections.

Theorem 3.5. *Let $\mathfrak{t}_0 \subset L_p$ be a torus such that $\mathfrak{h} := \mathfrak{c}_L(\mathfrak{t}_0)$ is nilpotent and $\mathfrak{t}$ be the maximal torus of $\mathfrak{t}_0 + \mathfrak{h}_p \subset L_p$. Let $\alpha \in \Gamma(L, \mathfrak{t})$. The following are equivalent*:

(i) *$L(\alpha)$ is solvable*;
(ii) *$\alpha([L_{i\alpha}, L_{-i\alpha}]) = 0$ for all $i \in \mathbb{F}_p^*$.*

Proof. Let $L(\alpha)' = \sum_{i \in \mathbb{F}_p^*}(L_{i\alpha} + [L_{i\alpha}, L_{-i\alpha}])$. Suppose $\alpha([L_{i\alpha}, L_{-i\alpha}]) = 0$ for all $i \in \mathbb{F}_p^*$. Then the union $\bigcup_{i \in \mathbb{F}_p^*} \operatorname{ad}[L_{i\alpha}, L_{-i\alpha}]$ consists of endomorphisms acting nilpotently on $L(\alpha)'$. By [19, (1.14)], $L(\alpha)'$ is solvable. Hence so is $L(\alpha) = \mathfrak{h} + L(\alpha)'$. Conversely, if there is $h \in \bigcup_{i \in \mathbb{F}_p^*}[L_{i\alpha}, L_{-i\alpha}]$ such that $\alpha(h) \neq 0$ then $\operatorname{ad} h$ acts invertibly on $\sum_{i \in \mathbb{F}_p^*} L_{i\alpha}$. Then $h \in L(\alpha)^{(\infty)}$ and $L(\alpha)$ is not solvable. □

Theorem 3.6. *Let $\mathfrak{t}_0 \subset L_p$ be a torus such that $\mathfrak{h} := \mathfrak{c}_L(\mathfrak{t}_0)$ is nilpotent and $\mathfrak{t}$ be the maximal torus of $\mathfrak{t}_0 + \mathfrak{h}_p \subset L_p$. Let $\alpha \in \Gamma(L, \mathfrak{t})$ be such that $L(\alpha)$ is nonsolvable. Then the following hold*:

(1) $\operatorname{rad} L(\alpha)$ *is $\mathfrak{t}$-invariant.*
(2) $L[\alpha] = L(\alpha)/\operatorname{rad} L(\alpha)$ *has a unique minimal ideal $S = L[\alpha]^{(\infty)}$.*

(3) *S is $\mathfrak{t}$-invariant and $\mathfrak{c}_S(\mathfrak{t})$ is a Cartan subalgebra of toral rank 1 in S.*
(4) *S is simple and isomorphic to one of $\mathfrak{sl}(2)$, $W(1;\underline{n})$, $H(2;\underline{n};\Psi)^{(2)}$.*

Proof. (a) Let $L(\alpha)' := \sum_{i\in\mathbb{F}_p^*}(L_{i\alpha} + [L_{i\alpha}, L_{-i\alpha}])$. Since $L(\alpha)$ is nonsolvable, Theorem 3.5 shows that there is $i_0 \in \mathbb{F}_p^*$ such that $\alpha([L_{i_0\alpha}, L_{-i_0\alpha}]) \neq 0$. Adjusting α, we may assume that $i_0 = 1$. Choose $h \in [L_\alpha, L_{-\alpha}]$ with $\alpha(h) \neq 0$ and let $t = h^{p^r} \in \mathfrak{t}$ be the semisimple part of h. Then $t \notin \ker\alpha$ yielding $\mathfrak{t} = Ft \oplus \ker\alpha$. Consequently,

$$[\mathfrak{t}, \operatorname{rad} L(\alpha)] = [Ft, \operatorname{rad} L(\alpha)] \subset (\operatorname{ad} h)^{p^r}(\operatorname{rad} L(\alpha)) \subset \operatorname{rad} L(\alpha).$$

This proves (1) and shows that $\operatorname{rad}_{\mathfrak{t}} L(\alpha) = \operatorname{rad} L(\alpha)$ is $\mathfrak{t} + \mathfrak{h}_p$-invariant. Thus $\mathfrak{t} + \mathfrak{h}_p$ acts on $L[\alpha] = L(\alpha)/\operatorname{rad} L(\alpha)$ giving a restricted homomorphism

$$\mathfrak{t} + \mathfrak{h}_p \ni x \mapsto \bar{x} \in \operatorname{Der} L[\alpha].$$

(b) Since $\mathfrak{t} \cap \ker\alpha$ acts trivially on $L(\alpha)$, we have that $\bar{\mathfrak{t}} = F\bar{t}$. We identify α with the corresponding root in $\Gamma(L[\alpha], \bar{\mathfrak{t}})$ so that $\alpha(\bar{t}) = \alpha(t)$. Then $L[\alpha] = \bar{\mathfrak{h}} \oplus \sum_{i\in\mathbb{F}_p^*} L[\alpha]_{i\alpha}$ is the root space decomposition of $L[\alpha]$ relative to $\bar{\mathfrak{t}}$. Since

$$(\operatorname{ad}_{L[\alpha]} \bar{\mathfrak{h}})^{p^r} = (\overline{\operatorname{ad}_{L(\alpha)} \mathfrak{h}})^{p^r}$$

for all r, the unique maximal torus of the p-envelope of $\bar{\mathfrak{h}}$ in $\operatorname{Der} L[\alpha]$ coincides with $F\bar{t}$, the image of the maximal torus of $\mathfrak{h}_p$ in $\operatorname{Der} L[\alpha]$.

(c) Let I be a minimal ideal of $L[\alpha]$. By the preceding remark, the toral element $\bar{t}$ acts on I which turns I into an $\mathbb{F}_p$-graded Lie algebra. If α vanishes on $\mathfrak{c}_I(\bar{\mathfrak{t}}) = \mathfrak{c}_I(\bar{t})$ then $\mathfrak{c}_I(\bar{t})$ acts nilpotently on I. By [19, (1.14)], this would imply that I is solvable. However, $L[\alpha]$ is semisimple. Thus there is $x \in \mathfrak{c}_I(\bar{\mathfrak{t}})$ with $\alpha(x) \neq 0$. As $\bar{\mathfrak{t}}$ is 1-dimensional, $F\bar{t} = F(\operatorname{ad}_{L[\alpha]}(x))^{p^r}$ for $r \gg 0$. Therefore, $L[\alpha] = I + \mathfrak{c}_{L[\alpha]}(\bar{\mathfrak{t}}) = I + \bar{\mathfrak{h}}$. As a consequence, $I = L[\alpha]^{(\infty)}$ is the *unique* minimal ideal in $L[\alpha]$. This description also shows that I is $\bar{\mathfrak{t}}$-invariant and $\operatorname{ad} L[\alpha]$ acts faithfully on I.

(d) Let $L[\alpha]_p$ and I_p denote the p-envelopes of $L[\alpha]$ and I in $\operatorname{Der} I$. Block's theorem says that there exist a simple Lie algebra S and $m \in \mathbb{N}_0$ such that $I \cong S \otimes A(m;\underline{1})$. It also yields a homomorphism of restricted Lie algebras $\Phi : L[\alpha]_p \to \operatorname{Der} I$ such that

$$S \otimes A(m;\underline{1}) \subset \Phi(L[\alpha]_p) \subset (\operatorname{Der} S) \otimes A(m;\underline{1}) + \operatorname{Id} \otimes W(m;\underline{1}).$$

Recall from part (c) that $L[\alpha] = I + \bar{\mathfrak{h}}$. This gives $\Phi(L[\alpha]) = S \otimes A(m;\underline{1}) + \Phi(\bar{\mathfrak{h}})$.

Suppose $m > 0$. Since $\bar{\mathfrak{t}}$ is spanned by an iterated pth power of $\operatorname{ad}_{L[\alpha]} x$ with $x \in I$, we then have $\Phi(\bar{\mathfrak{t}}) \subset \Phi(I_p) \subset (\operatorname{Der} S) \otimes A(m;\underline{1})$. In this situation Φ can be chosen such that $\Phi(\bar{\mathfrak{t}}) = \lambda_1(\bar{\mathfrak{t}}) \otimes F$ where $\lambda_1 : \mathfrak{t} \to \operatorname{Der}\ S$ is an injective restricted homomorphism (see [9, Theorem 2.6]). It follows that $\mathfrak{c}_S(\lambda_1(\bar{\mathfrak{t}}_1)) \otimes A(m;\underline{1}) \subset \Phi(\bar{\mathfrak{h}})$.

Let $\kappa \in \Gamma(L, \mathfrak{t})$ be such that $\kappa(\mathfrak{h}) \neq 0$. By Proposition 3.4, $[L_\kappa, L_{-\kappa}]^3 \subset \operatorname{nil} \mathfrak{h}_p$. Then for any $y \in [L_\kappa, L_{-\kappa}]$ one has

$$(\operatorname{ad} y)^3(\mathfrak{h}) \subset (\operatorname{nil} \mathfrak{h}_p) \cap \mathfrak{h}.$$

Let π_2 be as before and suppose there is $y \subset [L_\kappa, L_{-\kappa}]$ such that $(\pi_2 \circ \Phi)(\bar{y}) \notin W(m; \underline{1})_{(0)}$. Then

$$\Phi(\bar{y}) = E + \mathrm{Id} \otimes \sum_{i=1}^{m} f_i \partial_i,$$

where $E \in \mathfrak{c}_{\operatorname{Der} S}((\lambda_1(\mathfrak{t})) \otimes A(m; \underline{1}),\ f_i \in A(m; \underline{1})$, and $f_{i_0}(0) \neq 0$ for some $i_0 \leqslant m$. Then $(\operatorname{ad} \Phi(\bar{y}))^3(\Phi(\bar{\mathfrak{h}}))$ contains $(\operatorname{ad} \Phi(\bar{y}))^3(\mathfrak{c}_S(\lambda_1(\bar{\mathfrak{t}})) \otimes x_{i_0}^3)$, so that

$$\mathfrak{c}_S\big(\lambda_1(\bar{\mathfrak{t}})\big) \otimes A(m; \underline{1}) \subset \mathfrak{c}_S\big(\lambda_1(\bar{\mathfrak{t}})\big) \otimes A(m; \underline{1})_{(1)} + \big(\operatorname{ad} \Phi(y)\big)^3\big(\Phi(\bar{\mathfrak{h}})\big) \subset \Phi(\bar{\mathfrak{h}}).$$

Since $(\operatorname{ad} \Phi(\bar{y}))^3(\Phi(\bar{\mathfrak{h}}))$ is contained in the p-nilpotent ideal $\Phi((\overline{\operatorname{nil} \mathfrak{h}_p}) \cap \bar{\mathfrak{h}})$ of $\Phi(\bar{\mathfrak{h}})$ this yields that $\mathfrak{c}_S(\lambda_1(\bar{\mathfrak{t}})) \otimes A(m; \underline{1})$ acts nilpotently on $\Phi(L[\alpha])$. As a consequence, $\mathfrak{c}_I(\bar{\mathfrak{t}})$ acts nilpotently on $L[\alpha]$. However, we have seen in part (c) that this not true. Thus

$$(\pi_2 \circ \Phi)(\bar{y}) \in W(m; \underline{1})_{(0)} \quad \big(\forall \kappa \in \Gamma(L, \mathfrak{t}) \text{ with } \kappa(\mathfrak{h}) \neq 0,\ \forall y \in [L_\kappa, L_{-\kappa}]\big).$$

Set $\Omega_1 = \{\kappa \in \Gamma(L, \mathfrak{t}) \mid \kappa(\mathfrak{h}) \neq 0\}$. As α is not solvable it lies in Ω_1 (Theorem 3.5). So $\Omega_1 \neq \emptyset$ whence $\mathfrak{h} = \sum_{\kappa \in \Omega_1} [L_\kappa, L_{-\kappa}]$, by Schue's lemma. Combining this with the above remark, we obtain the inclusion $(\pi_2 \circ \Phi)(\bar{\mathfrak{h}}) \subset W(m; \underline{1})_{(0)}$. But then

$$\Phi\big(L[\alpha]\big) \subset (\operatorname{Der} S) \otimes A(m; \underline{1}) + \mathrm{Id} \otimes W(m; \underline{1})_{(0)}$$

implying that $S \otimes A(m; \underline{1})_{(1)}$ is a solvable ideal of $\Phi(L[\alpha])$. As Φ is injective this contradicts the semisimplicity of $L[\alpha]$. Thus $m = 0$ and $I \cong S$ is simple.

(e) Recall that $\bar{\mathfrak{t}} = F(\operatorname{ad} x)^{p^r}$ for some $x \in \mathfrak{c}_I(\bar{\mathfrak{t}})$. From this it is immediate that $\mathfrak{c}_I(\bar{\mathfrak{t}})$ is self-normalizing, hence a Cartan subalgebra of I. Also, $1 \leqslant TR(\mathfrak{c}_I(\bar{\mathfrak{t}}), I) \leqslant TR(\bar{\mathfrak{h}}, L[\alpha]) = 1$ (see [19, Theorem 1.7(1)]). Since $I \cong S$ this proves (3). Since S is simple and possesses a Cartan subalgebra of toral rank 1, one now derives (4) from Theorem 3.2. □

Corollary 3.7. *Let T be a torus of maximal dimension in L_p, and $\alpha \in \Gamma(L, T)$. Then* $\operatorname{rad} L(\alpha)$ *is T-invariant and $L[\alpha]$ is restricted. Moreover, either*

$$L[\alpha] \in \big\{(0), \mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)}, H(2; \underline{1})^{(1)}\big\}$$

or $p = 5$, L_p possesses a nonstandard torus of maximal dimension, and

$$L[\alpha] \cong H(2; \underline{1})^{(2)} \oplus F x_1^4 \partial_2.$$

Proof. The T-invariance of rad $L[\alpha]$ follows immediately from Theorem 3.6. It shows that $L(\alpha)_p$ acts on $L[\alpha]$ as derivations.

Let $L[\alpha]$ be nonsolvable. As T is a torus of maximal dimension $TR(L[\alpha]) = 1$ (by [19, Theorems 1.9, 1.7]). Let S denote the socle of $L[\alpha]$. By Theorem 3.6, S is simple, while [19, Theorems 1.9, 1.7]) show that $TR(S) = 1$. Then S is one of $\mathfrak{sl}(2)$, $W(1;\underline{1})$, $H(2;\underline{1})^{(2)}$ (see [7, Theorem 2]). Moreover, $S \subset L[\alpha] \subset \operatorname{Der} S$.

Suppose $S \subsetneq L[\alpha]$. Then $S = H(2;\underline{1})^{(2)}$ and $L[\alpha]$ contains a nonzero element

$$D = ax_1^{p-1}\partial_2 + bx_2^{p-1}\partial_1 + c(x_1\partial_1 + x_2\partial_2) + E$$

with $a, b, c \in F$ and $E \in H(2;\underline{1})^{(1)}$. Since $TR(L[\alpha]) = 1$ it must be that $c = 0$, so that $L[\alpha] \subset H(2;\underline{1})$. Jacobson's formula now shows that $L[\alpha]$ is restricted.

If $L[\alpha] \subset H(2;\underline{1})^{(1)}$ then we are done because $\dim H(2;\underline{1})^{(1)}/H(2;\underline{1})^{(2)} = 1$. So from now on we may assume that $D \neq E$. Then $a \neq 0$ or $b \neq 0$. Applying the automorphism σ_1 of $H(2;\underline{1})$ induced by the rule $\sigma_1(x_1) = x_2$, $\sigma_1(x_2) = -x_1$, we may assume that $a \neq 0$. Applying the automorphism σ_2 of $H(2;\underline{1})$ induced by the rule $\sigma_2(x_1) = x_1 - (b/a)^{1/p}x_2$, $\sigma_2(x_2) = x_2$, we may assume $b = 0$. Thus we may assume that $D = x_1^{p-1}\partial_2 + dD_H(x_1^{p-1}x_2^{p-1}) + E'$ where $E' \in H(2;\underline{1})^{(2)}$ and $d \in F$. Applying the automorphism σ_3 of $H(2;\underline{1})$ induced by the rule $\sigma_3(x_1) = x_1 + dx_2^{p-1}$, $\sigma_3(x_2) = x_2$, we may assume further that $d = 0$.

In other words, it can be assumed that $D = x_1^{p-1}\partial_2$. Note that

$$x_1^{p-1}\partial_2 \equiv (1 + x_1)^{p-1}\partial_2 \pmod{H(2;\underline{1})^{(2)}}.$$

It follows that $\mathfrak{c} := \mathfrak{c}_{L[\alpha]}(D_H((1 + x_1)x_2))$ contains $v_{-1} := (1 + x_1)^{p-1}\partial_2$. For $0 \leqslant i \leqslant p - 2$, put $v_i := (i + 1)^{-1}D_H((1 + x_1)^{i+1}x_2^{i+1})$. It is easy to check that $v_0, \ldots, v_{p-2}$ pairwise commute and

$$[v_{-1}, v_i] = iv_{i-1}, \quad 0 \leqslant i \leqslant p - 2.$$

Moreover, $L[\alpha]$ contains all v_i's with $0 \leqslant i \leqslant p - 3$. Since v_0 is a toral element this implies that $\mathfrak{c}$ is a nontriangulable Cartan subalgebra in $L[\alpha]$. Since $L(\alpha)_p$ is a restricted Lie algebra it contains a toral element, t say, which acts on $L[\alpha]$ as v_0. Then $T' := Ft \oplus (T \cap \ker\alpha)$ is a nonstandard torus of maximal dimension in L_p. Applying [7, Theorem 1] now yields $p = 5$.

Finally, suppose $L[\alpha] \supsetneq H(2;\underline{1})^{(2)} \oplus Fx_1^4\partial_2$. Then $\mathfrak{c}$ contains an element $u = \lambda x_2^4\partial_1 + \mu v_3$ with $\lambda \neq 0$ or $\mu \neq 0$. Observe that

$$[u, v_{-1}] \equiv -\lambda v_3 \pmod{H(2;\underline{1})^{(2)}}.$$

From this it is immediate that $v_3 \in \mathfrak{c}$ while from our earlier remarks it follows that $(\operatorname{ad} v_{-1})^3(v_3) \in \mathfrak{c}$ is not nilpotent. Let V denote the subspace in $H(2;\underline{1})$ spanned by u and all v_i's with $0 \leqslant i \leqslant 3$. It follows from the above discussion that $\mathfrak{c} \cap V$ is an abelian ideal of codimension 1 in $\mathfrak{c}$ acting nilpotently on $L[\alpha]$.

Let $\pi : L(\alpha) \to L[\alpha]$ denote the canonical homomorphism, $H' := \mathfrak{c}_L(T')$, and

$$\Omega' := \{\gamma' \in \Gamma(L, T') \mid \gamma'(H') \neq 0\}.$$

By Schue's lemma, $H' = \sum_{\gamma' \in \Omega'} [L_{\gamma'}, L_{-\gamma'}]$, hence there exist $\kappa' \in \Omega'$ and $x \in [L_{\kappa'}, L_{-\kappa'}]$ such that $\pi(x) \equiv v_{-1} \pmod{\mathfrak{c} \cap V}$. Then

$$(\operatorname{ad} v_{-1})^3(v_3) \in \pi\big([L_{\kappa'}, L_{-\kappa'}]^3\big) + \mathfrak{c} \cap V.$$

However, due to Proposition 3.4 and the Engel–Jacobson theorem, the subalgebra on the right acts nilpotently on $L[\alpha]$. This contradiction shows that the case we are considering is impossible. This completes the proof of the corollary. □

Remark. Theorem 3.6 and Corollary 3.7 extend [14, Theorem 4.1, Corollary 4.2] to our present situation.

Corollary 3.7 enables us to generalize the notion of a root being solvable, classical, Witt or Hamiltonian to the case where T is an arbitrary (not necessarily standard) torus of maximal dimension in L_p. It also allows us to generalize the notion of a distinguished maximal subalgebra to this situation.

Let $\alpha \in \Gamma(L, T)$. If α is solvable or classical, we set $Q(\alpha) := L(\alpha)$. If α is Witt, we define $Q(\alpha)$ to be the unique subalgebra of codimension 1 in $L(\alpha)$ containing $\operatorname{rad} L(\alpha)$. If α is Hamiltonian, we define $Q(\alpha)$ to be the inverse image of $L[\alpha] \cap H(2; \underline{1})_{(0)}$ under the canonical homomorphism $\pi : L(\alpha) \to L[\alpha]$ (in this case $Q(\alpha)$ has codimension 2 in $L(\alpha)$). We sometimes write $Q(\alpha) = Q(L(\alpha))$ in order to distinguish between $Q(L(\alpha))$ and $Q(L(\alpha)_p)$. The latter is defined analogously for the p-envelope of $L(\alpha)$ in L_p. By Corollary 3.7, $L[\alpha]$ is restricted, so that $L(\alpha)_p = L(\alpha) + \operatorname{rad}(L(\alpha)_p) = L(\alpha) + \widetilde{H} \cap \operatorname{rad}(L(\alpha)_p)$ where $\widetilde{H} = \mathfrak{c}_{L_p}(T)$. Thus

$$Q\big(L(\alpha)_p\big) = Q\big(L(\alpha)\big) + \operatorname{rad}\big(L(\alpha)_p\big)$$

and $\dim L(\alpha)/Q(L(\alpha)) = \dim L(\alpha)_p/Q(L(\alpha)_p)$. We call α *proper* if the subalgebra $Q(\alpha)$ is T-invariant, and *improper* otherwise. Note that if α is proper then $Q(\alpha)$ contains $H = \mathfrak{c}_L(T)$.

Proposition 3.8. *Let $\alpha \in \Gamma(L, T)$ be a proper root satisfying $\alpha(H) \neq 0$. Then*

$$\bigcup_{i \in \mathbb{F}_p^*} \Big(\big(\operatorname{rad} L(\alpha)\big)_{i\alpha} \cup \big[Q(\alpha)_{i\alpha}, \big(\operatorname{rad} L(\alpha)\big)_{-i\alpha}\big]\Big)$$

consists of p-nilpotent elements of L_p unless α is classical and there is $\beta \in \Gamma(L, T)$ with $L[\alpha, \beta] \cong \mathfrak{g}(1, 1)$. In the latter case

$$\bigcup_{i \in \mathbb{F}_p^*} \Big(\big(\operatorname{rad} L(\alpha)\big)_{i\alpha} \cup \big[\big(\operatorname{rad} L(\alpha)\big)_{i\alpha}, \big(\operatorname{rad} L(\alpha)\big)_{-i\alpha}\big]\Big)$$

consists of p-nilpotent elements of L_p.

Proof. Suppose the claim is not true. Define $W(L(\alpha)) := \operatorname{rad} L(\alpha)$ if α is classical and there is β with $L[\alpha,\beta] \cong \mathfrak{g}(1,1)$, and $W(L(\alpha)) := Q(\alpha)$ otherwise. Adjusting α, we may assume that there exists $u \in (\operatorname{rad} L(\alpha))_\alpha$ or $h_1 \in [W(L(\alpha))_\alpha, (\operatorname{rad} L(\alpha))_{-\alpha}]$ which is not p-nilpotent. Define

$$\Omega_1 := \{\gamma \in \Gamma(L,T) \mid \gamma(u^p) \neq 0\} \quad \text{or} \quad \Omega_1 := \{\gamma \in \Gamma(L,T) \mid \gamma(h_1) \neq 0\},$$

in the respective cases. Then $\Omega_1 \neq \emptyset$. By Schue's lemma, $H = \sum_{\gamma \in \Omega_1} [L_\gamma, L_{-\gamma}]$. Since $\alpha(H) \neq 0$ and each $[L_\gamma, L_{-\gamma}]$ is an ideal of H, the Engel–Jacobson theorem shows that there is $\beta \in \Omega_1$ such that $\alpha([L_\beta, L_{-\beta}]) \neq 0$. Choose $h_2 \in [L_\beta, L_{-\beta}]$ with $\alpha(h_2) \neq 0$. Since $h_1 \in H \cap \operatorname{rad} L(\alpha) \cap [L_\alpha, L_{-\alpha}]$ one has $\alpha(h_1) = 0$ (this is obvious if α is nonsolvable and follows from Theorem 3.5 otherwise). Thus the assumptions of Proposition 3.3 are satisfied. Set

$$\mathfrak{g} := \sum_{\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)\setminus\{0\}} \big(L_\gamma + [L_\gamma, L_{-\gamma}]\big).$$

Then $\bar{\mathfrak{g}} := \pi(\mathfrak{g}) = \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is simple and $\pi(u) \neq 0$ (respectively $\pi(h_1) \neq 0$). Hence there is an element in $(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_\alpha$ or in $[W(\bar{\mathfrak{g}}(\alpha))_\alpha, (\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{-\alpha}]$ which does not act nilpotently on $\bar{\mathfrak{g}}$. The semisimple parts of $\pi(u)$ and $\pi(h_2)$ (respectively $\pi(h_1)$ and $\pi(h_2)$) in $\operatorname{Der}\bar{\mathfrak{g}}$ span a 2-dimensional torus in $\bar{\mathfrak{g}}_p$ which we denote by $\mathfrak{t}$. Note that $\mathfrak{t} \subset \bar{\mathfrak{g}}_p$ coincides with the image of T in $\operatorname{Der}\bar{\mathfrak{g}}$. Since T has maximal dimension, we have

$$2 \leqslant TR(\bar{\mathfrak{g}}) \leqslant TR(\mathfrak{g}) \leqslant TR\big(L(\alpha,\beta)\big) \leqslant 2$$

(see [19, Theorems 1.7, 1.9]). Thus $\bar{\mathfrak{g}}$ is isomorphic to one of the simple Lie algebras listed in [10, Theorem 1.1]. Since α is proper in $\Gamma(L,T)$ it must be proper in $\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ as well (one should take into account that $Q(\alpha)$ contains H). Since at least one of the subspaces $(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{\pm\alpha}$ is nonzero $\bar{\mathfrak{g}}$ cannot be classical.

Suppose $\bar{\mathfrak{g}}$ is a restricted Lie algebra of Cartan type. Then $\bar{\mathfrak{g}}$ is one of

$$W(1;\underline{2}), \quad S(3;\underline{1})^{(1)}, \quad H(4;\underline{1})^{(1)}, \quad K(3;\underline{1}).$$

By [2, Lemma 5.8.2], $Q(\bar{\mathfrak{g}}(\alpha)) \subset \bar{\mathfrak{g}}_{(0)} + \mathfrak{t} \cap \ker\alpha$ (note that [2, Lemma 5.8.2] only relies on the classification of toral elements in restricted Lie algebras of Cartan type, hence holds for $p > 3$). Since $\bar{\mathfrak{g}}_{(1)}$ acts nilpotently on $\bar{\mathfrak{g}}$, we are reduced to examine the $\mathfrak{t}$-invariant quotient $\bar{\mathfrak{g}}_0 = \bar{\mathfrak{g}}_{(0)}/\bar{\mathfrak{g}}_{(1)}$. Since this quotient is classical reductive, we have $(\operatorname{rad}\bar{\mathfrak{g}}_0(\alpha))_{\pm\alpha} = (0)$. Then $\operatorname{rad}\bar{\mathfrak{g}}(\alpha) \subset \bar{\mathfrak{g}}_{(1)} + \mathfrak{t} \cap \ker\alpha$ which implies that the case we consider cannot occur.

Suppose $\bar{\mathfrak{g}} \cong W(1;\underline{2})$. Since $\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ contains a proper root the torus $\mathfrak{t}$ is optimal in $\bar{\mathfrak{g}}_p$, see [18, Section V.4]. By [18, Theorem V.4], all solvable roots in $\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ vanish on $\mathfrak{c}_{\bar{\mathfrak{g}}}(\mathfrak{t})$. So $\alpha \in \Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ is nonsolvable (for $\alpha(\pi(h_2)) \neq 0$). Then $(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{\pm\alpha} = (0)$, again by [18, Theorem V.4]. So this case cannot occur either. Since $\mathfrak{c}_{\bar{\mathfrak{g}}}(\mathfrak{t}) \neq (0)$, we also have that $\bar{\mathfrak{g}} \ncong H(2;\underline{1};\Phi(\tau))^{(1)}$, by [18, Theorem VII.3]. Proposition 2.1(2) shows that $\bar{\mathfrak{g}} \ncong H(2;(2,1))^{(2)}$ while Proposition 2.2 ensures that $\bar{\mathfrak{g}} \ncong H(2;\underline{1};\Delta)$.

Finally, suppose $\bar{\mathfrak{g}}$ is isomorphic to the restricted Melikian algebra $\mathfrak{g}(1,1)$. We have already mentioned that all derivations of $\mathfrak{g}(1,1)$ are inner. So $\mathfrak{t}$ can be identified with a 2-dimensional torus in $\mathfrak{g}(1,1)$ and $\bar{\mathfrak{g}}(\alpha)$ with the centralizer of a toral element in $\mathfrak{t}$. The conjugacy classes of toral elements in $\mathfrak{g}(1,1)$ are determined in [13, Theorem 3.1]. The centralizers of toral elements are described in [13, Theorem 4.1]. It follows from this description that no root in $\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ is solvable and the union $\bigcup_{i\in\mathbb{F}_p^*}(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{i\alpha}$ consists of nilpotent elements of $\bar{\mathfrak{g}}$. Moreover, if $\alpha\in\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ is Hamiltonian then $(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{\pm\alpha}=(0)$. If $\alpha\in\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ is Witt then it follows from [13, Theorem 4.1(3), (4)] that the union $\bigcup_{i\in\mathbb{F}_p^*}[Q(\bar{\mathfrak{g}}(\alpha))_{i\alpha},Q(\bar{\mathfrak{g}}(\alpha))_{-i\alpha}]$ consists of nilpotent elements of $\bar{\mathfrak{g}}$ (see also the proof of [13, Proposition 6.2]). But then $[W(\bar{\mathfrak{g}}(\alpha))_\alpha,(\operatorname{rad}\bar{\mathfrak{g}}(\alpha))_{-\alpha}]$ consists of nilpotent elements of $\bar{\mathfrak{g}}$. Thus $\alpha\in\Gamma(\bar{\mathfrak{g}},\mathfrak{t})$ must be classical. But then α is classical in $\Gamma(L,T)$ (this is immediate from Corollary 3.7 and the equality $L(\alpha)^{(\infty)}=\mathfrak{g}(\alpha)^{(\infty)}$).

