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Hochschild cohomology rings of *d*-Koszul algebras

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ABSTRACT

In this paper we determine the multiplicative structure of Hochschild cohomology rings of d-Koszul algebras with the Yoneda products based on an explicit construction of a chain map $\Delta : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$ lifting the identity with $\mathbb{P}$ a minimal projective bimodule resolution of a d-Koszul algebra A.

1. Introduction

Let K be a commutative Noetherian ring. Let $A = A_0 + A_1 + A_2 + \cdots$ be an associative graded K-algebra generated in degree 0 and 1 (that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \le i, j < \infty$). Assume that A_0 is a finitely generated semisimple K-algebra, and A_1 is a finitely generated K-module. Recall from [13] that A is a d-Koszul algebra if there exists a minimal A-projective resolution of A_0

$$\mathbb{P} : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

such that for each $n \ge 0$, P_n can be generated in exactly one degree $\chi(n)$, and the jump map $\chi(n)$ is defined by

$$\chi(n) = \begin{cases} \frac{n}{2}d, & \text{if } n \text{ is even} \\ \frac{n-1}{2}d + 1, & \text{if } n \text{ is odd.} \end{cases}$$

We note that if $d = 2$, then A is a Koszul algebra, which may be understood to be a positively graded algebra that is "as close to semisimplicity as possible". For $d \ge 3$, A is not a Koszul algebra.

The local d-Koszul algebras (that is, $A_0 = K$) have been extensively studied by many authors. R. Berger first generalized the Koszulity for quadratic algebras to algebras whose relations are homogeneous of degree $N > 2$ (N-homogeneous algebras) [6]. Artin–Schelter regular algebras of global dimension 3 which are generated in degree 1 are such algebras [3]. This generalization follows along the definition given by Beilinson, Ginzburg and Soergel for $N = 2$ [9]. In [6], generalized Koszulity is connected to lattice distributivity and to confluence, and the bimodule version of the generalized Koszul resolution is studied and applied to compute its Hochschild homology. On the other hand, following Manin's monograph for quadratic algebras in [17], R. Berger, M. Dubois-Violette and M. Wambst have given an alternative set-up of generalized Koszulity based on N-complexes [7]. Moreover, R. Berger and N. Marconnet showed that if an N-homogeneous algebra is generalized Koszul, AS-Gorenstein and of finite global dimension, then there is a Poincaré duality between its Hochschild homology and cohomology [8]. Recently, E.L. Green, E.N. Marcos, R. Martínez-Villa and Pu Zhang generalized in [13] the local d-Koszul algebras to nonlocal case, and presented a characterization in Ext-algebra being generated in degree in 0, 1 and 2.

Hochschild cohomology, as a branch of homological algebra, was introduced by Hochschild in 1945 [14] and developed by Cartan and Eilenberg [11]. In recent years, the Hochschild cohomology and Hochschild cohomology ring have been

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extensively studied, and play an important role in many branches of mathematics and physics. However, for most finite dimensional algebras, little is known about the Hochschild cohomology groups and even less is known about the Hochschild cohomology rings.

Our purpose in this paper is to determine the multiplicative structure of Hochschild cohomology rings of d-Koszul algebras. A first step was achieved in [5,1,10]. R. Buchweitz, E.L. Green, N. Snashall and Ø. Solberg described the multiplicative structure of Hochschild cohomology rings of Koszul algebras based on a "comultiplicative" structure of a minimal projective bimodule resolution [10]. For the case of monomial d-Koszul algebras, G. Ames, L. Cagliero and P. Tirao gave a clear description of the Yoneda product and completely determined the structure of Hochschild cohomology rings of truncated quiver algebras by the explicit constructions of comparison morphisms between the two different resolutions [1]. In [19], Siegel and Witherspoon defined a cup product of two elements η in $\mathrm{HH}^n(A)$ and θ in $\mathrm{HH}^m(A)$ using the composition $\mathbb{P} \xrightarrow{\Delta} \mathbb{P} \otimes_A \mathbb{P} \xrightarrow{\eta\otimes\theta} A \otimes_A A \xrightarrow{\nu} A$ which coincides with the ordinary cup product, and is independent of the projective resolution $\mathbb{P}$ of A and the chain map Δ. The difficulty is to describe the chain map Δ explicitly. For instance, the "comultiplicative" structure of a minimal projective bimodule resolution in [10] describes in fact such a chain map in the case of Koszul algebras.

In Section 2 we generalize the bimodule Koszul resolutions $\mathbb{P}$ of local d-Koszul algebras to nonlocal case. Section 3 is central for the paper and is devoted to giving an explicit construction of a chain map $\Delta : \mathbb{P} \longrightarrow \mathbb{P} \otimes_A \mathbb{P}$ lifting the identity. It took us a long time to look for the correct definition of the chain map. Applying it we obtain a clear description of the cup product on the minimal projective bimodule resolution and determine the multiplicative structure of Hochschild cohomology rings of d-Koszul algebras. Finally, we apply our results to monomial d-Koszul algebras and the well-known cubic Artin–Schelter regular algebra of global dimension 3 of type A in Section 4. As a consequence, we reobtain the description of the multiplicative structure of Hochschild cohomology rings of truncated quiver algebras in a different way [1].

2. Bimodule Koszul resolutions

Let $A = A_0 + A_1 + A_2 + \cdots$ be a d-Koszul algebra. Recall that A is a quotient of the tensor algebra $T_{A_0}(A_1) = A_0 + A_1 + A_1^{\otimes^2_{A_0}} + \cdots$, say $A = T_{A_0}(A_1)/I$ with I generated by homogeneous elements of degree d. Let $R = I \cap A_1^{\otimes^d_{A_0}}$. Note that R is an A_0–A_0-submodule of $A_1^{\otimes^d_{A_0}}$. Throughout we always assume that K is a field and that A_0 is not only semisimple, but, as a ring, A_0 is $K \times K \times \cdots \times K$.

In [13], a generalized Koszul complex of A is defined as follows. Let $H_0 = A_0$, $H_1 = A_1$, and for $n \geq 2$,

$$H_n = \bigcap_{i+j+d=n} A_1^{\otimes^i_{A_0}} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes^j_{A_0}}.$$

To simplify notation, we will denote $A_1^{\otimes^i_{A_0}}$ as simply A_1^i and write $\otimes_{A_0}$ as simply $\otimes$. Since $R \subseteq A_1^d$, $H_n \subseteq A_1^n$, and thus we can write elements of H_n as $\sum x_1 \otimes x_2 \otimes \cdots \otimes x_n$, where the x_i's are in A_1. It was shown in [13] (see also [6]) that the d-Koszulity of A implies that $(R \otimes A_1^i) \cap (A_1^i \otimes R) \subseteq A_1^{i-1} \otimes R \otimes A_1$ for $2 \leq i < d$, and using this fact they obtained the following proposition:

