Periodic Solutions of Duffing’s Equations with Superquadratic Potential

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This paper is devoted to the study of harmonic and subharmonic solutions for the second order scalar nonlinear Duffing’s equation \( x'' + g(x) = p(t, x, x') \), where \( g \) and \( p \) are continuous functions with \( p \) bounded and periodic in the first variable and \( g \) satisfying the assumption \( g(x) \) \( \text{sign}(x) \rightarrow +\infty \), as \( |x| \rightarrow +\infty \). Among other results, we prove the existence of infinitely many harmonic and subharmonic solutions (of any order) if \( p = p(t) \) and if the potential \( G(x) \) of \( g(x) \) satisfies certain conditions of superquadratic growth at \( \infty \). The new existence results can be applied to situations in which the more classical superlinear growth condition \( g(x)/x \rightarrow +\infty \), as \( |x| \rightarrow +\infty \), is not satisfied. In this manner, various preceding theorems are improved and sharpened (see the “Introduction” for more details). Proofs are based on a generalized version of the Poincaré–Birkhoff “twist” theorem due to W. Ding. © 1992 Academic Press, Inc.

1. INTRODUCTION

The problem of existence and multiplicity of \( T \)-periodic solutions for the second order nonlinear ordinary differential equation

\[
x'' + g(x) = p(t),
\]

(Duffing’s equation) has been widely investigated in the literature because of its significance for the applications as well as for its intrinsic interest as

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a good model for testing the effectiveness of various technical tools of non-linear analysis.

Throughout the paper, we suppose that \( g: \mathbb{R} \to \mathbb{R} \) is a continuous function and \( p: \mathbb{R} \to \mathbb{R} \) is \( T \)-periodic, where \( T > 0 \) is a fixed period.

One of the major difficulties which is encountered when the nonlinear restoring term \( g(x) \) grows faster than \( x \) at infinity, lies in the fact that in this case, there are no a priori bounds for the periodic solutions.

In [20], S. Fučík and V. Lovicar obtained the existence of at least one \( T \)-periodic solution of equation (D), for any \( T \)-periodic forcing term \( p \in L^1_{\text{loc}} \), assuming the superlinear growth condition

\[
\lim_{|x| \to +\infty} \frac{g(x)}{x} = +\infty.
\]

Their proof, based on a fixed point theorem for the Poincaré map associated to a planar system equivalent to (D), was first extended by Struwe in [40] and, subsequently (with a similar approach), by Shekhter in [39], for the solvability of the equation

\[
x'' + g(x) = p(t, x, x'),
\]

with \( p(\cdot, \cdot, \cdot) \) possibly unbounded in \( x' \). A further generalization of the theorems of Fučík and Lovicar and Struwe has been recently obtained by Capietto, Mawhin, and Zanolin in [6], using a continuation approach which allows extensions of the existence results to systems as well.

In [14], W. Ding, generalizing preceding theorems of Cesari, Harvey, Morris, and Micheletti (see, respectively, [7, 24, 32, 31]), proved the existence of infinitely many \( T \)-periodic solutions of arbitrarily large norm in \( C^1[0, T] \), for Eq. (D), assuming, besides the above considered superlinear growth condition on \( g \), that \( g \) is locally lipschitzian and \( p \) is continuous (in order to obtain the uniqueness of the solutions for the Cauchy problems). The proof of such result is based on an extended version of the Poincaré–Birkhoff “twist” theorem which is contained in the same paper (see [14]).

Another multiplicity result was obtained, using variational methods, by A. Bahri and H. Berestycki in [2] for differential systems in \( \mathbb{R}^n \), assuming \( g \) continuously differentiable, \( p \in L^1_{\text{loc}} \) and the growth condition

\[
0 < G(x) \leq \kappa(g(x)|x|) , \quad \forall |x| \geq R > 0 ,
\]

with \( 0 < \kappa < \frac{1}{2} \) and \( g(x) = \text{grad} G(x) \).

However, it can easily be seen (after an integration), that the above assumption implies that there are positive constants \( a \) and \( b \), such that

\[
a |x|^{1/\kappa} - b \leq G(x) , \quad \forall x \in \mathbb{R}^n.
\]
Thus, in the one-dimensional case, the condition of Bahri and Berestycki implies
\[
\lim_{|x| \to +\infty} \frac{(g(x) \text{sign}(x))}{|x|^2} = \lim_{|x| \to +\infty} \frac{G(x)}{|x|^{2+1}} = +\infty,
\]
for some \(\alpha > 1\),

so that, for the scalar Eq. (D), such result is strictly contained in the previously mentioned theorem of W. Ding (except for what concerns the different regularity assumptions on \(g\) and \(p\)).

Let us consider now the second order autonomous equation associated to PI,

\[
x'' + g(x) = 0, \quad (A)
\]

and assume
\[
\lim_{|x| \to +\infty} g(x) \text{sign}(x) = +\infty. \quad (g_0)
\]

In this case, we have that for each sufficiently large positive constant \(c\), there is a unique solution \(x_c(\cdot)\) of (A), satisfying the initial condition \(x_c(0) = c, \quad x_c'(0) = 0\).

Moreover, it can be seen that such solution \(x_c(\cdot)\) (for \(c\) large), is a (non-trivial) periodic function whose minimal period, denoted by \(\tau_0(c)\) can be computed by the formula
\[
\tau_0(c) = \sqrt{2} \int_{c^{-}}^{c} \frac{d\xi}{\sqrt{G(c) - G(\xi)}},
\]
where \(c^{-} := \min\{x_c(t): t \in \mathbb{R}\}\), and \(G'(x) = g(x)\).

Hence, it turns out that a simple consequence of the (above considered) superlinear growth condition of Fučík and Lovicar and W. Ding is that
\[
\lim_{c \to +\infty} \tau_0(c) = 0. \quad (\tau_0)
\]

On the other hand, it is clear that the assumption \((\tau_0)\) (paired with \((g_0)\)) is itself sufficient to guarantee the existence of a sequence of infinitely many \(T\)-periodic solutions to Eq. (A) (for any fixed period \(T\)). Then, the natural problem arises whether it is possible to obtain the same conclusion for the more general forced equation (D).

Another independent but related question consists in the search of (non-trivial) subharmonic solutions for (D). This latter problem is not solved in the previously quoted papers of W. Ding and of Bahri and Berestycki, where only the case of harmonic solutions is considered.
In the present work, we give a positive answer to both such problems by means of the following result.

**THEOREM.** Assume \((g_0)\) and \((\tau_0)\). Then, for every \(p \in L^1_{\text{loc}}\) and for each integer \(m \geq 1\), there is a sequence \(\{x_j(\cdot)\}\) of \(mT\)-periodic solutions (which are not \(kT\)-periodic, for \(1 \leq k < m\)) for Eq. \((D)\), such that

\[
\lim_{j \to +\infty} \min_{t \in \mathbb{R}} \{|x_j(t)| + |x'_j(t)|\} = +\infty.
\]

*In particular, if \(p(\cdot)\) has \(T > 0\) as minimal period, then, for each \(m \geq 1\), there are infinitely many periodic solutions of Eq. \((D)\), having \(mT\) as minimal period.*

The proof of this theorem follows from some steps which are contained in different sections, in order to clarify the exposition. More precisely, the plan of the paper is the following.