In order to reach a contradiction it will now suffice to show that $L[\alpha,\beta]\cong\mathfrak{g}(1,1)$. By [10, Corollary 2.10] and [7, Section 4], we have $|\Gamma(\bar{\mathfrak{g}},\mathfrak{t})|=p^2-1$. So any $\gamma\in\mathbb{F}_p\alpha+\mathbb{F}_p\beta\setminus\{0\}$ is a root of $\bar{\mathfrak{g}}$. Since $L(\gamma)^{(\infty)}=\mathfrak{g}(\gamma)^{(\infty)}$, it follows from [13, Corollary 4.3] that all roots of $L(\alpha,\beta)$ relative to T are nonsolvable. Combining Corollary 3.7 with Demushkin's theorem, it is now easy to observe that any root of $L(\alpha,\beta)$ relative to T vanishes on H^4. But then all elements in the union $\bigcup_{h\in H}(\operatorname{ad}h)^4(H\cap\mathfrak{g})$ act nilpotently on $\bar{\mathfrak{g}}$. As a consequence, $\operatorname{rad}\mathfrak{g}$ is H-invariant (Proposition 3.3(3)).

Thus $L(\alpha,\beta)$ acts on $\bar{\mathfrak{g}}$ as derivations. Since $\bar{\mathfrak{g}}=\operatorname{Der}\bar{\mathfrak{g}}$, there is an ideal I of $L(\alpha,\beta)$ such that $L(\alpha,\beta)/I\cong\mathfrak{g}(1,1)$. By our earlier remarks, $TR(L(\alpha,\beta))=TR(\mathfrak{g}(1,1))=2$. So [19, Theorem 1.7] shows that I is nilpotent. Then $I=\operatorname{rad}L(\alpha,\beta)$ and our proof is complete. □

Corollary 3.9. *The following are true*:

(1) $H^4\subset\operatorname{nil}\widetilde{H}$;
(2) *all roots in* $\Gamma(L,T)$ *are linear on* H.

Proof. (1) Suppose $H^4\not\subset\operatorname{nil}\widetilde{H}$. Then

$$\Omega:=\{\gamma\in\Gamma(L,T)\mid\gamma(H^4)\neq 0\}$$

is nonempty. If $\gamma\in\Gamma(L,T)$ is nonsolvable then Corollary 3.7 (combined with Demushkin's theorem) shows that $\gamma(H^4)=0$. Thus all roots in Ω are solvable. Let $\kappa\in\Omega$. Proposition 3.8 now says that the ideal $[L_\kappa,L_{-\kappa}]$ of $\widetilde{H}$ acts nilpotently on L. Combining Schue's lemma with the Engel–Jacobson theorem, we then obtain

$$H=\sum_{\kappa\in\Omega}[L_\kappa,L_{-\kappa}]\subset\operatorname{nil}\widetilde{H}.$$

But then $H^4\subset\operatorname{nil}\widetilde{H}$, a contradiction.

(2) We denote by h_s the semisimple part of $h\in H$. By part (1), $H^4\subset\operatorname{nil}\widetilde{H}$. Given $h_1,h_2\in H$, we then have

$$(h_1+h_2)^{p^r}\equiv h_1^{p^r}+h_2^{p^r}\pmod{\operatorname{nil}\widetilde{H}},\ \forall r\in\mathbb{N}_0$$

(by Jacobson's formula). Therefore, $(h_1+h_2)_s = h_{1,s}+h_{2,s}$ for all $h_1, h_2 \in H$. This is the same as to say that all roots in $\Gamma(L,T)$ are linear. $\square$

One of the key results of the classification theory for $p>7$ is [15, Theorem 3.1] which says that for any torus T of maximal dimension in L_p the Cartan subalgebra $\mathfrak{c}_{L_p}(T)$ of L_p acts triangulably on L (and L_p). We now come to extending this result to our present situation where $p>3$. As Skryabin pointed out to the second author, the proof of [15, Corollary 2.5] is incorrect (in the notation of [15], the implication $(\lambda,\mu)\in\Omega \Rightarrow (\mu,\lambda)\in\Omega$) is false). In [15], Corollary 2.5 is used in the proof of Theorem 3.1 and only there.

This problem is resolved easily for Lie algebras of rank 2 and has no effect on [9,10] (see [8, pp. 424–426]). Moreover, passing to rank-two sections allows one to salvage [15, Corollary 2.5] relying only on information available at the time when [15] was written. Thus what follows aims at both, a correct proof of [15, Theorem 3.1] for $p>7$ based only on that information and a partly different proof for $p>3$.

Recall that $T\subset L_p$ is a torus of maximal dimension, $H=\mathfrak{c}_L(T)$, and $\widetilde{H}=\mathfrak{c}_{L_p}(T)$. By [7, Theorem 1], if $p>5$ then T is standard.

Lemma 3.10. *If T is standard then $[H,\widetilde{H}]\subset \operatorname{nil}\widetilde{H}$.*

Proof. (a) Let $\alpha\in\Gamma(L,T)$ and $x\in L_\alpha$. If $p>7$ then [15, Lemma 3.2] says that $\alpha([x^p,H)=0$. The proof of this lemma is correct but relies on several results proved for $p>7$ in [14]. Theorems 3.1, 3.5, and 3.6, and Corollary 3.7 provide suitable substitutes for all these results. Thus the equality $\alpha([x^p,H])=0$ still holds under our present assumption on p.

(b) Next we are going to prove the stronger statement that

$$[x^p,H]\subset \operatorname{nil}\widetilde{H},\quad \forall x\in \bigcup_{\alpha\in\Gamma(L,T)} L_\alpha,$$

which constitutes the first part of [15, Lemma 3.3]. The proof will require some minor changes, even for $p>7$.

First assume $\alpha(H)=0$ and let $y\in H\cup\bigcup_{i\in\mathbb{F}_p^*}L_{i\alpha}$. Let y_s be the semisimple part of y in L_p. Since y_s lies in the restricted subalgebra generated by $y^p\in\widetilde{H}$ and T is a maximal torus in L_p, we have $y_s\in T$. If $y\in H$ then $\alpha(y_s)=0$ by our assumption, whereas if $y\in L_{i\alpha}$ for $i\in\mathbb{F}_p^*$ then $i\alpha(y_s)y=[y_s,y]=0$. The Engel–Jacobson theorem now yields that $L(\alpha)$ is nilpotent. By [14, Theorem 3.5(1)] for $p>7$ and by Theorem 3.1 in the general case, $L(\alpha)^{(1)}$ acts nilpotently on L. This shows $[x^p,H]\subset[L_\alpha,L_{-\alpha}]\subset\operatorname{nil}\widetilde{H}$.

Now assume that $\alpha(H)\neq 0$ and put $\Omega:=\{\gamma\in\Gamma(L,T)\mid \gamma([x^p,H])\neq 0\}$. If $\Omega=\emptyset$ then $[x^p,H]$ acts nilpotently on L, hence is contained in $\operatorname{nil}\widetilde{H}$. So suppose $\Omega\neq\emptyset$. Since L is simple, we then have $H=\sum_{\gamma\in\Omega}[L_\gamma,L_{-\gamma}]$, by Schue's lemma. As $H^{(1)}$ acts nilpotently on L, all roots in $\Gamma(L,T)$ are linear on H. So there is $\beta\in\Omega$ such that $\alpha([L_\beta,L_{-\beta}])\neq 0$. We thus have

$$\alpha\big([x^p,H]\big)=0,\qquad \beta\big([x^p,H]\big)\neq 0,\qquad \alpha\big([L_\beta,L_{-\beta}]\big)\neq 0.$$

As $[x^p, H] \subset [L_\alpha, L_{-\alpha}]$ the assumptions of Proposition 3.3 are satisfied. (Note that the proof of Proposition 3.3 is elementary in nature!) Let

$$\mathfrak{g} := \sum_{\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)\setminus\{0\}} \big(L_\gamma + [L_\gamma, L_{-\gamma}]\big).$$

As $H^{(1)}$ acts nilpotently on L, Proposition 3.3(3) shows that $\operatorname{rad}\mathfrak{g}$ is H-invariant. Let $\pi : \mathfrak{g} + H \to (\mathfrak{g} + H)/\operatorname{rad}\mathfrak{g}$ be the canonical homomorphism, and $\bar{\mathfrak{g}} := \pi(\mathfrak{g})$. By Proposition 3.3(2), $\bar{\mathfrak{g}}$ is a simple Lie algebra and $\pi(H \cap \mathfrak{g})$ is a Cartan subalgebra of toral rank 2 in $\bar{\mathfrak{g}}$. Since T has maximal dimension in L_p, we have $TR(\mathfrak{g}) \leqslant 2$. Then $2 = TR(\pi(H \cap \mathfrak{g}), \bar{\mathfrak{g}}) \leqslant TR(\mathfrak{g}) \leqslant 2$ (see [19, Theorems 1.7, 1.9]). Therefore, the p-envelope of $\pi(H \cap \mathfrak{g})$ in $\bar{\mathfrak{g}}_p \subset \operatorname{Der}\bar{\mathfrak{g}}$ contains a unique 2-dimensional torus, $\mathfrak{t}$ say, which coincides with the image of T in $\operatorname{Der}\bar{\mathfrak{g}}$ (the torus T acts on $\bar{\mathfrak{g}}$ by Proposition 3.3(1)).

Starting from this point the original proof in [15] goes through for $p > 7$. Let

$$M_\beta^\alpha := \big\{y \in \bar{\mathfrak{g}}_\beta \;\big|\; \alpha\big([y, \bar{\mathfrak{g}}_{-\beta}]\big) = 0\big\}.$$

Since $\alpha([L_\beta, L_{-\beta}]) \neq 0$ and $\operatorname{rad}\mathfrak{g}$ acts nilpotently on $\mathfrak{g}$ (by Proposition 3.3(3)), one has $\bar{\mathfrak{g}}_\beta \neq M_\beta^\alpha$. So the pair $(\mathfrak{g}, \mathfrak{t})$ satisfies all assumptions of [2, Proposition 5.5.2] except a restrictedness condition which can be dropped in view of [15, Lemma 2.4]. Then [2, Corollary 5.5.3] yields $\dim \bar{\mathfrak{g}}_\beta/M_\beta^\alpha \leqslant 7$. Since $H^{(1)} \subset \operatorname{nil}\widetilde{H}$ and $\alpha([x^p, H]) = 0$, the subspace M_β^α is invariant under $\mathfrak{h}_x := \operatorname{ad}\pi(H) + (\operatorname{ad}\pi(x))^p$. Then $\bar{\mathfrak{g}}_\beta/M_\beta^\alpha$ is a nonzero $\mathfrak{h}_x$-module of dimension $< p$ (for $p > 7$). Since $\mathfrak{h}_x$ is nilpotent (as a homomorphic image of a subalgebra of $\widetilde{H}$), all composition factors of this $\mathfrak{h}_x$-module are 1-dimensional. But then $(\mathfrak{h}_x)^{(1)}$ acts nilpotently on $\bar{\mathfrak{g}}_\beta/M_\beta^\alpha$. This, in turn, implies that $\operatorname{ad}[x^p, H]$ consists of endomorphisms acting noninvertibly on L_β. Then $\beta([x^p, H]) = 0$, a contradiction. For $p = 5$ this argument is no longer valid but it still works for $p = 7$ because we know, from [9, Lemma 1.4, Theorem 8.6], that $\dim \bar{\mathfrak{g}}_\beta/M_\beta^\alpha \leqslant 6$.

The main result of [10] enables us now to argue the general case differently and include the remaining case $p = 5$ into considerations. Since $\bar{\mathfrak{g}}$ is simple and has absolute toral rank 2, it is isomorphic to one of the Lie algebras listed in [10, Theorem 1.1]. Suppose $\bar{\mathfrak{g}}$ is restricted. Then there is $h \in H \cap \mathfrak{g}$ with $\pi(x)^p = \pi(h)$, so that

$$\pi\big(\big[(\operatorname{ad}x)^p, H\big]\big) = \big[\pi(x)^p, \pi(H)\big] = \big[\pi(h), \pi(H)\big] \subset \pi\big(H^{(1)}\big).$$

As before, this implies that $\beta([x^p, H]) = 0$ contrary to our choice of β. If $\bar{\mathfrak{g}}$ is one of $W(1; \underline{2})$, $H(2; \underline{1}; \Delta)$, $H(2; \underline{1}; \Phi(\tau))^{(1)}$ then all root spaces of $\bar{\mathfrak{g}}$ relative to $\mathfrak{t}$ are 1-dimensional (by [10, Corollary 2.10] and the results of [18]). Then again $\bar{\mathfrak{g}}_\beta/M_\beta^\alpha$ has dimension $1 < p$ and we are done.

So we are now left with the case where $\bar{\mathfrak{g}} \cong H(2; (2, 1))^{(2)}$. Recall that $\dim \mathfrak{t} = 2$. By Proposition 2.1(1), $\bar{\mathfrak{g}}_p = \bar{\mathfrak{g}} + \mathfrak{t}$. Thus there exist $t \in \mathfrak{t}$ and $h \in H \cap \mathfrak{g}$ such that $\pi(x)^p = t + \pi(h)$. Then

$$\begin{aligned}\pi\big((\operatorname{ad}x)^p(H)\big) &= \big[\pi(x)^p, \pi(H)\big] = \big[t + \pi(h), \pi(H)\big]\\ &= \big[\pi(h), \pi(H)\big] \subset \pi(H)^{(1)} = \pi\big(H^{(1)}\big).\end{aligned}$$

Again this implies that all elements in $[x^p, H] = (\operatorname{ad} x)^p(H)$ act noninvertibly on L_β. This contradicts our choice of β.

(c) Now observe that $\widetilde{H}' := \{h \in \widetilde{H} \mid [h, H] \subset \operatorname{nil} \widetilde{H}\}$ is a restricted subalgebra of $\widetilde{H}$ containing H and all x^p with $x \in \bigcup_{\alpha\in\Gamma(L,T)} L_\alpha$. By Jacobson's formula, $\widetilde{H}' = \widetilde{H}$. This completes the proof of the lemma. □

Lemma 3.11 (cf. [15, Lemma 3.4]). *If T is standard then $\alpha(\widetilde{H}^{(1)}) = 0$ for any nonsolvable root $\alpha \in \Gamma(L, T)$.*

Proof. No changes in the proof of [15, Lemma 3.4] are needed to obtain the result. □

We now come to our first main result.

Theorem 3.12 (cf. [15, Theorem 3.1]). *Let T be a torus of maximal dimension in L_p and suppose that T is standard. Then $\widetilde{H} = \mathfrak{c}_{L_p}(T)$ acts triangulably on L_p.*

Proof. Suppose there is $\alpha \in \Gamma(L, T)$ such that

$$\alpha\bigl(\widetilde{H}^{(1)}\bigr) \neq 0 \quad \text{and} \quad [L_\alpha, L_{-\alpha}] \not\subset \operatorname{nil} \widetilde{H}.$$

As $\widetilde{H}^{(1)} \subset H$ we have $\alpha(H) \neq 0$. Set $\Omega_1 := \{\kappa \in \Gamma(L, T) \mid \kappa([L_\alpha, L_{-\alpha}]) \neq 0\}$. As $\Omega_1 \neq \emptyset$, Schue's lemma yields $H = \sum_{\kappa\in\Omega_1}[L_\kappa, L_{-\kappa}]$. As α vanishes on $H^{(1)}$ but not on H there is $\beta \in \Gamma(L, T)$ such that

$$\beta\bigl([L_\alpha, L_{-\alpha}]\bigr) \neq 0 \quad \text{and} \quad \alpha\bigl([L_\beta, L_{-\beta}]\bigr) \neq 0.$$

By Lemma 3.11, α is a solvable root. Then $\alpha([L_\alpha, L_{-\alpha}]) = 0$, by Theorem 3.5. Consequently, case (a) of Proposition 3.3 applies to

$$\mathfrak{g} := \sum_{\gamma\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}} \bigl(L_\gamma + [L_\gamma, L_{-\gamma}]\bigr).$$

Lemma 3.10 enables us to apply Proposition 3.3(3) with $\mathfrak{h}' = \widetilde{H}$ which yields that $\operatorname{rad} \mathfrak{g}$ is $\widetilde{H}$-invariant and $\widetilde{H}$ acts on $\bar{\mathfrak{g}} = \mathfrak{g}/\operatorname{rad} \bar{\mathfrak{g}}$ as derivations. Since $\operatorname{rad} \mathfrak{g}$ acts nilpotently on $\mathfrak{g}$, we then have $\bar{\mathfrak{g}}_\beta \neq M_\beta^\alpha$, where the notation is suitably adopted from the proof of Lemma 3.10. Since $[\widetilde{H}, H] \subset \operatorname{nil} \widetilde{H}$ the subspace M_β^α is $\widetilde{H}$-stable. Moreover, as in part (b) of the proof of Lemma 3.10 we have $\dim \bar{\mathfrak{g}}_\beta/M_\beta^\alpha \leqslant 6$. Since $\widetilde{H}$ is nilpotent this forces $\alpha(\widetilde{H}^{(1)}) = 0$ for $p > 5$.

To settle the remaining case $p = 5$ we again invoke [10, Theorem 1.1]. It should be clear by now that the element x^p from the proof of Lemma 3.10 can be replaced by any element in $\widetilde{H}$. So the argument from the proof of Lemma 3.10 relying on [10, Theorem 1.1] yields that $\alpha(\widetilde{H}^{(1)}) = 0$ in all cases. Since this contradicts our choice of α we must have

$$[L_\alpha, L_{-\alpha}] \subset \operatorname{nil} \widetilde{H} \quad \text{whenever } \alpha\bigl(\widetilde{H}^{(1)}\bigr) \neq 0.$$

Now set $\Omega_2 := \{\gamma \in \Gamma(L,T) \mid \gamma(\widetilde{H}^{(1)}) \neq 0\}$. If $\Omega_2 \neq \emptyset$ then $H = \sum_{\gamma\in\Omega_2}[L_\gamma, L_{-\gamma}]$, by Schue's lemma, forcing $H \subset \operatorname{nil}\widetilde{H}$. But then $\widetilde{H}^{(1)} \subset H \subset \operatorname{nil}\widetilde{H}$ and $\Omega_2 = \emptyset$, a contradiction. Thus $\gamma(\widetilde{H}^{(1)}) = 0$ for all $\gamma \in \Gamma(L,T)$ which is the same as to say that $\widetilde{H}^{(1)}$ acts nilpotently on L_p. □

4. Two-sections

Now we are ready to begin our investigation of the 2-sections of L relative to T. Let $\alpha, \beta \in \Gamma(L,T)$ be such that $L(\alpha,\beta)$ is nonsolvable and denote by $\operatorname{rad}_T L(\alpha,\beta)$ the maximal T-invariant solvable ideal of $L(\alpha,\beta)$. Put

$$L[\alpha,\beta] := L(\alpha,\beta)/\operatorname{rad}_T L(\alpha,\beta),$$

and let $\widetilde{S} = \widetilde{S}[\alpha,\beta]$ be the T-socle of $L[\alpha,\beta]$, the sum of all minimal T-invariant ideals of $L[\alpha,\beta]$. Then $\widetilde{S} = \bigoplus_{i=1}^{r}\widetilde{S}_i$ where each $\widetilde{S}_i$ is a minimal T-invariant ideal of $L[\alpha,\beta]$. It is easily seen that T and $L(\alpha,\beta)_p$ act on $L[\alpha,\beta]$ as derivations and preserve $\widetilde{S}$. Thus there is a natural restricted homomorphism $T + L(\alpha,\beta)_p \to \operatorname{Der}\widetilde{S}$ which will be denoted by $\Psi_{\alpha,\beta}$. In what follows we identify $L[\alpha,\beta]$ with $\Psi_{\alpha,\beta}(L(\alpha,\beta))$ (as we may), denote the torus $\Psi_{\alpha,\beta}(T) \subset \operatorname{Der}\widetilde{S}$ by $\overline{T}$, and put $\overline{H} := \Psi_{\alpha,\beta}(H)$.

Note that $r \leqslant TR(\widetilde{S}) \leqslant TR(L[\alpha,\beta]) \leqslant TR(L(\alpha,\beta)) \leqslant 2$, by [14, Theorem 2.6] and [19, Theorem 1.7]. Applying [19, Theorem 1.7(8)] to $L = \overline{T} + L[\alpha,\beta]$ and $K = \widetilde{S}$ and taking p-envelopes in $\operatorname{Der}\widetilde{S}$, we get

$$\dim(\overline{T} \cap \widetilde{S}_p) = TR(\widetilde{S})$$

(one should also keep in mind that $\overline{T} + L[\alpha,\beta]_p \subset \operatorname{Der}\widetilde{S}$ is centerless). In particular, if $TR(\widetilde{S}) = 2$ then $\overline{T} \subset \widetilde{S}_p$. If $r > 1$ then $r = 2 = TR(\widetilde{S})$ and $TR(\widetilde{S}_i) = 1$ for $i = 1, 2$. Moreover, in this case $\widetilde{S}_p = (\widetilde{S}_1)_p + (\widetilde{S}_2)_p \subset \operatorname{Der}(\widetilde{S}_1 \oplus \widetilde{S}_2)$.

Theorem 4.1. *If $r = 2$ then there are $\mu_1, \mu_2 \in \Gamma(L,T)$ such that*

$$L[\mu_1]^{(1)} \oplus L[\mu_2]^{(1)} \subset L[\alpha,\beta] \subset L[\mu_1] \oplus L[\mu_2].$$

Proof. As each $\widetilde{S}_i$ is perfect, $\operatorname{Der}(\widetilde{S}_1 \oplus \widetilde{S}_2) = (\operatorname{Der}\widetilde{S}_1) \oplus (\operatorname{Der}\widetilde{S}_2)$. Therefore, $\widetilde{S}_p \cong (\widetilde{S}_1)_p \oplus (\widetilde{S}_2)_p$ where $(\widetilde{S}_i)_p \subset \operatorname{Der}\widetilde{S}_i$. Applying [19, Theorem 1.7(8)] with $L = \widetilde{S}$ and $K = \widetilde{S}_i$, we get $\dim \overline{T}/\overline{T} \cap (\widetilde{S}_i)_p = 2 - TR(\widetilde{S}_i) = 1$. Hence $\dim \overline{T} \cap (\widetilde{S}_i)_p = 1$ for $i = 1, 2$, and

$$\overline{T} = \big(\overline{T} \cap (\widetilde{S}_1)_p\big) \oplus \big(\overline{T} \cap (\widetilde{S}_2)_p\big).$$

Pick $\mu_i \in \Gamma(L,T)$ with $\mu_i(\overline{T} \cap (\widetilde{S}_i)_p) \neq 0$. Then $L[\alpha,\beta] = (\widetilde{S}_1(\mu_1) \oplus \widetilde{S}_2(\mu_2)) + \overline{H}$ and $\widetilde{S}_i = \widetilde{S}(\mu_i)$ for $i = 1, 2$. Let $\pi_i : L[\alpha,\beta](\mu_i) \twoheadrightarrow L[\mu_i]$ denote the canonical homomorphism and observe that $\operatorname{rad}_T L(\alpha,\beta) \cap L(\mu_i) \subset \operatorname{rad} L(\mu_i)$. Then $\pi_i(\widetilde{S}_i)$ is a nonzero ideal of $L[\mu_i]$

satisfying $\pi_i(\widetilde{S}_i) = \pi_i(\widetilde{S}_i)^{(1)}$. According to Corollary 3.7, $\pi_i(\widetilde{S}_i) = L[\mu_i]^{(1)}$. Observe that $(\ker \pi_i) \cap \widetilde{S}_i = (0)$, being a $\overline{T}$-invariant solvable ideal of $L[\alpha, \beta]$. Thus $\widetilde{S}_i \cong L[\mu_i]^{(1)}$ for $i = 1, 2$.

For $i = 1, 2$, the adjoint action of $L[\alpha, \beta]$ on its ideal $\widetilde{S}_i$ gives rise to a homomorphism $\psi_i : L[\alpha, \beta] \to \operatorname{Der} \widetilde{S}_i$ with $\ker \psi_i \subset L[\alpha, \beta](\mu_{3-i})$. Our discussion above implies that $\psi_i(L[\alpha, \beta]) \cong L[\mu_i]$. Let $\psi = \psi_1 \oplus \psi_2 : L[\alpha, \beta] \to L[\mu_1] \oplus L[\mu_2]$. If $x \in \ker \psi$ then $[x, \widetilde{S}_i] \subset (\ker \psi) \cap \widetilde{S}_i = (0)$. Since $L[\alpha, \beta]$ is isomorphic to a subalgebra of $\operatorname{Der}(\widetilde{S}_1 \oplus \widetilde{S}_2)$, we thus have $x = 0$. So ψ is injective and our proof is complete. □

Theorem 4.2. *If $r = 1$ and $TR(\widetilde{S}) = 2$ then $\widetilde{S}$ simple and the following hold*:

1) *If $\widetilde{S}$ is restricted then $L[\alpha, \beta] = \widetilde{S}$.*
2) *If $\widetilde{S}$ is nonrestricted then $\widetilde{S} \subset L[\alpha, \beta] \subset \widetilde{S} + \overline{T} = \widetilde{S}_p$ unless $\widetilde{S} \cong H(2; (2, 1))^{(2)}$ in which case $H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1))_p$.*

Proof. Given a Lie subalgebra M in $L[\alpha, \beta]$, we denote by M_p the p-envelope of M in $\operatorname{Der} \widetilde{S}$. Note that the p-envelope $L[\alpha, \beta]_p$ is semisimple.

(a) By Block's theorem, there are a simple Lie algebra $\mathfrak{s}$ and $m \in \mathbb{N}_0$ such that $\widetilde{S} \cong \mathfrak{s} \otimes A(m; \underline{1})$. Then

$$\mathfrak{s} \otimes A(m; \underline{1}) \subset \overline{T} + L[\alpha, \beta]_p \subset \big((\operatorname{Der} \mathfrak{s}) \otimes A(m; \underline{1})\big) \oplus \big(\operatorname{Id} \otimes W(m; \underline{1})\big),$$

where $\pi_2(\overline{T} + L[\alpha, \beta]_p)$ is a transitive subalgebra of $W(m; \underline{1})$. Let $\mathcal{S}$ denote the (semisimple) p-envelope of $\mathfrak{s}$ in $\operatorname{Der} \mathfrak{s}$. Our assumption on $TR(\widetilde{S})$ (combined with an earlier remark) shows that $\overline{T} \subset \widetilde{S}_p$. Since $\widetilde{S}_p \subset (\operatorname{Der} \mathfrak{s}) \otimes A(m; \underline{1})$ it follows from [10, Theorem 2.6] that we can choose $\widetilde{S} \xrightarrow{\sim} \mathfrak{s} \otimes A(m; \underline{1})$ such that $\overline{T} \subset (\operatorname{Der} \mathfrak{s}) \otimes F$. Since $(\mathfrak{s} \otimes A(m; \underline{1}))_p = \mathfrak{s} \otimes A(m; 1)_{(1)} + \mathcal{S} \otimes F$ we have $\overline{T} \subset \mathcal{S} \otimes F$. Then $\overline{T} = \mathfrak{t} \otimes F$ where $\mathfrak{t}$ is a 2-dimensional torus in $\mathcal{S}$, forcing $2 \leqslant TR(\mathfrak{s}) \leqslant TR(\widetilde{S}) = 2$ and $\sum_{\gamma \neq 0} L[\alpha, \beta]_\gamma \subset \mathfrak{s} \otimes A(m; \underline{1})$. As a consequence,

$$L[\alpha, \beta] = \overline{H} + \widetilde{S} \quad \text{and} \quad \overline{T} + L[\alpha, \beta]_p = \overline{H}_p + \widetilde{S}_p,$$

which implies that $\operatorname{rad}_T L(\alpha, \beta) = \operatorname{rad} L(\alpha, \beta)$. Besides, the subalgebra $\pi_2(\overline{H}_p) = \pi_2(\overline{T} + L[\alpha, \beta]_p)$ is transitive in $W(m; \underline{1})$ and $\mathfrak{c}_{\mathfrak{s}}(\mathfrak{t}) \otimes A(m; 1) \subset \overline{H}$.

(b) Suppose $m \neq 0$. Then there exists $h \in \overline{H}$ such that $\pi_2(h) = \sum_{i=1}^m a_i \partial_i + E$ where $a_i \in F$, $a_{i_0} \neq 0$, and $E \in W(m; \underline{1})_{(0)}$. Since

$$\mathfrak{c}_{\mathfrak{s}}(\mathfrak{t}) \otimes F \subset (\operatorname{ad} h)^{p-1}\big(\mathfrak{c}_{\mathfrak{s}}(\mathfrak{t}) \otimes x_{i_0}^{p-1}\big) + \mathfrak{s} \otimes A(m; \underline{1})_{(1)}$$

and $H^4 \subset \operatorname{nil} \widetilde{H}$, by Corollary 3.9, the subalgebra $\mathfrak{c}_{\mathfrak{s}}(\mathfrak{t}) \otimes F$ must act nilpotently on $\widetilde{S}$. By the Engel–Jacobson theorem, each 1-section $\widetilde{S}(\gamma)$ relative to $\overline{T}$ must be solvable. From this it is immediate that $L(\gamma) = \operatorname{rad} L(\gamma)$ for any $\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\}$.

Suppose there is $\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\}$ with $\gamma(H) = 0$ and let $\mathfrak{t}_0 := T \cap \ker\gamma$. As T is a maximal torus in L_p, the 1-section $L(\gamma)$ is nilpotent and the maximal torus of the p-envelope of $\mathfrak{t}_0 + L(\gamma)$ is contained in $T \cap \ker\gamma = \mathfrak{t}_0$. Note that $L(\alpha, \beta)$ is a 1-section relative to $\mathfrak{t}_0$. By Theorem 3.6, the unique minimal ideal $\widetilde{S}$ of $L(\alpha,\beta)/\operatorname{rad} L(\alpha,\beta) = L[\alpha,\beta]$ is simple. But then $m = 0$ contrary to our assumption. Therefore, $\gamma(H) \neq 0$ for all $\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\})$. According to Proposition 3.8, all elements in the union

$$\bigcup_{\gamma \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)\setminus\{0\}} \big(L_\gamma \cup [L_\gamma, L_{-\gamma}]\big)$$

are p-nilpotent in L_p. But then $L(\alpha, \beta)$ is solvable (again by the Engel–Jacobson theorem). This contradiction shows that $m = 0$.