Proposition 2.1 *([13, Corollary 8.2]). For $n \geq 0$, we have*

$$\begin{aligned} H_{dn+1} = &(R \otimes A_1^{(n-1)d+1}) \cap (A_1 \otimes R \otimes A_1^{(n-1)d}) \\ &\cap (A_1^d \otimes R \otimes A_1^{(n-2)d+1}) \cap (A_1^{d+1} \otimes R \otimes A_1^{(n-2)d}) \\ &\cap \cdots \\ &\cap (A_1^{(n-1)d} \otimes R \otimes A_1) \cap (A_1^{(n-1)d+1} \otimes R) \end{aligned}$$

and

$$\begin{aligned} H_{dn} = &(R \otimes A_1^{(n-1)d}) \\ &\cap (A_1^{d-1} \otimes R \otimes A_1^{(n-2)d+1}) \cap (A_1^d \otimes R \otimes A_1^{(n-2)d}) \\ &\cap (A_1^{2d-1} \otimes R \otimes A_1^{(n-3)d+1}) \cap (A_1^{2d} \otimes R \otimes A_1^{(n-3)d}) \\ &\cap \cdots \\ &\cap (A_1^{(n-1)d-1} \otimes R \otimes A_1) \cap (A_1^{(n-1)d} \otimes R). \end{aligned}$$

Thus one can define $Q_n = A \otimes_{A_0} H_{\chi(n)}$ as a projective left A-module for $n \geq 0$ and $\sigma_n : Q_n \longrightarrow Q_{n-1}$ for $n \geq 1$. Where if $n = 2k$, for $\sum a \otimes x_1 \otimes \cdots \otimes x_{kd} \in Q_n$,

$$\sigma_n \left(\sum a \otimes x_1 \otimes \cdots \otimes x_{kd} \right) = \sum a x_1 \cdots x_{d-1} \otimes x_d \otimes \cdots \otimes x_{kd}.$$

If $n = 2k+1$, for $\sum a \otimes x_1 \otimes \cdots \otimes x_{kd+1} \in Q_n$,

$$\sigma_n\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{kd+1}\right) = \sum ax_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1}.$$

Furthermore, it was shown in [13] that A is a d-Koszul algebra if and only if

$$(\mathbb{Q}, \sigma) : \cdots \longrightarrow Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \longrightarrow Q_1 \xrightarrow{\sigma_1} Q_0 \xrightarrow{\sigma_0} A_0 \longrightarrow 0$$

is a minimal graded projective A-resolution of A_0.

The aim of this section is to give a bimodule characterization for d-Koszulity. For the local d-Koszul algebras such a characterization has been obtained by Berger and Marconnet in [6,8].

Similar to R. Berger work, we can also define a minimal projective bimodule resolution of A. For $n \geq 0$, define $K_{L,n} = A \otimes H_n$, and the A-linear map $\delta_L : K_{L,n} \longrightarrow K_{L,n-1}$ is defined by the natural inclusion $H_n \hookrightarrow A \otimes H_{n-1}$. Then $(\delta_L)^d = 0$ and thus (K_L, δ_L) is a d-complex. Define analogously the d-complex (K_R, δ_R). Thus $K_{L-R} = K_L \otimes A = A \otimes K_R$ is a bimodule d-complex for $\delta'_L = \delta_L \otimes 1_A$ ($\delta'_R = 1_A \otimes \delta_R$ respectively), and δ'_L and δ'_R commute. Now we define $P_n = A \otimes_{A_0} H_{\chi(n)} \otimes_{A_0} A$, and a new differential $d_n : P_n \longrightarrow P_{n-1}$ as follows: if n is odd, then

$$d_n = \delta'_L - \delta'_R,$$

if n is even, then

$$d_n = \delta'^{d-1}_L + \delta'^{d-2}_L\delta'_R + \cdots \delta'_L\delta'^{d-2}_R + \delta'^{d-1}_R.$$

More precisely, if $n = 2k+1$, for $\sum a \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes b \in P_n$, then

$$d_n\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes b\right) = \sum(ax_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1} \otimes b - a \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes x_{kd+1}b);$$

and if $n = 2k$, for $\sum a \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes b \in P_n$, then

$$d_n\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes b\right) = \sum\left(\sum_{j=1}^{d} ax_1 \cdots x_{j-1} \otimes x_j \otimes \cdots \otimes x_{j+(k-1)d} \otimes x_{j+(k-1)d+1} \cdots x_{kd}b\right).$$

It is easy to check that $d_{n-1}d_n = 0$, so $(\mathbb{P}, d)$ is a complex.

In order to transfer acyclicity from $(\mathbb{Q}, \sigma)$ to $(\mathbb{P}, d)$, we need the following result which is the nonlocal version of [8, Prop. 4.1].

Proposition 2.2. *Let $A = A_0 + A_1 + \cdots$ be a graded algebra. Assume that the complex $L \xrightarrow{f} M \xrightarrow{g} N$ is formed of graded-free modules, with L bounded below. Then this complex is exact if the following is exact:*

$$A_0 \otimes_A L \xrightarrow{1_{A_0} \otimes_A f} A_0 \otimes_A M \xrightarrow{1_{A_0} \otimes_A g} A_0 \otimes_A N.$$

We can now state the bimodule characterization for d-Koszulity which is the nonlocal version of [8, Thm. 4.4] or [6, Thm. 5.6].

Theorem 2.3. *Let $A = T_{A_0}(A_1)/I$ with I generated by homogeneous elements of degree d. Then the following statements are equivalent:*

(a) *A is d-Koszul;*
(b) *The complex*

$$(\mathbb{P}, d) : \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$$

is a minimal projective A^e-resolution of A.

Proof. Assume that A is d-Koszul. Applying the functor $- \otimes_A A_0$ to $(\mathbb{P}, d)$, we get the Koszul resolution of A_0 in A-grMod, and we conclude that (b) holds by Proposition 2.2.

Conversely, assume $(\mathbb{P}, d)$ is exact. Viewing $(\mathbb{P}, d)$ as a projective resolution of A in grMod-A and comparing it with the identity map of A, one sees that there are in grMod-A two morphisms of resolutions $f : (\mathbb{P}, d) \longrightarrow A$ and $g : A \longrightarrow (\mathbb{P}, d)$ (where A is considered as a complex concentrated in degree 0), and a chain homotopy $s : (\mathbb{P}, d) \longrightarrow (\mathbb{P}, d)$ such that

$$1_{(\mathbb{P},d)} - gf = sd + dS.$$

Clearly, $1_{(\mathbb{P},d)} = sd + ds$ in degree > 0. Applying $- \otimes_A A_0$, we draw

$$1_{(\mathbb{Q},\sigma)} = (s \otimes_A 1_{A_0})\sigma + \sigma(s \otimes_A 1_{A_0})$$

in degree > 0, which implies that $(\mathbb{Q}, \sigma)$ is exact in degree > 0 and then A is d-Koszul. This completes the proof. $\square$

3. Cup products

Denote by A^e the enveloping algebra of A, i.e., the tensor product $A \otimes_k A^{op}$ of the algebra A and its opposite A^{op}. By the Cartan–Eilenberg formula [11] the n-th Hochschild cohomology group of A is

$$\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A,\ A)$$

and the Hochschild cohomology ring $\mathrm{HH}^*(A)$ is defined to be

$$\mathrm{HH}^*(A) = \bigoplus_{n\geq 0} \mathrm{HH}^n(A)$$

whose multiplication is given by the multiplication induced by the Yoneda product. It is also well known that the Yoneda product of $\mathrm{HH}^*(A)$ coincides with the cup product defined on the cohomology of $\mathrm{Hom}_{A^e}(\mathbb{B},\ A)$, where $(\mathbb{B},\ b)$ is the standard projective A^e-resolution of A given by $B^n = A^{\otimes_{A_0}(n+2)}$ and $b^n : B^n \longrightarrow B^{n-1}$ by

$$b^n(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = \sum_{i=0}^{n} (-1)^i \lambda_0 \otimes \cdots \otimes \lambda_i\lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1}.$$

When $A_0 = K$, this is the usual bar resolution. Note that if $A_0 = K \times \cdots \times K$, then A_0 is not central in general, so that A is not necessarily an algebra over A_0.