In Section 2, we introduce some technical tools and present all the auxiliary results. In particular, we develop a phase-plane analysis for a class of systems of the form

\[
x' = y, \quad y' = -g(x) + q(t, x, y),
\]

where \(q(\cdot, \cdot, \cdot)\) belongs to a set \(P\) of continuous functions which are all bounded by a suitable constant \(M\). This approach permits us (via a standard approximation and compactness argument) to obtain periodic solutions for the given differential equation, whenever periodic solutions are achieved for sufficiently regular (e.g., locally lipschitzian) differential equations which belong to the same class \(P\), provided that the needed estimates depend only on \(M\).

Then, in Section 3, we give a general existence and multiplicity result for \(T\)-periodic solutions of the Duffing's equation, in which, as a main hypothesis, we have a condition on the number of the rotations of the solutions of the considered planar system. The proof is based on the search of fixed points for the Poincaré's operator and makes use of the results by Fučík and Lovičar and W. Ding.

In Section 4, we state the main result of the paper (Theorem 2) which guarantees (under the assumptions \((g_0)\) and \((\tau_0)\)) the existence of infinitely many \(T\)-periodic solutions for Eq. \((D)\), for every continuous \(p(\cdot)\). The possibility of more general forcing terms, like \(p = p(t, x, x')\), is considered as well. However, in this situation, we can ensure just the existence of at least one solution, since no multiplicity result is available in the general case of a dependence of \(p\) on \(x'\) (in this connection, see the counterexamples contained in [23, 40]).
Clearly, Theorem 2 generalizes the previously mentioned theorem of W. Ding [14], principally in the use of the new condition \((\tau_0)\), but also by allowing less regularity on the function \(g\).

Then, we give some corollaries in which more explicit assumptions on \(G(x)\) are proposed in order to have the validity of \((\tau_0)\). In this manner, we also produce some examples which show the applicability of our result even in situations in which

\[
\lim \inf_{|x| \to +\infty} \frac{g(x)}{x} < \lim \sup_{|x| \to +\infty} \frac{g(x)}{x} = +\infty.
\]

We finally end the section by a discussion about the relationships between \((\tau_0)\) and other superquadratic growth conditions on \(G(x)\). Thus, as a by-product of our investigation of the properties of the time-map for Eq. (A), we also obtain some improvements of preceding results by Z. Opial in [36].

After Section 5, which is completely devoted to the proof of Theorem 2, we consider, in Section 6, the case of subharmonic solutions for Eq. (D). Accordingly, we need to extend our main tool, which is Corollary 1 of Section 3, to a theorem which guarantees the existence of fixed points for the \(m\)th iterate of the Poincaré map, which are not fixed points for the \(k\)th iterate of such operator, for any \(k < m\). Clearly, this is the only delicate point of the proof, since all the rest is just a straightforward repetition of the arguments developed through the preceding sections. To this end, we use in a crucial way the generalized Poincaré–Birkhoff “twist” theorem due to W. Ding in [14].

We observe that the result concerning the existence of infinitely many subharmonic solutions is new, even with respect to the use of the more classical superlinear growth restriction on \(g(x)/x\) (which is a particular case of \((\tau_0)\)). In this connection, we recall a recent paper by F. Nakajima [34], where a result of this type is achieved under the more restrictive assumption on \(g(x)/x\) and with supplementary conditions on \(p(t)\) or \(g(x)\) (e.g., \(p\) even, or \(p\) odd and \(g\) odd).

Section 6 ends with a final remark concerning the possibility of extending all the results to the case of \(p \in L^1_{\text{loc}}\). Such generalization is only outlined, since it comes by just adapting the main arguments to the equivalent system

\[
x' = y + \bar{P}(t), \quad y' = -g(x) + \bar{p},
\]

where

\[
\bar{p} := \frac{1}{T} \int_0^T p(t) \, dt \quad \text{and} \quad \bar{P}(t) := \int_0^t (p(s) - \bar{p}) \, ds.
\]
Anyhow, for sake of completeness, we note, step by step, which are the pertinent changes which have to be performed in the corresponding proofs in order to check that everything still works in such slightly more general situation.

We remark that all the existence and multiplicity results are obtained without the standard regularity hypotheses (e.g., \( p(t) \) continuous and \( g(x) \) locally Lipschitzian) which are usually assumed using phase-plane analysis techniques.

Putting together all these partial steps, we are finally able to guarantee the validity of the theorem stated before (see Theorems 5 and 6 in Section 6).

The paper ends with an Appendix (Section 7) where the main tools (from [20, 14]) are recalled for the reader's convenience.

2. SETTING OF THE PROBLEM AND TECHNICAL RESULTS

We consider the problem of existence and multiplicity of \( T \)-periodic \textit{(harmonic)} and \( mT \)-periodic \textit{(subharmonic)} solutions for the second order scalar Duffing's equation

\[
x'' + g(x) = p(t, x, x'),
\]

where \( g: \mathbb{R} \to \mathbb{R} \) and \( p: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions, with \( p \), \( T \)-periodic in the first variable (\( T > 0 \) is fixed).

Throughout the paper, the following assumptions are made

\[
\lim_{|x| \to \infty} g(x) \text{sign}(x) = +\infty, \tag{g_0}
\]

\[
\exists M_0 > 0: |p(t, x, y)| \leq M_0, \quad \text{for each } t \in [0, T] \text{ and } x, y \in \mathbb{R}. \tag{p_0}
\]

Given \( M_0 \) as in \( p_0 \), we fix a constant \( M \), with

\[
M > M_0,
\]

and denote by \( p \) the set of all continuous functions \( q: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) which are \( T \)-periodic in the first variable and satisfy the boundedness condition

\[
|q(t, x, y)| \leq M, \quad \text{for each } t \in [0, T] \text{ and } x, y \in \mathbb{R}. \tag{2}
\]

By definition, we clearly have \( p \in p \). We also set

\[
G(x) := \int_0^x g(s) \, ds,
\]
and observe that, from \((g_0)\), we have that \(G(x) \to +\infty\), as \(|x| \to +\infty\). Hence \(G\) is bounded from below and achieves its absolute minimum

\[
G_{\text{min}} := \min\{G(x) : x \in \mathbb{R}\}
\]
on \(\mathbb{R}\).

In what follows, we perform a phase-plane analysis for the first order differential system

\[
x' = y, \quad y' = -g(x) + q(t, x, y),
\]

with \(q \in \mathbb{R}\), which is equivalent to the equation

\[
x'' + g(x) = q(t, x, x').
\]

Any solution \(z(t) = (x(t), y(t))\) of (3) is supposed to be noncontinuable, i.e., defined on a maximal interval of existence. Moreover, from \((g_0)\) and the boundedness of \(q\), we can show that global existence occurs, that is, \(z(t)\) is defined for all \(t \in \mathbb{R}\).