(c) It follows from parts (a) and (b) that $\widetilde{S}$ is simple with $TR(\widetilde{S}) = 2$, and $\overline{T} + L[\alpha,\beta] \subset \operatorname{Der}\widetilde{S}$. Then $\widetilde{S}$ is listed in [10, Theorem 1.1].

If $\widetilde{S}$ is classical or one of $W(2;\underline{1})$, $K(3;\underline{1})$, $\mathfrak{g}(1,1)$ then $\operatorname{Der}\widetilde{S} \cong \widetilde{S}$ (see [19,22]). If $\widetilde{S}$ is non-restricted Cartan-type and $\widetilde{S} \ncong H(2;(2,1))^{(2)}$, then $\operatorname{Der}\widetilde{S} = \widetilde{S} + \overline{T} = \widetilde{S}_p$ (see [2,18]). Thus in order to finish the proof it remains to consider the case where $\widetilde{S} \in \{S(3;\underline{1})^{(1)}, H(4;\underline{1})^{(1)}, H(2;(2,1))^{(2)}\}$.

Suppose $\widetilde{S} = S(3;\underline{1})^{(1)}$ and $\overline{H} \not\subset \widetilde{S}$. We have

$$\operatorname{Der}\widetilde{S} = Fx_1^{p-1}x_2^{p-1}\partial_3 \oplus Fx_1^{p-1}x_3^{p-1}\partial_2 \oplus Fx_2^{p-1}x_3^{p-1}\partial_1 \oplus Ft_0 \oplus \widetilde{S},$$

where $t_0 = x_1\partial_1 + x_2\partial_2 + x_3\partial_3$. If $\overline{H}$ contains $t_0 + \alpha_1 x_2^{p-1}x_3^{p-1}\partial_1 + \alpha_2 x_1^{p-1}x_3^{p-1}\partial_2 + \alpha_3 x_1^{p-1}x_2^{p-1}\partial_3 + E$ for some $E \in \widetilde{S}$ then $TR(L[\alpha,\beta]) \geqslant 3$ which is not true. Thus

$$\overline{H} \subset \widetilde{S} + Fx_2^{p-1}x_3^{p-1}\partial_1 + Fx_1^{p-1}x_3^{p-1}\partial_2 + Fx_1^{p-1}x_2^{p-1}\partial_3.$$

We may assume (by symmetry) that $\overline{H} \not\subset \widetilde{S} \oplus Fx_2^{p-1}x_3^{p-1}\partial_1 \oplus Fx_1^{p-1}x_3^{p-1}\partial_2 =: \widetilde{S}'$. Let $z_1 := (1+x_1)$, $z_2 := (1+x_2)$, and $\mathfrak{t}' := F(z_1\partial_1 - z_2\partial_2) \oplus F(z_1\partial_1 - x_3\partial_3)$, a 2-dimensional torus in $\widetilde{S}$. The restricted Lie algebra $L(\alpha,\beta)_p$ contains a torus of maximal dimension T' with $\Psi_{\alpha,\beta}(T') = \mathfrak{t}'$. Let $H' := \mathfrak{c}_L(T')$ and $\overline{H}' = \Psi_{\alpha,\beta}(H')$. Since $L[\alpha,\beta]/\widetilde{S}$ is a trivial $\widetilde{S}$-module, we have $L[\alpha,\beta] = \widetilde{S} + \overline{H}' \not\subset \widetilde{S}'$. It follows that $\overline{H}' \not\subset \widetilde{S}'$.

Since $\mathfrak{t}' \subset \overline{H}'$, we have $H' \not\subset \operatorname{nil}\widetilde{H}$. So $\Omega' := \{\gamma' \in \Gamma(L, T') \mid \gamma'(H') \neq 0\} \neq \emptyset$ whence $H' = \sum_{\gamma' \in \Omega'}[L_{\gamma'}, L_{-\gamma'}]$, by Schue's lemma. Therefore, there is $y \in [L_{\kappa'}, L_{-\kappa'}]$ for some $\kappa' \in \Omega'$ such that

$$\bar{y} := \Psi_{\alpha,\beta}(y) = z_1^{p-1}z_2^{p-1}\partial_3 + \beta_1 z_2^{p-1}x_3^{p-1}\partial_1 + \beta_2 z_1^{p-1}x_3^{p-1}\partial_2 + E,$$

where $E \in \widetilde{S} \cap \overline{H}'$. Recall that $\widetilde{S} = S(3;\underline{1})^{(1)}$ has dimension $2(p^3-1)$ and is spanned by the elements $D_{i,j}(f)$ with $f \in A(3;\underline{1})$ and $1 \leqslant i < j \leqslant 3$ (see [22, (4.3)] for example). Since any root space of $\widetilde{S}$ relative to $\mathfrak{t}'$ has dimension $2p$ (this is explained in [10, p. 284]), we have that $\dim \overline{H}' \cap \widetilde{S} = 2(p-1)$. It follows that $\overline{H}' \cap \widetilde{S}$ has basis consisting of all $D_{1,2}(z_1^{k+1}z_2^{k+1}x_3^k) = (k+1)z_1^k z_2^k x_3^k(z_1\partial_1 - z_2\partial_2)$ and $D_{1,3}(z_1^{k+1}z_2^k x_3^{k+1}) =$

$(k+1)z_1^k z_2^k x_3^k(z_1\partial_1 - x_3\partial_3)$ with $0 \leqslant k \leqslant p-2$. As a consequence, $\overline{H}' \cap \widetilde{S}$ is abelian. From this it is immediate that

$$\big[\bar{y}, (z_1z_2x_3)^i(z_1\partial_1 - x_3\partial_3)\big] = i(z_1z_2x_3)^{i-1}(z_1\partial_1 - x_3\partial_3)$$

for all $i \leqslant p-2$ Therefore,

$$(\operatorname{ad}\bar{y})^{p-2}\big(D_{1,3}\big(z_1^{p-1}z_2^{p-2}x_3^{p-1}\big)\big) = (p-2)!(z_1\partial_1 - x_3\partial_3).$$

Since the element on the right is toral in $\widetilde{S}$, we have $[L_{\kappa'}, L_{-\kappa'}]^3 \not\subset \operatorname{nil}(H')_p$. Since $p \geqslant 5$, this contradicts Proposition 3.4.

Suppose $\widetilde{S} = H(4;\underline{1})^{(2)}$ and $\overline{H} \not\subset \widetilde{S}$. One has

$$\operatorname{Der}\widetilde{S} = \widetilde{S} \oplus \sum_{i=1}^{4} F D_H\big(x_i^{(p)}\big) \oplus F\left(\sum_{i=1}^{4} x_i D_i\right).$$

Since $TR(\widetilde{S}) = 2$ it must be that $\overline{H} \subset \widetilde{S} \oplus \sum_{i=1}^4 F D_H(x_i^{(p)})$. No generality will be lost by assuming that $\overline{H} \not\subset \widetilde{S} + \sum_{i=2}^4 F D_H(x_i^{(p)})$. This time we set

$$\mathfrak{t}' := F\big((1+x_1)\partial_1 - x_3\partial_3\big) \oplus F(x_2\partial_2 - x_4\partial_4).$$

Clearly, $\mathfrak{t}'$ is a 2-dimensional torus in $\widetilde{S}$, hence there exists a torus of maximal dimension $T' \subset L(\alpha,\beta)_p$ such that $\Psi_{\alpha,\beta}(T') = \mathfrak{t}'$. As before, we set $H' := \mathfrak{c}_L(T')$ and $\overline{H}' = \Psi_{\alpha,\beta}(H')$. It is straightforward to check that $\bar{H}' \cap \widetilde{S}$ is spanned by all $D_H((1+x_1)^i x_2^j x_3^i x_4^j)$, with $0 \leqslant i, j \leqslant p-1$ and $0 < i+j < 2p-2$. This implies that $\overline{H}' \cap \widetilde{S}$ is abelian (see [22, Lemma 4.3(2)]). Arguing as in the previous case, we find a root $\kappa' \in \Gamma(L,T')$ with $\kappa'(H') \neq 0$ and an element $y \in [L_{\kappa'}, L_{-\kappa'}]$ with

$$\bar{y} := \Psi_{\alpha,\beta}(y) = (1+x_1)^{p-1}\partial_3 + \beta_2 x_2^{p-1}\partial_4 + \beta_3 x_3^{p-1}\partial_1 + \beta_4 x_4^{p-1}\partial_2 + E$$

for some $E \in \widetilde{S} \cap \overline{H}'$. Applying $(\operatorname{ad}\bar{y})^{p-2}$ to the element

$$D_H\big((1+x_1)^{p-1}x_3^{p-1}\big) = -(1+x_1)^{p-2}x_3^{p-1}\partial_3 + (1+x_1)^{p-1}x_3^{p-2}\partial_1,$$

we obtain a nonzero multiple of $(1+x_1)\partial_1 - x_3\partial_3$. Since $p \geqslant 5$, this contradicts Proposition 3.4.

Finally, suppose $\widetilde{S} = H(2;(2,1))^{(2)}$. Then $H(2;(2,1))_p = H(2;(2,1)) + FD_1^p$ and $\operatorname{Der}\widetilde{S} = H(2;(2,1))_p \oplus F(x_1D_1 + x_2D_2)$ (see, e.g., [2, Theorem 2.1.8]). Since $2 = TR(\widetilde{S}) = TR(L[\alpha,\beta])$ and $L[\alpha,\beta]$ is semisimple, the restricted quotient $L[\alpha,\beta]_p/\widetilde{S}_p$ must be p-nilpotent. This yields $L[\alpha,\beta]_p \subset H(2;(2,1))_p$ completing the proof. □

Lemma 4.3. *Let $\alpha \in \Gamma(L,T)$ be a proper root with $\alpha(H) \neq 0$, and $y \in L_\alpha$. Then* $(\operatorname{ad} y)^{2p}(H \cap \operatorname{rad} L(\alpha)) \subset \operatorname{nil}\widetilde{H}$.

Proof. (a) Suppose that $(\operatorname{ad} y)^{2p}(H \cap \operatorname{rad} L(\alpha))$ contains an element which is not p-nilpotent in L_p. Then the set

$$\Omega := \{\gamma \in \Gamma(L,T) \mid \gamma((\operatorname{ad} y)^{2p}(H \cap \operatorname{rad} L(\alpha))) \neq 0\}$$

is not empty. By Schue's lemma, we then have $H = \sum_{\gamma\in\Omega}[L_\gamma, L_{-\gamma}]$. Since $\alpha(H) \neq 0$ and α is a linear function on H, by Corollary 3.9, there is $\beta \in \Omega$ with $\alpha([L_\beta, L_{-\beta}]) \neq 0$. If $\alpha(H \cap \operatorname{rad} L(\alpha)) \neq 0$ then α is solvable, hence vanishes on $\sum_{i\in\mathbb{F}_p^*}[L_{i\alpha}, L_{-i\alpha}]$ by Theorem 3.5. Therefore,

$$\alpha((\operatorname{ad} y)^{2p}(H \cap \operatorname{rad} L(\alpha))) \subset \alpha\left(H \cap \operatorname{rad} L(\alpha) \cap \sum_{i\in\mathbb{F}_p^*}[L_{i\alpha}, L_{-i\alpha}]\right) = 0$$

in all cases. As a consequence, there are $h_1 \in [L_\alpha, L_{-\alpha}]$ and $h_2 \in [L_\beta, L_{-\beta}]$ such that $\alpha(h_1) = 0$, $\beta(h_1) \neq 0$, and $\alpha(h_2) \neq 0$. But then Proposition 3.3 applies to

$$\mathfrak{g} := \sum_{\gamma\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}} (L_\gamma + [L_\gamma, L_{-\gamma}])$$

showing that $\bar{\mathfrak{g}} := \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is simple and $\operatorname{rad}\mathfrak{g}$ is H-invariant (one should also take into account Corollary 3.9). The semisimple parts $h_{1,s}, h_{2,s} \in T$ of h_1 and h_2 are linearly independent. Then $T = Fh_{1,s} \oplus Fh_{2,s} \oplus (T \cap \ker\alpha \cap \ker\beta)$ forcing $\operatorname{rad}_T L(\alpha,\beta) = \operatorname{rad} L(\alpha,\beta)$. Since $L(\alpha,\beta) = H + \mathfrak{g}$ and $\operatorname{rad}\mathfrak{g}$ is H-stable, we also obtain that $\operatorname{rad}\mathfrak{g} = \mathfrak{g} \cap \operatorname{rad} L(\alpha,\beta)$. This entails that $\bar{\mathfrak{g}}$ is nothing but $\widetilde{S} = \widetilde{S}[\alpha,\beta]$, the T-socle of $L[\alpha,\beta]$. Since $2 \leqslant TR(\bar{\mathfrak{g}}) \leqslant TR(\mathfrak{g}) \leqslant TR(L(\alpha,\beta)) \leqslant 2$, we get $TR(\widetilde{S}) = 2$. Therefore, Theorem 4.2 is applicable to $L[\alpha,\beta]$. Given $x \in L(\alpha,\beta)_p$, we set $\bar{x} := \Psi_{\alpha,\beta}(x)$. As $TR(\widetilde{S}) = 2$, the simple Lie algebra $\widetilde{S}$ is listed in [10, Theorem 1.1].

Suppose $\widetilde{S}$ is restricted. Then $\widetilde{S} = L[\alpha,\beta]$, by Theorem 4.2. Moreover, $\bar{y}^p = z$ for some $z \in \overline{H}$. Therefore,

$$\overline{(\operatorname{ad} y)^{2p}(H)} = (\operatorname{ad}\bar{y})^{2p}(\overline{H}) = (\operatorname{ad}\bar{y}^p)^2(\overline{H}) = [z,[z,\overline{H}]] \subset \overline{H}^3.$$

As β does not vanish on $\overline{(\operatorname{ad} y)^{2p}(H)}$, we deduce that $\overline{T}$ is a nonstandard torus in $L[\alpha,\beta]_p$. Then [7, Theorem 1] says that $L[\alpha,\beta]$ is isomorphic to the restricted Melikian algebra. By [7, Section 4], all roots in $\Gamma(L[\alpha,\beta],\overline{T})$ are then improper. However, α is still proper when viewed as a root of $L[\alpha,\beta]$ (by our assumption). Thus $\widetilde{S}$ is non-restricted. If $\widetilde{S}$ is isomorphic to one of $W(1;\underline{2})$, $H(2;\underline{1};\Delta)$, $H(2;\underline{1};\Phi(\tau))^{(1)}$ then $\dim\mathfrak{c}_{\widetilde{S}}(\overline{T}) \leqslant 1$ (by [10, Corollary 2.10] and [18]). But $\bar{h}_1, \bar{h}_2 \in \mathfrak{c}_{\widetilde{S}}(\overline{T})$ are linearly independent. Thus it must be that $\widetilde{S} \cong H(2;\underline{(2,1)})^{(2)}$. Then $\widetilde{S}_p = \overline{T} + \widetilde{S}$, by Lemma 2.1(1). Since $L[\alpha,\beta] = \overline{H} + \widetilde{S}$, we also have $\bar{y} \in \widetilde{S}$. So there are $z \in \mathfrak{c}_{\widetilde{S}}(\overline{T})$ and $t \in \overline{T}$ such that $\bar{y}^p = z + t$. Arguing as before, we now get

$$\overline{(\operatorname{ad} y)^{2p}(H)} = [z+t,[z+t,\overline{H}]] \subset [\mathfrak{c}_{\widetilde{S}}(\overline{T}), \mathfrak{c}_{\widetilde{S}}(\overline{T})].$$

Since $\mathfrak{c}_{\widetilde{S}}(\overline{T})$ is triangulable in $\widetilde{S}$, by [7, Theorem 1], this contradicts our choice of β thereby completing the proof. □

Theorem 4.4. *If $r = 1$ and $TR(\widetilde{S}) = 1$ then one of the following occurs*:

(1) $L[\alpha,\beta] = L[\mu]$ *for some* $\mu \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\}$. *Moreover,* $\widetilde{S} = L[\mu]^{(1)}$ *and* $\dim \Psi_{\alpha,\beta}(T) = 1$.

(2) $\widetilde{S} = H(2;\underline{1})^{(2)}$ *and* $L[\alpha,\beta] = H(2;\underline{1})^{(2)} \oplus FD$ *where either* $D = 0$ *or* $D = D_H(x_1^{p-1}x_2^{p-1})$ *or* $p = 5$ *and* $D = x_1^4\partial_2$. *Moreover,* $\dim \Psi_{\alpha,\beta}(T) = 2$.

(3) $\widetilde{S} = S \otimes A(1;\underline{1})$ *where S is one of* $\mathfrak{sl}(2),\ W(1;\underline{1}),\ H(2;\underline{1})^{(2)}$. *Moreover,* $L[\alpha,\beta] \subset (\operatorname{Der} S) \otimes A(1;\underline{1})$ *and* $\Psi_{\alpha,\beta}(T) = (Fh_0 \otimes 1) \oplus (F\operatorname{Id} \otimes (1 + x_1)\partial_1)$ *where h_0 is a nonzero toral element in S.*

(4) $\widetilde{S} = S \otimes A(m;\underline{1})$ *where S is one of* $\mathfrak{sl}(2),\ W(1;\underline{1}),\ H(2;\underline{1})^{(2)}$ *and* $m > 0$. *There exists a classical root* $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ *such that*

$$L[\alpha,\beta] = S \otimes A(m;\underline{1}) + L[\alpha,\beta](\mu); \qquad \pi_2\big(L[\alpha,\beta](\mu)\big) \cong \mathfrak{sl}(2);$$
$$L[\mu,\nu] \cong \mathfrak{g}(1,1) \quad \textit{for some } \nu \in \Gamma(L,T).$$

(5) $\widetilde{S} = S \otimes A(1;\underline{1})$ *where S is one of* $\mathfrak{sl}(2),\ W(1;\underline{1}),\ H(2;\underline{1})^{(2)}$, *and* $L[\alpha,\beta]$ *is a subalgebra in* $(\operatorname{Der} S) \otimes A(1;\underline{1}) + \operatorname{Id} \otimes W(1;\underline{1})$ *such that*

$$\pi_2\big(L[\alpha,\beta]\big) = \pi_2\big(L[\mu]\big) = W(1;\underline{1})$$

for some Witt root $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$.

(6) $\widetilde{S} = S \otimes A(2;\underline{1})$ *where S is one of* $\mathfrak{sl}(2),\ W(1;\underline{1}),\ H(2;\underline{1})^{(2)}$, *and* $L[\alpha,\beta]$ *is a subalgebra in* $(\operatorname{Der} S) \otimes A(2;\underline{1}) + \operatorname{Id} \otimes W(2;\underline{1})$ *such that*

$$H(2;1)^{(2)} \subset \pi_2\big(L[\alpha,\beta]\big) = \pi_2\big(L[\mu]\big) \subset H(2;\underline{1})$$

for some Hamiltonian root $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$.

Proof. As before we denote by $\overline{T}$ the torus $\Psi_{\alpha,\beta}(T) \subset \operatorname{Der} \widetilde{S}$. Since $r = 1$, the T-socle $\widetilde{S}$ is a minimal ideal of $\overline{T} + L[\alpha,\beta]$. By Block's theorem there exists a simple algebra S and a nonnegative integer m such that $\widetilde{S} \cong S \otimes A(m;\underline{1})$ (under an isomorphism φ). As in the present case $0 \neq TR(S) \leqslant TR(\widetilde{S}) = 1$ the Lie algebra S is one of $\mathfrak{sl}(2),\ W(1;\underline{1}),\ H(2;\underline{1})^{(2)}$ (see [19, Theorem 7] and [7]). The isomorphism φ gives rise to a restricted homomorphism

$$\Phi : \Psi_{\alpha,\beta}\big(T + L(\alpha,\beta)_p\big) \longrightarrow \operatorname{Der}\big(S \otimes A(m;\underline{1})\big) = (\operatorname{Der} S) \otimes A(m;\underline{1}) \oplus \big(\operatorname{Id} \otimes W(m;\underline{1})\big).$$

By [9, Theorem 2.6], we may choose φ such that

$$\Phi(\overline{T}) = (Fh_0 \otimes 1) \oplus F(d \otimes 1 + \operatorname{Id} \otimes t_0),$$

where $t_0 \in W(m;\underline{1})$ is a toral element, Fh_0 is a maximal torus of the restricted Lie algebra S, and either $d = 0$ or $S = H(2;\underline{1})^{(2)}$ and $Fh_0 \oplus Fd$ is a maximal torus in $\operatorname{Der} S$. In what follows we choose φ as above and identify $\overline{T}$ with $\Phi(\overline{T})$. Given $y \in L(\alpha,\beta)_p$, we set $\bar{y} := \Phi(\Psi_{\alpha,\beta}(y))$.

(a) Suppose $m = 0$ and $d = 0$. Then $\overline{T} = Fh_0$ acts on $\widetilde{S} = S$ as a 1-dimensional torus. Let $\mu \in T^* \setminus \{0\}$ be such that $\widetilde{S}_\mu \neq (0)$ (recall that $h_0 \in S$ and $\mu(h_0) \neq 0$). Then $L[\alpha,\beta] = \widetilde{S}(\mu) + \overline{H} = L(\mu)/\operatorname{rad}_T L(\mu)$ and $L[\mu]^{(1)} = \widetilde{S}$ (see Corollary 3.7). This is case (1) of the theorem.

(b) Suppose $m = 0$ and $d \neq 0$. Then $S = H(2;\underline{1})^{(2)}$ and $\overline{T} = Fh_0 \oplus Fd$ is a 2-dimensional torus in $\operatorname{Der} S$. By [1, Theorem 1.18.4] (which does not require the assumption that $p > 7$), any 2-dimensional torus of $\operatorname{Der} H(2;\underline{1})^{(2)}$ is conjugate under an automorphism of $H(2;\underline{1})^{(2)}$ to one of $Fx_1\partial_1 \oplus Fx_2\partial_2$, $F(1+x_1)\partial_1 \oplus Fx_2\partial_2$, $Fx_1\partial_1 \oplus F(1+x_2)\partial_2$, $F(1+x_1)\partial_1 \oplus F(1+x_2)\partial_2$. Thus we may assume that

$$h_0 = z_1\partial_1 - z_2\partial_2, \quad d = z_1\partial_1 + z_2\partial_2, \quad z_i \in \{x_i, 1+x_i\}.$$

The eigenspaces of $\operatorname{Der} H(2;\underline{1})^{(2)}$ with respect to $\overline{T}$ are described in [18, Proposition III.1]. As $L[\alpha,\beta] \subset \operatorname{Der} H(2;\underline{1})^{(2)}$, it follows from this description that $\overline{H} \subset \overline{T}$. Suppose $\overline{H} = \overline{T}$ and consider the torus $\mathfrak{t} = F(1+x_1)\partial_1 \oplus F(1+x_2)\partial_2$. Note that $\mathfrak{t} \subset H(2;\underline{1})^{(2)} + \overline{H} \subset L[\alpha,\beta]$. Let $T' \subset T + L(\alpha,\beta)_p$ be a torus of maximal dimension satisfying $\Psi_{\alpha,\beta}(T') = \mathfrak{t}$, and define $\mu' \in (T')^*$ by setting

$$\mu'(T' \cap \ker\alpha \cap \ker\beta) = 0, \qquad \mu'\big((1+x_1)\partial_1\big) = \mu'\big((1+x_2)\partial_2\big) = 1.$$

As $(H(2;\underline{1})^{(2)})(\mu')$ is abelian, $L[\alpha,\beta](\mu')$ is solvable. Therefore, so is the 1-section $L(\mu')$. As $\mu'(\mathfrak{t}) \neq 0$ and $\mathfrak{t} \subset L[\alpha,\beta]$, Proposition 3.8 shows that every $x \in L_{2\mu'}$ is p-nilpotent in L_p. However,

$$D_H\big((1+x_1)^2(1+x_2)^2\big) = 2(1+x_1)(1+x_2)^2\partial_1 - 2(1+x_1)^2(1+x_2)\partial_2$$

is a vector of $\mathfrak{t}$-weight $2\mu'$ which is not p-nilpotent in $H(2;\underline{1})^{(2)}$. This contradiction shows that $\overline{H} \neq \overline{T}$. Thus $\overline{H} = Fh_0 \subset H(2;\underline{1})^{(2)}$. Therefore, the $\overline{T}$-module $L[\alpha,\beta]/H(2;\underline{1})^{(2)}$ has no zero weight. Consequently,

$$L[\alpha,\beta] \subset H(2;\underline{1})^{(2)} \oplus Fz_1^{p-1}\partial_2 \oplus Fz_2^{p-1}\partial_2 \oplus FD_H\big(z_1^{p-1}z_2^{p-1}\big).$$

Now define $\mu \in T^*$ by setting

$$\mu(T \cap \ker\alpha \cap \ker\beta) = 0, \quad \mu(h_0) = 0, \ \mu(d) = 1,$$

and let $\mathfrak{t}_0 := T \cap \ker\mu$ and $\mathfrak{h} := L(\mu)$. Since $\mu(\overline{H}) = 0$ the 1-section $L(\mu) \supsetneq H$ is nilpotent, hence $\mathfrak{t}_0$ coincides with the unique maximal torus in the p-envelope of $\mathfrak{t}_0 + \mathfrak{h}$ in L_p. Moreover, there exists $\Lambda \in \mathfrak{t}_0^*$ such that $L(\alpha,\beta)$ coincides with the 1-section $L(\Lambda)$ relative to $\mathfrak{t}_0$. By Theorem 3.6, the unique minimal ideal of the quotient $L[\Lambda] = L(\Lambda)/\operatorname{rad} L(\Lambda)$ is simple and coincides with $L[\Lambda]^{(\infty)}$. Note that $L[\Lambda]$ is a homomorphic image of $L[\alpha,\beta]$ and the unique minimal ideal of $L[\alpha,\beta]$ is simple on its own right. Then $L[\alpha,\beta]\cong L[\Lambda]$ so that $L[\Lambda]^{(\infty)}\cong H(2;\underline{1})^{(2)}$. Since there is a torus T_0 in $T+L(\alpha,\beta)_p$ with $T\cap\ker\alpha\cap\ker\beta\subset T_0$ and $\overline{T}_0=\mathfrak{t}_0$, we can replace T by T_0 in the final part of the proof of Corollary 3.7 to conclude that $L[\alpha,\beta]\cong H(2;\underline{1})^{(2)}\oplus FD$ where D is either 0 or $D_H(x_1^{p-1}x_2^{p-1})$ or $p=5$ and $D=x_1^4\partial_2$. This is case (2) of the theorem.

(c) Suppose that $m>0$ and there is $\nu\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}$ with $\nu(H)=0$. Set $\mathfrak{t}_0':=T\cap\ker\nu$ and $\mathfrak{h}':=L(\nu)$. Then again $\mathfrak{h}'$ is nilpotent and $\mathfrak{t}_0'$ is the unique maximal torus in the p-envelope of $\mathfrak{t}_0'+\mathfrak{h}'$ in L_p. Moreover, there exists $\Lambda'\in(\mathfrak{t}_0')^*$ such that $L(\alpha,\beta)$ coincides with the 1-section $L(\Lambda')$ relative to $\mathfrak{t}_0'$. By Theorem 3.6, $L[\Lambda']=L(\Lambda')/\operatorname{rad}L(\Lambda')$ has a unique minimal ideal which is simple and coincides with $L[\Lambda']^{(\infty)}$. Let

$$\psi: L[\alpha,\beta]\longrightarrow L[\Lambda']=L[\alpha,\beta]/\operatorname{rad}L[\alpha,\beta]$$

denote the canonical homomorphism. Since $S\otimes A(m;\underline{1})$ is perfect the image $\psi(S\otimes A(m;\underline{1}))$ coincides with the minimal ideal of $L[\Lambda']$, hence is simple. Therefore $S\otimes A(m;\underline{1})_{(1)}=S\otimes A(m;\underline{1})\cap\ker\psi$ is an ideal of $L[\alpha,\beta]$. Then it is an ideal of $L[\alpha,\beta]_p$ as well. This yields $\pi_2(L[\alpha,\beta]_p)\subset W(m;\underline{1})_{(0)}$. On the other hand, $\pi_2(L[\alpha,\beta]_p)$ is an ideal of $\pi_2(L[\alpha,\beta]_p+\overline{T})$ and the latter subalgebra is transitive in $W(m;\underline{1})$. So it must be that $\pi_2(L[\alpha,\beta]_p)=(0)$. Then $\pi_2(\overline{T})=\pi_2(F(d\otimes 1+\mathrm{Id}\otimes t_0))$ is a transitive subalgebra of $W(m;\underline{1})$. This means that there is a toral element $t\in\overline{T}$ such that $\pi_2(t)\notin W(m;\underline{1})_{(0)}$ and $\pi_2(Ft)$ is transitive in $W(m;\underline{1})$. Since $\pi_2(t)$ is conjugate under $\operatorname{Aut}A(m;\underline{1})$ to $(1+x_1)\partial_1$, by Demushkin's theorem, we conclude that $m=1$. This is case (3) of the theorem.

(d) From now on suppose that $m>0$ and $\gamma(H)\neq 0$ for all $\gamma\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}$ (this implies that $T\subset H_p$). Fix $\mu\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}$ with $\mu(h_0)=0$. Then

$$\begin{aligned}L[\alpha,\beta]&=L[\alpha,\beta](\mu)+\widetilde{S},\\ L[\alpha,\beta](\mu)&\subset\mathfrak{c}_{\operatorname{Der}S}(h_0)\otimes A(m;\underline{1})+\mathrm{Id}\otimes\pi_2\big(L[\alpha,\beta](\mu)\big),\\ Fh_0\otimes A(m;\underline{1})&\subset\operatorname{rad}_T\big(L[\alpha\beta](\mu)\big).\end{aligned}$$

Since $T\subset H_p$, we also have that $\pi_2(L[\alpha,\beta]_p+\overline{T})=\pi_2(\overline{L(\mu)_p})$. As a consequence, $\pi_2(\overline{L(\mu)_p})$ is a transitive subalgebra of $W(m;\underline{1})$.