For the standard projective resolution $(\mathbb{B},\ b)$ there is a chain map $\Delta' : \mathbb{B} \longrightarrow \mathbb{B} \otimes_A \mathbb{B}$ given by

$$\Delta'(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = \sum_{i=0}^{n} (\lambda_0 \otimes \cdots \otimes \lambda_i \otimes 1) \otimes_A (1 \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1}).$$

If η and θ in $\mathrm{HH}^n(A)$ and $\mathrm{HH}^m(A)$, respectively, are represented by $\eta : B^n \longrightarrow A$ and $\theta : B^m \longrightarrow A$, then the cup product $\eta \sqcup \theta$ in $\mathrm{HH}^{n+m}(A)$ is given by the following composition of maps

$$\mathbb{B} \xrightarrow{\Delta'} \mathbb{B} \otimes_A \mathbb{B} \xrightarrow{\eta\otimes\theta} A \otimes_A A \xrightarrow{\nu} A,$$

where $\upsilon : A \otimes_A A \longrightarrow A$ is the multiplication map. We see that the cup product is

$$(\eta \sqcup \theta)(\lambda_0 \otimes \cdots \otimes \lambda_{n+m+1}) = \eta(\lambda_0 \otimes \cdots \otimes \lambda_n \otimes 1)\theta(1 \otimes \lambda_{n+1} \otimes \cdots \otimes \lambda_{n+m+1}).$$

In [19] it was shown that any projective A^e-resolution $\mathbb{X}$ of A gives rise to a "cup product", which coincides with the ordinary 'cup product'. Let $\mathbb{X}$ be a projective A^e-resolution of A. There exists a chain map $\Delta : \mathbb{X} \longrightarrow \mathbb{X} \otimes_A \mathbb{X}$ lifting the identity, which is unique up to homotopy. Siegel and Witherspoon defined a cup product of two elements η in $\mathrm{HH}^n(A)$ and θ in $\mathrm{HH}^m(A)$ as above using the composition

$$\mathbb{X} \xrightarrow{\Delta} \mathbb{X} \otimes_A \mathbb{X} \xrightarrow{\eta\otimes\theta} A \otimes_A A \xrightarrow{\nu} A,$$

and note that it is independent of the projective resolution $\mathbb{X}$ of A and the chain map Δ. Now we give an explicit formula for Δ for the minimal projective A^e-resolution of A constructed in Section 2.

Recall that $\mathbb{P} \otimes_A \mathbb{P}$ is still a projective A^e-resolution of A which is given by

$$(\mathbb{P} \otimes_A \mathbb{P})_n = \coprod_{i+j=n} P_i \otimes_A P_j$$

and $D_n : (\mathbb{P} \otimes_A \mathbb{P})_n \longrightarrow (\mathbb{P} \otimes_A \mathbb{P})_{n-1}$ is given by

$$D_n = \sum_{i=0}^{n-1} ((-1)^i \otimes d_{n-i} + d_{i+1} \otimes 1).$$

Now we define $\Delta : (\mathbb{P}, d) \longrightarrow (\mathbb{P} \otimes_A \mathbb{P}, D)$ as follows:

Definition 3.1. The linear map $\Delta_n : P_n \longrightarrow (\mathbb{P}\otimes_A\mathbb{P})_n$ is defined as follows: for $n = 2k+1$, $\sum 1\otimes x_1\otimes x_2\otimes\cdots\otimes x_{kd+1}\otimes 1 \in P_{2k+1}$,

$$\begin{aligned}\Delta_{2k+1}&\left(\sum 1 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1} \otimes 1\right)\\ &= \sum \sum_{r=0}^{2k+1} (1 \otimes x_1 \otimes \cdots \otimes x_{\chi(r)} \otimes 1) \otimes_A (1 \otimes x_{\chi(r)+1} \otimes \cdots \otimes x_{kd+1} \otimes 1).\end{aligned}$$

and for $n = 2k$, $\sum 1 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{kd} \otimes 1 \in P_{2k}$,

$$\Delta_{2k}\left(\sum 1 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{kd} \otimes 1\right) = \sum \sum_{r=0}^{k} (1 \otimes x_1 \otimes \cdots \otimes x_{rd} \otimes 1) \otimes_A (1 \otimes x_{rd+1} \otimes \cdots \otimes x_{kd} \otimes 1)$$

$$+ \sum \sum_{r=0}^{k-1} \sum_{j+m+l=d-2} (\underbrace{x_1 \cdots x_j}_{j} \otimes x_{j+1} \otimes \cdots \otimes x_{j+rd+1} \otimes 1)$$

$$\otimes_A (\underbrace{x_{j+rd+2} \cdots x_{j+rd+m+1}}_{m} \otimes x_{j+rd+m+2} \otimes \cdots \otimes x_{j+m+(k-1)d+2} \otimes \underbrace{x_{j+m+(k-1)d+3} \cdots x_{kd}}_{l})$$

where $x_0 = 1$.

Remark. If the algebra A is Koszul (i.e. $d = 2$), then the map $\Delta = (\Delta_n)$ is just the "comultiplication" defined in [10].

In order to prove the map $\Delta = (\Delta_n) : \mathbb{P} \longrightarrow \mathbb{P} \otimes_A \mathbb{P}$ is a chain map, the first step is to show the following diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{d_1} & P_0 \\ \Delta_1 \big\downarrow & & \Delta_0 \big\downarrow \\ (\mathbb{P} \otimes_A \mathbb{P})_1 & \xrightarrow{D_1} & (\mathbb{P} \otimes_A \mathbb{P})_0 \end{array}$$

is commutative. Let $p = 1 \otimes x_1 \otimes 1 \in P_1$, where $x_1 \in A_1$. Since

$$\Delta_0 d_1(p) = \Delta_0(x_1 \otimes 1 - 1 \otimes x_1) = (x_1 \otimes 1) \otimes_A (1 \otimes 1) - (1 \otimes 1) \otimes_A (1 \otimes x_1)$$

and

$$\begin{aligned} D_1\Delta_1(p) &= D_1[(1 \otimes 1) \otimes_A (1 \otimes x_1 \otimes 1) + (1 \otimes x_1 \otimes 1) \otimes_A (1 \otimes 1)] \\ &= (1 \otimes d_1)[(1 \otimes 1) \otimes_A (1 \otimes x_1 \otimes 1)] + (d_1 \otimes 1)[(1 \otimes x_1 \otimes 1) \otimes_A (1 \otimes 1)] \\ &= (1 \otimes 1) \otimes_A (x_1 \otimes 1 - 1 \otimes x_1) + (x_1 \otimes 1 - 1 \otimes x_1) \otimes_A (1 \otimes 1) \\ &= 1 \otimes x_1 \otimes 1 - (1 \otimes 1) \otimes_A (1 \otimes x_1) + (x_1 \otimes 1) \otimes_A (1 \otimes 1) - 1 \otimes x_1 \otimes 1 \\ &= \Delta_0 d_1(p). \end{aligned}$$

So we have $D_1\Delta_1 = \Delta_0 d_1$.