Indeed, to prove this claim, we introduce the Liapunov-like function

\[
V(a, b) = (G(a) - G_{\text{min}}) + \frac{1}{2}(b^2 + 1)
\]
and observe that \(V(a, b) > 0\), for each \((a, b) \in \mathbb{R}^2\) and \(V(a, b) \to +\infty\) as \(a^2 + b^2 \to +\infty\). Then, we consider the derivative of \(V\) along a solution \(z(t) = (x(t), y(t))\) of Eq. (3). Accordingly, we set

\[
v(t) := V(z(t)) = V(x(t), y(t)), \quad \text{with} \quad v(t_0) = v_0 := V(z(t_0))
\]
and we evaluate \(v'(t)\). By standard inequalities, we obtain, using (2),

\[
|v'(t)| = |g(x(t)) x'(t) + y(t) y'(t)|
= |q(t, x(t), y(t))| \cdot |y(t)| \leq M |y(t)| \leq Mv(t).
\]

Hence, using elementary differential inequalities, we have that

\[
v(t) \leq v_0 \exp(M |t - t_0|), \quad \text{(5)}
\]
for all \(t\) in a maximal interval containing \(t_0\), where \(z(t)\) is defined.

In this manner, we have proved that \(V(z(t))\) is bounded for \(t\) in any bounded interval. Thus we see that there is no blow-up at finite time and the global existence follows from the fundamental theory of ODEs (see [22]). Q.E.D.
Let

\[ T_0 \geq T, \]

be an arbitrary (but fixed) constant. Thanks to the above remark, from now on, every solution

\[ z(t) = (x(t), y(t)) \]

of system (3), will be supposed to be defined on the interval \([0, T_0] \supseteq [0, T]\). However, we do not assume, unless when explicitly stated, the uniqueness of the solutions for the Cauchy problems.

Proceeding further with the previous argument, we also can prove the following:

**Lemma 1.** For each \( R_1 > 0 \), there is \( R_2 \geq R_1 \), such that, for any \( q \in \mathcal{P} \) and any \( z(\cdot) \) solution of (3), the following conclusions hold:

\[
|z(0)| \leq R_1 \Rightarrow |z(t)| \leq R_2, \quad \text{for all} \quad t \in [0, T_0],
\]

\[
|z(0)| \geq R_2 \Rightarrow |z(t)| \geq R_1, \quad \text{for all} \quad t \in [0, T_0].
\]

By \(|w| = |(a, b)| := \sqrt{a^2 + b^2}\), we denote the usual euclidean norm of the point (vector) \( w = (a, b) \in \mathbb{R}^2 \); \( B(R) \) and \( B[R] \) are, respectively, the open and closed discs in the plane (i.e., two-dimensional balls), with center the origin \( O \) and radius \( R > 0 \).

**Proof.** Let \( R_1 > 0 \) be given. We set, for \( w = (a, b) \in \mathbb{R}^2 \),

\[
c_1 := 1 + \sup \{ V(w) : |w| \leq R_1 \}, \quad c_2 := c_1 \exp(MT_0),
\]

\[
R_2 := 1 + \sup \{ |w| : V(w) \leq c_2 \}.
\]

Then, using (5), it is easy to check that the constant \( R_2 \) above defined is suitable for the fulfillment of Lemma 1. Q.E.D.

(For analogous computations, see [34, Lemma 1]).

From Lemma 1, we immediately obtain that there is a constant \( r_0 > 0 \), such that, for each \( q \in \mathcal{P} \) and each \( z(\cdot) \) solution of (3), with

\[
|z(0)| \geq r_0,
\]

we have

\[ z(t) \neq O, \quad \text{for every} \quad t \in [0, T_0], \]

where \( O = (0, 0) \), is the origin in \( \mathbb{R}^2 \).
Then we can introduce a system of polar coordinates \((\theta, \rho)\), such that any solution \(z(\cdot) = (x(\cdot), y(\cdot))\) of (3), with \(q \in \mathbb{P}\), satisfying (6), may be written as

\[
x(t) = \rho(t) \cos \theta(t), \quad y(t) = \rho(t) \sin \theta(t),
\]

where

\[
\rho(t) = |z(t)|, \quad \text{and} \quad \theta(t) = \text{Arg}\left(\frac{x(t) + i y(t)}{\rho(t)}\right).
\]

More precisely, let \(H^+ = \{(\theta, \rho); \rho > 0\}\) be the upper half-plane in the \((\theta, \rho)\)-plane and let \(\Pi: H^+ \to \mathbb{R}^2 \setminus \{0\}\) be the covering projection given by \(\Pi(\theta, \rho) = (\rho \cos \theta, \rho \sin \theta)\). Then, the function \(z(\cdot): [0, T_0] \to \mathbb{R}^2 \setminus \{0\}\), admits a lifting \((\theta(\cdot), \rho(\cdot)): [0, T_0] \to H^+\), such that,

\[
\Pi(\theta(t), \rho(t)) = z(t), \quad \forall t \in [0, T_0].
\]

Note that, with respect to the fixed covering projection \(\Pi\), if \((\theta_1(\cdot), \rho_1(\cdot))\) is another lifting of \(z(\cdot)\), then, \(\rho_1(t) = \rho(t)\) and \(\theta_1(t) = \theta(t) + 2k\pi\), for some \(k \in \mathbb{Z}\) (see, e.g., [21]).

With the above positions, \(\rho(\cdot)\) and \(\theta(\cdot)\) are differentiable functions and, from system (3), we obtain

\[
\theta'(t) = -\frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2},
\]

\[
= -\sin^2 \theta(t) - \frac{1}{\rho^2(t)} \left( g(x(t))x(t) - q(t, x(t), y(t))x(t) \right),
\]

for all \(t \in [0, T_0]\). (7)

In what follows, we denote, respectively, by

\[
\Theta(t; z) \quad \text{and} \quad q(t; z),
\]

the polar coordinates (angular and radial), evaluated at the time \(t \in [0, T_0]\), which are associated (as described above) to a solution \(z(\cdot)\) of Eq. (3), for some \(q \in \mathbb{P}\), with \(|z(0)| \geq r_0\). However, the symbol \(z\) in parenthesis, will be omitted when no confusion occurs. (Once more, we note that uniqueness of the solutions for the Cauchy problems is not assumed at this stage of the paper.)

**Lemma 2.** There is a constant \(d > r_0\), such that, for any \(q \in \mathbb{P}\) and any \(z(\cdot)\) solution of (3), we have

\[
|z(0)| = r \geq d \Rightarrow \frac{d}{dt} \Theta(t; z) < 0, \quad \text{for all} \quad t \in [0, T_0].
\]
Moreover, there exists a nondecreasing and continuous function \( \beta : [d, +\infty) \to \mathbb{R}^+ = ]0, +\infty) \), such that, for any \( q \in \mathcal{P} \) and any \( z(\cdot) \) solution of (3), we have

\[
|z(0)| = r \geq d \Rightarrow \frac{d}{dt} \Theta(t; z) \geq -\beta(r), \quad \text{for all } t \in [0, T_0].
\]

Roughly speaking, Lemma 2 asserts that the solutions of (3) turn clockwise around the origin (as \( t \) increases), provided that the initial points are sufficiently far from the origin. Moreover, the number of such revolutions, during the time \( T_0 \), is upper bounded, provided that the initial points belong to a compact set.