Suppose all roots in $\mathbb{F}_p^*\mu\cap\Gamma(L,T)$ (if any) are proper. Let $y\in\bigcup_{i\in\mathbb{F}_p^*}(\operatorname{rad}L(\mu))_{i\mu}$. Proposition 3.8 shows that the subspace $[y,(\Phi\circ\Psi_{\alpha,\beta})^{-1}(Fh_0\otimes A(m;\underline{1}))]$ consists of p-nilpotent elements of L_p. Then all elements in $[\bar{y},Fh_0\otimes A(m;\underline{1})]$ act nilpotently on $\widetilde{S}$ forcing $\pi_2(\bar{y})\in W(m;\underline{1})_{(0)}$. Now let $y\in H$. Then $\bar{y}\in\mathfrak{c}_{\operatorname{Der}S}(h_0)\otimes A(m;\underline{1})+\mathrm{Id}\otimes\pi_2(\bar{y})$, hence $[\bar{y},h_0\otimes f]=h_0\otimes\pi_2(\bar{y})f$ for all $f\in A(m;\underline{1})$. Since $Fh_0\otimes A(m;\underline{1})$ is $\overline{T}$-stable, we can combine Proposition 3.8 and Corollary 3.9 to deduce that all elements in $(\operatorname{ad}\bar{y})^3(Fh_0\otimes A(m;\underline{1}))$ act nilpotently on $\widetilde{S}$. But then again $\pi_2(\bar{y})\in W(m;\underline{1})_{(0)}$.

Thus we have proved that $\pi_2(\overline{\operatorname{rad}L(\mu)})\subset W(m;\underline{1})_{(0)}$. Since $\pi_2(\overline{\operatorname{rad}L(\mu)})$ is an ideal in $\pi_2(\overline{L(\mu)_p})$, a transitive subalgebra of $W(m;\underline{1})$ we conclude that $\pi_2(\overline{\operatorname{rad}L(\mu)})=(0)$. If

$\pi_2(\overline{L(\mu)}) = (0)$ then $\pi_2(\overline{L(\mu)_p}) = (0)$, because $\pi_2 \circ \Phi \circ \Psi_{\alpha,\beta}$ is a restricted homomorphism. However, $m > 0$ and $\pi_2(\overline{L(\mu)_p})$ is transitive in $W(m;\underline{1})$. So $L(\mu)$ is nonsolvable, in particular, $\mathbb{F}_p^*\mu \cap \Gamma(L,T) \neq \emptyset$. Since $(\ker \pi_2)(\mu) \subset \mathfrak{c}_{\mathrm{Der}\, S}(h_0) \otimes A(m;\underline{1})$ is solvable, we also obtain

$$\pi_2\big(L[\alpha,\beta]\big) = \pi_2\big(\overline{L(\mu)}\big) \cong L(\mu)/\operatorname{rad} L(\mu) = L[\mu].$$

(e) We continue assuming that all roots in $\mathbb{F}_p^*\mu$ are proper. Since $\mathbb{F}_p^*\mu \cap \Gamma(L,T) \neq \emptyset$, by part (d), we may assume without loss that μ is a root. If μ is classical and fits into a Melikian 2-section then we are in case (4) of the theorem. So suppose μ is not of this type. Proposition 3.8 then says that the union $\bigcup_{i\in\mathbb{F}_p^*}[Q(\mu)_{i\mu}, (\operatorname{rad} L(\mu))_{-i\mu}]$ consists of p-nilpotent elements of L_p. Arguing as in part (d), we are now able to deduce that $\pi_2(\bar{y}) \in W(m;\underline{1})_{(0)}$ for all $y \in H \cup (\bigcup_{i\in\mathbb{F}_p^*} Q_{i\mu})$. Now let $y \in \bigcup_{i\in\mathbb{F}_p^*} L_{i\mu}$. Then

$$\overline{y^p} \in \mathfrak{c}_{\mathrm{Der}\, S}(h_0) \otimes A(m;\underline{1}) + \mathrm{Id} \otimes \pi_2(\overline{y^p}).$$

Since $Fh_0 \otimes A(m;\underline{1})$ is $\overline{T}$-stable, we now combine Lemma 4.3 and Corollary 3.9 to deduce that all elements in $(\operatorname{ad} \overline{y^p})^2(Fh_0 \otimes A(m;\underline{1}))$ act nilpotently on $\widetilde{S}$. Then again $\pi_2(\overline{y^p}) \in W(m;\underline{1})_{(0)}$. Thus

$$\pi_2\left(\overline{Q(\mu)} + \sum_{i\in\mathbb{F}_p^*}\sum_{r>0} \overline{L_{i\mu}^{p^r}}\right) \subset W(m;\underline{1})_{(0)},$$

which implies that $\pi_2(\overline{Q(L(\mu)_p)}) \subset W(m;\underline{1})_{(0)}$. This enables us to conclude that

$$\begin{aligned} m = \dim W(m;\underline{1})/W(m;\underline{1})_{(0)} &= \dim \pi_2\big(\overline{L(\mu)_p}\big)/W(m;\underline{1})_{(0)} \cap \pi_2\big(\overline{L(\mu)_p}\big) \\ &\leqslant \dim \pi_2\big(\overline{L(\mu)_p}\big)/\pi_2\big(\overline{Q\big(L(\mu)_p\big)}\big) \\ &= \dim L(\mu)_p/Q\big(L(\mu)_p\big) \\ &= \dim L[\mu]/Q\big(L[\mu]\big) \end{aligned}$$

(one should keep in mind that the solvable ideal $(\ker \pi_2 \circ \Phi \circ \Psi_{\alpha,\beta})(\mu)$ of $L(\mu)_p$ is contained in $Q(L(\mu)_p)$). As a first consequence, $m \leqslant 2$. More precisely, μ is Witt if and only if $m = 1$, and μ is Hamiltonian if and only if $m = 2$ (since $m > 0$, μ is not classical).

(f) Finally, suppose $\mu \in \Gamma(L,T)$ is improper. This case will involve toral switchings; we refer to [9, pp. 218–222] for related material and notation. It follows from [2, (1.9)] that there are $\xi \in \{\varkappa \in \operatorname{Hom}_{\mathbb{F}_p}(F,F) \mid \varkappa^p - \varkappa = \mathrm{Id}_F\}$ and $u \in \bigcup_{i\in\mathbb{F}_p^*} L_{i\mu}$ such that the torus $T_u = \{t - \mu(t)\sum_{i=1}^{m(u)} u^{p^i} \mid t \in T\}$ has the property that any root $i\mu_{u,\xi} \in \Gamma(L,T_u)$ with $i \in \mathbb{F}_p^*$ is proper. Since $u \in L(\mu)$, we have that $L(\mu) = L(\mu_{u,\xi})$. Since $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$, we have that $L(\alpha,\beta) = L(\alpha_{u,\xi},\beta_{u,\xi})$. Since the generalized Winter exponentials $E_{\pm u,\xi}$ preserve all ideals of $L(\alpha,\beta)$, we also have that $\operatorname{rad}_T L(\alpha,\beta) = \operatorname{rad}_{T_u} L(\alpha_{u,\xi},\beta_{u,\xi})$. Then

$$L[\alpha,\beta] = L(\alpha,\beta)/\operatorname{rad}_T L(\alpha,\beta) = L(\alpha_{u,\xi},\beta_{u,\xi})/\operatorname{rad}_{T_u} L(\alpha_{u,\xi},\beta_{u,\xi}) = L[\alpha_{u,\xi},\beta_{u,\xi}].$$

As a consequence, $\Psi_{\alpha,\beta} = \Psi_{\alpha_{u,\xi},\beta_{u,\xi}}$. Moreover, $L[\alpha_{u,\xi}, \beta_{u,\xi}]$ has a unique minimal ideal isomorphic to $S \otimes A(m; \underline{1})$ with S and m unchanged. Thus we can choose the same embedding Φ for both $L[\alpha, \beta]$ and $L[\alpha_{u,\xi}, \beta_{u,\xi}]$. Note that $h_0 \in \Phi(\Psi_{\alpha,\beta}(T)) \cap \ker \mu = \Phi(\Psi_{\alpha_{u,\xi},\beta_{u,\xi}}(T_u)) \cap \ker \mu_{u,\xi}$ and

$$(\pi_2 \circ \Phi \circ \Psi_{\alpha,\beta})\big(L(\mu)\big) = (\pi_2 \circ \Phi \circ \Psi_{\alpha_{u,\xi},\beta_{u,\xi}})\big(L(\mu_{u,\xi})\big).$$

Since all roots in $\mathbb{F}_p^*\mu_{u,\xi}$ are proper we can apply parts (d) and (e) of this proof to conclude that we are in case (5) (respectively (6)) of the theorem when $\mu_{u,\xi}$ is Witt (respectively Hamiltonian). To finish the proof it now remains to mention that $\mu_{u,\xi}$ is Witt (Hamiltonian) if and only if μ is. □

5. Some remarks on Block algebras of dimension $p^2 - 1$

In this section, we are going to revise [17] in order to extend the results there to our present situation. It is assumed in [17, §1–3] that $p > 3$ but at the beginning of §4 it is imposed that $p > 5$. We will go through the proofs and check for their validity in characteristic 5. All our references to [17] will be boldfaced. Recall that $H = H(2; \underline{1}; \Phi(\tau))^{(1)}$ is a Block algebra of dimension $p^2 - 1$, $\Lambda = 1 - x^{p-1}y^{p-1} \in H$, $\Theta = -y^{p-1}\partial_x \in \operatorname{Der} H$, and M is an irreducible H-module of dimension $\leqslant p^2$. By Proposition **2.2**, the semisimple p-envelope H_p of H (which is isomorphic to $\operatorname{Der} H$) acts naturally on M. The p-character of the H_p-module M is denoted by μ. From now on we use the notation of [17] without further comment.

Lemmas **4.1–4.3** hold for $p = 5$. Lemma **4.4(1)** needs a new proof given below.

Let M_0 denote an irreducible $H_{(0)}$-submodule of M. According to (4.2), M_0 is an irreducible module for $Fx^2 + Fxy + Fy^2$. Pick a nonzero $u \in M_0$ with $\rho(xy) \cdot u \in Fu$. The set

$$\big\{\rho(x)^i\rho(y)^j u \;\big|\; 0 \leqslant i \leqslant 4,\ 0 \leqslant j \leqslant 2\big\} \cup \big\{\rho(x)^i\rho(y)^j\rho(x^2)u \;\big|\; 0 \leqslant i \leqslant 4,\ 0 \leqslant j \leqslant 2\big\}$$

consists of 30 elements. As $\dim M \leqslant 5^2$ we have a nontrivial relation

$$\left(\sum_{i=0}^{4}\sum_{j=0}^{2}\big(\alpha_{ij}\rho(x)^i\rho(y)^j + \beta_{ij}\rho(x)^i\rho(y)^j\rho\big(x^2\big)\big)\right) \cdot u = 0.$$

Put $k := \max\{i + j \mid \alpha_{ij} \neq 0 \text{ or } \beta_{ij} \neq 0\}$, $s := \max\{j \mid \alpha_{k-j,j} \neq 0 \text{ or } \beta_{k-j,j} \neq 0\}$, and $r := k - s$. Obviously, $k \leqslant 6$ and $s \leqslant 2$. If $k < 6$ then argue as in the original proof. If $k = 6$ then $s = 2$, $r = 4$, and

$$\begin{aligned} 0 &= \big(\rho(\Lambda) - \mu(\Lambda)\mathrm{Id}\big)\left(\sum_{i,j}\big(\alpha_{i,j}\rho(x)^i\rho(y)^j + \beta_{ij}\rho(x)^i\rho(y)^j\rho\big(x^2\big)\big)u\right) \\ &= \sum_{i,j}\alpha_{ij}\big[\rho(\Lambda), \rho(x)^i\rho(y)^j\big]u + \sum_{i,j}\beta_{ij}\big[\rho(\Lambda), \rho(x)^i\rho(y)^j\big]\rho\big(x^2\big)u \end{aligned}$$

$$= \alpha_{4,2}\big[\rho(\Lambda), \rho(x)^4\rho(y)^2\big]u + \beta_{4,2}\big[\rho(\Lambda), \rho(x)^4\rho(y)^2\big]\rho\big(x^2\big)u$$
$$= \frac{1}{2}\alpha_{4,2}\rho\big(x^2\big)u + \frac{1}{2}\beta_{4,2}\rho\big(x^2\big)^2u.$$

As $\rho(x^2)u$ and $\rho(x^2)^2u$ belong to distinct eigenspaces of $\rho(xy)$ it must be that $\alpha_{4,2}\rho(x^2)u = \beta_{4,2}\rho(x^2)^2u = 0$. As $k = 6$ one of these coefficients has to be nonzero implying $\rho(x^2)^2u = 0$. The rest of Lemma **4.4** now follows from this result.

To extend Lemma **4.5** we consider the set

$$\big\{\rho(\Theta)^i\rho(y)^ju \;\big|\; 0 \leqslant i, j \leqslant 3\big\} \cup \big\{\rho(\Theta)^i\rho(y)^j\rho(x)u \mid 0 \leqslant i, j \leqslant 3\big\}$$

and then proceed as in the original proof. The proofs of Theorems **4.6–6.3** only require Lemmas **4.4** and **4.5**, and $(p > 3)$-arguments. So these theorems hold for $p > 3$. However, we will need a better estimate in Proposition **6.4(1)**.

Proposition 6.4(1) (improved). *Let G be a central extension of $H(2; \underline{1}; \Phi(\tau))^{(1)}$ and M be a finite-dimensional G-module. If $G^{(1)} \cap C(G)$ acts non-nilpotently on M then* $\dim M > p^{(p-1)/2}$.

Proof. Suppose $\dim M \leqslant p^{(p-1)/2}$. The proof in [17] shows that M is an irreducible G-module of dimension $p^{(p-1)/2}$ and the monomials $\rho(y)^{k_1}\rho(y^2)^{k_2}\cdots\rho(y^{p-1})^{k_{p-1}}$ with $0 \leqslant k_i \leqslant p-1$ form a basis of $\operatorname{End} M$. Then

$$\rho(\Lambda) = \sum_{\mathbf{k}=(k_1,k_2,\dots,k_{p-1})} \alpha_{\mathbf{k}}\rho(y)^{k_1}\rho\big(y^2\big)^{k_2}\cdots\rho\big(y^{p-1}\big)^{k_{p-1}}$$

for some $\alpha_{\mathbf{k}} \in F$. The central extensions of $H(2; \underline{1}; \Phi(\tau))^{(1)}$ are described in Theorem **6.3**. It is immediate from this description that $\rho(\Lambda)$ commutes with $\rho(y^k)$ for $k = 2, \dots, p-1$. Applying the operators $\operatorname{ad}\rho(y^{p-1}), \operatorname{ad}\rho(y^{p-2}), \dots, \operatorname{ad}\rho(y^2)$ to the above expression, one derives that $\alpha_{\mathbf{k}} = 0$ unless $\mathbf{k} = (0, \dots, k_{p-1})$. It follows that $\rho(\Lambda)$ is a linear combination of $\rho(y^{p-1})^j$ with $0 \leqslant j \leqslant p-1$. Since $\rho(\Lambda)$ commutes with $\rho(xy)$, this yields that $\rho(\Lambda) \subset F\mathrm{Id}_M$. Since $H(2; \underline{1}; \Phi(\tau))^{(1)}$ is simple this contradicts our general assumption on M. □

The improved Proposition **6.4(1)** enables us to extend Theorem **6.5(1)** after which all arguments used in [17, Sections 6, 7] go through for $p > 3$. We conclude that all results of [17] hold for $p > 3$.

In the sequel, we will need two additional results on $H(2; \underline{1}; \Phi(\tau))^{(1)}$.

Proposition 5.1. *Let χ be a linear function on the Lie algebra* $\operatorname{Der} H(2; \underline{1}; \Phi(\tau))^{(1)}$ *vanishing on $H(2; \underline{1}; \Phi(\tau))^{(1)}{}_{(0)}$ and T be a* 2*-dimensional torus in* $\operatorname{Der} H(2; \underline{1}; \Phi(\tau))^{(1)}$. *Let $u, v \in H(2; \underline{1}; \Phi(\tau))^{(1)}$ be root vectors for T corresponding to roots α and β, respectively. Then*

$$\chi\big([u, v]\big)^p = -\alpha\big(v^p\big)\chi(u)^p + \beta\big(u^p\big)\chi(v)^p.$$

Proof. Recall from [17, Section 1] that $H(2;\underline{1};\Phi(\tau))^{(1)}$ is the derived subalgebra of a Poisson Lie algebra. More precisely, $H(2;\underline{1};\Phi(\tau))^{(1)} \cong (A(2;\underline{1}),\{\cdot,\cdot\})^{(1)}$ where the Poisson bracket $\{\cdot,\cdot\}$ on the commutative algebra $A(2;\underline{1})$ is given by $\{x,y\}=\Lambda$. According to [18, Theorem VII.3], there is a generating set $\{y_1,y_2\}\subset A(2;\underline{1})_{(1)}$ of the commutative algebra $A(2;\underline{1})$ such that

$$\{y_1,y_2\}=(1+y_1)(1+y_2) \quad \text{and} \quad T=(1+y_1)\partial/\partial y_1 \oplus F(1+y_2)\partial/\partial y_2.$$

For $i=1,2$, set $z_i:=(1+y_i)$. Since all T-root spaces of the Poisson algebra $A(2;\underline{1})$ are 1-dimensional, we may assume, after rescaling, that $u=z_1^a z_2^b$ and $v=z_1^c z_2^d$ for some $a,b,c,d\in\mathbb{F}_p$ with $(0,0)\notin\{(a,b),(c,d)\}$. As $\{z_1^a z_2^b, z_1^c z_2^d\}=(ad-bc)z_1^{a+c}z_2^{b+d}$ we derive that

$$\beta\big(u^p\big)v=(\operatorname{ad}u)^p\big(z_1^c z_2^d\big)=(ad-bc)^p v,$$
$$\alpha\big(v^p\big)u=(\operatorname{ad}v)^p\big(z_1^a z_2^b\big)=(bc-ad)^p u.$$

Since $z_1^i z_2^j\equiv iy_1+jy_2 \pmod{A(2;\underline{1})_{(2)}+F\Lambda}$, it follows from [17, Proposition 1.2(1)] that

$$z_1^i z_2^j\equiv iz_1+jz_2 \quad \big(\operatorname{mod} H\big(2;\underline{1};\Phi(\tau)\big)^{(1)}{}_{(0)}\big) \quad (\forall i,j\in\mathbb{F}_p).$$

As χ vanishes on $H(2;\underline{1};\Phi(\tau))^{(1)}{}_{(0)}$, we then have

$$\begin{aligned}
&\chi\big([u,v]\big)^p+\alpha\big(v^p\big)\chi(u)^p-\beta\big(u^p\big)\chi(v)^p\\
&\quad=\chi\big((ad-bc)z_1^{a+c}z_2^{b+d}\big)^p+(bc-ad)^p\chi\big(z_1^a z_2^b\big)^p-(ad-bc)^p\chi\big(z_1^c z_2^d\big)^p\\
&\quad=(ad-bc)^p\big(\big((a+c)\chi(z_1)+(b+d)\chi(z_2)\big)^p-\big(a\chi(z_1)+b\chi(z_2)\big)^p\\
&\qquad-\big(c\chi(z_1)+d\chi(z_2)^p\big)\big)=0,
\end{aligned}$$

completing the proof. □

By [2, Proposition 2.1.8(b)], the derivation algebra $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$ is naturally identified with a restricted subalgebra of $W(2;\underline{1})$. The standard maximal subalgebra of $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$ is defined as $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}\cap W(2;\underline{1})_{(0)}$. It is obviously restricted and has codimension 2 in $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$.

Proposition 5.2. *Let $\mathcal{L}$ be a restricted Lie algebra with $MT(\mathcal{L})=2$ such that*

(a) $\operatorname{rad}\mathcal{L}$ *is abelian and* $\mathcal{L}/\operatorname{rad}\mathcal{L}\cong \operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$,
(b) $\operatorname{rad}\mathcal{L}\cong H(2;\underline{1};\Phi(\tau))^{(1)}$ *as* $(L/\operatorname{rad}L)$*-modules.*

Let T be a 2-dimensional torus in $\mathcal{L}$, let $\mathcal{L}_{(0)}$ denote the preimage of the standard maximal subalgebra of $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$ *in $\mathcal{L}$, and suppose that*

$$\operatorname{rad}\mathcal{L} \not\subset [\mathcal{L}_{(0)}, \mathcal{L}_{(0)}].$$

Then there exists a restricted subalgebra $\mathcal{M}$ in $\mathcal{L}$ with $T \subset \mathcal{M}$ and $\mathcal{L} = \mathcal{M} \oplus \operatorname{rad}\mathcal{L}$. In particular, $\mathcal{L}$ is a split extension.

Proof. (1) Since both the standard maximal subalgebra of $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$ and $\operatorname{rad}\mathcal{L}$ are restricted so is $\mathcal{L}_{(0)}$. It is immediate from assumption (b) that the Lie algebra $\mathcal{L}$ is centerless, while from the description in [17, Proposition 1.2(1)] it follows that

$$\operatorname{rad}\mathcal{L}_{(0)} = \left\{x \in \mathcal{L}_{(0)} \;\middle|\; x + \operatorname{rad}\mathcal{L} \in \operatorname{Der} H\big(2;\underline{1};\Phi(\tau)\big)^{(1)} \cap W(2;\underline{1})_{(1)}\right\}$$

and $\mathcal{L}_{(0)}/\operatorname{rad}\mathcal{L}_{(0)} \cong \mathfrak{sl}(2)$. Given $x \in \operatorname{rad}\mathcal{L}_{(0)}$ one has $x^{[p]^r} \in \operatorname{rad}\mathcal{L}$ for $r \gg 0$ forcing $x^{[p]^{r+1}} \in C(\mathcal{L}) = (0)$. Thus $\operatorname{rad}\mathcal{L}_{(0)}$ is p-nilpotent.

(2) By our assumption, $\operatorname{rad}\mathcal{L} \not\subset [\mathcal{L}_{(0)}, \mathcal{L}_{(0)}]$. So there exist a subspace V in $\mathcal{L}_{(0)}$ and a nonzero $c \in \operatorname{rad}\mathcal{L}$ such that $[\mathcal{L}_{(0)}, \mathcal{L}_{(0)}] \subset V$ and $\mathcal{L}_{(0)} = V \oplus Fc$. Clearly, V is an ideal of $\mathcal{L}_{(0)}$. Let λ denote the linear function on $\mathcal{L}_{(0)}$ with $\ker\lambda = V$ and $\lambda(c) = 1$. Let $F_\lambda = F1_\lambda$ be a 1-dimensional vector space over F. The map $\rho_\lambda : \mathcal{L}_{(0)} \to \mathfrak{gl}(F_\lambda)$ given by $\rho_\lambda(x) \cdot 1_\lambda = \lambda(x)1_\lambda$, is a representation of $\mathcal{L}_{(0)}$. It is well known (and easily seen) that there exists a linear function μ on $\mathcal{L}$ such that

$$\mu(x)^p = \lambda(x)^p - \lambda\big(x^{[p]}\big) \quad \forall x \in \mathcal{L}_{(0)}.$$

Let $u(\mathcal{L}, \mu)$ denote the reduced enveloping algebra of $\mathcal{L}$ corresponding to $\mu \in \mathcal{L}^*$, and

$$M := u(\mathcal{L}, \mu) \otimes_{u(\mathcal{L}_{(0)}, \mu)} F_\lambda,$$

a p^2-dimensional induced $\mathcal{L}$-module with p-character μ. Let M' be a composition factor of M. It is immediate from assumption (b) that $[\mathcal{L}, \operatorname{rad}\mathcal{L}] = \operatorname{rad}\mathcal{L}$. Due to the choice of λ the radical of $\mathcal{L}$ does not act nilpotently on M' (indeed, as $c^{[p]} \in C(\mathcal{L}) = (0)$, the only eigenvalue of c on M' equals $\mu(c) = \lambda(c) = 1$). Theorem **6.5** now shows that $\dim M' = p^2$. As $\operatorname{rad}\mathcal{L}$ is a minimal ideal of $\mathcal{L}$, we deduce that M is irreducible and faithful. Therefore, so is the dual $\mathcal{L}$-module M^*. It is well known that the $\mathcal{L}$-module M^* is coinduced. More precisely, the antipode of the universal enveloping algebra $U(\mathcal{L})$ induces a natural isomorphism

$$M^* \cong \operatorname{Hom}_{u(\mathcal{L}_{(0)}, -\mu)}\big(u(\mathcal{L}, -\mu), F_\lambda^*\big).$$

(3) We now observe that the discussion in [8, Section 2] applies to the $\mathcal{L}$-module M^*. Recall that the restricted $\mathcal{L}$-module $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)}) = \mathrm{Hom}_{u(\mathcal{L}_{(0)})}(u(\mathcal{L}), F)$ carries a natural commutative algebra structure and $\mathcal{L}$ acts on $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$ as derivations. The algebra $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$ acts on M^* via the comultiplication in $U(\mathcal{L})$ and this action is compatible with the action of $\mathcal{L}$ on $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$ and M^*. Furthermore, M^* is a free module of rank 1 over $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$.

Since $\mathcal{L}_{(0)}$ has codimension 2 in $\mathcal{L}$, we have that $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)}) \cong A(2; \underline{1})$ as algebras. The (tautological) semidirect product $\mathfrak{W}(2; \underline{1}) := W(2; \underline{1}) \oplus A(2; \underline{1})$ acts faithfully and restrictedly on $A(2; \underline{1})$. It follows from the preceding remark that after a proper identification of the free $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$-modules M^* and $A(2; \underline{1})$ the initial representation $\mathcal{L} \to \mathfrak{gl}(M^*)$ will factor as

$$\mathcal{L} \xrightarrow{\sigma} \mathfrak{W}(2; \underline{1}) \longrightarrow \mathfrak{gl}\big(A(2; \underline{1})\big)$$

for some injective homomorphism σ (see [8, p. 428] for more detail). Let π denote the canonical projection from $\mathfrak{W}(2; \underline{1})$ onto $W(2; \underline{1})$. Since M^* is $\mathcal{L}$-irreducible, the subalgebra $\pi(\mathcal{L})$ is transitive in $W(2; \underline{1})$. Since $\mathcal{L}_{(0)}$ preserves the unique maximal ideal of $\mathcal{F}(\mathcal{L}, \mathcal{L}_{(0)})$, it follows from our construction that $(\pi \circ \sigma)(\mathcal{L}_{(0)}) \subset W(2; \underline{1})_{(0)}$. Since $(\pi \circ \sigma)(\operatorname{rad} \mathcal{L}) \subset W(2; \underline{1})_{(0)}$ is an ideal of $(\pi \circ \sigma)(\mathcal{L})$, the transitivity of $(\pi \circ \sigma)(\mathcal{L})$ yields $(\pi \circ \sigma)(\operatorname{rad} \mathcal{L}) = (0)$. As a consequence,

$$(\pi \circ \sigma)(\mathcal{L}) \cong \mathcal{L}/\operatorname{rad} \mathcal{L} \cong \operatorname{Der} H\big(2; \underline{1}; \Phi(\tau)\big)^{(1)}.$$

(4) We now look at $\sigma(T) \subset \mathfrak{W}(2; \underline{1})$. Clearly, $T = Ft_1 \oplus Ft_2$ for some toral elements $t_1, t_2 \in \mathcal{L}$. Since M^* has p-character $-\mu$, we have that $\sigma(t_i)^p = \sigma(t_i^{[p]}) - \mu(t_i)^p 1$ where 1 is the unity in $A(2; \underline{1}) \subset \mathfrak{W}(2; \underline{1})$. Therefore, if $\mu(t_i) \neq 0$ for some i then $\sigma(T)$ is not closed under taking pth powers in $\mathfrak{W}(2; \underline{1})$. Choose $\lambda_i \in F$ with $\lambda_i^p - \lambda_i = -\mu(t_i)^p$ where $i = 1, 2$, and set

$$\mathfrak{t} := F\big(\sigma(t_1) + \lambda_1 1\big) \oplus F\big(\sigma(t_2) + \lambda_2 1\big).$$

Then $\mathfrak{t}$ is a torus in $\mathfrak{W}(2; \underline{1})$ and $\pi(\mathfrak{t}) = (\pi \circ \sigma)(T)$ is a torus in $(\pi \circ \sigma)(\mathcal{L})$. By [8, Theorem 3.3], there is $f \in A(2; \underline{1})_{(1)}$ such that $(\exp \operatorname{ad} f)(\mathfrak{t}) \subset W(2; \underline{1}) \oplus F1$. Thus we may assume without loss of generality that $\sigma(T) \subset W(2; \underline{1}) \oplus F1$. Since $C(\mathcal{L}) = (0)$ and $(\pi \circ \sigma)(\operatorname{rad} \mathcal{L}) = (0)$, we have $\sigma(\operatorname{rad} \mathcal{L}) \cap (W(2; \underline{1}) \oplus F1) = (0)$. Since $\operatorname{rad} \mathcal{L}$ has dimension $p^2 - 1$, we now get

$$\mathfrak{W}(2; \underline{1}) = \sigma(\operatorname{rad} \mathcal{L}) \oplus \big(W(2; \underline{1}) \oplus F1\big).$$

Set $\mathcal{M} := \sigma^{-1}(\sigma(\mathcal{L}) \cap (W(2; \underline{1}) \oplus F1))$. By construction, $\mathcal{L} = \mathcal{M} \oplus \operatorname{rad} \mathcal{L}$ and $T \subset \mathcal{M}$. If $x \in \mathcal{M}$ then $\sigma(x)^p \in W(2; \underline{1}) \oplus F1$ as $W(2; \underline{1}) \oplus F1$ is a restricted subalgebra in $\mathfrak{W}(2; \underline{1})$. But then $\sigma(x^{[p]}) = \sigma(x)^p + \mu(x)^p 1 \in W(2; \underline{1}) \oplus F1$. Thus $\mathcal{M}$ is a restricted subalgebra of $\mathcal{L}$ and our proof is complete. □

As a consequence of Propositions 5.1 and 5.2 we obtain the following proposition.