Proposition 3.1. *The map $\Delta : \mathbb{P} \longrightarrow \mathbb{P} \otimes_A \mathbb{P}$ defined above is a chain map.*

Proof. It suffices to show that the following diagram

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & P_{n-1} \\ \Delta_n \big\downarrow & & \Delta_{n-1} \big\downarrow \\ (\mathbb{P} \otimes_A \mathbb{P})_n & \xrightarrow{D_n} & (\mathbb{P} \otimes_A \mathbb{P})_{n-1} \end{array}$$

is commutative for all $n \geq 1$.

(Case A) If $n = 2k + 1$, let $\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes 1 \in P_{2k+1}$. By the linearity of Δ and d, we consider the summand $p = 1 \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes 1$, where each $x_i \in A_1$. Since

$$(\mathbb{P} \otimes_A \mathbb{P})_{2k} = \coprod_{r=0}^{2k} P_r \otimes_A P_{2k-r},$$

we can prove $\Delta_{2k}d_{2k+1}(p) = D_{2k+1}\Delta_{2k+1}(p) \in (P \otimes P)_{2k}$ componentwise, i.e., we will prove that the r-th component of $\Delta_{2k}d_{2k+1}(p)$ is equal to that of $D_{2k+1}\Delta_{2k+1}(p)$ for each $0 \leq r \leq 2k$.

(A1) We first consider the case that $r = 2s$ is even. Since

$$d_{2k+1}(p) = x_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1} \otimes 1 - 1 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1},$$

applying Δ_{2k} to the equality above, we get the r-th component of $\Delta_{2k}d_{2k+1}(p)$ is

$$\begin{aligned} &(x_1 \otimes x_2 \otimes \cdots \otimes x_{sd+1} \otimes 1) \otimes_A (1 \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1) \\ &-(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (1 \otimes x_{sd+1} \otimes \cdots \otimes x_{kd} \otimes x_{kd+1}) \end{aligned}$$

which belongs to $P_r \otimes P_{2k-r} = P_{2s} \otimes P_{2(k-s)}$.

On the other hand, note that the preimage of $P_r \otimes P_{2k-r}$ under the map D_{2k+1} is $P_{2s} \otimes P_{2(k-s)+1} \coprod P_{2s+1} \otimes P_{2(k-s)}$, so the terms in $\Delta_{2k+1}(p)$ which belong to $P_{2s} \otimes P_{2(k-s)+1}$ and $P_{2s+1} \otimes P_{2(k-s)}$ are

$$(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (1 \otimes x_{sd+1} \otimes \cdots \otimes x_{kd+1} \otimes 1) \tag{1}$$

and

$$(1 \otimes x_1 \otimes \cdots \otimes x_{sd+1} \otimes 1) \otimes_A (1 \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1) \tag{2}$$

respectively. By the definition of $D_{2k+1} = \sum_{r=0}^{2k}((-1)^r \otimes d_{2k+1-r} + d_{r+1} \otimes 1)$, we may apply $(-1)^r \otimes d_{2k+1-r} = 1 \otimes d_{2(k-s)+1}$ to (1) and $d_{r+1} \otimes 1 = d_{2s+1} \otimes 1$ to (2) respectively, and get the r-th component of $D_{2k+1}\Delta_{2k+1}(p)$ is

$$\begin{aligned}
&(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (x_{sd+1} \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1 - 1 \otimes x_{sd+1} \otimes \cdots \otimes x_{kd+1}) \\
&\quad + (x_1 \otimes x_2 \otimes \cdots \otimes x_{sd+1} \otimes 1 - 1 \otimes x_1 \otimes \cdots \otimes x_{sd+1}) \otimes_A (1 \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1) \\
&= 1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes x_{sd+1} \otimes \cdots \otimes x_{kd+1} \otimes 1 \\
&\quad - (1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (1 \otimes x_{sd+1} \otimes \cdots \otimes x_{kd} \otimes x_{kd+1}) \\
&\quad + (x_1 \otimes x_2 \otimes \cdots \otimes x_{sd+1} \otimes 1) \otimes_A (1 \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1) \\
&\quad - 1 \otimes x_1 \otimes \cdots \otimes x_{sd+1} \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1 \\
&= \text{the } r\text{th component of } \Delta_{2k} d_{2k+1}(p).
\end{aligned}$$

(A2) If $r = 2s + 1$ is odd, the r-th component in

$$\Delta_{2k} d_{2k+1}(p) = \Delta_{2k}(x_1 \otimes x_2 \otimes \cdots \otimes x_{kd+1} \otimes 1 - 1 \otimes x_1 \otimes \cdots \otimes x_{kd+1})$$

is

$$\begin{aligned}
&\left(\sum_{j=1}^{d-1} x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+1+sd} \otimes 1\right) \otimes_A \left(\sum_{m=1}^{d-j} x_{j+2+sd} \cdots x_{j+m+sd}\right. \\
&\quad \left.\otimes x_{j+m+sd+l} \otimes \cdots \otimes x_{j+m+(k-1)d+1} \otimes x_{j+m+(k-1)d+2} \cdots x_{kd+1}\right) \\
&- \left(\sum_{i=1}^{d-1} x_1 \cdots x_{i-1} \otimes x_i \otimes \cdots \otimes x_{i+sd} \otimes 1\right) \otimes_A \left(\sum_{l=1}^{d-i} x_{i+1+sd} \cdots x_{i+l-1+sd}\right. \\
&\quad \left.\otimes x_{i+l+sd} \otimes \cdots \otimes x_{i+l+(k-1)d} \otimes x_{i+l+(k-1)d+1} \cdots x_{kd+1}\right)
\end{aligned}$$

where we view the product $x_i x_j$ as 1 if $i = j + 1$. For simplicity we denote by $\mathrm{I}_{(j,m)}$ the following term in the first sum

$$\begin{aligned}
&(x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+1+sd} \otimes 1) \otimes_A (x_{j+2+sd} \cdots x_{j+m+sd} \\
&\quad \otimes x_{j+m+sd+l} \otimes \cdots \otimes x_{j+m+(k-1)d+1} \otimes x_{j+m+(k-1)d+2} \cdots x_{kd+1})
\end{aligned}$$

and similarly, by $\mathrm{II}_{(i,l)}$ the following term in the second sum

$$\begin{aligned}
&(x_1 \cdots x_{i-1} \otimes x_i \otimes \cdots \otimes x_{i+sd} \otimes 1) \otimes_A (x_{i+1+sd} \cdots x_{i+l-1+sd} \\
&\quad \otimes x_{i+l+sd} \otimes \cdots \otimes x_{i+l+(k-1)d} \otimes x_{i+l+(k-1)d+1} \cdots x_{kd+1}).
\end{aligned}$$