Lemma 2 is an almost standard consequence of the global existence of solutions (i.e., Lemma 1), properties \((g_0)\) and \((2)\), and formula \((7)\); accordingly, it is more or less explicitly contained in various preceding papers dealing with similar problems (see, e.g., \([23, 20, 40, 14, 8, 13] \)). We give just a sketch of the proof for the reader's convenience.

**Proof.** From \((g_0)\) it follows that there is a constant \( L \geq 1 \), such that

\[
(g(a) - q(t, a, b))a \geq g(a) a \quad M |a| \geq |a| - L,
\]

holds for every \( t \in [0, T_0] \), \((a, b) \in \mathbb{R}^2 \) and each \( q \in \mathcal{P} \). Let \( z(\cdot) \) be a solution of (3) (for a given \( q \in \mathcal{P} \)), with \( |z(0)| \geq r_0 \) and consider the associated angular function

\[
\theta(t) := \Theta(t; z),
\]

and radial function

\[
\rho(t) := \varrho(t; z),
\]

for \( t \in [0, T_0] \). From \((7)\) and \((8)\), we obtain

\[
\theta' \leq -\sin^2 \theta - \frac{1}{\rho} |\cos \theta| + \frac{L}{\rho^2} \leq - \left( 1 - \frac{1}{\rho} \right) (1 - |\cos \theta|) - \frac{\rho - L}{\rho^2} ,
\]

so that \( \theta'(t) < 0 \) on \([0, T_0]\), provided that \( \rho(t) > L \geq 1 \), for all \( t \in [0, T_0] \). Now, we employ Lemma 1 according to which there is a constant \( d > r_0 \) such that \( |z(t)| \geq L + 1 \), for all \( t \in [0, T_0] \), provided that \( |z(0)| \geq d \) (use the second inference in Lemma 1, with \( R_2 = d \), for \( R_1 = L + 1 \)). With such choice of the constant \( d \), the first part of the lemma is proved.

Second, we define, for each \( r \geq d \),

\[
\gamma(r) := \max \{|g(s)|: -r \leq s \leq r\}
\]
and recall that, by the above discussion, \( \rho(t) = q(t; z) > L \), when \( |z(0)| = r \geq d \). Hence, from (7), by standard inequalities, we obtain for each \( q \in P \) and \( z(\cdot) \) solution of (3), with \( |z(0)| = r \geq d \),

\[
\theta' \geq -\sin^2 \theta - \frac{1}{\rho} (|g(x)| + |q(t, x, y)|) \geq -1 - L^{-1}(|g(x)| + M)
\]

\[
\geq -\alpha(r) := -(1 + L^{-1}(\gamma(R(r)) + M)),
\]

where \( R(r) \geq r \) is such that \( |z(t)| = \rho(t) = q(t; z) \leq R(r) \), \( \forall t \in [0, T_0] \), if \( |z(0)| \leq r \) (according to the first inference in Lemma 1). Without restriction (from the proof of Lemma 1), we can suppose that the map \( r \mapsto R(r) \) is nondecreasing and therefore \( \alpha: [d, + \infty) \rightarrow R^+ \) is nondecreasing, too.

Finally, we take \( \beta: [d, + \infty) \rightarrow R^+ \), a continuous and nondecreasing function such that \( \alpha(r) \leq \beta(r) \), \( \forall r \geq d \). For such a choice of \( \beta \), we have that \( \theta'(t) \geq -\beta(r) \) on \([0, T_0]\) and thus the proof of Lemma 2 is complete.

Q.E.D.

Our program is now the following. We find \( T \)-periodic solutions (harmonics) for Eq. (1) as fixed points of the Poincaré map, which, in our case, is a multivalued function. To this end, we consider a family of approximating equations like (3), with forcing term \( q \in P \), such that the unique solvability of the associated Cauchy problems is ensured. Moreover, we have that the solutions of (3) are defined on the interval \([0, T]\) and we can apply the previous lemmas, since \( T_0 \geq T \). On the other hand, in the search of \( mT \)-periodic solutions (subharmonics of order \( m \)) for Eq. (1), we follow a similar procedure, except that we have to deal with the \( m \)th iterate of the Poincaré map. In this case, we require \( T_0 \geq mT \), in order to fit into the hypotheses of Lemmas 1 and 2.

Accordingly, for sake of clarifying the subsequent exposition, we first consider the \( T \)-periodic boundary value problem. The case of subharmonic solutions is then treated in a final section, where we outline the relevant changes with respect to the main argument.

3. Fixed Points for the Poincaré's Operator

Through this section, we consider the same framework of Section 2, with

\[
T_0 := T.
\]

Correspondingly, we recall the constants \( r_0 \) and \( d \), with \( 0 < r_0 < d \), coming from (6) and Lemma 2.
By virtue of Lemma 2, we are now in a position to define, for each \( r \in [d, + \infty) \), the following nonnegative integers \( n_*(r) \) and \( n^*(r) \) which play a special role in our existence results.

Let \( Z_+ \) denote the set of nonnegative integers. For any \( r \geq d \), we set

\[
\begin{align*}
n_*(r) &:= \max \left\{ n \in Z_+: n \leq \inf \left\{ \frac{\| \Theta(T; z) - \Theta(0; z) \|}{2\pi}: |z(0)| = r \right\} \right\}, \\
n^*(r) &:= \min \left\{ n \in Z_+: n \geq \sup \left\{ \frac{\| \Theta(T; z) - \Theta(0; z) \|}{2\pi}: |z(0)| = r \right\} \right\},
\end{align*}
\]

where the \( \inf \{ \ldots \} \) is taken over all \( q \in \mathbb{P} \) and all solutions \( z(\cdot) \) of (3);

\[
\begin{align*}
n_*(r) &:= \max \left\{ n \in Z_+: n \leq \inf \left\{ \frac{\| \Theta(T; z) - \Theta(0; z) \|}{2\pi}: |z(0)| = r \right\} \right\}, \\
n^*(r) &:= \min \left\{ n \in Z_+: n \geq \sup \left\{ \frac{\| \Theta(T; z) - \Theta(0; z) \|}{2\pi}: |z(0)| = r \right\} \right\},
\end{align*}
\]

where the \( \sup \{ \ldots \} \) is taken over all \( q \in \mathbb{P} \) and all solutions \( z(\cdot) \) of (3).

From these definitions and Lemma 2, it is clear that, for each \( r \in [d, + \infty) \), we have

\[
0 \leq n_*(r) \leq n^*(r) \leq 1 + \beta(r) T/2\pi.
\]

The geometric meaning of the integers \( n_* \) and \( n^* \) is the following: every solution of (3), for any \( q \in \mathbb{P} \), with initial value on the circle \( \sqrt{a^2 + b^2} = r \), makes at least \( n_*(r) \) and at most \( n^*(r) \) clockwise rotations around the origin, during the time \( T \).