Proposition 5.3. *Let L, $\mathcal{L}_{(0)}$, and T be as in Proposition* 5.2, *and let χ be a linear function on $\mathcal{L}$ with $\chi([\mathcal{L}_{(0)}, \mathcal{L}_{(0)}]) = 0$. Let u and v be root vectors in $\mathcal{L}$ corresponding to* (*nonzero*) *T-roots α and β. Then*

$$\chi\big([u,v]\big)^p = -\alpha\big(v^{[p]}\big)\chi(u)^p + \beta\big(u^{[p]}\big)\chi(v)^p.$$

Proof. (1) By Proposition 5.2, $\mathcal{L} = \mathcal{M} \oplus \operatorname{rad}\mathcal{L}$ and $T \subset \mathcal{M}$. Since $\mathcal{M}^{(1)} \cong H(2;\underline{1};\Phi(\tau))^{(1)}$ and $[\mathcal{L}, \operatorname{rad}\mathcal{L}] = \operatorname{rad}\mathcal{L}$ it must be that $\mathcal{L}^{(1)} = \mathcal{M}^{(1)} \oplus \operatorname{rad}\mathcal{L}$. So $\mathcal{L}_\gamma = \mathcal{M}_\gamma \oplus (\operatorname{rad}\mathcal{L})_\gamma \subset \mathcal{L}^{(1)}$ for any $\gamma \in \Gamma(\mathcal{L}, T)$. Besides, T has no zero weight on $\mathcal{L}^{(1)}$. In view of Jacobson's formula, the latter implies that the function $x \mapsto \gamma(x^{[p]})$ is p-linear on $\mathcal{L}_\delta$ for any $\delta \in \Gamma(\mathcal{L}, T)$. But then the function

$$(x,y) \longmapsto \chi\big([x,y]\big)^p + \alpha\big(x^{[p]}\big)\chi(y)^p - \beta\big(y^{[p]}\big)\chi(x)^p$$

is p-bilinear on $\mathcal{L}_\alpha \times \mathcal{L}_\beta$.

(2) Let $\mathcal{M}_{(0)} := \mathcal{L}_{(0)} \cap \mathcal{M}$. Clearly, $\mathcal{M}_{(0)}$ is isomorphic to the standard maximal subalgebra of $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$. According to [17, Proposition 1.2(2d)], ${\mathcal{M}_{(0)}}^{(1)} = {\mathcal{M}^{(1)}}_{(0)} \cong {H(2;\underline{1};\Phi(\tau))^{(1)}}_{(0)}$. By our assumption, χ vanishes on ${\mathcal{M}_{(0)}}^{(1)}$. Thus if $u, v \in \mathcal{M}$ then the desired result follows from Proposition 5.1.

Recall that $\operatorname{rad} L$ is p-nilpotent (see the proof of Proposition 5.2). So if $u, v \in \operatorname{rad}\mathcal{L}$ then $[u,v] = 0$ and $\alpha(u^{[p]}) = \beta(u^{[p]}) = 0$. Thus in this case we are done as well. Due to our discussion in part (1), we can now assume that $u \in \mathcal{M}$, $v \in \operatorname{rad}\mathcal{L}$. To finish the proof we will need to show that

$$\chi\big([u,v]\big)^p = \beta\big(u^{[p]}\big)\chi(v)^p.$$

(3) We identify $\mathcal{M}^{(1)} \cong H(2;\underline{1};\Phi(\tau))^{(1)}$ with the derived subalgebra of the Poisson algebra $(A(2;\underline{1}), \{\cdot,\cdot\})$. As in the proof of Proposition 5.1, we choose generators $y_1, y_2 \in A(2;\underline{1})_{(1)}$ such that

$$\{y_1, y_2\} = (1+y_1)(1+y_2) \quad \text{and} \quad T = (1+y_1)\partial/\partial y_1 \oplus F(1+y_2)\partial/\partial y_2$$

(this is possible because $T \subset \mathcal{M} = \operatorname{Der}\mathcal{M}^{(1)}$). We now fix an $\mathcal{M}$-module isomorphism $\eta : H(2;\underline{1};\Phi(\tau))^{(1)} \xrightarrow{\sim} \operatorname{rad}\mathcal{L}$ and set $z_i = 1 + y_i$ for $i = 1, 2$. Since all T-root spaces of the Poisson algebra $A(2;\underline{1})$ are 1-dimensional it can be assumed, after rescaling, that $u = z_1^a z_2^b$ and $v = \eta(z_1^c z_2^d)$ for some $(a,b), (c,d) \in \mathbb{F}_p^2 \setminus (0,0)$. Since $z_1^2 - 2z_1 + 1, z_2^2 - 2z_2 + 1 \in A(2;\underline{1})_{(2)} \subset \mathcal{M}_{(0)}$, the subspace $[\mathcal{M}_{(0)}, \operatorname{rad}\mathcal{L}]$ contains all elements

$$\big\{z_1^2 - 2z_1 + 1, \eta\big(z_1^i z_2^j\big)\big\} = 2j\big(\eta\big(z_1^{i+2} z_2^j\big) - \eta\big(z_1^{i+1} z_2^j\big)\big),$$
$$\big\{z_2^2 - 2z_2 + 1, \eta\big(z_1^i z_2^j\big)\big\} = -2i\big(\eta\big(z_1^i z_2^{j+2}\big) - \eta\big(z_1^i z_2^{j+1}\big)\big).$$

From this it is easy to deduce that $\eta(z_1^i z_2^j) - \eta(z_1 z_2) \in [\mathcal{M}_{(0)}, \operatorname{rad}\mathcal{L}]$ for all $(i,j) \in \mathbb{F}_p^2 \setminus (0,0)$. As a consequence, $\eta(z_1^a z_2^b) \equiv \eta(z_1^{a+c} z_2^{b+d}) \pmod{[\mathcal{M}_{(0)}, \operatorname{rad}\mathcal{L}]}$ unless $(a,b) =$

$(-c, -d)$. As χ vanishes on ${\mathcal{L}_{(0)}}^{(1)}$ and $[\mathcal{M}_{(0)}, \operatorname{rad}\mathcal{L}] \subset {\mathcal{L}_{(0)}}^{(1)}$, the preceding remark shows that $\chi(\eta(z_1^a z_2^b)) = \chi(\eta(z_1^{a+c} z_2^{b+d}))$. In the course of the proof of Proposition 5.1 we have established that $\beta(u^{[p]}) = (ad - bc)^p$. So if $(a, b) = (-c, -d)$ then $[u, v] = 0$ and $\beta(u^{[p]}) = 0$. Hence

$$\chi\big([u, v]\big)^p - \beta\big(u^{[p]}\big)\chi(v)^p = (ad - bc)^p \chi\big(\eta\big(z_1^{a+c} z_2^{b+d}\big)\big)^p - (ad - bc)^p \chi\big(\eta\big(z_1^c z_2^d\big)\big)^p$$
$$= 0,$$

as required. □

6. Case (A): Lie algebras without nonsolvable 1-sections

In this section, we assume that

all 1*-sections of* L *relative to* T *are solvable.*

Recall that the general case of the classification problem in characteristic $p > 7$ was split by the second author into four special cases known as Cases (A)–(D). The simple Lie algebras L satisfying the assumption above fall into Case (A) which was solved for $p > 7$ in [16, Section 2] and [20]. Our goal in this section is to solve this case for $p > 3$.

Proposition 6.1. *Then the following are true*:

(1) $H \subset \operatorname{nil} \widetilde{H}$.
(2) *Each* 1*-section of* L *relative to* T *is nilpotent and acts triangulably on* L.

Proof. Suppose $H \not\subset \operatorname{nil} \widetilde{H}$. Then there is $\alpha \in \Gamma(L, T)$ such that $\alpha(H) \neq 0$. Set

$$\Omega := \big\{\kappa \in \Gamma(L, T) \;\big|\; \kappa(H) \neq 0\big\}.$$

As $\Omega \neq \emptyset$, by our assumption, Schue's lemma yields $H = \sum_{\kappa \in \Omega} [L_\kappa, L_{-\kappa}]$. Since all roots in $\Gamma(L, T)$ are solvable, Proposition 3.8 shows that $[L_\kappa, L_{-\kappa}] \subset \operatorname{nil} \widetilde{H}$ for all $\kappa \in \Omega$. But then $H \subset \operatorname{nil} \widetilde{H}$, a contradiction. Since T is a maximal torus in L_p, this argument (in conjunction with the Engel–Jacobson theorem) also yields that T is standard and each $L(\alpha)$ is nilpotent. Then Theorem 3.1 applies (with $\mathfrak{t}_0 = T$) showing that each $L(\alpha)$ acts triangulably on L. □

Recall that in prime characteristics there is a natural way to extend domain of root functions. Let $\alpha \in \Gamma(L, T)$. Given $x \in L_\gamma$ with $\gamma \in \Gamma(L, T)$ one has $x^p \in \widetilde{H}$. We define

$$\alpha(x) := \sqrt[p]{\alpha(x^p)} \quad \forall x \in L_\gamma.$$

Thus α is defined on the union $\widetilde{H} \cup \bigcup_{\gamma \in \Gamma(L,T)} L_\gamma$. Since in our case $[L_\alpha, L_{-\alpha}] \subset H \subset \operatorname{nil} \widetilde{H}$, by Proposition 6.1, it follows from Jacobson's formula that $(x + y)^p \equiv$

$x^p + y^p \pmod{\operatorname{nil} \widetilde{H}}$ for all $x, y \in L_\gamma$ and all $x, y \in \widetilde{H}$. Therefore, α is a linear function not only on H but also on $\widetilde{H}$ and any root space L_γ.

Lemma 6.2. *Let G be a simple Lie algebra with $TR(G) \leqslant 2$ and $\mathfrak{t} \subset \operatorname{Der} G$ be a torus such that the centralizer $\mathfrak{c}_G(\mathfrak{t})$ acts nilpotently on G and each 1-section of G relative to $\mathfrak{t}$ is nilpotent. Then $G \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$, $\dim \mathfrak{t} = 2$, and $\mathfrak{t} \subset G_p = \operatorname{Der} G$.*

Proof. (a) Let $\mathfrak{t}' \supset \mathfrak{t}$ be a maximal torus of $\operatorname{Der} G$. If every 2-section of G relative to $\mathfrak{t}'$ is solvable then so is G [19, Theorem 1.16]. Since this is not the case, there are $\kappa', \lambda' \in \Gamma(G, \mathfrak{t}')$ such that the 2-section $M := \mathfrak{g}(\kappa', \lambda')$ is nonsolvable. For $i \in \mathbb{F}_p$, set

$$M(i) := \sum_{j \in \mathbb{F}_p} M_{i\kappa' + j\lambda'}.$$

Clearly, the decomposition $M = \bigoplus_{i \in \mathbb{F}_p} M(i)$ is an $\mathbb{F}_p$-grading of M. As M is nonsolvable, $M(0)$ does not act nilpotently on M (see [19, Proposition 1.14]). Since $\mathfrak{c}_G(\mathfrak{t}') \subset \mathfrak{c}_G(\mathfrak{t})$ does act nilpotently on G (by our assumption), the Engel–Jacobson theorem shows that there is $x \in M_{r\lambda'}$ for some $r \in \mathbb{F}_p^*$ such that $\operatorname{ad}_M x$ is not nilpotent. Since the torus $\mathfrak{t}'$ is maximal, we have that $\lambda'(x) = 0$ and $\kappa'(x) \neq 0$. Interchanging the roles of κ' and λ' in this argument, we find $y \in L_{s\kappa'}$ for some $s \in \mathbb{F}_p^*$ such that $\kappa'(y) = 0$ and $\lambda'(y) \neq 0$. Since $\mathfrak{t}'$ is a maximal torus, the semisimple parts x_s and y_s of x, y in $M_p \subset \operatorname{Der} G$ lie in the torus $\mathfrak{t}_0' := \mathfrak{t}' \cap M_p \subset G_p$. By construction, x_s and y_s are linearly independent. Since $\dim \mathfrak{t}_0' \leqslant MT(G_p) = TR(G)$, our assumption on G implies that $\mathfrak{t}_0' = Fx_s + Fy_s$.

(b) Suppose that for some $\Lambda \in \Gamma(G, \mathfrak{t}_0')$ the 1-section $G(\Lambda)$ is nonsolvable. Clearly, $G(\Lambda) = G(\alpha_1', \dots, \alpha_l')$ for some $\alpha_i' \in \Gamma(G, \mathfrak{t}')$. So $G(\Lambda)$ is a section of G relative to $\mathfrak{t}'$. Since $G(\Lambda)$ is assumed to be nonsolvable it contains a nonsolvable 2-section relative to $\mathfrak{t}'$, say M' (again by [19, Theorem 1.16]). We now repeat the argument from part (a) with M replaced by M' (and with $\mathfrak{t}'$ unchanged) to observe that the p-envelope $G(\Lambda)_p \subset G_p$ contains a 2-dimensional torus, say $\mathfrak{t}_1'$ which acts faithfully on $G(\Lambda)$. But then $\mathfrak{t}_1' \oplus (\mathfrak{t}_0' \cap \ker \Lambda)$ is a 3-dimensional torus in G_p violating our assumption that $TR(G) = 2$. Thus all 1-sections of G relative to the 2-dimensional torus $\mathfrak{t}_0'$ in G_p are solvable.

(c) As $TR(G) = 2$, the Lie algebra G is listed in [10, Theorem 1.1]. Thanks to part (b) of this proof, Proposition 6.1 applies to G implying that $\mathfrak{c}_G(\mathfrak{t}_0')$ contains no nonzero p-semisimple elements of G_p. It follows that G is non-restricted (for otherwise $\mathfrak{c}_G(\mathfrak{t}_0')$ would contain $\mathfrak{t}_0'$ which is impossible). If G is isomorphic to one of $W(1; \underline{2})$, $H(2; \underline{1}; \Delta)$, $H(2; (2, 1))^{(2)}$ then G has codimension 1 in G_p. Since $\mathfrak{t}_0'$ is 2-dimensional, $(0) \neq \mathfrak{t}_0' \cap G \subset \mathfrak{c}_G(\mathfrak{t}_0')$, a contradiction. We conclude that $G \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$. So $\operatorname{Der} S = S_p$, by [2, Theorem 2.1.8(b)], implying that $\dim \mathfrak{t} \leqslant TR(S) = 2$. Then $\mathfrak{t} = \mathfrak{t}' \supset \mathfrak{t}_0' \neq (0)$ is 2-dimensional. $\square$

Proposition 6.3. *If $L(\alpha,\beta)$ is nonsolvable for some $\alpha,\beta\in\Gamma(L,T)$ then*

$$L[\alpha,\beta]\cong H\big(2;\underline{1};\Phi(\tau)\big)^{(1)}.$$

Proof. Suppose that $L[\alpha,\beta]$ is nonsolvable and let $(0)\neq\widetilde{S}=\bigoplus_{i=1}^{r}\widetilde{S}_i$ be the sum of all minimal T-invariant ideals of the T-semisimple Lie algebra $L[\alpha,\beta]$. The structure of $\widetilde{S}$ is described in Theorems 4.1, 4.2, 4.4. The algebras described in Theorems 4.1 and 4.4 do not occur in our case since, as is easily seen, they all possess nonsolvable 1-sections relative to $\overline{T}$. Thus $L[\alpha,\beta]$ is described in Theorem 4.2, so that $\widetilde{S}$ is simple and $TR(\widetilde{S})=2$. As all 1-sections of L relative to T are solvable so are all 1-sections of $\widetilde{S}$ relative to $\overline{T}$. Lemma 6.2 now says $\widetilde{S}\cong H(2;\underline{1};\Phi(\tau))^{(1)}$. Then $\widetilde{S}\subset L[\alpha,\beta]\subset \operatorname{Der}\widetilde{S}=\widetilde{S}\oplus\overline{T}$, by Theorem 4.2(2). Since $L[\alpha,\beta]\cap\overline{T}=(0)$, by Lemma 6.1, we must have that $L[\alpha,\beta]\cong\widetilde{S}$. □

Corollary 6.4. *Let T' be an arbitrary torus of maximal dimension in L_p. Then all roots in $\Gamma(L,T')$ are solvable.*

Proof. According to [6, Theorem 1], the torus T' is obtained from T by a finite sequence of successive elementary switchings. Easy induction on the number of elementary switchings involved shows that it suffices to prove the corollary under the assumption that $T'=T_z$ where $z\in L_\alpha$ and α is an arbitrary root in $\Gamma(L,T)$ (for the terminology related to toral switchings, see the end of Section 4). Fix $\xi\in\operatorname{Hom}_{\mathbb{F}_p}(F,F)$ with $\xi^p-\xi=\operatorname{Id}_F$. Any 1-section of L relative to T_z has the form $L(\beta_{z,\xi})=E_{z,\xi}(L(\beta))$ for some $\beta\in\Gamma(L,T)$; see [9, p. 221] for example. Clearly, $E_{z,\xi}(L(\beta))\subset L(\alpha,\beta)$. So if $L(\alpha,\beta)$ is solvable we are done.

Suppose $L(\alpha,\beta)$ is nonsolvable. Then $L[\alpha,\beta]\cong H(2;\underline{1};\Phi(\tau))^{(1)}$, by Proposition 6.3, while Lemma 6.2 shows that

$$\Psi_{\alpha,\beta}\big(T+L(\alpha,\beta)_p\big)=\Psi_{\alpha,\beta}(T)\oplus L[\alpha,\beta].$$

Since $E_{z,\xi}$ is invertible and preserves both $L(\alpha,\beta)$ and $\operatorname{rad}(L(\alpha,\beta)_p)$, we get

$$\Psi_{\alpha,\beta}\big(T+L(\alpha,\beta)_p\big)=\Psi_{\alpha,\beta}(T_z)\oplus L[\alpha,\beta].$$

Thus the image of T_z in $\operatorname{Der}H(2;\underline{1};\Phi(\tau))^{(1)}$ is 2-dimensional. So it follows from [15, Theorem VII.3] that all 1-sections of $H(2;\underline{1};\Phi(\tau))^{(1)}$ relative to the image of T_z are abelian. But then all 1-sections of $L(\alpha,\beta)$ relative to T_z are solvable. In particular, this applies to $L(\beta_{z,\xi})$ completing the proof. □

Next we are going to determine 3-sections.

Theorem 6.5. *Let $\alpha,\beta,\gamma\in\Gamma(L,T)$. Then one of the following holds*:

(1) $L[\alpha,\beta,\gamma]=(0)$.

(2) *There exist* $\delta_1, \delta_2 \in \Gamma(L, T)$ *such that*

$$L[\delta_1, \delta_2] = L[\alpha, \beta, \gamma] \cong H\big(2; \underline{1}; \Phi(\tau)\big)^{(1)}.$$

(3) $H(2; \underline{1}; \Phi(\tau))^{(1)} \otimes A(m; \underline{1}) \subset L[\alpha, \beta, \gamma] \subset \operatorname{Der}(H(2; \underline{1}; \Phi(\tau))^{(1)} \otimes A(m; \underline{1}))$ *for some* $m \in \mathbb{N}_0$. *There exists* $\mu \in \Gamma(L, T) \cup \{0\}$ *such that*

$$L[\alpha, \beta, \gamma] = H\big(2; \underline{1}; \Phi(\tau)\big)^{(1)} \otimes A(m; \underline{1}) + L[\alpha, \beta, \gamma](\mu).$$

Moreover, the image of the torus T *in* $\operatorname{Der} L[\alpha, \beta, \gamma]$ *is* 3*-dimensional.*

(4) *There exists a simple Lie algebra* S *with* $TR(S) = 3$ *such that*

$$S \subset L[\alpha, \beta, \gamma] \subset \operatorname{Der} S.$$

Proof. (1) If $\widetilde{G} := L(\alpha, \beta, \gamma)$ is solvable then we are in case (1) of the theorem. So assume from now that $L(\alpha, \beta, \gamma)$ is not solvable. Then $G := L[\alpha, \beta, \gamma] = L(\alpha, \beta, \gamma)/\operatorname{rad}_T L(\alpha, \beta, \gamma)$ is a nonzero T-semisimple Lie algebra. Let I be a minimal T-invariant ideal of G. By Block's theorem, there exist a simple algebra S and $m \in \mathbb{N}_0$ such that $I \cong S \otimes A(m; \underline{1})$, under a Lie algebra isomorphism φ. Since $T + L(\alpha, \beta, \gamma)_p$ preserves I, the isomorphism φ induces a restricted Lie algebra homomorphism

$$\Phi : T + L(\alpha, \beta, \gamma)_p \longrightarrow (\operatorname{Der} S) \otimes A(m; \underline{1}) + \big(\operatorname{Id} \otimes W(m; \underline{1})\big).$$

Let $\mathfrak{t} := \Phi(T)$ and identify I with $S \otimes A(m; \underline{1})$. By [9, Theorem 2.6], we can choose Φ such that

$$\mathfrak{t} = \left(\sum_{i=1}^{s} F \operatorname{Id}_S \otimes (1 + x_i)\partial_i \right) \oplus \mathfrak{t}_0,$$

where $\mathfrak{t}_0$ is the normalizer of $S \otimes A(m; \underline{1})_{(1)}$ in $\mathfrak{t}$ and $s = \dim \mathfrak{t}/\mathfrak{t}_0$. Moreover, $\mathfrak{t}_0 = \{\lambda_1(t) \otimes 1 + \operatorname{Id}_S \otimes \lambda_2(t) \mid t \in \mathfrak{t}_0\}$ where $\lambda_1 : \mathfrak{t}_0 \to \operatorname{Der} S$ and $\lambda_2 : \mathfrak{t}_0 \to \sum_{i=s+1}^{m} F x_i \partial_i$ are restricted homomorphisms. Put $\mathfrak{t}_1 := \lambda_1(\mathfrak{t}_0) \subset \operatorname{Der} S$, a torus in $\operatorname{Der} S$. For $\gamma \in \mathfrak{t}_1^*$, define $\tilde{\gamma} \in \mathfrak{t}^*$ by setting

$$\tilde{\gamma}\big((1 + x_i)\partial_i\big) = 0 \quad (1 \leqslant i \leqslant s), \qquad \tilde{\gamma}\big(\lambda_1(t) \otimes 1 + \operatorname{Id} \otimes \lambda_2(t)\big) = \gamma\big(\lambda_1(t)\big) \quad (t \in \mathfrak{t}_0).$$

Let γ be any root in $\Gamma(S, \mathfrak{t}_1)$. Then $S_\gamma \otimes F \subset \Phi(\widetilde{G})_{\tilde{\gamma}}$ yielding $\tilde{\gamma} \in \Gamma(\Phi(\widetilde{G}), \mathfrak{t})$. In view of Proposition 6.1, $\mathfrak{c}_S(\mathfrak{t}_1) \otimes F \subset \mathfrak{c}_{\Phi(\widetilde{G})}(\mathfrak{t})$ acts nilpotently on S and $S(\gamma) \otimes F \subset (\Phi(\widetilde{G}))(\tilde{\gamma})$ is nilpotent. Since G is a homomorphic image of the 3-section $\widetilde{G}$, we have, by [19, Theorem 1.9], that $0 < TR(S) \leqslant TR(G) \leqslant 3$.

Suppose $TR(S) \leqslant 2$. Since all 1-sections of S relative to $\mathfrak{t}_1 \subset \operatorname{Der} S$ are nilpotent, Lemma 6.2 yields that $S \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$ and $\mathfrak{t}_1 \subset S_p$ is 2-dimensional. In particular, $TR(S) = 2$ in this case.

Now let $\widetilde{S}=\bigoplus_{i=1}^{r}\widetilde{S}_i$ be the sum of all minimal T-invariant ideals of G. From the preceding remark it follows that $2r\leqslant\sum_{i=1}^{r}TR(\widetilde{S}_i)=TR(\widetilde{S})\leqslant 3$ (see [19, Theorem 1.7(6)]). Thus $r=1$ and I is the unique T-invariant minimal ideal in G.

(2) Suppose $TR(S)=2$. Then $\widetilde{S}=S\cong H(2;\underline{1};\Phi(\tau))^{(1)}$. If $m=0$ then Φ maps $T+L(\alpha,\beta,\gamma)_p$ onto $\operatorname{Der} S=(H(2;\underline{1};\Phi(\tau))^{(1)})_p$ which gives $\dim\overline{T}\leqslant TR(S)=2$. Also, $\mathfrak{t}=\mathfrak{t}_1$ is 2-dimensional (Lemma 6.2). So there exist $\mathbb{F}_p$-independent T-roots $\delta_1,\delta_2\in\mathbb{F}_p\alpha+\mathbb{F}_p\beta+\mathbb{F}_p\gamma$ such that $\Phi(\widetilde{G})=\Phi(L(\delta_1,\delta_2))$. Since G is T-semisimple, Proposition 6.3 gives $G=L[\delta_1,\delta_2]\cong H(2;\underline{1};\Phi(\tau))^{(1)}$. This is case (2) of the theorem.

(3) Now suppose that $TR(S)=2$ and $m>0$. Let I_p denote the p-envelope of I in $\operatorname{Der} I$. Since $S\cong H(2;\underline{1};\Phi(\tau))^{(1)}$ we have that $\operatorname{Der} S=S_p$. Therefore, $I_p=(\operatorname{Der} S)\otimes F+S\otimes A(m;\underline{1})$. By [19, Theorem 1.7(8)], $\dim\overline{T}\cap I_p=2$ (one should keep in mind that $\overline{T}+I_p\subset\operatorname{Der} I$ is centerless). Since $\mathfrak{t}\cap I_p=\mathfrak{t}_0\cap I_p=\ker\lambda_2$, we deduce that $\mathfrak{t}=(\mathfrak{t}_2\otimes F)\oplus Ft$ where $\mathfrak{t}_2$ is a 2-dimensional torus in S_p and t is a toral element (possibly zero). Then

$$\Phi(\widetilde{G})=S\otimes A(m;\underline{1})+\bigl(\Phi(\widetilde{G})\bigr)(\mu)$$

for some $\mu\in\Gamma(\Phi(G),\overline{T})\cup\{0\}$ with $\mu(\mathfrak{t}_2\otimes F)=0$. If $\mathfrak{t}=\mathfrak{t}_2\otimes F$ then G is a homomorphic image of a 2-section in L. Since G is T-semisimple and $m>0$, this contradicts Proposition 6.3. Thus we are in case (3) of the theorem.

(4) Finally, suppose $TR(S)=3$. Then $TR(I)=3$ and hence $\dim\mathfrak{t}\cap I_p=3$; see [19, Theorem 1.7(8)]. Therefore, $\mathfrak{t}\subset I_p$. By [7, Lemma 2.5], we can choose Φ such that $\mathfrak{t}=\mathfrak{t}_2\otimes F$ for some 3-dimensional torus $\mathfrak{t}_2\subset S_p$. As a consequence,

$$\Phi\bigl(T+L(\alpha,\beta,\gamma)_p\bigr)=I_p+\Phi(H_p).$$

Since I is a minimal ideal in $\Phi(T+L(\alpha,\beta,\gamma)_p)$, the subalgebra $\pi_2(\Phi(H_p))$ is transitive in $W(m;\underline{1})$. Suppose $m>0$. Then there exists $h\in\Phi(H_p)$ with $\pi_2(h)\notin W(m;\underline{1})_{(0)}$. Note that $S_\gamma\otimes A(m;\underline{1})\subset\Phi(\widetilde{G})_{\tilde{\gamma}}$ for any $\gamma\in\Gamma(S,\mathfrak{t}_2)$. Therefore,

$$\begin{aligned}S_\gamma\otimes A(m;\underline{1})&=S_\gamma\otimes A(m;\underline{1})_{(1)}+\bigl[h,S_\gamma\otimes A(m;\underline{1})\bigr]\\&\subset S(\gamma)\otimes A(m;\underline{1})_{(1)}+\bigl(\Phi(\widetilde{G})\bigr)(\tilde{\gamma})^{(1)}.\end{aligned}$$

It is immediate from Proposition 6.1(2) and the definition of $\widetilde{G}$ that $(\Phi(\widetilde{G}))(\tilde{\mu})^{(1)}$ acts nilpotently on I. But then $S_\gamma\otimes A(m;\underline{1})$ acts nilpotently on I, too (one should take into account that the last summand in the displayed formula is stable under the action of $S(\gamma)\otimes A(m;\underline{1})_{(1)}$). Since the above applies to any $\gamma\in\Gamma(S,\mathfrak{t}_2)$, we now combine Proposition 6.1(1) with the Engel–Jacobson theorem to deduce that I acts nilpotently on itself. This contradicts the simplicity of S proving that $m=0$. Thus we are in case (4) of the theorem. □

In the $(p > 7)$-theory, the role of the result we just proved is played by [16, Theorem 2.7]. It should be mentioned here that our result is slightly weaker than the result in [16] where it is stated, in case (3), that the minimal ideal I is isomorphic to $H(2; \underline{1}, \Phi(\tau))^{(1)} \otimes A(1, \underline{1})$. Theorem 2.7 of [16] is only used in [20] to deduce [20, Lemma 6.3]. Now that lemma follows easily from our weaker version of [16, Theorem 2.7].

It seems that relying on the information available at the time when [St 91/1] was written, one can only prove our weaker version of [16, Theorem 2.7]. This oversight has no effect on the classification for $p > 7$. Indeed, one of the consequences of [20] is that all solvable 1-sections in case (A) are triangulable; using this result, one can recover the original version of case (3) in [16] from our slightly weaker version.

We are now going to take a closer look at the Lie algebras which appear in case (4) of Theorem 6.5. To streamline our exposition we often impose in the rest of this section that $TR(L) = 3$.