Thus the above sum can be written as

$$\sum_{j=1}^{d-1}\sum_{m=1}^{d-j} \mathrm{I}_{(j,m)} - \sum_{i=1}^{d-1}\sum_{l=1}^{d-i} \mathrm{II}_{(i,l)}.$$

Clearly, $\mathrm{I}_{(j,m)} - \mathrm{II}_{(j+1,m)} = 0$ for $1 \leq j \leq d-2$ and $1 \leq m \leq d-(j+1)$. So the above sum changes into

$$\sum_{j=1}^{d-1} \mathrm{I}_{(j,d-j)} - \sum_{l=1}^{d-1} \mathrm{II}_{(1,l)}. \tag{3}$$

On the other hand, applying Δ_{2k+1} to p, we get the terms in $\Delta_{2k+1}(p)$ which belong to $P_{2s+1} \otimes P_{2(k-s)}$ and $P_{2s+2} \otimes P_{2(k-s)-1}$

are

$$(1 \otimes x_1 \otimes \cdots \otimes x_{sd+1} \otimes 1) \otimes_A (1 \otimes x_{sd+2} \otimes \cdots \otimes x_{kd+1} \otimes 1) \tag{4}$$

and

$$(1 \otimes x_1 \otimes \cdots \otimes x_{(s+1)d} \otimes 1) \otimes_A (1 \otimes x_{(s+1)d+1} \otimes \cdots \otimes x_{kd+1} \otimes 1) \tag{5}$$

respectively. Applying $(-1)^r \otimes d_{2k+1-r} = -1 \otimes d_{2(k-s)}$ to (4) and $d_{r+1} \otimes 1 = d_{2(s+1)} \otimes 1$ to (5), we obtain the terms which lie in $P_{2s+1} \otimes P_{2(k-s)-1}$ are

$$\begin{aligned} &-(1 \otimes x_1 \otimes \cdots \otimes x_{sd+1} \otimes 1) \otimes_A \left(\sum_{l=1}^{d} x_{sd+2} \cdots x_{sd+l} \otimes x_{sd+l+1} \otimes \cdots \otimes x_{(k-1)d+l+1} \otimes x_{(k-1)d+l+2} \cdots x_{kd+1} \right) \\ &+ \left(\sum_{j=0}^{d-1} x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+1+sd} \otimes x_{j+2+sd} \cdots x_{(s+1)d} \right) \otimes_A (1 \otimes x_{(s+1)d+1} \otimes \cdots \otimes x_{kd+1} \otimes 1). \end{aligned}$$

That is,

$$-\sum_{l=1}^{d} II_{(1,l)} + \sum_{j=0}^{d-1} I_{(j,d-j)}. \tag{6}$$

Note that $-\mathrm{II}_{(1,d)} + \mathrm{I}_{(0,d)} = 0$, we have Formula (3) = Formula (6). Thus the r-th component of $D_{2k+1}\Delta_{2k+1}(p)$ is equal to the r-th component of $\Delta_{2k} d_{2k+1}(p)$.

So we have shown that the diagram is commutative in the case that n is odd.

(Case B) If n is even, say $n = 2k$, for $\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes 1 \in P_{2k}$, consider the summand $q = 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes 1$, where each $x_i \in A_1$. Then

$$d_{2k}(q) = \sum_{j=0}^{d-1} x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+1+(k-1)d} \otimes x_{j+2+(k-1)d} \cdots x_{kd}.$$

Similar to the case A, we will show that the r-th component of $D_{2k}\Delta_{2k}(q)$ is equal to the r-th component of $\Delta_{2k-1} d_{2k}(q)$ for each $0 \le r \le 2k-1$.

(B1) If r is even, say $r = 2s$, we apply Δ_{2k-1} to $d_{2k}(q)$ and obtain the terms which belong to $P_r \otimes P_{2k-1-r} = P_{2s} \otimes P_{2(k-s)-1}$ is

$$\sum_{j=0}^{d-1} (x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+sd} \otimes 1) \otimes_A (1 \otimes x_{j+1+sd} \otimes \cdots \otimes x_{j+1+(k-1)d} \otimes x_{j+2+(k-1)d} \cdots x_{kd}). \tag{7}$$

On the other hand, applying Δ_{2k} to q, we get the term in $P_{2s} \otimes P_{2(k-s)}$

$$(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (1 \otimes x_{sd+1} \otimes \cdots \otimes x_{kd} \otimes 1) \tag{8}$$

and the terms in $P_{2s+1} \otimes P_{2(k-s)-1}$

$$\begin{aligned} &\left(\sum_{j=1}^{d-1} x_1 \cdots x_{j-1} \otimes x_j \otimes \cdots \otimes x_{j+sd} \otimes 1 \right) \otimes_A \left(\sum_{m=1}^{d-j} x_{j+1+sd} \cdots x_{j+m-1+sd} \right. \\ &\quad \left. \otimes x_{j+m+sd} \otimes \cdots \otimes x_{j+m+(k-1)d} \otimes x_{j+m+1+(k-1)d} \cdots x_{kd} \right) \end{aligned} \tag{9}$$

respectively. Then we apply $(-1)^r \otimes d_{2k-r} = 1 \otimes d_{2(k-s)}$ to (8) and $d_{r+1} \otimes 1 = d_{2s+1} \otimes 1$ to (9), we get the terms which lie in $P_{2s} \otimes P_{2(k-s)-1}$ are

$$
\begin{aligned}
&(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A \left(\sum_{t=0}^{d-1} x_{sd+1} \cdots x_{sd+t} \right. \\
&\quad \left. \otimes\ x_{sd+t+1} \otimes \cdots \otimes x_{(k-1)d+t+1} \otimes x_{(k-1)d+t+2} \cdots x_{kd} \right) \\
&+ \left(\sum_{j=1}^{d-1} x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+sd} \otimes 1 \right) \otimes_A \left(\sum_{m=1}^{d-j} x_{j+1+sd} \cdots x_{j+m-1+sd} \right. \\
&\quad \left. \otimes\ x_{j+m+sd} \otimes \cdots \otimes x_{j+m+(k-1)d} \otimes x_{j+m+1+(k-1)d} \cdots x_{kd} \right) \\
&- \left(\sum_{i=1}^{d-1} x_1 \cdots x_{i-1} \otimes x_i \otimes \cdots \otimes x_{i+sd} \right) \otimes_A \left(\sum_{l=1}^{d-i} x_{i+1+sd} \cdots x_{i+l-1+sd} \right. \\
&\quad \left. \otimes\ x_{i+l+sd} \otimes \cdots \otimes x_{i+l+(k-1)d} \otimes x_{i+l+1(k-1)d} \cdots x_{kd} \right).
\end{aligned}
$$