It is important to remark that the integers \( n_*(r) \) and \( n^*(r) \) do not depend on the particular choice of the polar coordinates system, since a different lifting of \( z(t) \) to \( H^+ \) does not change the value of \( |\Theta(T; z) - \Theta(0, z)| \). This claim can also be checked directly, by observing that

\[
|\Theta(T; z) - \Theta(0; z)| = \left| \int_0^T \frac{y'(t) x(t) - x'(t) y(t)}{x(t)^2 + y(t)^2} \, dt \right|,
\]

for \( z(t) = (x(t), y(t)) \).

Next, we give a result for the existence of \( T \)-periodic solutions to Eq. (1). For the proof, we find fixed points for the Poincaré's operator associated to the planar system

\[
x' = y, \quad y' = -g(x) + p(t, x, y) \quad (3_p)
\]

which is equivalent to (1). However, since we do not assume the unique solvability for the corresponding Cauchy problems, we have to work with a family of approximating equations satisfying the uniqueness property for the initial value problems. The validity of such approach is then ensured by the fact that all the estimates obtained in Lemmas 1 and 2 are uniform with respect to \( q \in \mathbb{P} \).
For each $k \in \mathbb{N}$, let $g_k : \mathbb{R} \to \mathbb{R}$ and $p_k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous locally lipschitzian functions, with $p_k$, $T$-periodic in the first variable, such that $g_k \to g$ and $p_k \to p$ (as $k \to +\infty$), uniformly on the compact subsets of their respective domains. (The existence of such $g_k$ and $p_k$ follows from the Weierstrass–Stone approximation theorem [38]).

Then we define

$$q_k(t, x, y) := \delta_M \left( g(x) - g_k(x) + p_k(t, x, y) \right),$$

where, for all $s \in \mathbb{R},$

$$\delta_M(s) := \max \{-M, \min \{s, M\}\}.$$

By construction, we have that $q_k \in \mathcal{P}$, $\forall k \in \mathbb{N}$ and $q_k \to p$ (as $k \to +\infty$), uniformly on the compact subsets of $\mathbb{R}^3$. Moreover, for any $R > 0$, there is

$$k_1 = k_1(R) \in \mathbb{N},$$

such that, for every $k \geq k_1$, we obtain

$$q_k(t, a, b) = g(a) - g_k(a) + p_k(t, a, b),$$

for all $t \in \mathbb{R}$ and $(a, b) \in B[R]$. (Recall that $|p(t, a, b)| \leq M_0 < M$, by $(p_0)$).

**Lemma 3.** For any compact set $K \subset \mathbb{R}^2$, there is $k_2 = k_2(K) \in \mathbb{N}$, such that, for every $k \geq k_2$ and $z_0 = (x_0, y_0) \in K$, there is a unique solution $z_k(\cdot; z_0) = (x_k(\cdot; z_0), y_k(\cdot; z_0))$ on $[0, T]$, to equation

$$x' = y, \quad y' = -g(x) + q_k(t, x, y), \quad (3_k)$$

satisfying the initial condition $z_k(0; z_0) = z_0$.

**Proof.** Let $R_1 > 0$ be such that $K \subset B[R_1]$ and let $R_2 \geq R_1$, be a constant corresponding to $R_1$, according to Lemma 1. Then $z(0) \in K$ implies that $|z(t)| \leq R_2$ on $[0, T]$, for all possible solutions $z(\cdot)$ of $(3)$, for any $q \in \mathcal{P}$. Now we take

$$k_2 := k_1(R_2),$$

with $k_1$ previously defined.

Clearly, if $k \geq k_2$ and $z(\cdot)$ is any solution of $(3_k)$ with $z(0) = z_0 \in K$, then $z(t) \in B[R_2]$ for all $t \in [0, T]$ and so, $z$ is a solution of

$$x' = y, \quad y' = -g_k(x) + p_k(t, x, y).$$

Then, the uniqueness of $z$ follows from the Lipschitz condition for $g_k$ and $p_k$. Q.E.D.
From Lemma 3, we thus have that, for any compact set \( K \in \mathbb{R}^2 \), the Poincaré map
\[
\varphi_k : K \to \mathbb{R}^2, \quad \varphi_k(w) := z_k(T; w),
\]
associated to system (3\(_k\)), is defined, provided that \( k \) is sufficiently large (namely, \( k \geq k_2(K) \)). Clearly, here \( z_k(\cdot; w) \) denotes the solution of (3\(_k\)) with \( z(0; w) = w \), according to Lemma 3.

In this situation, \( \varphi_k \) is also continuous (by the uniqueness of the solutions for the Cauchy problems). We also note that, if \( q_k \) is independent of \( y \), i.e.,
\[
q_k = q_k(t, x),
\]
then (3\(_k\)) is a Hamiltonian system and so (by Liouville theorem [1]), \( \varphi_k : K \to \varphi_k(K) \) is an area-preserving homeomorphism.

**Lemma 4.** Let \( K \subset \mathbb{R}^2 \) be compact and suppose that there is \( k_0 \in \mathbb{N} \), such that, for every \( k \geq k_0 \), there exists \( w_k \in K \), with \( \varphi_k(w_k) = w_k \). Then Eq. (1) has at least one \( T \)-periodic solution \( x(\cdot) \), with \( (x(0), x'(0)) \in K \).

The proof of Lemma 4 is omitted, since it is a standard application of Lemma 1 and the compactness principle. We only observe that a limit process implies the existence of a solution \( z(\cdot) \) to system (3\(_p\)), with \( z(0) = z(T) \in K \). Then, by the periodicity of \( p \) in the first variable, it is sufficient to extend \( z(\cdot) \) from \([0, T]\) to \( \mathbb{R} \) by \( T \)-periodicity and the statement is proved.

With the notations introduced along this section (the constant \( d \) coming from Lemma 2), we have now the following:

**Theorem 1.** Suppose that there are two constants \( r_1, r_2 \in [d, + \infty) \), with \( r_1 \neq r_2 \), such that
\[
\mathbf{n}(r_2) - \mathbf{n}(r_1) \geq 2. \tag{\text{n}_0}
\]
Then, Eq. (1) has at least one \( T \)-periodic solution \( x(\cdot) \), with
\[
\sqrt{x(0)^2 + x'(0)^2} \leq \max \{r_1, r_2\}.
\]
If, moreover, \( p = p(t, x) \), then Eq. (1) has at least one \( T \)-periodic solution \( x(\cdot) \), with
\[
\min \{r_1, r_2\} \leq \sqrt{x(0)^2 + x'(0)^2} \leq \max \{r_1, r_2\}.
\]

**Proof.** We consider only the case when
\[
r_1 < r_2, \tag{9}
\]
which, by hypothesis \( (n_0) \), corresponds to the situation when the number of the revolutions of the trajectories around the origin increases. Condition (9) actually occurs in all the subsequent applications to the superlinear problem. The study of the case in which \( r_1 > r_2 \) can be performed by easy changes of the argument developed along this proof and therefore it is omitted.