Theorem 6.6. *Let $TR(L) = 3$ and suppose that either $L \cong H(2; \underline{n}; \Psi)^{(2)}$ or $L \cong S(3; \underline{n}; \Psi)^{(1)}$. Let T be a 3-dimensional torus in the p-envelope $L_p \subset \operatorname{Der} L$ with the property that $L(\alpha)$ is solvable for any $\alpha \in \Gamma(L, T)$. Then L is isomorphic to one of $H(2; (2, 1); \Phi(\tau))^{(1)}$ or $S(3; \underline{1}; \Phi(\tau))^{(1)}$. Furthermore, the following hold*:

(1) *$H = (0)$ and no root vector for T act nilpotently on L.*
(2) *Every solvable 2-section of L relative to T is abelian.*
(3) *$\Gamma(L, T) \cup \{0\}$ is an elementary abelian p-group of order p^3.*
(4) *If $x \in L_\alpha$ and $y \in L_\beta$ then $[x, y]^p = -\alpha(y^p)x^p + \beta(x^p)y^p$.*

Proof. For $p > 7$, this is proved in [20, Propositions 5.4, 5.5]. The proof follows from some explicit computations involving the Lie algebras $H(2; \underline{n}; \Psi)^{(2)}$ and $S(3; \underline{n}; \Psi)^{(1)}$ with property $(A; 3)$ (see [20, Definition 2]). That property holds for our L due to Corollary 6.4, while the computations themselves go through for $p > 3$. The result follows. □

Proposition 6.7. *If $TR(L) = 3$ and L contains a solvable 2-section $L(\alpha, \beta)$ with $\mathbb{F}_p$-independent $\alpha, \beta \in \Gamma(L, T)$, then $L \cong H(2; (2, 1); \Phi(\tau))^{(1)}$.*

Proof. Let M be a maximal T-invariant subalgebra of L containing $L(\alpha, \beta)$.

(a) Suppose M is not solvable. Then M contains a nonsolvable 2-section relative to T; see [19, Theorem 1.16]. Since $M(\alpha, \beta) = L(\alpha, \beta)$ is solvable there exist $\mathbb{F}_p$-independent $\gamma, \delta \in \Gamma(L, T)$ with $\gamma \notin \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ such that $M(\gamma, \delta)$ is nonsolvable. For $i \in \mathbb{F}_p$, set

$$M(\gamma, \delta)_i := \sum_{j \in \mathbb{F}_p} M_{i\delta + j\gamma}.$$

Then $M(\gamma, \delta) = \bigoplus_{i \in \mathbb{F}_p} M(\gamma, \delta)_i$ is $\mathbb{F}_p$-graded. By [19, Proposition 1.14], the subalgebra $M(\gamma, \delta)_0$ does not act nilpotently on $M(\gamma, \delta)$. By Proposition 6.1, $H \subset M(\gamma, \delta)_0$ acts nilpotently on L. The Engel–Jacobson theorem now yields that there exists $x \in M_{j\gamma}$, for some $j \in \mathbb{F}_p^*$, with $\delta(x) \neq 0$. Let $L_1(x)$ denote the Fitting 1-component for $\operatorname{ad} x$. Since $\operatorname{ad} x$ is not nilpotent, $L_1(x) = (\operatorname{ad} x)^{p^r}(L) \neq (0)$ (here r is a big enough positive integer). By [19, Proposition 1.12], we then have $L = L_1(x) + [L_1(x), L_1(x)]$. Hence $L_1(x) \not\subset M$. Note that $L_1(x) = (\operatorname{ad} x)^{p^r}(L)$ is T-invariant. So the complement $L_1(x) \setminus M$ contains a root vector, say $u \in L_\mu$. Since $\operatorname{ad} x$ acts invertibly on $L_1(x)$ it also acts invertibly of the factor space $(L_1(x) + M)/M$. From this it follows that $\mu + \mathbb{F}_p\gamma \subset \Gamma^w(L/M, T)$. Now $\mu = m\alpha + n\beta + r\gamma$ for some $m, n, r \in \mathbb{F}_p$. By our preceding remark, $m\alpha + n\beta$ is a T-weight of L/M. However $L(\alpha, \beta) \subset M$, a contradiction.

(b) As a consequence, M is solvable. Now [21, Corollary 6.34] (which generalizes earlier work of Kuznetsov [3], Weisfeiler [24], and Skryabin [11]) says that L is one of $\mathfrak{sl}(2)$, $W(1; \underline{n})$, $H(2; \underline{n}; \Phi)^{(2)}$ for some $\underline{n}$ and Φ. As $TR(L) = 3$ we have $L \not\cong \mathfrak{sl}(2)$. The semisimple p-envelope $W(1; \underline{n})_p$ is nothing but $W(1; \underline{n}) + \sum_{i=1}^{n-1} FD^{p^i} \subset W(n; \underline{1})$. It is well known that $TR((W(1; \underline{n})) = n$ (see [18, Section V] for example). So $W(1; \underline{n})$ intersects with any torus of maximal dimension in $W(1; \underline{n})_p$. As $H \subset \operatorname{nil} \widetilde{H}$, by Proposition 6.1(1), L is not of Witt type. Now Theorem 6.6 gives $L \cong H(2; (2, 1), \Phi(\tau))^{(1)}$ completing the proof. □

Lemma 6.8. *Suppose that $L = L(\alpha, \beta, \gamma)$ has absolute toral rank 3. Then $\Gamma(L, T) = (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta \oplus \mathbb{F}_p\gamma) \setminus \{0\}$ and there is $k \in \mathbb{N}$ such that $\dim L_\delta = k$ for all $\delta \in \Gamma(L, T)$.*

Proof. (1) Assume first that for any pair (δ_1, δ_2) of $\mathbb{F}_p$-independent roots in $\Gamma(L, T)$ the 2-section $L(\delta_1, \delta_2)$ is nonsolvable. Proposition 6.3 then says that $L[\delta_1, \delta_2]$ is isomorphic to $H(2; 1; \Phi(\tau))^{(1)}$. By [18, Theorem VII.3], any root vector in $L[\delta_1, \delta_2]$ acts non-nilpotently on $L[\delta_1, \delta_2]$. So there exist $x \in L_{\delta_1}$ and $y \in L_{\delta_2}$ with $\delta_2(x) \neq 0$, $\delta_1(x) = 0$ and $\delta_1(y) \neq 0$, $\delta_2(y) = 0$. Thus $\dim L_{i\delta_2} = \dim L_{i\delta_1 + j\delta_2} = \dim L_{j\delta_2}$ for all $\mathbb{F}_p$-independent $\delta_1, \delta_2 \in \Gamma(L, T)$ and all $i, j \in \mathbb{F}_p^*$. This implies that all elements in $(\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta \oplus \mathbb{F}_p\gamma) \setminus \{0\}$ are roots and all root spaces are of the same dimension.

(2) Now assume that L contains a solvable 2-section relative to T. Then L is isomorphic to $H(2; (2, 1); \Phi(\tau))^{(1)}$, by Proposition 6.7. Root space decomposition in $L = H(2; (2, 1); \Phi(\tau))^{(1)}$ relative to a 3-dimensional torus in L_p has been investigated in [20, Proposition 5.4]. Inspection shows that the computations in [20] involving $H(2; (2, 1); \Phi(\tau))^{(1)}$ go through for $p > 3$. They imply, again, that all elements in $(\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta \oplus \mathbb{F}_p\gamma) \setminus \{0\}$ are roots and all root spaces are of the same dimension. □

Lemma 6.9. *Let $\mathfrak{g} = \mathfrak{g}_{(-r)} \supset \cdots \supset \mathfrak{g}_{(0)} \supset \cdots \supset \mathfrak{g}_{(s)}$ be a filtered Lie algebra and let $\mathfrak{b}$ be a triangulable subalgebra of $\mathfrak{g}$. Then $\operatorname{gr} \mathfrak{b}$ is a triangulable subalgebra of $\operatorname{gr} \mathfrak{g}$.*

Proof. Let $G = \operatorname{gr} \mathfrak{g}$, $B = \operatorname{gr} \mathfrak{b}$, and assume that $B^{(1)}$ does not act nilpotently on G. Clearly, $B^{(1)} = \sum_{i,j \in \mathbb{Z}} [B_i, B_j]$ where

$$B_i := \operatorname{gr}_i \mathfrak{b} = (\mathfrak{b} \cap \mathfrak{g}_{(i)} + \mathfrak{g}_{(i+1)})/\mathfrak{g}_{(i+1)} \subset \mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)} = \operatorname{gr}_i \mathfrak{g}.$$

For $i \neq 0$, the subspace $\mathrm{ad}_G B_i$ consists of nilpotent endomorphisms. So the Engel–Jacobson theorem implies that the subalgebra $\sum_{i\in\mathbb{Z}}[B_i, B_{-i}]$ does not act nilpotently on G. Since the set $\bigcup_{i\in\mathbb{Z}}[B_i, B_{-i}]$ is weakly closed, there is $k \in \mathbb{Z}$ such that the subalgebra $[B_k, B_{-k}]$ does not act nilpotently on G. Then there exist $u_1, \ldots, u_l \in \mathfrak{b} \cap \mathfrak{g}_{(-k)}$ and $v_1, \ldots, v_l \in \mathfrak{b} \cap \mathfrak{g}_{(k)}$ such that the coset $\sum[u_i, v_i] + \mathfrak{g}_{(1)}$ contains an element which does not act nilpotently on $\mathfrak{g}$. But this is impossible as $\sum[u_i, v_i] \in \mathfrak{b}^{(1)} \cap \mathfrak{g}_{(0)}$ acts nilpotently on $\mathfrak{g}$. □

Our next result will be crucial for the rest of this section. Its proof illustrates well some of the classification methods.

Proposition 6.10. *If $TR(L) = 3$ then $H = (0)$ and $(L_\delta)^p \subset T$ for all $\delta \in \Gamma(L, T)$. Moreover, no root vector for T is p-nilpotent in L_p.*

Proof. (1) Suppose the theorem is not true and let L be a counterexample of minimal dimension to it. Let $\mathcal{N}(L_p)$ denote the set of all p-nilpotent elements in L_p. By Proposition 6.1(2), all 1-sections $L(\delta)$ relative to T are nilpotent and have the property that $L(\delta)^{(1)} \subset \mathcal{N}(L_p)$. Let $n(\delta)$ denote the nilpotency class of $L(\delta)$ and let $\alpha \in \Gamma(L, T)$ be such that $n(\alpha) = \max\{n(\delta) \mid \delta \in \Gamma(L, T)\}$. If $n = n(\alpha) \geqslant 3$ then $(0) \neq L(\alpha)^{n-1} \subset \mathcal{N}(L_p)$. Since $L(\alpha)^{n-1}$ is T-invariant, either $H \cap L(\alpha)^{n-1} \neq (0)$ or $L_{i\alpha} \cap L(\alpha)^{n-1} \neq (0)$ for some $i \in \mathbb{F}_p^*$. If $n(\alpha) \geqslant 3$, we let w be any nonzero element in the union $(H \cap L(\alpha)^{n-1}) \cup (\bigcup_{i\in\mathbb{F}_p^*} L(\alpha)^{n-1} \cap L_{i\alpha})$.

Now suppose that all 1-sections of L relative to T are abelian. Then $H = (0)$ (as L is centerless). Since L is a counterexample, there is a nonzero $x \in L_\alpha$, for some $\alpha \in \Gamma(L, T)$, such that either $x^p = 0$ or x^p is not p-semisimple in $\widetilde{H}$. If $x^p = 0$ for some nonzero $x \in L_\alpha$, we set $w := x$. If $H = (0)$ and x^p is not p-semisimple in $\widetilde{H}$ for some $x \in L_\alpha$, we let w be the p-nilpotent part of x^p in $\widetilde{H}$. Clearly, $w = f(x^p)$ for some p-polynomial $f \in F[t]$ without constant term.

Thus in all cases we can find $w \in \widetilde{H} \cup (\bigcup_{i\in\mathbb{F}_p^*} L_{i\alpha})$, for some $\alpha \in \Gamma(L, T)$, such that

$$w \in \mathcal{N}(L_p) \setminus \{0\} \quad \text{and} \quad \big[w, L(\alpha)\big] = (0).$$

Furthermore, if $w \in \widetilde{H}$ then either $w \in H$ or all 1-sections of L relative to T are abelian and w is the p-nilpotent part of x^p for some $x \in L_\alpha$. From now we fix such a w and denote by M a maximal T-invariant subalgebra of L containing the centralizer of w in L.

Let $\delta \in \Gamma(L, T)$. Since $\sum_{i\in\mathbb{F}_p} L_{\delta+i\alpha}$ is invariant under the nilpotent endomorphism $\mathrm{ad}\, w$, there exists $j = j(\delta)$ with $L_{\delta+j\alpha} \cap \mathfrak{c}_L(w) \neq (0)$. Since this holds for all roots δ we can find $\mathbb{F}_p$-independent $\alpha, \beta, \gamma \in \Gamma(L, T)$ with

$$L(\alpha) \subset M, \qquad M_\beta \neq (0), \qquad M_\gamma \neq (0). \tag{1}$$

(2) We identify L with $\mathrm{ad}\, L \subset \mathrm{Der}\, L$ and consider $T + L_p$, a Lie subalgebra of $\mathrm{Der}\, L$. If J is an ideal of $T + L$ then $[J, L]$ is an ideal of L. So either $[J, L] = (0)$ or $[J, L] = L$. In the first case $J = (0)$, for $J \subset \mathrm{Der}\, L$, while in the second case $J \supset L$. Thus $T + M$

contains no nontrivial ideals of $T + L$. The T-maximality of M in L implies that $T + M$ is a maximal subalgebra of $T + L$.

Choose a subspace $(T+L)_{(-1)}$ in $T+L$ which contains $T+M$ and has the property that $(T+L)_{(-1)}/(T+M)$ is $(T+M)$-irreducible. Let $\{(T+L)_{(i)} \mid i \in \mathbb{Z}\}$ denote the standard filtration associated with the pair $((T+L)_{(-1)}, (T+L)_{(0)})$ where $(T+L)_{(0)} = T+M$. By the above, this filtration is exhaustive and separating, that is

$$T + L = (T + L)_{(-r)} \supset \cdots \supset (T + L)_{(s+1)} = (0),$$

where $r > 0$ and $s \geqslant 0$ are finite. Let G denote the associated graded Lie algebra $\operatorname{gr}(T+L)$ and let $\widetilde{M}(G)$ be the maximal ideal of G contained in $\sum_{i \leqslant -1} G_i$. It is well known (and easy to see) that $\widetilde{M}(G)$ is a graded ideal of G. So the quotient $\overline{G} := G/\widetilde{M}(G)$ inherits from G a graded Lie algebra structure. If $G_1 \neq (0)$ then the graded Lie algebra $\overline{G} = \bigoplus_{i \in \mathbb{Z}} \overline{G}_i$ satisfies the standard conditions (g1)–(g4) (see [10, p. 246] for example). If $G_1 = (0)$ then, of course, $\widetilde{M}(G) = \sum_{i<0} G_i$ and $\overline{G} \cong G_0$.

Let $\overline{T}$ denote the image of T in $\overline{G}$. By construction, $\overline{T}$ acts on $\overline{G}$ as a torus of derivations. Since M carries three $\mathbb{F}_p$-independent T-roots, by (1), and since $\widetilde{M}(G) \cap \sum_{i \geqslant 0} G_i = (0)$, the image of $\overline{T}$ in $\operatorname{Der} \overline{G}$ is 3-dimensional. In other words, $\overline{G}$ carries three $\mathbb{F}_p$-independent $\overline{T}$-roots. As a consequence, $TR(\overline{T}, \overline{G}) = 3$. Combining Skryabin's result [12, Theorem 5.1] with [19, Theorem 1.7], we now get $3 = TR(\overline{T}, \overline{G}) \leqslant TR(\overline{G}) \leqslant TR(G) \leqslant TR(L) = 3$, forcing $TR(\overline{G}) = 3$.

(3) Suppose $G_1 = (0)$. Then $\widetilde{M}(G) = \sum_{i<0} G_i$ which entails that

$$\overline{G} = \overline{G}_0 = (T + L)_{(0)}/(T + L)_{(1)} \cong T + M$$

as Lie algebras. Thanks to (1), we have $\alpha, \beta, \gamma \in \Gamma(M, T)$. Let A be an abelian ideal of $T + M$. By what we just said, any element $x \in A_\delta$, for $\delta \in \Gamma(M, T) \cup \{0\}$, has the property that $\alpha(x) = \beta(x) = \gamma(x) = 0$. Then any element in A_δ acts nilpotently on L_p, and hence on $G_{-1} = (T + L)_{(-1)}/(T + L)_{(0)}$. The Engel–Jacobson theorem now shows that A acts nilpotently on G_{-1}. The irreducibility of G_{-1} yields that A annihilates G_{-1} implying $A \subset (T + L)_{(1)} = (0)$. Thus $\overline{G}$ is semisimple. Since the grading of $\overline{G}$ is trivial in this case, all minimal ideals of $\overline{G}$ are obviously graded.

If $G_1 \neq (0)$, then Weisfeiler's theorem [23] says that $\overline{G}$ is semisimple and contains a unique minimal ideal which is graded. Thus $\overline{G}$ is semisimple in all cases, and any minimal ideal of $\overline{G}$ is graded.

Since $TR(\overline{G}) = \dim \overline{T} = 3$, we can identify $\overline{T}$ with a torus of maximal dimension in the (semisimple) p-envelope $\overline{G}_p \subset \operatorname{Der} \overline{G}$. Let I be a minimal ideal of $\overline{G}$. By Block's theorem, there exist a simple algebra S and $m \in \mathbb{N}_0$ such that

$$I \xrightarrow{\sim} S \otimes A(m; \underline{1}),$$

under an isomorphism φ. We suppress φ by identifying I with $S \otimes A(m; \underline{1})$, and denote by I_p the p-envelope of I in $\operatorname{Der} I$. The adjoint action of $\overline{G}_p$ on I gives rise to a restricted homomorphism

$$\Phi : \overline{G}_p \longrightarrow \operatorname{Der} I = (\operatorname{Der} S) \otimes A(m; \underline{1}) \oplus \bigl(\mathrm{Id} \otimes W(m; \underline{1})\bigr).$$

Since $\overline{G}_p$ is restricted, Block's theorem yields that the subalgebra $(\pi_2 \circ \Phi)(\overline{G}_p)$ is transitive in $W(m; \underline{1})$. Since both $\Phi(\overline{G}_p)$ and I_p contain I (or rather $\operatorname{ad} I$), they must be centerless. Then [19, Theorem 1.7(8)] shows that $\dim \Phi(\overline{T}) \cap I_p = TR(I)$ and, as a consequence, $\Phi(\overline{T}) \cap I_p$ is a torus of maximal dimension in I_p.

Put $\mathfrak{t} := \Phi(\overline{T})$ and let $\mathfrak{t}_0$ denote the normalizer of $S \otimes A(m; \underline{1})_{(1)}$ in $\mathfrak{t}$. According to [9, Theorem 2.6], we can choose φ such that

$$\mathfrak{t} = \left(\sum_{i=1}^{s} F \otimes (1 + x_i)\partial_i\right) \oplus \mathfrak{t}_0 \quad \text{and} \quad \mathfrak{t}_0 = \bigl\{\lambda_1(t) \otimes 1 + \mathrm{Id} \otimes \lambda_2(t) \bigm| t \in \mathfrak{t}_0\bigr\},$$

for some restricted homomorphisms $\lambda_1 : \mathfrak{t}_0 \to \operatorname{Der} S$ and $\lambda_2 : \mathfrak{t}_0 \to \sum_{i=s+1}^{m} F x_i \partial_i$. Let $\mathfrak{t}_1 := \mathfrak{t} \cap I_p$, a subtorus in $\mathfrak{t}_0$. Since $I_p = I + S_p \otimes F$, where S_p is the p-envelope of S in $\operatorname{Der} S$, it is straightforward to see that λ_2 vanishes on $\mathfrak{t}_1$ and $\lambda_1(\mathfrak{t}_1) \subset S_p \otimes F$. Combined with our discussion above, this shows that $\mathfrak{t}_1 = \mathfrak{t}_1' \otimes F$ where $\mathfrak{t}_1'$ is a torus of maximal dimension in S_p.

Let $\mathfrak{t}_0' := \lambda_1(\mathfrak{t}_0)$, a torus in $\operatorname{Der} S$. Given $\delta \in (\mathfrak{t}_0')^*$ we let $\tilde{\delta}$ denote the linear function on $\mathfrak{t}$ given by

$$\tilde{\delta}\bigl((1 + x_i)\partial_i\bigr) = 0 \quad (1 \leqslant i \leqslant s), \qquad \tilde{\delta}\bigl(\lambda_1(t) \otimes 1 + \mathrm{Id} \otimes \lambda_2(t)\bigr) = \delta\bigl(\lambda_1(t)\bigr) \quad (t \in \mathfrak{t}_0).$$

Since the image of T in $G = \operatorname{gr} L$ lies in G_0, each 1-section $G(\delta)$ in G has the form $\operatorname{gr} L(\delta)$, hence is nilpotent. Therefore, so is each 1-section of $\overline{G}(\delta)$ relative to $\overline{T}$. This, in turn, yields that all 1-sections of $\Phi(\overline{G})$ relative to $\mathfrak{t}$ are nilpotent. Now $S(\delta) \otimes F \subset (\Phi(\overline{G}))(\tilde{\delta})$ for all $\delta \in \Gamma(S, \mathfrak{t}_0')$. This means that all 1-section of S relative to $\mathfrak{t}_0'$ are nilpotent as well. Applying Lemma 6.2, we now deduce that $TR(S) \geqslant 2$.

If $\overline{G}$ has two minimal ideals, say $I_1 \cong S_1 \otimes A(m_1, \underline{1})$ and $I_2 \cong A(m_2, \underline{1})$, then the above discussion yields $3 = TR(\overline{G}) \geqslant TR(I_1 \oplus I_2) = TR(S_1) + TR(S_2) \geqslant 4$, a contradiction. This enables us to conclude that $I = \operatorname{Soc} \overline{G}$ is the only minimal ideal in $\overline{G}$. As a consequence, Φ is injective (otherwise $[\ker \Phi, \overline{G}]$ would contain a minimal ideal of $\overline{G}$ commuting with I). In particular, $\dim \mathfrak{t} = 3$.

(4) Suppose $TR(S) = 2$.

(a) By Lemma 6.2, $S \cong H(2; 1; \Phi(\tau))^{(1)}$, $\dim \mathfrak{t}_0' = 2$, and $\mathfrak{t}_0' \subset S_p$. Note that $S \otimes F$ is $\mathfrak{t}$-stable; moreover, $\mathfrak{t}$ acts on $S \otimes F$ as $\mathfrak{t}_0' \otimes F$. The kernel of this action is contained in $\mathrm{Id} \otimes W(m; \underline{1})$ and has dimension 1 (because $\dim \mathfrak{t} = 3$). Since $\mathfrak{t}_1 = \mathfrak{t} \cap I_p$ is a torus of maximal dimension in I_p, by part (3), we also have that $\mathfrak{t}_0' = \mathfrak{t}_1'$. It follows that

$$\mathfrak{t} = \left(\mathfrak{t}_0' \otimes F\right) \oplus (F\operatorname{Id} \otimes d) \tag{2}$$

for some nonzero toral derivation $d \in W(m; \underline{1})$. As a consequence, $m > 0$. Since $\mathfrak{t}_0'$ is 2-dimensional it follows from (2) that $\Phi(\overline{G}) = (S \otimes A(m; \underline{1})) \oplus (\Phi(\overline{G}))(\mu)$ for some nonzero $\mu \in \mathfrak{t}^*$ vanishing on $\mathfrak{t}_0' \otimes F$ and taking values in $\mathbb{F}_p$ on all toral elements of $\mathfrak{t}$ (such a μ is unique up to a nonzero scalar multiple in $\mathbb{F}_p$).

Given $\eta \in \mathfrak{t}^*$ we will denote by η^* the pull-back of η in T. Note that η^* is uniquely determined by η and has the property that $\eta^*(t) = \eta(\Phi(\operatorname{gr} t))$ for all $t \in T$. Our goal in this part is to show that the pull-back μ^* of μ is a multiple of α from the first part of this proof.

Since $\operatorname{Der} S = \mathfrak{t}_0' \oplus S$, by [2, Theorem VII.3], we have the inclusion

$$\left(\Phi(\overline{G})\right)(\mu) \subset \left(\mathfrak{t}_0' \otimes A(m; \underline{1})\right) \oplus \left(\operatorname{Id} \otimes W(m; \underline{1})\right).$$

It follows easily from Demushkin's theorem that the spectrum of the toral derivation d on $A(m; \underline{1})$ equals $\mathbb{F}_p$ and all its eigenspaces have dimension p^{m-1}. According to [18, Theorem VII.3], there exist $\mathbb{F}_p$-independent $\kappa, \nu \in \Gamma(S, \mathfrak{t}_0')$ such that

$$S = \sum_{\delta \in (\mathbb{F}_p\kappa + \mathbb{F}_p\nu) \setminus \{0\}} S_\delta.$$

Moreover, $\Gamma^w(S \otimes F, \mathfrak{t}) = (\mathbb{F}_p\kappa + \mathbb{F}_p\nu) \setminus \{0\}$ and each root space S_δ is 1-dimensional. Each subspace $S_\delta \otimes A(m; \underline{1})$ is $\mathfrak{t}$-invariant and $\Gamma^w(S_\delta \otimes A(m; \underline{1}), \mathfrak{t}) = \tilde{\delta} + \mathbb{F}_p\mu$. As a consequence,

$$\left(S \otimes A(m; \underline{1})\right) \cap \left(\Phi(\overline{G})\right)(\mu) = (0).$$

It follows from the definition of μ that the $\mathfrak{t}$-roots $\mu, \tilde{\kappa}, \tilde{\nu}$ are linearly independent. Combining this with our earlier remarks gives $\Gamma(\Phi(\overline{G}), \mathfrak{t}) \supset (\mathbb{F}_p\tilde{\kappa} + \mathbb{F}_p\tilde{\nu} + \mathbb{F}_p\mu) \setminus \mathbb{F}_p\mu$, and

$$\dim \Phi(\overline{G})_\eta = p^{m-1} \quad \left(\forall \eta \in \Gamma\left(\Phi(\overline{G}), \mathfrak{t}\right) \setminus \mathbb{F}_p\mu\right). \tag{3}$$

(b) Suppose that $G_1 \neq (0)$ and we are in the nondegenerate case of Weisfeiler's theorem [23]. Then $\overline{G}_{-1} \subset I$ and $I \cap \overline{G}_i = S_i \otimes A(m; \underline{1})$ for some grading $S = \bigoplus_{i \in \mathbb{Z}} S_i$ of S. Note that [9, Theorem 2.6] is applicable in this graded setting, that is it can be assumed that φ is a graded isomorphism and (2) holds for $\mathfrak{t} = \Phi(\overline{T})$. Since $\overline{T} \subset \overline{G}_0$, the torus $\mathfrak{t}_0'$ preserves S_{-1}. But then all root vectors for $\mathfrak{t}_0'$ contained in S_{-1} are p-nilpotent in S_p. Since this contradicts [18, Theorem VII.3(3)], we conclude that this case cannot occur.

(c) Now suppose $G_1 = (0)$. Recall that G_{-1} is an irreducible and faithful G_0-module. Identify the 3-dimensional tori $\overline{T}$ and $\operatorname{gr} T \subset G_0$. Suppose there is $\eta \in \Gamma(\Phi(\overline{G}), \mathfrak{t}) \setminus \mathbb{F}_p\mu$ such that $\eta^* \notin \bigcup_{i>0} \Gamma(G_{-i}, \overline{T})$. In view of (3), we then have $\dim L_{\eta^*} = \dim \Phi(\overline{G})_\eta = p^{m-1}$. Since all root spaces of L relative to T have the same dimension, by Lemma 6.8, this implies that $\dim L_{\eta^*} = p^{m-1}$ for all $\eta \in \Gamma(\Phi(\overline{G}), \mathfrak{t})$. So (3) now yields that $\bigcup_{i>0} \Gamma(G_{-i}, \overline{T}) \subset \mathbb{F}_p\mu^*$. But then $\overline{T}$ does not act faithfully on G_{-1}, a contradiction.

Thus $(\mathbb{F}_p\tilde{\kappa}^* + \mathbb{F}_p\tilde{\nu}^* + \mathbb{F}_p\mu^*) \setminus \mathbb{F}_p\mu^* \subset \bigcup_{i>0}\Gamma(G_{-i}, \overline{T})$, by our final remark in part (4a). Since $\mathbb{F}_p\alpha \cap (\bigcup_{i>0}\Gamma(G_{-i}, \overline{T})) = \emptyset$, by (1), we deduce that in the present case $\mathbb{F}_p\alpha = \mathbb{F}\mu^*$, as desired.

(d) Finally, suppose that $G_1 \neq (0)$ and we are in the degenerate case of Weisfeiler's theorem [23]. Then $\overline{G}_2 = (0)$ and $I \cap \overline{G}_1 = (0)$.

Let $x_1, \ldots, x_m$ be a generating set in $A(m; \underline{1})_{(1)}$. Given a subset $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, m\}$ we denote by $A(x_{i_1}, \ldots, x_{i_k})$ the unital subalgebra of $A(m; \underline{1})$ generated $x_{i_1}, \ldots, x_{i_k}$, a truncated polynomial algebra in k variables. By [23, Theorem 3.1], there exists a nonnegative $e < m$ such that

$$\varphi(I_{-j}) = \bigl\{S \otimes A(x_1, \ldots, x_e) \cdot f \bigm| f \in A(x_{e+1}, \ldots, x_m),\ \deg f = j\bigr\},$$

for all $j > 0$, and

$$\Phi(\overline{G}_0) \subset \operatorname{Der}\bigl(S \otimes A(x_1, \ldots, x_e)\bigr) + \operatorname{Id} \otimes \Biggl(\sum_{e+1 \leqslant i, j \leqslant m} A(x_1, \ldots, x_e) \cdot x_i \partial_j\Biggr).$$

To show that (2) is still valid in our present (more restrictive) setting we will apply [9, Theorem 2.6] to the graded subalgebra $I' := S \otimes A(x_{e+1}, \ldots, x_m)$ of I. We first observe that $I \cong I' \otimes A(x_1, \ldots, x_e)$ as graded Lie algebras. Since

$$\operatorname{Der}_0 I' = (\operatorname{Der} S) \otimes F + \operatorname{Id} \otimes \Biggl(\sum_{e+1 \leqslant i, j \leqslant m} F x_i \partial_j\Biggr),$$

we have that

$$\Phi(\overline{T}) \subset \Phi(\overline{G}) \subset (\operatorname{Der}_0 I') \otimes A(x_1, \ldots, x_e) + \bigl(\operatorname{Id} \otimes \operatorname{Der} A(x_1, \ldots, x_e)\bigr).$$

Combining [9, Theorem 2.6] with the fact that all maximal tori in $\sum_{e+1 \leqslant i, j \leqslant m} F x_i \partial_j \cong \mathfrak{gl}(m-e)$ are conjugate under the adjoint action of $\mathrm{GL}(m-e)$, we deduce that the graded map φ can be chosen such that the torus $\mathfrak{t} = \Phi(\overline{T})$ has the form described in part (3). Then our remarks at the beginning of part (4a) show that (2) is still valid for $\mathfrak{t}$ in the present case.