where we also view the product $x_i x_j$ as 1 if $i = j + 1$. Similarly, we denote by $\mathrm{III}_{(0,t)}$ the following term in the first summand

$$(1 \otimes x_1 \otimes \cdots \otimes x_{sd} \otimes 1) \otimes_A (x_{sd+1} \cdots x_{sd+t} \otimes x_{sd+t+1} \otimes \cdots \otimes x_{(k-1)d+t+1} \otimes x_{(k-1)d+t+2} \cdots x_{kd}),$$

by $\mathrm{IV}_{(j,m)}$ the following term in the second summand

$$(x_1 \cdots x_j \otimes x_{j+1} \otimes \cdots \otimes x_{j+sd} \otimes 1) \otimes_A (x_{j+1+sd} \cdots x_{j+m-1+sd} \otimes x_{j+m+sd} \otimes \cdots \otimes x_{j+m+(k-1)d} \otimes x_{j+m+1+(k-1)d} \cdots x_{kd})$$

and by $\mathrm{V}_{(i,l)}$ the following term in the third summand

$$(x_1 \cdots x_{i-1} \otimes x_i \otimes \cdots \otimes x_{i+sd}) \otimes_A (x_{i+1+sd} \cdots x_{i+l-1+sd} \otimes x_{i+l+sd} \otimes \cdots \otimes x_{i+l+(k-1)d} \otimes x_{i+l+1+(k-1)d} \cdots x_{kd}).$$

Thus the above sum can be written as

$$\sum_{t=0}^{d-1} \mathrm{III}_{(0,t)} + \sum_{j=1}^{d-1} \sum_{m=1}^{d-j} \mathrm{IV}_{(j,m)} - \sum_{i=1}^{d-1} \sum_{l=1}^{d-i} \mathrm{V}_{(i,l)}.$$

It is clear that $\mathrm{IV}_{(j,m)} - \mathrm{V}_{(j+1,m-1)} = 0$ for $1 \le j \le d-2$ and $m \ge 2$, and $\mathrm{III}_{(0,t)} - \mathrm{V}_{(1,t)} = 0$ for $1 \le t$, so the above sum changes into

$$\mathrm{III}_{(0,0)} + \sum_{j=1}^{d-1} \mathrm{IV}_{(j,1)} = \sum_{j=0}^{d-1} \mathrm{IV}_{(j,1)} \tag{10}$$

which is just formula (7). Thus the r-th component of $D_{2k}\Delta_{2k}(q)$ is equal to the r-th component of $\Delta_{2k-1}d_{2k}(q)$ for $r = 2s$.

(B2) The proof of the case $r = 2s + 1$ is similar to that of the case (B1), we omit it here.

Thus we prove the diagram is also commutative for the case that n is even. And hence we can see that Δ is a chain map. $\square$

We shall now use the chain map Δ to describe the cup product on the minimal resolution $\mathbb{P}$ of A.

Theorem 3.2. *Let A be a d-Koszul algebra over a field K with a minimal projective A^e-resolution $(\mathbb{P}, d)$. Suppose that $\eta : P_n \longrightarrow A$ and $\theta : P_m \longrightarrow A$ represent elements in $\mathrm{HH}^*(A)$, then we have*

(1) *if $n = 2k$ and $m = 2h + 1$, then for $\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes y_1 \otimes \cdots \otimes y_{hd+1} \otimes 1 \in P_{n+m}$,*

$$
\begin{aligned}
&(\eta \sqcup \theta) \left(\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes y_1 \otimes \cdots \otimes y_{h+1} \otimes 1 \right) \\
&\quad = \sum \eta(1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes 1)\theta(1 \otimes y_1 \otimes \cdots \otimes y_{hd+1} \otimes 1);
\end{aligned}
$$

(2) *if $n = 2k + 1$ and $m = 2h$, then for $\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1 \in P_{n+m}$,*

$$
\begin{aligned}
&(\eta \sqcup \theta) \left(\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1 \right) \\
&\quad = \sum \eta(1 \otimes x_1 \otimes \cdots \otimes x_{kd+1} \otimes 1)\theta(1 \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1);
\end{aligned}
$$

(3) *if* $n = 2k$ *and* $m = 2h$, *then for* $\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1 \in P_{n+m}$,

$$(\eta \sqcup \theta)\left(\sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1\right) = \sum \eta(1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes 1)\theta(1 \otimes y_1 \otimes \cdots \otimes y_{hd} \otimes 1);$$

(4) *if* $n = 2k+1$ *and* $m = 2h+1$, *then for* $\sum 1 \otimes z_1 \otimes \cdots \otimes z_{(k+h+1)d} \otimes 1 \in P_{n+m}$,

$$(\eta \sqcup \theta)\left(\sum 1 \otimes z_1 \otimes \cdots \otimes z_{(k+h+1)d} \otimes 1\right) = \sum \sum_{j+m+l=d-2} \eta(\underbrace{z_1 \cdots z_j}_{j} \otimes z_{j+1} \otimes \cdots \otimes z_{j+1+kd} \otimes 1)$$
$$\theta(\underbrace{z_{j+2+kd} \cdots z_{j+m+1+kd}}_{m} \otimes z_{j+m+2+kd} \otimes \cdots \otimes z_{j+m+2+(k+h)d} \otimes \underbrace{z_{j+m+3+(k+h)d} \cdots z_{(k+h+1)d}}_{l}).$$

Proof. By definition $\eta \sqcup \theta = \nu(\eta \otimes_A \theta)\Delta$.

(1) Assume that $n = 2k$ and $m = 2h+1$. Let $s = k+h$ and $\sum 1 \otimes p \otimes 1 = \sum 1 \otimes z_1 \otimes \cdots \otimes z_{sd+1} \otimes 1 = \sum 1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes y_1 \otimes \cdots \otimes y_{hd+1} \otimes 1 \in P_{n+m}$. Then

$$\Delta_{2s+1}\left(\sum 1 \otimes p \otimes 1\right) = \sum \sum_{r=0}^{2s+1} (1 \otimes z_1 \otimes \cdots \otimes z_{\chi(r)} \otimes 1) \otimes_A (1 \otimes z_{\chi(r)+1} \otimes \cdots \otimes z_{sd+1} \otimes 1).$$

Since $\eta : P_n \longrightarrow A$ and $\theta : P_m \longrightarrow A$, we have

$$\begin{aligned}(\eta \sqcup \theta)\left(\sum 1 \otimes p \otimes 1\right) &= \nu(\eta \otimes_A \theta)\Delta_{2s+1}\left(\sum 1 \otimes p \otimes 1\right) \\ &= \nu(\eta \otimes_A \theta)\left(\sum \sum_{r=0}^{2s+1} (1 \otimes z_1 \otimes \cdots \otimes z_{\chi(r)} \otimes 1) \otimes_A (1 \otimes z_{\chi(r)+1} \otimes \cdots \otimes z_{sd+1} \otimes 1)\right) \\ &= \sum \eta(1 \otimes x_1 \otimes \cdots \otimes x_{kd} \otimes 1)\theta(1 \otimes y_1 \otimes \cdots \otimes y_{hd+1} \otimes 1).\end{aligned}$$