First, we apply Lemma 3 with \( K = B[r_2] \) and thus we find an index \( k_2 = k_2(K) \) such that, for each \( k \geq k_2 \), the uniqueness of the solutions for the Cauchy problems of (3\(_k\)), with initial value in \( B[r_2] \), is guaranteed.

Let \( q_* \in \{ q_k : k \geq k_2 \} \) be fixed and denote by

\[ \varphi : B[r_2] \to \mathbb{R}^2, \quad \varphi(w) := z(T; w), \]

the Poincaré map associated to the system

\[ \begin{align*}
x' &= y, \\
y' &= -g(x) + q_*(t, x, y),
\end{align*} \tag{3*} \]

where \( z(t; w) = (x(t; w), y(t; w)) \) is the unique solution of (3\(_*\)), defined in \([0, T]\), with \( z(0; w) = w \in B[r_2] \). The polar coordinates corresponding to \( z(t; w) \) are denoted, respectively, by \( \Theta(t; w) \) and \( q(t; w) \) and, from the notations introduced for Lemma 2, we have

\[ \Theta(t; w) := \Theta(t; z(\cdot; w)), \quad q(t; w) := q(t; z(\cdot; w)) = |z(t; w)|. \]

In particular, for \( w \in B[r_2] \), we obtain

\[ \varphi(w) = (q(T; w) \cos \Theta(T; w), q(T; w) \sin \Theta(T; w)). \tag{10} \]

Now we observe that, from the definition of \( n_* \) and \( n^* \), and by hypothesis \( (n_0) \), we can find an integer \( n_* \in \mathbb{N} \), with

\[ n^*(r_1) < n_* < n^*(r_2) \]

and a constant \( \varepsilon_0 \), with

\[ 0 < \varepsilon_0 < \frac{1}{2}, \]

such that

\[ |\Theta(T; w) - \Theta(0; w)| \leq 2\pi(n_* - \varepsilon_0), \quad \text{for all } |w| = r_1, \tag{11} \]

\[ |\Theta(T; w) - \Theta(0; w)| \geq 2\pi(n_* + \varepsilon_0), \quad \text{for all } |w| = r_2. \tag{12} \]

(The proof of the existence of such \( \varepsilon_0 \), follows from a standard compactness argument and therefore it is left out).

Then we consider the closed annulus

\[ \mathcal{A} := \{ w \in \mathbb{R}^2 : r_1 \leq |w| \leq r_2 \} = B[r_2] \setminus B(r_1) \]
and the continuous map
\[ \Psi : A \to \mathbb{R}^+, \quad \Psi(w) := |\Theta(T; w) - \Theta(0; w)|. \]

Finally, let \( M \) be the compact subset of \( \mathbb{R}^2 \), defined by
\[ M := \{ w \in A : \varphi(w) = \lambda w, \text{ for some } \lambda \in \mathbb{R} \}. \]

We note that
\[ M \supseteq \{ w \in A : \Psi(w) = 2\pi n, \text{ for some } n \in \mathbb{Z} \}, \]
so that \( M \) is nonempty (by the continuity of \( \Psi \), the intermediate value theorem and conditions (11) and (12)).

We also claim that the origin 0 is contained in a bounded component of the complement of \( M \) in \( \mathbb{R}^2 \). The proof of this claim is omitted, since it makes use of exactly the same argument as in [20] (see also [40, p. 294]).

Now we are in a position to apply a fixed point theorem of S. Fučík and V. Lovišar (see [20; 19, Chap. 39; 40, p. 294]) which ensures the existence of a fixed point \( w_* \) for \( \varphi \), with \( w_* \) belonging to the convex hull of \( M \) (which, in turns, is contained in \( B[r_2] \)). (To be precise, we have to note that Fučík and Lovišar suppose that the map, for which the fixed point is found, is defined in \( \mathbb{R}^* \). However, from the proof of the fixed point theorem in [19, Chap. 39], one can easily check that it is sufficient to require that the map is defined on a closed ball containing the set \( M \) and the origin. This is precisely the case for the map \( \varphi \) in our result). The theorem of Fučík and Lovišar is recalled in the Appendix (Section 7) for the reader’s convenience.

Since \( \varphi \) is the Poincaré map associated to Eq. (3_k) for an arbitrary but fixed \( k \geq k_2 \), we conclude that, for each \( k \geq k_2 \), there exists \( w_k \in B[r_2] \), such that \( \varphi_k(w_k) = w_k \). Then we can use Lemma 4 and the first part of Theorem 1 is proved.

For the second part of the theorem, we suppose that \( p = p(t, x) \) is independent of \( y \). In this case, we can choose a sequence \( p_k = p_k(t, x) \) converging to \( p \) (as \( k \to +\infty \)), uniformly on the compact subsets of \( \mathbb{R}^2 \), with \( p_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), continuous, locally lipschitzian and \( T \)-periodic in the first variable. We also set, for \( g_k \) as above,
\[ q_k(t, x) := \delta_M(g(x) - g_k(x) + p_k(t, x)), \]
so that the sequence of differential Eqs. (3_k) approximating (3_p), takes the form
\[ x' = y, \quad y' = -g(x) + q_k(t, x) \]
and, correspondingly, (3_\varphi) becomes
\[ x' = y, \quad y' = g(x) + q_\varphi(t, x). \]

Then, as observed before,
\[ \varphi: B[r_2] \to \varphi(B[r_2]) \subset \mathbb{R}^2 \]
is an area-preserving homeomorphism and
\[ \varphi: A \to \varphi(A) \subset \mathbb{R}^2 \setminus \{O\}. \]

Moreover, we remark that (as \( r_1 \geq d \)),
\[ \varphi^{-1}(O) \in B(r_1). \]

(Indeed, if \( \varphi(w_0) = z(T; w_0) = 0 \), for some \( w_0 \in \mathbb{R}^2 \), then, by the second inference in Lemma 1 (with \( T_0 = T \)) and the choice \( d > r_0 \), we obtain that \( |\varphi^{-1}(O)| = |z(0; w_0)| = |w_0| < r_0 < d \leq r_1 \).

Finally, we recall (see Lemma 2) that \( (d/dt) \Theta(t; w) < 0 \), for all \( t \in [0, T] \) and \( w \in A \), so that
\[ |\Theta(T; w) - \Theta(0; w)| = \Theta(0; w) - \Theta(T; w), \quad \text{for all} \quad w \in A. \]

Hence, we can assert that (11) and (12) imply the fulfillment of "twist condition" for inner and outer boundaries of \( A \) (see [3, 5, 14]).

Indeed, to be precise, let \( H^+ = \{ (\theta, \rho): \rho > 0 \} \) be the upper half plane in the \( (\theta, \rho) \)-plane and let \( \Pi: H^+ \to \mathbb{R}^2 \setminus \{O\} \) be the previously considered covering projection given by
\[ \Pi(\theta, \rho) = (\rho \cos \theta, \rho \sin \theta). \]

Then, the map \( \varphi: A \to \varphi(A) \subset \mathbb{R}^2 \setminus \{O\} \) admits a lifting \( \varphi^0: A^0 \to \varphi^0(A^0) \subset H^+ \), defined on the universal covering space \( A^0 = \pi^{-1}(A) = \{ (\theta, \rho): r_1 \leq \rho \leq r_2 \} \) of \( A \). (See also [21].)