As $S \otimes F \subset I_0$ and $[I_0, \Phi(\overline{G}_1)] \subset I \cap \Phi(\overline{G}_1) = (0)$, the p-envelope $S_p \otimes F$ of $S \otimes F$ in $\operatorname{Der}\Phi(\overline{G})$ annihilates $\Phi(\overline{G}_1)$. Since $\mathfrak{t}_0' \subset S_p$, we now get $\Phi(\overline{G_1}) = (\Phi(\overline{G_1}))(\mu)$.

Since $[\operatorname{Id} \otimes d, I_0] \subset I_0$, the derivation d must preserve $A(x_1, \ldots, x_e)$. If d acts nontrivially of $A(x_1, \ldots, x_e)$ then, as in part (4a), the spectrum of d on $A(x_1, \ldots, x_e)$ equals $\mathbb{F}_p$. As a consequence, $\Gamma(S_\delta \otimes A(x_1, \ldots, x_e)x_m, \mathfrak{t}) = \tilde{\delta} + \mathbb{F}_p\mu$ for any $\delta \in \Gamma(S, \mathfrak{t}_0')$. Since $S \otimes A(x_1, \ldots, x_e)x_m \subset \overline{G}_{-1}$, we deduce, as at the end of part (4a), that $(\mathbb{F}_p\tilde{\kappa} + \mathbb{F}_p\tilde{\nu} + \mathbb{F}_p\mu) \setminus \mathbb{F}_p\mu \subset \Gamma(G_{-1}, \mathfrak{t})$. Then $\mathbb{F}_p\mu^* = \mathbb{F}_p\alpha$ as desired.

Now suppose d acts trivially on $A(x_1, \ldots, x_e)$. If $e + 1 \leqslant i \leqslant m$ then $d(x_i) = a_i x_i$ for some $a_i \in \mathbb{F}_p$. As $d \neq 0$, at least one a_i is nonzero. So we may assume, after renumbering, that $a_m \neq 0$. Clearly, $\Gamma(S_\delta \otimes A(x_1, \dots, x_e)x_m^k, \mathfrak{t}) = \tilde{\delta} + k\mu$ for all $k \leqslant p-1$. From this it is immediate that

$$\bigcup_{i>0} \Gamma(G_{-i}, \overline{T}) \supset \Gamma' := (\mathbb{F}_p\tilde{\kappa}^* + \mathbb{F}_p\tilde{\nu}^* + \mathbb{F}_p\mu^*) \setminus \big((\mathbb{F}_p\tilde{\kappa}^* + \mathbb{F}_p\tilde{\nu}^*) \cup \mathbb{F}_p\mu^*\big).$$

Moreover, $\dim L_{\delta^*} \geqslant p^e$ for any $\delta^* \in \Gamma'$. Suppose $\mathbb{F}_p\alpha \neq \mathbb{F}_p\mu^*$. Then $\alpha = \eta^*$ for some $\eta \in (\mathbb{F}_p\tilde{\kappa}^* + \mathbb{F}_p\tilde{\nu})^* \setminus \{0\}$. Hence $\dim L_\alpha = \dim G_{0,\alpha} = \dim \Phi(\overline{G}_0)_\eta = p^e$. But then all root spaces of L relative to T have this dimension; see Lemma 6.8. It also follows that $a_i = 0$ for $e+1 \leqslant i < m$ (indeed, if $a_l \neq 0$ for some $l < m$ then $\eta \in \Gamma(S \otimes A(x_1, \dots, x_e)x_l^c x_m, \mathfrak{t})$ for some $c \leqslant p-1$, a contradiction). Thus it can be assumed that $d = x_m\partial_m$.

Let $\overline{H}$ denote the image of $\operatorname{gr} H$ in $\overline{G}$. Combining [23, Theorem 3.1] with our earlier remarks it is easy to observe that $(\pi_2 \circ \Phi)(\overline{G})$ is contained in the free $A(x_1, \dots, x_e)$-module generated by all ∂_i and $x_j\partial_k$ with $1 \leqslant i \leqslant m$ and $e+1 \leqslant j, k \leqslant m$. Therefore, $(\pi_2 \circ \Phi)(\overline{G})$ decomposes into eigenspaces for d as follows:

$$(\pi_2 \circ \Phi)(\overline{G}) = (\pi_2 \circ \Phi)(\overline{G})_0 \oplus (\pi_2 \circ \Phi)(\overline{G})_{-1} \oplus (\pi_2 \circ \Phi)(\overline{G})_1,$$

where $(\pi_2 \circ \Phi)(\overline{G})_0 = (\pi_2 \circ \Phi)(\overline{H})$ and

$$(\pi_2 \circ \Phi)(\overline{G})_{-1} \subset A(x_1, \dots, x_{m-1})\partial_m,$$
$$(\pi_2 \circ \Phi)(\overline{G})_1 \subset \sum_{i \leqslant m-1} A(x_1, \dots, x_{m-1})x_m\partial_i.$$

As a consequence, $[(\pi_2 \circ \Phi)(\overline{G})_{\pm 1}]^p = (0)$. Jacobson's formula now gives

$$(\pi_2 \circ \Phi)(\overline{G}_p) = (\pi_2 \circ \Phi)(\overline{H}_p) \oplus (\pi_2 \circ \Phi)(\overline{G})_{-1} \oplus (\pi_2 \circ \Phi)(\overline{G})_1.$$

Suppose $m > 1$. Since $(\pi_2 \circ \Phi)(\overline{G}_p)$ is a transitive subalgebra of $W(m; \underline{1})$, the subalgebra $(\pi_2 \circ \Phi)(\overline{H}_p)$ acts transitively on $A(x_1, \dots, x_{m-1})$. Then there is $h \in \overline{H}$ with $(\pi_2 \circ \Phi)(h))(x_k) \notin A(m; \underline{1})_{(1)}$ for some $k \leqslant e$. From the description of $(\Phi(\overline{G}))(\mu)$ given in part (4a) we deduce that $[\Phi(h), S_\delta \otimes x_k)] \not\subset S_\delta \otimes A(m; \underline{1})_{(1)}$ for any $\delta \in \Gamma(S, \mathfrak{t}_0')$. However, this means that some of the 1-sections of L relative to T are not triangulable, contradicting Proposition 6.1(2).

Thus $m = 1$ forcing $e = 0$. As a consequence, $\dim L_\delta = 1$ for all $\delta \in \Gamma(L, T)$. This, in turn, gives $\widetilde{H} = T$. Theorem 3.1 of [23] shows that $G_0(\mu^*) \cong \overline{G}_0(\mu^*) = \overline{T}$ and $G_1(\mu^*) \cong \Phi(\overline{G}_1)(\mu) = F\partial_1 \cong G_{1,-k\mu^*}$ for some $k \in \mathbb{F}_p^*$. Therefore, $M_{k\mu^*}$ is a 1-dimensional ideal of M. By [18, Theorem VII.3], all 1-sections of S relative to $\mathfrak{t}_0'$ are abelian and no root vector of S relative to $\mathfrak{t}_0'$ act nilpotently on S. In conjunction with Proposition 6.1(2), this implies that Φ induces an embedding

$$M/M_{k\mu^*} \hookrightarrow (S \otimes F) \oplus (F\mathrm{Id} \otimes d).$$

From this it is immediate that all 1-sections of M relative to T are abelian. But then so is $L(\alpha)$. Due to the choice of α, all 1-sections of L relative to T are abelian (see part (1) of this proof). Since $\operatorname{nil}\widetilde{H} = (0)$, our choice of w in part 1 now shows that $\mathcal{N}(L_p) \cap (\bigcup_{i\in\mathbb{F}_p^*} L_{i\alpha}) \neq \emptyset$. Since this contradicts [18, Theorem VII.3], we conclude, at last, that $\mathbb{F}_p\alpha = \mathbb{F}_p\mu^*$ in all cases.

(5) (a) We will need some subtle estimates for the dimensions of root spaces. From our discussion in part (4) we know that either $G_1 = (0)$ or we are in the degenerate case of Weisfeiler's theorem [23]. We also know that in either case φ is a graded map and (2) holds for $\mathfrak{t} = \Phi(\overline{T})$. Set

$$K := \sum_{i\in\mathbb{F}_p^*} \big((\pi_2 \circ \Phi)(\overline{G}_{i\alpha}) + \big[(\pi_2 \circ \Phi)(\overline{G}_{i\alpha}), (\pi_2 \circ \Phi)(\overline{G}_{-i\alpha})\big]\big),$$

and let K_p denote the p-envelope of K in $W(m; \underline{1})$.

Let $v \otimes f \in S_\delta \otimes A(m; \underline{1})$ be a root vector for $\mathfrak{t}$. Since all components of our filtration are T-invariant, any 1-section $G(\eta)$ of G relative to $\operatorname{gr} T$ has the form $\operatorname{gr} L(\eta)$. In conjunction with Proposition 6.1(2) and Lemma 6.9 this yields that any 1-section $G(\eta)$ is triangulable. But then so is any 1-section $\overline{G}(\eta)$. Since $v \in S_\delta$ is non p-nilpotent in S_p the preceding remark shows that $[\Phi(\overline{H}), v \otimes f] \in S_\delta \otimes A(m; \underline{1})_{(1)}$. As this holds for all root vectors in $S_\delta \otimes A(m; \underline{1})$ it must be that $[\Phi(\overline{H}), S_\delta \otimes A(m; \underline{1})] \subset S_\delta \otimes A(m; \underline{1})_{(1)}$. This forces $(\pi_2 \circ \Phi)(\overline{H}) \subset W(m; \underline{1})_{(0)}$. Since

$$(\pi_2 \circ \Phi)(\overline{G}_p) = \big[(\pi_2 \circ \Phi)(\overline{G})\big]_p = \big[(\pi_2 \circ \Phi)\big(\overline{G}(\alpha)\big)\big]_p$$

is transitive in $W(m; \underline{1})$, we now deduce that the subalgebra $Fd + K_p \subset W(m; \underline{1})$ is transitive as well. Recall from part (4a) that

$$(\ker \pi_2) \cap \Phi\big(\overline{G}(\alpha)\big) \subset \mathfrak{t}_0' \otimes A(m; \underline{1}).$$

So any $y \in (\ker \pi_2) \cap \Phi(\overline{G}_{i\alpha})$ can be written as $y = t_1 \otimes f + t_2 \otimes g$ where $t_1, t_2 \in \mathfrak{t}_0'$ are linearly independent and $f, g \in A(m; \underline{1})$.

Suppose $d \in W(m; \underline{1})_{(0)}$. Then K_p is still transitive in $W(m; \underline{1})$. Since $L(\alpha)$ acts triangulably on L, all elements in $[K_p, y]$ act nilpotently on $\Phi(\overline{G})$, by Lemma 6.9. Since $\mathfrak{t}_0'$ is a torus, it must be that $f, g \in F$. Then $y \in (t_0 \otimes A(m; \underline{1})) \cap \Phi(\overline{G}_{i\alpha}) = (0)$. As a consequence,

$$d \in W(m; \underline{1})_{(0)} \quad\Longrightarrow\quad (\ker \pi_2) \cap \Phi(\overline{G}_{i\alpha}) = (0) \quad (\forall i \in \mathbb{F}_p^*). \tag{4}$$

Suppose $d \notin W(m; \underline{1})_{(0)}$. Then it can be assumed further that $d = (1 + x_1)\partial_1$, by [9, Theorem 2.6]. Then $y = t_1 \otimes (1+x_1)^k f_1 + t_2 \otimes (1+x_1)^k g_1$ for some truncated polynomials $f_1, g_1 \in A(m; \underline{1})$ in $x_2, \ldots, x_m$. Since $Fd + K_p$ is transitive in $W(m; \underline{1})$, we have that $\partial_2, \ldots, \partial_m \in K_p + W(m; \underline{1})_{(0)}$. Arguing as before, we now obtain that $f_1, g_1 \in F$. Thus

$$d \notin W(m; \underline{1})_{(0)} \quad\Longrightarrow\quad \dim\big((\ker \pi_2) \cap \Phi(\overline{G}_{i\alpha})\big) \leqslant 2 \quad (\forall i \in \mathbb{F}_p^*). \tag{5}$$

(b) We claim that K_p consists of nilpotent elements of $W(m;\underline{1})$. So assume for a contradiction that it does not and consider the weight space decomposition $K=\sum_{i\in\mathbb{F}_p}K_i$ of K relative to d. By Jacobson's formula, $K_p=K+\sum_{i\in\mathbb{F}_p}\sum_{j>0}K_i^{p^j}$. Since $L(\alpha)$ is nilpotent, so is K. Suppose one of the K_i's contains a non-nilpotent element of $W(m;\underline{1})$. Then $C(K_p)$ contains a nonzero toral element t commuting with d. By the definition of K, we have $K\neq K_0$. Hence d does not centralize K. So $d\notin Ft$, that is K_p+Fd contains a 2-dimensional torus. But then $\overline{G}$ contains a 4-dimensional torus, a contradiction. Thus all K_i's consist of nilpotent elements and our claim follows in view of the Engel–Jacobson theorem.

Thus $(Fd+K_p)^{(1)}=K\subset K_p$ acts nilpotently on $A(m;\underline{1})$. So we can apply [8, Theorem 3.2] to $Fd+K$. That theorem yields a restricted embedding

$$\sigma: Fd+K_p\hookrightarrow \sum_{i=1}^m Fx_i\partial_i+\sum_{i=1}^m A(x_1,\ldots,x_{i-1})\partial_i.$$

Since $\sigma(K)$ is p-nilpotent, it must lie in $\sum_{i=1}^m A(x_1,\ldots,x_{i-1})\partial_i$. Consequently,

$$\sum_{i\in\mathbb{F}_p^*}\dim\bigl((\pi_2\circ\Phi)(\overline{G}_{i\alpha})\bigr)\leqslant\sum_{i=1}^m p^{i-1}=\frac{p^m-1}{p-1}. \tag{6}$$

Recall that all root spaces of L relative to T occur and have the same dimension, by Lemma 6.8. Also, $G_{i\alpha}\cap\widetilde{M}(G)=(0)$, by our choice of M in part (1). So (3) yields that

$$\dim\overline{G}_{i\alpha}\geqslant p^{m-1}\quad(\forall i\in\mathbb{F}_p^*). \tag{7}$$

Combining (7), (6), (5), and (4), we now get

$$(p-1)p^{m-1}\leqslant\sum_{i\in\mathbb{F}_p^*}\dim\overline{G}_{i\alpha}\leqslant\frac{p^m-1}{p-1}+2(p-1)$$

which gives $m=1$.

(c) Suppose $d\in W(1;\underline{1})_{(0)}$. Then $(\ker\pi_2)\cap\Phi(\overline{G}_{i\alpha})=(0)$ for all $i\in\mathbb{F}_p^*$; see (4). But then (7) leads to a contradiction:

$$(p-1)\leqslant\sum_{i\in\mathbb{F}_p^*}\dim\overline{G}_{i\alpha}=\sum_{i\in\mathbb{F}_p^*}\dim\bigl((\pi_2\circ\Phi)(\overline{G}_{i\alpha})\bigr)\leqslant 1.$$

Thus $d\notin W(1;\underline{1})_{(0)}$, and hence it can be assumed that $d=(1+x)\partial$.

Suppose there exists $u\in\overline{G}_{i\alpha}$, for some $i\in\mathbb{F}_p^*$, such that $(\pi_2\circ\Phi)(u)\neq(0)$. By our concluding remark in part (5a),

$$\Phi(u)=\lambda_1 t\otimes(1+x)^k+\lambda_2\mathrm{Id}\otimes(1+x)^{k+1}\partial$$

for some $\lambda_1 \in F$, $\lambda_2 \in F^*$, and $t \in \mathfrak{t}_0'$. By (7), there exists a nonzero $v \in \overline{G}_{2i\alpha}$. Our assumption in conjunction with (6) shows that $\sum_{i\in\mathbb{F}_p^*} \dim((\pi_2 \circ \Phi)(\overline{G}_{i\alpha})) = 1$. Since u and v have different weights for d, it must be that $(\pi_2 \circ \Phi)(v) = 0$. Then $\Phi(v) = t' \otimes (1+x)^{2k}$ with $t' \in \mathfrak{t}_0' \setminus \{0\}$. Therefore,

$$\left[\Phi(u), \Phi(v)\right] = 2k\lambda_2 t' \otimes (1+x)^{3k}.$$

We have already mentioned that all 1-sections of $\overline{G}$ relative to $\overline{T}$ are triangulable. Thus $k = 0$ necessarily holds. But then $v \in \overline{H} \cap \overline{G}_{2i\alpha} = (0)$, a contradiction. We conclude that $\sum_{i\in\mathbb{F}_p^*} \Phi(\overline{G}_{i\alpha}) \subset \mathfrak{t}_0' \otimes A(1;\underline{1})$. Our discussion at the end of part (4d) (together with [23, Theorem 3.1(v)]) now shows that we are not in the degenerate case of Weisfeiler's theorem. As a result, $G_1 = (0)$ and hence $L(\alpha)$ injects into $\mathfrak{t}_0' \otimes A(1;\underline{1}) + (F\mathrm{Id} \otimes d)$. As $H \subset \operatorname{nil} \widetilde{H}$, it must be that $L(\alpha) \hookrightarrow \sum_{i\in\mathbb{F}_p^*} \Phi(\overline{G}_{i\alpha})$. But then $L(\alpha)$ is abelian and $\mathcal{N}(L_p) \cap (\bigcup_{i\in\mathbb{F}_p^*} L_{i\alpha}) = \{0\}$. Moreover, either $\dim L_\alpha = 1$ or $\dim L_\alpha = 2$.

(d) Suppose $\dim L_\alpha = 1$. Then $\dim L_\delta = 1$ for all $\delta \in \Gamma(L,T)$, which implies that $\widetilde{H} = T$. It is easy to see that this contradicts our choice of the p-nilpotent element $w \in L_p(\alpha)$ in part (1). Thus $\dim L_\alpha = 2$ whence $\dim L_\delta = 2$ for all $\delta \in \Gamma(L,T)$. Since $G_1 = (0)$, it also follows that $T + M \cong G_0 \cong \overline{G}$. Recall that $S \cong H(2;\underline{1};\Phi(\tau))^{(1)}$. Therefore, the minimal p-envelope of $\overline{T} + S \otimes A(1;\underline{1})$ is nothing but $\overline{T} + S \otimes A(1;\underline{1}) + S_p \otimes F = \overline{T} + S \otimes A(1;\underline{1})$. This shows that the Lie algebra $\overline{G}$ is restrictable. Let $[p] : \overline{G} \to \overline{G}$ denote the pth power map of $\overline{G}$. Since $L(\alpha)$ is abelian, $H = (0)$, and no root vector in $L(\alpha)$ act nilpotently on L, our discussion in part (1) shows that there is $x \in L_\alpha$ such that w is equal to the p-nilpotent part of $x^p \in \widetilde{H}$. Identify $T + M$ with $\overline{G}$ and observe that $\overline{T}$ is self-centralizing in $\overline{G}$. Therefore, $x^{[p]} \in T$ is $[p]$-semisimple. Since M carries three $\mathbb{F}_p$-independent T-roots and $x^p - x^{[p]}$ centralizes $T + M$, it must be that $x^p - x^{[p]} \in \widetilde{H}$ is p-nilpotent. But then $(x^p)_s = x^{[p]}$ and $w = (x^p)_n = x^p - x^{[p]}$. As a result, $[w, T + M] = (0)$. Since both $T + L$ and $T + M$ carry $p^3 - 1$ roots and $L(\alpha) \subset T + M$, our present assumption on $\dim L_\alpha$ implies that $\dim(T + L)/(T + M) = p(p^2 - 1)$. On the other hand, G_{-1} is an irreducible and faithful $\overline{G}$-module. By the toral rank considerations, this module is also restricted (for otherwise the centralizer of G_0 in the p-envelope of G in $\operatorname{Der} G$ would contain a nonzero semisimple element and this would eventually result in the inequality $TR(G) \geqslant 4$, a contradiction). Applying [9, Theorem 1.7] with $I = S \otimes A(1;\underline{1})$ now yields that there is a nontrivial S-module U such that $G_{-1} \cong U \otimes A(1;\underline{1})$ as vector spaces. By the preceding remark, $\dim U \leqslant p^2 - 1$. Recall that we have already reinstated all results of [17] under our present assumption on p; see Section 5. It is immediate from [17, Theorems 4.6, 4.9] that U is S-irreducible of dimension $p^2 - 1$. This leaves no room for G_{-2} forcing $T + L = (T + L)_{(-1)}$. As $\operatorname{ad} w$ commutes with $T + M$ and acts nilpotently on $T + L$, it acts trivially on the factor space $(T + L)/(T + M)$, by Schur's lemma. This gives $(\operatorname{ad} w)^2(L + T) = (0)$. Let y be an arbitrary element in $T + L$ and put $W := \operatorname{ad}_{T+L} w$, $Y := \operatorname{ad}_{T+L} y$. Since $[W, [W, Y]] = \operatorname{ad}_{T+L}[w, [w, y]] = 0$ and $W^2 = 0$, we have

$$0 = \left[W, [W, Y]\right] = W^2 Y - 2WYW + YW^2 = -2WYW.$$

Therefore, $WYW(z) = 0$ for all $z \in T + L$ implying that $[w, T + L] \subset T + M$ is an abelian ideal in $T + M$. As $T + M$ is semisimple we now get $w = 0$, a contradiction. The case $TR(S) \leqslant 2$ is thus impossible.

(6) Suppose $TR(S) = 3$.

(a) Recall from part (3) that $\mathfrak{t}_1'$ is a torus of maximal dimension in S_p. Therefore, $\dim \mathfrak{t}_1' = 3$ giving $\mathfrak{t} = \mathfrak{t}_1$. Our discussion in part (3) now shows that $\mathfrak{t} = \mathfrak{t}_0' \otimes F \subset S_p \otimes F$. Then $\Phi(\overline{G}) = I + \mathfrak{c}_{\Phi(\overline{G})}(\mathfrak{t}) = I + \Phi(\overline{H}) + \mathfrak{t}$ (as before, $\overline{H}$ stands for the image of $\operatorname{gr} H$ in $\overline{G}$).

(b) As explained in part (3), all 1-sections of S relative to $\mathfrak{t}_0'$ are nilpotent. Suppose S is not a counterexample to our theorem. Then no root vector of S relative to $\mathfrak{t}_0'$ act nilpotently on S. Combining our discussion in part (6a) with Jacobson's formula, we observe that

$$\Phi(\overline{G}_p) = \Phi(\overline{G})_p = S \otimes A(m; \underline{1}) + S_p \otimes F + \Phi(\overline{H})_p + \mathfrak{t}$$
$$\subset (\operatorname{Der} S) \otimes A(m; \underline{1}) + \Phi(\overline{H})_p.$$

Consequently, $(\pi_2 \circ \Phi)(\overline{H})_p = (\pi_2 \circ \Phi)(\overline{G}_p)$ is a transitive subalgebra of $W(m; \underline{1})$. Suppose $m > 0$ and let δ be any root in $\Gamma(S, \mathfrak{t}_0')$. Then $[\Phi(\overline{H}), S_\delta \otimes A(m; \underline{1})]$ contains non-nilpotent elements of $\Phi(\overline{G}_p)$. In view of Lemma 6.9, this contradicts the triangulability of $L(\tilde{\delta}^*)$, however. Thus $m = 0$. But then $[\Phi(\overline{H}), S_\delta] = (0)$, by a similar reasoning. Since this holds for all roots δ we derive that $\Phi(\overline{H})$ is a nilpotent ideal in $\Phi(\overline{G})$. Since $\Phi(\overline{G})$ is semisimple, by part (3), we now get $\Phi(\overline{H}) = (0)$. Then $H = (0)$, $\overline{G} = I + \overline{T}$, and $\overline{G}^{(1)} = (I + \overline{T})^{(1)} = I \cong S$ is simple.

Recall that I is a graded ideal of $\overline{G}$. Since all graded components of I are $\overline{T}$-invariant and no root vector in I relative to $\overline{T} \subset I_p$ is p-nilpotent in I_p, it must be that $I = I \cap \overline{G}_0$. But then $\overline{G} = \overline{T} + I = \overline{G}_0$. This shows that $G_1 = (0)$ and $S + \mathfrak{t}_0' \cong M + T$ as Lie algebras. By Lemma 6.8, $|\Gamma(S, \mathfrak{t}_0')| = p^3 - 1 = |\Gamma(L, T)|$ and all root spaces of S relative to $\mathfrak{t}_0'$ (and of L relative to T) are of the same dimension. Since $H = (0)$ and $L_\alpha \subset M$ this gives $\dim M = (p^3 - 1) \dim L_\alpha = \dim L$. But then $L = M$. This contradiction shows that S is a counterexample to our theorem.

(c) It follows from part (6b) and our choice of L that $\dim S = \dim L$. As $\overline{\operatorname{gr} L}$ is an ideal of $\overline{G}$ containing $I \cong S \otimes A(m; \underline{1})$ we get $m = 0$ and $(\operatorname{gr} L) \cap \widetilde{M}(G) = (0)$. It is now straightforward to see that $\widetilde{M}(G) = (0)$. Then $G \cong \overline{G} = \overline{T} + S$ and $\overline{T}$ is a 3-dimensional torus in S_p. Besides, $S = \overline{G}^{(1)}$ is graded and $\overline{T} \subset \operatorname{Der}_0 S$. For $i \in \mathbb{Z}$, we let S_i denote the ith graded component of S.

If S contains a solvable 2-section $S(\eta, \delta)$, for some $\mathbb{F}_p$-independent $\eta, \delta \in \Gamma(S, \overline{T})$, then $S \cong H(2; (2, 1); \Phi(\tau))^{(1)}$, by Proposition 6.7. By Theorem 6.6, $\mathfrak{c}_S(\overline{T}) = (0)$ and no root vector for $\overline{T}$ act nilpotently on S in this case. Also, $\mathfrak{c}_S(\overline{T}) = (0)$. Since $\overline{T}$ preserves all graded components of S, this entails $G = G_0$. But then, again, $L = M$, a contradiction.

Thus no 2-section $S(\eta, \delta)$ with $\mathbb{F}_p$-independent η and δ is solvable. By Proposition 6.3, we then have $S[\eta, \delta] \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$ whenever η and δ are $\mathbb{F}_p$-independent. Recall that no root vector in $H(2;\underline{1};\Phi(\tau))^{(1)}$ act nilpotently on $H(2;\underline{1};\Phi(\tau))^{(1)}$. From this it is immediate that $S_i \cap S(\eta,\delta) \subset \operatorname{rad} S(\eta,\delta)$ for all $i \neq 0$. This shows that

$$S_0[\nu,\gamma] := S_0(\eta,\delta)/\operatorname{rad} S_0(\eta,\delta) \cong S[\eta,\delta] \cong H\big(2;\underline{1};\Phi(\tau)\big)^{(1)}$$

for all $\mathbb{F}_p$-independent $\eta,\delta \in \Gamma(S,\overline{T})$.

Recall from part (1) that the roots α,β,γ are $\mathbb{F}_p$-independent and $S_{-1,i\alpha} = (0)$ for all $i \in \mathbb{F}_p$. Since $G \neq G_0$, we have $S_{-1} \neq (0)$. Let $\nu \in \Gamma(S,\overline{T})$ be such that $S_{1,\nu} \neq (0)$. We may assume without loss of generality that ν is $\mathbb{F}_p$-independent of γ. Then $S_0[\nu,\gamma] \cong H(2;1;\Phi(\tau))^{(1)}$, hence there is $x \in S_{0,\gamma}$ with $\nu(x) \neq 0$. Then

$$S_{-1,\nu+j\gamma} = (\operatorname{ad} x)^j(S_{-1,\nu}) \neq (0) \quad (\forall j \in \mathbb{F}_p).$$

Consequently, $\Gamma(S_{-1},\overline{T})$ contains $r\alpha + s\beta$ for some $r \in \mathbb{F}_p$ and $s \in \mathbb{F}_p^*$. Since α and $r\alpha + s\beta$ are $\mathbb{F}_p$-independent, $S_0[\alpha, r\alpha+s\beta] \cong H(2;\underline{1};\Phi(\tau))^{(1)}$. We now proceed as before to obtain $\Gamma(S_{-1},\overline{T}) \cap \mathbb{F}_p\alpha \neq \emptyset$. This contradiction finally completes the proof of the proposition. □

Proposition 6.11. *Suppose* $TR(L) = 3$. *Then*

$$[u,v]^p = -\alpha\big(v^p\big)u^p + \beta\big(u^p\big)v^p \quad (\forall u \in L_\alpha,\ \forall v \in L_\beta).$$

Proof. (1) Proposition 6.10 in conjunction with Jacobson's formula implies that $L_p = L \oplus T$. Given a subalgebra M of L, we denote by M_p the p-envelope of M in L_p. Let T_M denote the set of all $t \in T$ such that $m + t \in M_p$ for some $m \in M$. Then $T_M = T \cap M_p$ is a subtorus of T.