The proofs of (2) and (3) are analogous. We prove the case (4). Assume that $n = 2k+1$ and $m = 2h+1$. Let $s = k+h+1$. Then by the definition of Δ, we have

$$\begin{aligned}(\eta \sqcup \theta)\left(\sum 1 \otimes z_1 \otimes \cdots \otimes z_{sd} \otimes 1\right) &= \nu(\eta \otimes_A \theta)\Delta_{2s}\left(\sum 1 \otimes z_1 \otimes \cdots \otimes z_{sd} \otimes 1\right) \\ &= \sum \sum_{j+m+l=d-2} \eta(\underbrace{z_1 \cdots z_j}_{j} \otimes z_{j+1} \otimes \cdots \otimes z_{j+1+kd} \otimes 1)\theta(\underbrace{z_{j+2+kd} \cdots z_{j+m+1+kd}}_{m} \\ &\quad \otimes z_{j+m+2+kd} \otimes \cdots \otimes z_{j+m+2+(k+h)d}\underbrace{z_{j+m+3+(k+h)d} \cdots Z_{(k+h+1)d}}_{l})\end{aligned}$$

as desired. □

4. Examples

We end this paper by applying Theorem 3.2 to some examples of some d-Koszul algebras for $d > 2$. We first consider monomial d-Koszul algebras which include all the truncated quiver algebras. The second example is the well-known cubic Artin–Schelter regular algebra of global dimension 3 of type A with generic coefficients.

4.1. Monomial d-Koszul algebras

In the subsection we restrict our attention to quotients of path algebras. Let $A = KQ/(\rho)$ be a monomial K-algebra with ρ a finite set of paths of length d in Q. Recall from [13] that A is a d-Koszul algebra if and only if ρ is d-covering, i.e. whenever $pq,\ qr \in \rho$ with q of length at least 1, then every subpath of pqr of length d is in ρ. In this case A has a minimal projective A^e-resolution $(\mathbb{P},\ d)$, where $P_n \cong A \otimes_{KQ_0} H_{\chi(n)} \otimes_{KQ_0} A$ and $H_{\chi(n)}$ is the free KQ_0-bimodule generated by the set $AP(n)$ of associated paths constructed by Bardzell in [4]. The following theorem follows directly from Theorem 3.2:

Theorem 4.1.1. *Let A be a monomial d-Koszul algebra over a field K with a minimal projective A^e-resolution $(\mathbb{P}, d)$. Suppose that $\eta : P_n \longrightarrow A$ and $\theta : P_m \longrightarrow A$ represent elements in $\mathrm{HH}^*(A)$, then we have*

(1) *if* $n = 2k$ *and* $m = 2h+1$, *then for* $x_1x_2 \cdots x_{kd}y_1y_2 \cdots y_{hd+1} \in AP(n+m)$,

$$(\eta \sqcup \theta)(1 \otimes x_1x_2 \cdots x_{kd}y_1y_2 \cdots y_{hd+1} \otimes 1) = \eta(1 \otimes x_1x_2 \cdots x_{kd} \otimes 1)\theta(1 \otimes y_1y_2 \cdots y_{hd+1} \otimes 1);$$

(2) *if $n = 2k + 1$ and $m = 2h$, then for $x_1x_2 \cdots x_{kd+1}y_1y_2 \cdots y_{hd} \in AP(n + m)$,*

$$(\eta \sqcup \theta)(1 \otimes x_1x_2 \cdots x_{kd+1}y_1y_2 \cdots y_{hd}\otimes 1) = \eta(1 \otimes x_1x_2 \cdots \otimes x_{kd+1} \otimes 1)\theta(1 \otimes y_1y_2 \cdots y_{hd}\otimes 1);$$

(3) *if $n = 2k$ and $m = 2h$, then for $x_1x_2 \cdots x_{kd}y_1y_2 \cdots y_{hd} \in AP(n + m)$,*

$$(\eta \sqcup \theta)(1 \otimes x_1x_2 \cdots x_{kd}y_1y_2 \cdots y_{hd}\otimes 1) = \eta(1 \otimes x_1x_2 \cdots x_{kd} \otimes 1)\theta(1 \otimes y_1y_2 \cdots y_{hd}\otimes 1);$$

(4) *if $n = 2k + 1$ and $m = 2h + 1$, then for $x_1x_2 \cdots x_{(k+h+1)d} \in AP(n + m)$,*

$$(\eta \sqcup \theta)(1 \otimes x_1x_2 \cdots x_{(k+h+1)d} \otimes 1) = \sum_{j+m+l=d-2} \eta(\underbrace{x_1 \cdots x_j}_{j} \otimes x_{j+1} \cdots x_{j+1+kd} \otimes 1)\theta(\underbrace{x_{j+2+kd} \cdots x_{j+m+1+kd}}_{m}$$
$$\otimes x_{j+m+2+kd} \cdots x_{j+m+2+(k+h)d} \otimes \underbrace{x_{j+m+3+(k+h)d} \cdots x_{(k+h+1)d}}_{l}).$$

In the particular case when ρ consists of all paths of length d in Q, namely, $A = kQ/J^d$ with J the arrow ideal, A is called a truncated quiver algebra, which is a monomial d-Koszul algebra [13]. Hochschild homology and cohomology of truncated quiver algebras have been extensively studied by many authors [1,2,5,12,16,15,20,22].

Let $A = KQ/J^d$ be a truncated quiver algebra with a K-basis $\mathcal{B} = \bigcup_{l=0}^{d-1} Q_l$. Here Q_l denotes the set of paths of length l in Q. Then A has a minimal projective A^e-resolution of the form

$$(\mathbb{P}, d) : \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}}, \ldots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0,$$

where

$$P_n = \begin{cases} A \bigotimes_{Q_0} KQ_{kd} \bigotimes_{Q_0} A & \text{if } n = 2k \\ A \bigotimes_{Q_0} KQ_{kd+1} \bigotimes_{Q_0} A & \text{if } n = 2k+1. \end{cases}$$

Note that $\mathrm{Hom}_{A^e}(P_n, A) \cong \mathrm{Hom}_{KQ_0}(KQ_{\chi(n)}, A) \cong K(\mathcal{B}//Q_{\chi(n)})$, where $\mathcal{B}//Q_{\chi(n)} = \{(\alpha, \tau) \in \mathcal{B} \times Q_{\chi(n)} | o(\alpha) = o(\tau),\ t(\alpha) = o(\tau)\}$ is the set of parallel paths in Q, $o(\alpha)$ and $t(\alpha)$ stand for the original and terminal of the path α.

As an immediate corollary of Theorem 4.1.1, we reobtain the description of the cup product of Hochschild cohomology ring of a truncated quiver algebra, which was first obtained by G. Ames, L. Cagliero and P. Tirao by an explicit constructions of comparison morphisms between the minimal projective resolution and the reduced bar resolution of a truncated quiver algebra (cf. [1, Thm. 7.6]).