We observe that, in our situation, it is possible to choose a lifting \( \varphi^0 \) of \( \varphi \) defined by
\[ \varphi^0(\theta, \rho) = (\theta_T, \rho_T), \quad \text{with} \quad \theta_T = \theta + h(\theta, \rho), \quad \rho_T = f(\theta, \rho) \]
and
\[ h(\theta, \rho) = \Theta(T; \Pi(\theta, \rho)) - \Theta(0; \Pi(\theta, \rho)) + 2\pi n_\varphi, \quad f(\theta, \rho) = |\varphi(\Pi(\theta, \rho))|. \]
Clearly, $h$ and $f$ are continuous functions defined in $A^0$ and $2\pi$-periodic in the $\theta$-variable. Moreover, we note that the above choice of $\varphi^0$ actually gives a lifting of $\varphi$ to $A^0$, since we have

$$\Pi \circ \varphi^0 = \varphi \circ \Pi.$$ 

Then (11) and (12) read respectively, as

$$h(\theta, r_1) \geq 2\pi v_0 > 0, \quad \forall \theta \in [0, 2\pi],$$

$$h(\theta, r_2) \leq -2\pi v_0 < 0, \quad \forall \theta \in [0, 2\pi]$$

and the "twist" condition for inner and outer boundaries of $A$ is rigorously proved. (For a correct explanation of the meaning of the "twist" condition, in the same setting as above, see, for instance, [4, 23, 15]).

Now we can apply the extended version of the Poincaré–Birkhoff "twist" theorem due to W. Ding [14] (see the Appendix), which ensures the existence of at least two fixed points $w_#, w^*$ for $\varphi$, with $w_#, w^* \in A$. Since $\varphi$ is the Poincaré map associated to $(3k)$ for an arbitrary but fixed $k \geq k_2$, we conclude that, for each $k \geq k_2$, there exists $w_k \in A$, such that $\varphi_k(w_k) = w_k$. Then we can use Lemma 4 and also the second part of Theorem 1 is proved.

The proof is complete. Q.E.D.

In the applications to the superlinear problem, the following corollary is used.

**Corollary 1.** Assume (besides $(g_0)$ and $(p_0)$)

$$\lim_{r \to +\infty} n_0(r) = +\infty. \quad (n_1)$$

Then, Eq. (1) has at least one $T$-periodic solution.

If, moreover, $p = p(t, x)$, then, Eq. (1) has a sequence $\{x_j(\cdot)\}$ of $T$-periodic solutions, such that

$$\lim_{j \to +\infty} \left(\min\{\sqrt{x_j(t)^2 + x'_j(t)^2}: t \in \mathbb{R}\}\right) = +\infty. \quad (13)$$

**Proof.** From Lemma 1, Lemma 2, and hypothesis $(n_1)$, we can find two increasing sequences $\{r_1^{(j)}\}$, $\{r_2^{(j)}\}$, such that the following conditions are satisfied:

$$d \leq r_1^{(j)} < r_2^{(j)} < r_1^{(j+1)}, \quad \forall j \in \mathbb{N};$$

$$\lim_{j \to +\infty} r_1^{(j)} = \lim_{j \to +\infty} r_2^{(j)} = +\infty;$$
for every $j \in \mathbb{N}, j \geq 2$, and each $x(\cdot)$ solution of (1),
\[ r_1^{(j)} \leq \sqrt{x(0)^2 + x'(0)^2} \leq r_2^{(j)} \Rightarrow r_2^{(j-1)} < \sqrt{x(t)^2 + x'(t)^2} < r_1^{(j+1)}, \quad \forall t \in [0, T]; \]
\[ n_*(r_2^{(j)}) - n^*(r_1^{(j)}) \geq 2, \quad \forall j \in \mathbb{N}. \]

Then the first part of Corollary 1 follows from the corresponding first statement in Theorem 1, just setting
\[ r_1 := r_1^{(j)}, \quad r_2 := r_2^{(j)}, \]
for any $j \in \mathbb{N}$. If $p = p(t, x)$, then for every $j \in \mathbb{N}$, we obtain the existence of a $T$-periodic solution $x_j(\cdot)$ of (1), with
\[ r_1^{(j)} < \sqrt{x_j(0)^2 + x_j'(0)^2} \leq r_2^{(j)} \]
(this follows from the second statement in Theorem 1). Moreover, by the above choice of $r_1^{(j)}$ and $r_2^{(j)}$, we have that, for each $j \geq 2$, and all $t \in [0, T], 
\[ r_2^{(j-1)} < \sqrt{x_j(t)^2 + x_j'(t)^2} < r_1^{(j+1)} \leq \sqrt{x_{j+1}(0)^2 + x_{j+1}'(0)^2} \leq r_2^{(j+1)}, \quad (14) \]
so that,
\[ x_i(\cdot) \neq x_j(\cdot), \quad \text{for} \quad i \neq j. \]

The fact that $|x_j(t)| + |x_j'(t)| \to +\infty$, as $j \to +\infty$, uniformly on $t$, follows from the above inequality (14). The proof is complete. Q.E.D.

In Section 6 we give a more general version of Corollary 1 in the conservative case $p = p(t, x)$, proving (under $(g_0)$, $(p_0)$, and $(n_1)$) that, for each $m \in \mathbb{N}, m \geq 2$, there is a sequence $\{x_j(\cdot)\}$ of subharmonic solutions of minimal period $m$, satisfying condition (13).

The next section is devoted to an application of Corollary 1 to the periodic boundary value problem for Eq. (1).

4. STATEMENT OF THE MAIN RESULT AND REMARKS

Consider again the Duffing's equation
\[ x'' + g(x) = p(t, x, x') \quad (1) \]
or, equivalently, the planar system
\[ x' = y, \quad y' = -g(x) + p(t, x, y), \quad (3_p) \]
with $g$ and $p$ satisfying the same assumptions $(g_0)$ and $(p_0)$ of Section 2.
As pointed out in the preceding discussion, for the applicability of Theorem 1 and Corollary 1 we have to evaluate the number of revolutions around the origin of the solutions of systems
\[ x' = y, \quad y' = -g(x) + q(t, x, y), \] (3)
with \( q \) continuous, \( T \)-periodic in \( t \) and such that
\[ q \in \mathcal{P} := \{ q : |q(t, x, y)| \leq M, \forall (t, x, y) \in [0, T] \times \mathbb{R}^2 \}. \]

For such purpose, we study (as a comparison system for what concerns the angular coordinates of the solutions), the associated autonomous system
\[ x' = y, \quad y' = -g(x). \] (15)

As is well known, all the orbits of (15) are contained in the level lines of the energy functional
\[ E(x, y) := G(x) + \frac{y^2}{2}, \quad \text{with} \quad G(x) := \int_x^x g(s) \, ds. \]

From (go) then it follows that all the level lines of \( E(x, y) \), passing outside a suitable closed disc containing the origin, are closed curves. Such situation may be precised as follows.