Suppose $\dim T_{L(\alpha,\beta)} < 3$. Lemma 6.8 identifies the set $\Gamma(L,T) \cup \{0\}$ with the $\mathbb{F}_p$-space dual to $T^{\mathrm{tor}} := \{t \in T \mid t^p = t\}$, a 3-dimensional $\mathbb{F}_p$-subspace of T. Since $T_{L(\alpha,\beta)}$ is spanned by its toral elements, there is $\gamma \in \Gamma(L,T)$ with $\gamma(T_{L(\alpha,\beta)}) = 0$. The 2-section $L(\alpha,\gamma)$ carries a natural $\mathbb{F}_p$-grading with graded components $L(\alpha,\gamma)_i = \sum_{j\in\mathbb{F}_p} L(\alpha,\gamma)_{i\gamma+j\alpha}$ for $i \in \mathbb{F}_p$. By the choice of γ, both α and γ vanish on $T_{L(\alpha)} \subset T_{L(\alpha,\beta)}$. This implies that $L(\alpha) = L(\alpha,\gamma)_0$ acts nilpotently on $L(\alpha,\gamma)$. But then $L(\alpha,\gamma)$ is solvable, by [19, Proposition 1.14]. According to Proposition 6.7 and Theorem 6.6(4), our result holds in this case.

(2) Thus we may assume from now that $L(\alpha,\beta)$ is nonsolvable and

$$L(\alpha,\beta)_p = L(\alpha,\beta) \oplus T.$$

Then

$$C\big(L(\alpha,\beta)\big) = C\big(L(\alpha,\beta)_p\big) \cap L(\alpha,\beta) \subset C_{L(\alpha,\beta)}(T) = (0),$$

by Proposition 6.10. So $L(\alpha,\beta)$ embeds into $\operatorname{Der} L(\alpha,\beta)$ via the adjoint representation. Let G denote the p-envelope of $L(\alpha,\beta)$ in $\operatorname{Der} L(\alpha,\beta)$. Since G contains $\operatorname{ad} L(\alpha,\beta)$, we have $C(G) = (0)$. But then G is a minimal p-envelope of $L(\alpha, \beta)$ in the sense of [22, (2.5)]. According to [22, Theorem 2.5], $L(\alpha, \beta)_p$ contains an isomorphic copy of G as an ideal. More precisely, we have a commutative diagram

$$\begin{array}{ccc} L(\alpha,\beta) & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ L(\alpha,\beta)_p & \xrightarrow{\sim} & G \oplus C \end{array}$$

such that C is central, all maps are injective Lie algebra homomorphisms, and the bottom-row isomorphism is restricted. We stress, however, that G is not a restricted ideal of $G \oplus C$. After identifying the restricted Lie algebras $G \oplus C$ and $L(\alpha, \beta)_p = L(\alpha, \beta) \oplus T$, we will have $C = T \cap \ker \alpha \cap \ker \beta = C(L(\alpha, \beta)_p)$ and $G \subset L(\alpha, \beta)_p$.

Let $[p] : G \to G$ denote the (unique) pth power map on G. We extend $[p]$ to a pth power map on $L(\alpha, \beta)_p$ by setting

$$(x + c)^{[p]} := x^{[p]} + c^p \quad (\forall x \in G,\ \forall c \in C).$$

By Proposition 6.10, $w^{[p]} \in w^p + C \subset T$ for all root vectors w contained in $L(\alpha, \beta)$. Therefore, $\kappa(w^{[p]}) = \kappa(w^p)$ for all $\kappa \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$. Let $T' := T \cap G$; then $G = L(\alpha, \beta) \oplus T'$. It is immediate from our earlier remarks that $C = Ft_0$ for some nonzero toral element $t_0 \in T \cap \ker \alpha \cap \ker \beta$. As a consequence, for any $x \in L(\alpha, \beta)_p$ we have that $x^p - x^{[p]} = \chi(x)^p t_0$ with $\chi(x) \in F$. It is well known that the function

$$\chi : L(\alpha, \beta) \longrightarrow F, \quad x \mapsto \chi(x),$$

is linear; see [22, Proposition 2.1(2)], for example. For $u \in L_\alpha$, $v \in L_\beta$, we now have

$$\begin{aligned} [u, v]^p + \alpha\big(v^p\big)u^p - \beta\big(u^p\big)v^p = {} & [u, v]^{[p]} + \chi\big([u, v]\big)^p t_0 + \alpha\big(v^{[p]}\big)u^{[p]} \\ & + \alpha\big(v^{[p]}\big)\chi(u)^p t_0 - \beta\big(u^{[p]}\big)v^{[p]} - \beta\big(u^{[p]}\big)\chi(v)^p t_0. \end{aligned}$$

Thus it suffices to establish the following two equalities:

$$[u, v]^{[p]} = -\alpha\big(v^{[p]}\big)u^{[p]} + \beta\big(u^{[p]}\big)v^{[p]}, \tag{8}$$

$$\chi\big([u, v]\big)^p = -\alpha\big(v^{[p]}\big)\chi(u)^p + \beta\big(u^{[p]}\big)\chi(v)^p. \tag{9}$$

(3) In this part, we will show that (8) holds. Put

$$\Delta := [e_\alpha, e_\beta]^{[p]} + \alpha\big(e_\beta^{[p]}\big)e_\alpha^{[p]} - \beta\big(e_\alpha^{[p]}\big)e_\beta^{[p]}.$$

We first suppose that $\dim L_\delta = 1$ for all $\delta \in \Gamma(L, T)$. Since $H = (0)$, by Proposition 6.10, and $L(\alpha, \beta)$ is nonsolvable, we then have $L(\alpha, \beta) \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$; see Proposition 6.3. Then $G \cong \operatorname{Der} H(2; \underline{1}; \Phi(\tau))^{(1)}$ as restricted Lie algebras, by [2, Proposition 2.1.8]. Due to [18, Theorem VII.3], the Lie algebra $L(\alpha,\beta)\cong H(2;\underline{1};\Phi(\tau))^{(1)}$ has a basis $\{e_\delta \mid \delta\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}\}$ consisting of root vectors for T and such that

$$[e_\lambda, e_\mu]=f(\lambda,\mu)e_{\lambda+\mu}\quad \bigl(\forall\lambda,\mu\in(\mathbb{F}_p\alpha+\mathbb{F}_p\beta)\setminus\{0\}\bigr),$$

where f is a skew-symmetric $\mathbb{F}_p$-bilinear form on $\mathbb{F}_p\alpha+\mathbb{F}_p\beta$. For $u=e_\alpha,\ v=e_\beta$, this gives

$$\begin{aligned}\bigl(\operatorname{ad}[e_\alpha,e_\beta]\bigr)^p(e_\mu)&=f(\alpha,\beta)^p(\operatorname{ad}e_{\alpha+\beta})^p(e_\mu)=f(\alpha,\beta)^pf(\alpha+\beta,\mu)^pe_\mu\\&=f(\alpha,\beta)^pf(\alpha,\mu)^pe_\mu+f(\alpha,\beta)^pf(\beta,\mu)^pe_\mu.\end{aligned}$$

Since $\beta(e_\alpha^{[p]})e_p=(\operatorname{ad}e_\alpha)^p(e_\beta)=f(\alpha,\beta)^pe_\beta$ and, similarly, $\alpha(e_\beta^{[p]})e_\alpha=f(\beta,\alpha)^pe_\alpha=-f(\alpha,\beta)^pe_\alpha$, we obtain that $\operatorname{ad}_{L(\alpha,\beta)}\Delta=0$. Since G acts faithfully on $L(\alpha,\beta)$ and $\Delta\in G$, we deduce (8) in the present case.

Now suppose that $\dim L_\delta\geqslant 2$ for all $\delta\in\Gamma(L,T)$ (recall that all root spaces for T have the same dimension). We still have $L[\alpha,\beta]\cong H(2;\underline{1};\Phi(\tau))^{(1)}$, by Proposition 6.3. So $\operatorname{rad}L(\alpha,\beta)$ is T-stable. From this it follows that $(L(\alpha,\beta)\oplus T')/\operatorname{rad}L(\alpha,\beta)$ is semisimple. As a consequence, $\operatorname{rad}G=\operatorname{rad}L(\alpha,\beta)$. Then $G/\operatorname{rad}G$ is a minimal p-envelope of $L[\alpha,\beta]$. The argument used in the former case now gives $\Delta\in\operatorname{rad}G$ while our earlier remarks yield $\Delta\in T$. But then $\Delta\in T\cap\operatorname{rad}G=T'\cap\operatorname{rad}G=(0)$. Thus (8) holds in all cases.

(4) In this part, we will show that (9) holds. We may assume that $\chi\neq 0$.

Choose a root γ independent of α and β, and let M be a composition factor of the $L(\alpha,\beta)_p$-module $\sum_{i,j\in\mathbb{F}_p}L_{\gamma+i\alpha+j\beta}$. Let ρ denote the corresponding representation of $L(\alpha,\beta)_p$. By the definition of $L(\alpha,\beta)_p$, this representation is restricted.

Since $t_0\in C$ is a nonzero toral element, we have that $\rho(t_0)=\gamma(t_0)\mathrm{Id}_M\neq 0$ and $\gamma(t_0)\in\mathbb{F}_p^*$. So, given $w\in G$ we have

$$\rho(w)^p-\rho\bigl(w^{[p]}\bigr)=\rho\bigl(w^p\bigr)-\rho\bigl(w^{[p]}\bigr)=\chi(w)^p\rho(t_0)=\bigl(\gamma(t_0)\chi(w)^p\bigr)\mathrm{Id}_M.$$

Thus $\rho:G\to\mathfrak{gl}(M)$ is a representation of $(G,[p])$ with p-character $\chi':=\gamma(t_0)\chi\neq 0$. Since $L(\alpha,\beta)_p=G\oplus Ft_0$, the restriction of ρ to G remains irreducible.

Suppose $\dim L_\delta=1$ for all $\delta\in\Gamma(L,T)$. Then $\dim M\leqslant p^2$. Since $\rho|_G$ is a non-restricted representation, [17, Theorem 4.9] shows M is induced from a 1-dimensional module Fu over the standard maximal subalgebra $G_{(0)}$ of $G=\operatorname{Der}H(2;\underline{1};\Phi(\tau))^{(1)}$ (see Section 5 for the definition of $G_{(0)}$). More precisely,

$$M\cong u(G,\chi')\otimes_{u(G_{(0)},\chi')}Fu.$$

Since Fu is 1-dimensional, ${G_{(0)}}^{(1)}$ annihilates u. By [17, Proposition 1.2(2d)], ${G_{(0)}}^{(1)}$ coincides with ${H(2;\underline{1};\Phi(\tau))^{(1)}}_{(0)}$, and hence is restricted. So χ' vanishes on ${G_{(0)}}^{(1)}$. Then Proposition 5.1 shows that (9) holds in the present case.

In view of Lemma 6.8, it remains to consider the case where $\dim L_\delta \geqslant 2$ for all $\delta \in \Gamma(L,T)$. By Proposition 6.10, the p-linear map $L_\delta \to T \cap \ker\delta$, $x \mapsto x^p$, is injective. Since $\dim T = 3$, we can assume that $\dim L_\delta = 2$ for all $\delta \in \Gamma(L,T)$. Since $2 \geqslant TR(L(\alpha,\beta) \geqslant TR(L[\alpha,\beta]) = TR(H(2;\underline{1};\Phi(\tau))^{(1)}) = 2$, the radical of $L(\alpha,\beta)$ is nilpotent; see [19, Theorem 1.7]. By our earlier remarks, $\operatorname{rad} L(\alpha,\beta) = \operatorname{rad} G$. Choose $n \geqslant 1$ such that $(\operatorname{rad} G)^n \neq (0)$ and $(\operatorname{rad} G)^{n+1} = (0)$. Then $N := (\operatorname{rad} G)^n$ is a module for the factor algebra $G' := G/\operatorname{rad} G \cong \operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$. Since $(G/\operatorname{rad} G)_\delta \neq (0)$ for each $\delta \in (\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\}$ (and since all root spaces of L are 2-dimensional and $H = (0)$), we have that $\dim(\operatorname{rad} G) < p^2$. If N is a trivial G-module, then T' annihilates N. But then $N \subset C_L(T) = (0)$, a contradiction. So N is a nontrivial G'-module of dimension $< p^2$. Thanks to [17, Theorems 4.6, 4.9], the G'-module N is isomorphic to the adjoint G'-module $H(2;\underline{1};\Phi(\tau))^{(1)}$. Then $N = \operatorname{rad} G$, by dimension reasons. As a consequence, $\operatorname{rad} G$ is abelian and isomorphic to $H(2;\underline{1};\Phi(\tau))^{(1)}$ as $(G/\operatorname{rad} G)$-modules.

We now look more closely at the irreducible G-module M with p-character χ'. Let y be a root vector for T contained in $\operatorname{rad} G$. From our earlier remarks it is immediate that $y^{[p]} = 0$ and $\alpha(y) = \beta(y) = 0$. As $y \neq 0$ we also have $\gamma(y) \neq 0$; see Proposition 6.10. It follows that $\chi'(y) \neq 0$. According to [22, Corollary 5.7.6],

$$M \cong u(G,\chi') \otimes_{u(G_0,\chi')} M_0$$

as G-modules, where $G_0 = \{x \in G \mid \chi'([x,\operatorname{rad} G]) = 0\}$ and M_0 is an irreducible $G_{(0)}$-submodule of M. Clearly, $\operatorname{rad} G \subset G_0$. Also $G_0 \neq G$, for otherwise χ' would vanish on $[G,\operatorname{rad} G] = \operatorname{rad} G$, which is not the case. Since $\dim M \leqslant 2p^2$, the restricted subalgebra G_0 has codimension $\leqslant 2$ in G.

Let $\pi : G \twoheadrightarrow \operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$ denote the canonical homomorphism. Recall that $H(2;\underline{1};\Phi(\tau))^{(1)}{}_{(0)}$ is the only proper subalgebra of maximal dimension in $H(2;\underline{1};\Phi(\tau))^{(1)}$ (see [19, Theorem 3.20] for example). If $\pi(G_0) \cap H(2;\underline{1};\Phi(\tau))^{(1)}$ had codimension $\leqslant 1$ in $H(2;\underline{1};\Phi(\tau))^{(1)}$, then $\pi(G_0)$ would contain $H(2;\underline{1};\Phi(\tau))^{(1)}$. Since G_0 is restricted, this would yield $G = G_0$, however. Thus $\pi(G_0)$ normalizes $H(2;\underline{1};\Phi(\tau))^{(1)}{}_{(0)}$ and, as a consequence, π maps G_0 onto the standard maximal subalgebra of $\operatorname{Der} H(2;\underline{1};\Phi(\tau))^{(1)}$. Hence G_0 has codimension 2 in G and $\dim M_0 \leqslant 2$.

Since $\dim M_0 \leqslant 2$, the image of G_0 in $\mathfrak{gl}(M_0)$ is either $\mathfrak{sl}(M_0)$ or $\mathfrak{gl}(M_0)$. This implies that $\operatorname{rad} G_0$ acts on M_0 as scalar operators. Then $[G_0,\operatorname{rad} G_0]$ acts trivially on M_0, that is M_0 is a module over $G_0/[G_0,\operatorname{rad} G_0]$. If $\operatorname{rad} G \subset [G_0,G_0]$ then computing traces yields that $\operatorname{rad} G$ acts trivially on M_0 (one should keep in mind that $\dim M_0 < p$ and $\operatorname{rad} G$ acts on M_0 as scalar operators). But we have already found $y \in \operatorname{rad} G$ with $\chi'(y) \neq 0$. Since $\operatorname{rad} G$ is a restricted ideal, this leads to a contradiction.

Thus $\operatorname{rad} G \not\subset [G_0,G_0]$. Proposition 5.2 now shows that G is split, that is

$$G = K \oplus \operatorname{rad} G, \qquad K \cong \operatorname{Der} H\big(2;1;\Phi(\tau)\big)^{(1)}.$$

Then $[G_0,G_0] = [K_{(0)},K_{(0)}] + [K_{(0)},\operatorname{rad} G]$. Besides, $[K_{(0)},K_{(0)}] \cong H(2;\underline{1};\Phi(\tau))^{(1)}{}_{(0)}$, by [17, Proposition 1.2(2d)]. In particular, $[K_{(0)},K_{(0)}]$ is $[p]$-closed. As $x^{[p]} = 0$ for all $x \in \operatorname{rad} G$, Jacobson's formula shows that $[G_0,G_0]$ is $[p]$-closed as well. Using [17, Proposition 1.2], one observes without difficulty that $[K_{(0)}, \operatorname{rad} K_{(0)}] = \operatorname{rad}[K_{(0)}, K_{(0)}]$. Since $\operatorname{rad}[K_{(0)}, K_{(0)}] \cong H(2; \underline{1}; \Phi(\tau))^{(1)}{}_{(1)}$ is $[p]$-closed, Jacobson's formula shows that so is

$$[G_0, \operatorname{rad} G_0] = [K_{(0)}, \operatorname{rad} K_{(0)}] + [K_{(0)}, \operatorname{rad} G] = \operatorname{rad}[K_{(0)}, K_{(0)}] + [K_{(0)}, \operatorname{rad} G].$$

Let y be any element in $[G_0, \operatorname{rad} G_0]$. Since $[G_0, \operatorname{rad} G_0]$ is $[p]$-closed and acts trivially on M_0, the central element $y^p - y^{[p]} \in Z(U(G))$ annihilates the induced module M. Therefore, χ' vanishes on $[G_0, \operatorname{rad} G_0]$. Note that

$$[G_0, G_0]/[G_0, \operatorname{rad} G_0] \cong [K_{(0)}, K_{(0)}]/\operatorname{rad}[K_{(0)}, K_{(0)}] \cong \mathfrak{sl}(2),$$

and M_0 is a restricted $\mathfrak{sl}(2)$ module (being irreducible of dimension $< p$). Since χ' vanishes on $[G_0, \operatorname{rad} G_0]$ and $[G_0, G_0]$ is $[p]$-closed, it must then be that χ' vanishes on $[G_0, G_0]$ as well. Proposition 5.3 now completes the proof. □

We now come to our first classification result for Lie algebras of an arbitrary rank:

Theorem 6.12. *Let L be a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ and suppose that the p-envelope of L in* Der L *contains a torus T of maximal dimension such that for every root $\alpha \in \Gamma(L, T)$ the 1-section $L(\alpha)$ is solvable. Then the set $A := \Gamma(L, T) \cup \{0\}$ is an $\mathbb{F}_p$-subspace in T^* and either $L \cong S(m; \underline{n}; \Phi(\tau))^{(1)}$ for some $m \geqslant 3$ and $\underline{n} \in \mathbb{N}^m$ or L is isomorphic to a Block algebra $L(A, 0, f)$ for some $\mathbb{F}_p$-bilinear mapping $f : A \times A \to F$. In all cases, each $L(\alpha)$ is abelian and $\mathfrak{c}_L(T) = (0)$.*

Proof. (a) Since $\mathfrak{c}_L(T)$ consists of p-nilpotent elements of L_p, by Lemma 6.1, the torus T is standard. Then Theorem 2.1 shows that every nilpotent section $L(\alpha_1, \ldots, \alpha_k)$ acts triangulably on L.

(b) Suppose in addition that $TR(L) = 3$. Then:

- no root vector for T act nilpotently on L (Proposition 6.10);
- each solvable 2-section relative to T is abelian (Proposition 6.7, Theorem 6.6);
- $\Gamma(L, T) \cup \{0\}$ is an $\mathbb{F}_p$-subspace in T^* (Lemma 6.8),
- $[x, y]^p = -\alpha(y^p)x^p + \beta(x^p)y^p$ whenever $x \in L_\alpha$ and $y \in L_\beta$ (Proposition 6.11).

Combined together, these results show that [20, Theorem 5.6] holds for $p > 3$.

(c) Lemmas 6.2–6.4 of [20] hold because their original proofs work when supplemented by our Theorem 6.5. Inspection of [20, Sections 6, 7] shows that only the results mentioned in parts (a)–(c) of this proof are used to establish of [20, Theorems 7.5, 7.8]. Thus these theorems continue to hold for $p > 3$, hence the result. □

7. Case (B): pushing the classical Lie algebras

In this section we will assume that $\Gamma(L,T)$ consists of solvable and classical roots and contains at least one classical root. Our results will parallel those obtained for $p>7$ in [16, Sections 3, 4]. Note that, due to our assumption, all roots in $\Gamma(L,T)$ are proper. In other words, T is an optimal torus in the sense of [2].

Proposition 7.1. *Let $\alpha,\beta\in\Gamma(L,T)$. Then one of the following occurs*:

(1) $L[\alpha,\beta]=(0)$;
(2) *there is a classical root $\mu\in\Gamma(L,T)$ such that $L[\alpha,\beta]=L[\mu]$*;
(3) $L[\alpha,\beta]=L[\delta_1]\oplus L[\delta_2]$ *for some classical roots $\delta_1,\delta_2\in\Gamma(L,T)$*;
(4) $L[\alpha,\beta]\cong\mathfrak{sl}(2)\otimes A(1;\underline{1})$;
(5) $L[\alpha,\beta]\cong H(2;\underline{1};\Phi(\tau))^{(1)}$;
(6) $L[\alpha,\beta]$ *is classical simple of type* A_2, C_2, *or* G_2.

Moreover, in cases (1)–(3), *and* (6), *we have that $\Psi_{\alpha,\beta}(T)\subset L[\alpha,\beta]$. In case* (4), *we have $\Psi_{\alpha,\beta}(T)=(Fh\otimes 1)\oplus(F\mathrm{Id}\otimes(1+x)\partial)$, while in case* (5), $L[\alpha,\beta]\cap\Psi_{\alpha,\beta}(T)=(0)$.

Proof. Suppose $L(\alpha,\beta)$ is nonsolvable. Then Theorems 4.1, 4.2, 4.4 apply. Since neither Witt nor Hamiltonian roots occur in $\Gamma(L,T)$, Theorem 4.1 yields the algebras listed in case (3) of our theorem. Suppose $L[\alpha,\beta]$ satisfies the conditions of Theorem 4.2. If $\widetilde{S}$ is classical then the equality $TR(\widetilde{S})=2$ implies that $L[\alpha,\beta]\cong\widetilde{S}$ where $\widetilde{S}$ is of type A_2, C_2 or G_2.

$\widetilde{S}$ cannot be a restricted Lie algebra of Cartan type because otherwise $L[\delta]$ would be of Cartan type for some $\delta\in\Gamma(L,T)$; see [2] or [18, Section IX] (these references apply in our case as T is optimal for L).

$\widetilde{S}$ cannot be isomorphic to the Melikian algebra $\mathfrak{g}(1,1)$ because otherwise $L[\delta]$ would be of Cartan type for some $\delta\in\Gamma(L,T)$, by [13, Theorem 5.2].

If $\widetilde{S}$ is a non-restricted Lie algebra of Cartan type, then we apply [18, Sections V, VI, VIII] and argue as before to show that $\widetilde{S}\cong H(2;\underline{1};\Phi(\tau))^{(1)}$. This is case (5) of our theorem.

Now assume that $L[\alpha,\beta]$ satisfies the conditions of Theorem 4.4. If $\widetilde{S}=L[\mu]^{(1)}$ for some $\mu\in\Gamma(L,T)$, then $L[\alpha,\beta]\cong\widetilde{S}\cong\mathfrak{sl}(2)$. This is case (2) of our theorem. If $\widetilde{S}\cong H(2;\underline{1})^{(2)}$ then a Witt root occurs in $\Gamma(L,T)$; see [18, Theorem III.5]. This case is therefore impossible. If $\widetilde{S}$ is as in case (3) of Theorem 4.4 then $\widetilde{S}\cong S\otimes A(1;\underline{1})$ and $\widetilde{S}\otimes F$ is contained in a 1-section of $L[\alpha,\beta]$. Then $S\cong\mathfrak{sl}(2)$. This brings case (4). Since no 2-section of L relative to T is Melikian, by [13, Theorem 5.2], case (4) of Theorem 4.4 is impossible.

Let $\Psi=\Psi_{\alpha,\beta}:T\to\operatorname{Der}L[\alpha,\beta]$ and $L[\alpha,\beta]_p$ be as in Section 4. If $TR(L[\alpha,\beta])=2$ then $T\subset L[\alpha,\beta]_p+C(T+L[\alpha,\beta]_p)$; see [19, Theorem 1.7(8)]. As $\Psi(T)+L[\alpha,\beta]_p$ is centerless (being a subalgebra of $\operatorname{Der}L[\alpha,\beta]$), we get $\Psi(T)\subset L[\alpha,\beta]_p$. Since in cases (1)–(3), (6) of our theorem, $L[\alpha,\beta]$ is restricted, the preceding remark shows that $\Psi(T)\subset L[\alpha,\beta]$ as claimed. If $L[\alpha,\beta]$ is as in case (4) of our theorem, then Theorem 4.4(3) says that $\Psi(T)=(Fh\otimes 1)\oplus(F\mathrm{Id}\otimes(1+x)\partial)$ for some nonzero toral element $h\in S$.

Finally, if $L[\alpha,\beta] \cong H(2;\underline{1};\Phi(\tau))^{(1)}$, then [18, Theorem VII.3] (together with our preceding remark) shows that $\Psi(T) \cap L[\alpha,\beta] = (0)$. □

Proposition 7.1 is an analogue of [16, Proposition 3.1]. It shows that (L,T) satisfies the conditions (B1)–(B4) of [16]. Inspection shows that the proofs of Theorem 3.2, Corollary 3.3, Proposition 4.1, Lemma 4.2, and Proposition 4.3 in [16] go through for $p > 3$. So all these results apply to our L (denoted by G in [16]).

Given $\alpha \in \Gamma(L,T)$ we let $E(\alpha)$ be the set of all solvable roots $\mu \in \Gamma(L,T)$ such that $L[\alpha,\mu] \cong \mathfrak{sl}(2) \otimes A(1;\underline{1})$. Following [16, Section 3], we now set

$$B(L,T) := \bigl\{\alpha \in \Gamma(L,T) \bigm| L(\alpha) \text{ is nonsolvable and } E(\alpha) \neq \emptyset\bigr\}.$$

The roots in $B(L,T)$ are called *bad*. According to [17, Lemma 4.2], for any bad α the set $E(\alpha) \cup \{0\}$ is an $\mathbb{F}_p$-subspace in T^*. Given $\alpha \in B(L,T)$, we set $P(\alpha) := H \oplus \sum_{\mu \in E(\alpha)} L_\mu$, a T-invariant subalgebra of L. Our next result is an analogue of [16, Theorem 4.4].

Proposition 7.2. *If $B(L,T) \neq \emptyset$ then $L[\alpha,\beta] \cong H(2;\underline{1};\Phi(\tau))^{(1)}$ for some $\alpha,\beta \in \Gamma(L,T)$.*

Proof. By [17, Proposition 4.3], there is $\alpha \in B(L,T)$ such that $P(\alpha)$ is nonsolvable. Since $E(\alpha) \cup \{0\}$ is an $\mathbb{F}_p$-subspace, $P(\alpha)$ is a T-section of L. By [19, Theorem 1.16], it contains a nonsolvable 2-section, $L(\beta,\gamma)$ say. By the definition of $E(\alpha)$, each 1-section $L(\delta)$ with $\delta \in (\mathbb{F}_p\beta + \mathbb{F}_p\gamma) \setminus \{0\}$ is solvable. Proposition 7.1 now yields that $L[\beta,\gamma] \cong H(2;\underline{1};\Phi(\tau))^{(1)}$ as desired. □

Our second classification result is as follows.

Theorem 7.3. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 3$ and assume that the p-envelope of L in* $\operatorname{Der} L$ *contains a torus T of maximal dimension such that all roots in $\Gamma(L,T)$ are either solvable or classical. Assume further that at least one root in $\Gamma(L,T)$ is classical. Then L is a classical Lie algebra, that is there exists a simple algebraic group G of adjoint type over F such that $L \cong (\operatorname{Lie} G)^{(1)}$. In particular, L is restricted.*

Proof. Assume the contrary. One observes, by inspection, that the proof of [16, Lemma 4.6] goes through for $p > 3$. This reduces the general case to the case where $TR(L) = 3$. More precisely, we can assume that L has the following properties:

(i) L is simple with $TR(L) = 3$ and T is a torus of maximal dimension in L_p;
(ii) there exists $\alpha \in B(L,T)$ such that $L(\delta) \cong \mathfrak{sl}(2)$ for all $\delta \in \Gamma(L,T) \setminus E(\alpha)$;
(iii) there are $\beta,\gamma \in \Gamma(L,T)$ such that $E(\alpha) = (\mathbb{F}_p\beta \oplus \mathbb{F}_p\gamma) \setminus \{0\}$;
(iv) $\delta(H) = 0$ for all $\delta \in E(\alpha)$.

As $\dim T = 3$ we have that $\Gamma(L,T) \subset \mathbb{F}_p\alpha + \mathbb{F}_p\beta + \mathbb{F}_p\gamma$. Let $\delta = i\alpha + j\beta + k\gamma$ be a root, and put $\mu = j\beta + k\gamma$. Then $\mu \in E(\alpha)$, by (iii), while from the definition of $E(\alpha)$ it follows that $L[\alpha,\mu]\cong\mathfrak{sl}(2)\otimes A(1;\underline{1})$. Suppose $i\notin\{0,\pm1\}$. It is clear from the description of $\Psi_{\alpha,\beta}(T)$ in Proposition 7.1 (case (4)) that $\delta\notin\Gamma(L[\alpha,\mu],\Psi_{\alpha,\beta}(T))$. As a consequence, $L_\delta\subset\operatorname{rad}_T L(\alpha,\mu)\subset\operatorname{rad}L(\delta)$. As $\delta(H)=i\alpha(H)\neq 0$, [16, Theorem 3.2] shows that L_δ is contained in a proper ideal of L. As L is simple, we now obtain

$$\Gamma(L,T)\subset\{0,\pm\alpha\}+\mathbb{F}_p\beta+\mathbb{F}_p\gamma.$$

Set $L_{\pm1}:=\sum_{i,j\in\mathbb{F}_p}L_{\pm\alpha+i\beta+j\gamma}$ and $L_0:=L(\beta,\gamma)$. Then the decomposition

$$L=L_{-1}\oplus L_0\oplus L_1$$

is a nontrivial short $\mathbb{Z}$-grading of L. So [4, Lemma 14] now yields that L is classical, forcing $\beta(H)\neq 0$. This contradiction proves the theorem. □

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