Corollary 4.1.2. *Let A be a truncated quiver algebra. If (α, π) and (β, τ) represent elements in $\mathrm{HH}^n(A)$ and $\mathrm{HH}^m(A)$ respectively under the above isomorphisms, then we have*

(1) *if n or m is even, then*

$$(\alpha, \pi) \sqcup (\beta, \tau) = (\alpha\beta, \pi\tau)$$

(2) *if n and m are odd, then*

$$(\alpha, \pi) \sqcup (\beta, \tau) = \sum_{\mu} (\gamma_\mu, \mu),$$

where the sum runs over all paths μ of length $\chi(n + m)$ containing π and τ as a subpath, and Γ_μ is the result of substituting π and τ by α and β respectively in μ. In particular, $(\alpha, \pi) \sqcup (\beta, \tau) = 0$ if $l(\alpha) + l(\beta) > 1$.

From the description of the cup product of Hochschild cohomology ring of a truncated quiver algebra, it is easy to see that the cup product of two odd-degree cohomology classes is zero, and if $f_1, \ldots, f_d$ are cohomology classes of positive degree, then the product $f_1 \sqcup \cdots \sqcup f_d = 0$. In particular, $\mathrm{HH}^*(A)/\mathcal{N} \cong K$ (where $\mathcal{N}$ is the ideal generated by homogeneous nilpotent elements) and hence the Snashall–Solberg conjecture holds true for truncated quiver algebras [1]. Furthermore, G. Ames, L. Cagliero and P. Tirao completely determined the multiplicative structure of Hochschild cohomology rings of two large classes of truncated quiver algebras, see [1] for details.

4.2. Artin–Schelter regular algebras

Regular algebras introduced by Artin and Schelter have been classified in the case of global dimension 3 in [3]. The Hochschild (co)homology, de Rham cohomology, cyclic and periodic cyclic homologies were computed by Michel van den Bergh for the AS-regular algebras of type A with quadratic relations in [21] and by N. Marconnet for these algebras with cubic relations in [18].

Let K be a field of characteristic zero. Recall that an AS-regular algebra of global dimension 3 of type A, with cubic relations is a graded algebra A of the form

$$A = K\langle x, y\rangle/(f_1, f_2)$$

with

$$f_1 = axy^2 + byxy + ay^2x + cx^3, \qquad f_2 = ayx^2 + bxyx + ax^2y + cy^3$$

where $(a : b : c) \in \mathbb{P}^2K\backslash S$ with $S = \{(a : b : c) \in \mathbb{P}^2K | a^2 = b^2 = c^2\} \cup \{(0 : 0 : 1),\ (0 : 1 : 0)\}$, and x, y have degree 1. Moreover, A has a central element of degree 4

$$C = b(c^2 - a^2)yxyx + a(a^2 - b^2)yx^2y - a(c^2 - a^2)x^2y^2 - c(a^2 - b^2)x^4.$$

Let $A_0 = K,\ A_1 = V$ be the vector space generated by x and y, R be the linear subspace of $V^{\otimes 3}$ generated by f_1 and f_2 (here we omit the symbol $\otimes$ in the summands). Then $A \cong T_{A_0}(A_1)/(R)$, which is a 3-Koszul algebra and has a minimal projective A^e-resolution $(\mathbb{P},\ d)$

$$0 \longrightarrow A \otimes J_4 \otimes A \xrightarrow{d_3} A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \longrightarrow 0 \tag{11}$$

where J_4 is the subspaces generated by $w = x \otimes f_1 + y \otimes f_2 = f_1 \otimes x + f_2 \otimes y$. d_0 is the multiplication map, and

$$\begin{aligned}
d_1(1 \otimes x \otimes 1) &= x \otimes 1 - 1 \otimes x; \qquad d_1(1 \otimes y \otimes 1) = y \otimes 1 - 1 \otimes y;\\
d_2(1 \otimes f_1 \otimes 1) &= 1 \otimes ax \otimes y^2 + 1 \otimes by \otimes xy + 1 \otimes ay \otimes yx + 1 \otimes cx \otimes x^2\\
&\quad + ax \otimes y \otimes y + by \otimes x \otimes y + ay \otimes y \otimes x + cx \otimes x \otimes x\\
&\quad + axy \otimes y \otimes 1 + byx \otimes y \otimes 1 + ay^2 \otimes x \otimes 1 + cx^2 \otimes x \otimes 1;\\
d_2(1 \otimes f_2 \otimes 1) &= 1 \otimes ay \otimes x^2 + 1 \otimes bx \otimes yx + 1 \otimes ax \otimes xy + 1 \otimes cy \otimes y^2\\
&\quad + ay \otimes x \otimes x + bx \otimes y \otimes x + ax \otimes x \otimes y + cy \otimes y \otimes y\\
&\quad + ayx \otimes x \otimes 1 + bxy \otimes x \otimes 1 + ax^2 \otimes y \otimes 1 + cy^2 \otimes y \otimes 1;\\
d_3(1 \otimes w \otimes 1) &= x \otimes f_1 \otimes 1 + y \otimes f_2 \otimes 1 - 1 \otimes f_1 \otimes x - 1 \otimes f_2 \otimes y.
\end{aligned}$$

Applying the functor $\mathrm{Hom}_{A^e}(-, A)$ to the resolution (11), and noting that $\mathrm{Hom}_{A^e}(A \otimes X \otimes A,\ A) \cong \mathrm{Hom}_K(X,\ A)$ for any linear space X, we obtain the cochain complex

$$0 \longrightarrow A \xrightarrow{d^1} \mathrm{Hom}_K(V,\ A) \xrightarrow{d^2} \mathrm{Hom}_K(R,\ A) \xrightarrow{d^3} \mathrm{Hom}_K(J_4,\ A) \longrightarrow 0.$$

It was shown in [18] that $\mathrm{HH}^0(A) = K[C]$ (here C is the central element of A), $\mathrm{HH}^1(A)$ is the free $k[C]$-module of rank 1 generated by the Euler derivation D, which is defined on each homogeneous component A_n of A by $D|_{A_n} = nid_{An}$; and $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$ are free $k[C]$-modules of rank 9.

It is clear that the cup product of $\mathrm{HH}^m(A)$ and $\mathrm{HH}^n(A)$ is trivial for $m,\ n \geq 2$. If $\eta \in \mathrm{Hom}_{A^e}(A \otimes R \otimes A,\ A) \cong \mathrm{Hom}_K(R,\ A)$ represents an element in $\mathrm{HH}^2(A)$, then, by Theorem 3.2, we have

$$\begin{aligned}
&(D \sqcup \eta)(w) = D(x)\eta(f_1) + D(y)\eta(f_2),\\
&(\eta \sqcup D)(w) = \eta(f_1)D(x) + \eta(f_2)D(y).
\end{aligned}$$

Thus the multiplicative structure of Hochschild cohomology ring of a cubic AS-regular algebra of type A is determined.

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