Let \( d^- < 0 < d^+ \) be such that (according to (go)),
\[ g(x)x > 0 \quad \text{for} \quad x \notin [d^-, d^+], \]
\[ G(x) < G(d^-) = G(d^+) \quad \text{for} \quad x \in [d^-, d^+]. \] (16)

Let us fix
\[ d_0 > \max \{ |d^-|, d^+ \} \] (17)
and consider, for any
\[ c \geq d_0, \]
the solution
\[ z_c(\cdot) = (x_c(\cdot), y_c(\cdot)) \]
of (15) with
\[ z_c(0) = (c, 0). \]
Such a solution is unique (see [37]: the continuity of $g$ is sufficient here) and globally defined. Moreover, the orbit path

$$I(c) := \{z_c(t); t \in \mathbb{R}\}$$

through $(c, 0)$ is a closed curve encircling the origin and $z_c(\cdot)$ is a periodic solution with fundamental period (minimal period)

$$\tau_0(c) := \sqrt{2} \int_{-l(c)}^{c} \frac{d\xi}{\sqrt{G(c) - G(\xi)}},$$

(18)

where $(-l(c), 0)$ is the intersection of $I(c)$ with the negative $x$-axis and

$$G(-l(c)) = G(c),$$

with $-l(c) \leq d^-$. We also observe that

$$\lim_{c \to +\infty} l(c) = +\infty$$

(19)

(see [22, 8], for more details).

In the sequel, we decompose $r_0(c)$ as

$$r_0(c) = r(c) + r(-l(c)),$$

where, for $|s| \geq d_0$,

$$r(s) := \sqrt{2} \left| \int_{0}^{s} \frac{d\xi}{\sqrt{G(s) - G(\xi)}} \right|.$$  

(20)

Observe that $r(s)$ and $r_0(c)$ are continuous for $|s| \geq d_0$ and $c \geq d_0$, respectively.

The use of estimates for the time-maps $r_0$ and $r$ of the autonomous system (15) in the search of $T$-periodic solutions of Eq. (1) has been already considered in the literature.

In [35], Z. Opial obtained the existence of $T$-periodic solutions for Eq. (1), with $p = p(t)$, assuming, besides $(g_0)$ and $(p_0)$, that

$$\left(\liminf_{s \to -\infty} r(s) + \liminf_{s \to +\infty} r(s)\right) > T.$$  

Opial's condition implies that, for initial points with sufficiently large norm, the solutions of (3) cannot complete a turn around the origin. In this manner, for the associated Poincaré map $\varphi$, we obtain $\varphi(w) \neq \lambda w$, for all $\lambda \geq 1$ and $|w| \geq R_0$ (with $R_0$ a large constant). Then the Poincaré–Bohl fixed point theorem (see [28]) permits us to conclude. Opial's theorem
essentially covers the case in which $g$ is sublinear at infinity (see also [16, 12, 17] for recent improvements in this direction).

In [8], T. Ding obtained the existence of infinitely many $T$-periodic solutions for Eq. (1), with $p = p(t)$, assuming, besides $(p_0)$, that $g$ is globally lipschitzian and also semilinear, i.e.,

$$0 < a \leq \frac{g(x)}{x} \leq b < +\infty, \quad \text{for } |x| \geq d_0.$$

Ding's theorem is proved through a comparison between the angular functions associated to systems $(3_p)$ and (15). In this case, the time-map $\tau_0$ is employed in the evaluation of the angular functions of (15). The existence and multiplicity results then follow from a generalization of the Poincaré–Birkhoff "twist" theorem due to W. Ding [15]. Further results and improvements in the semilinear case can be found in [11]. New applications of T. Ding's theorem are included in the recent paper [13].

However, we remark that none of the above quoted results can be applied to the case when the orbits of the autonomous system (15) satisfy

$$\lim_{t \to +\infty} (c) = 0. \quad \text{(d c' for +c)}$$

The main goal of the next theorem is to fill this gap. Namely, we have the following result.

**Theorem 2.** Assume $(g_0)$, $(p_0)$, and $(\tau_0)$. Then Eq. (1) has at least one $T$-periodic solution. If, moreover, $p = p(t, x)$, then Eq. (1) has a sequence $\{x_j(\cdot)\}$ of $T$-periodic solutions, such that

$$\lim_{j \to +\infty} \left( \min \left\{ \sqrt{x_j'(t)^2 + x_j(t)^2} : t \in \mathbb{R} \right\} \right) = +\infty. \quad (13)$$

The proof of Theorem 2 is postponed to the next section.

Note that from (19) it follows that condition $(\tau_0)$ is equivalent to

$$\lim_{|c| \to +\infty} \tau(c) = 0,$$

with $\tau$ defined by (20).

Condition $(\tau_0)$ is a peculiar property of superlinear nonlinearities (see [23, 5, 20, 40, 14, 6]), since it is satisfied whenever

$$\lim_{|x| \to +\infty} \frac{g(x)}{x} = +\infty \quad \text{(21)}$$
holds. Nevertheless, it can be seen that \((\tau_0)\) is fulfilled under more general assumptions than \((21)\). In this connection, we observe that the validity of \((g_0)\) and \((\tau_0)\) implies the following condition of superquadratic growth for the potential \(G(x)\) of \(g(x)\):

\[
\lim_{|x| \to +\infty} \frac{G(x)}{x^2} = +\infty. \tag{22}
\]

Indeed, it is sufficient to recall \((16)\) and observe that, for \(|c| > d_0\),

\[
\tau(c) \geq \frac{c}{\sqrt{G(c)}} - \frac{2d_0}{\sqrt{G(c) - G(d_0)}}.
\]

Then we use the fact that \(G(x) \to +\infty\) as \(|x| \to +\infty\) (which is a consequence of \((g_0)\)) and \((22)\) easily follows from \((\tau_0)\). Q.E.D.

Thus, the above discussion can be summarized by "\((21) \Rightarrow (\tau_0) \Rightarrow (22)\)." On the other hand, we have that "\((22) \not\Rightarrow (\tau_0) \not\Rightarrow (21)\)" (see, respectively, Examples 2 and 1, below). Consequently, if we want to look for explicit assumptions on the nonlinearity \(G\), in order to obtain \((\tau_0)\), we have to add further conditions to \((22)\) (since this hypothesis, alone, is not sufficient).

As a partial step in this direction, we give the next result.

For any constant \(L > 0\), we define the function

\[
\phi_L(s) := G(s) - Ls^2, \quad s \in \mathbb{R}.
\]

Then we have the following:

**Lemma 5.** Assume \((g_0)\) and suppose that, for each \(L > 0\), there exist \(R \geq d_0\) and \(D > 0\) such that

\[
\phi_L(s_1) < \phi_L(s_2), \quad \forall s_1, s_2: R \leq s_1 \leq s_2 - D \text{ and } \forall s_1, s_2: -R \geq s_1 \geq s_2 + D.
\]

\((G_0)\)

Then \((\tau_0)\) holds.

**Proof.** Let \(L > 0\) be an arbitrary but fixed constant. For \(R\) and \(D\) chosen according to the hypothesis and for \(s\) and \(c\) such that \(R \leq s \leq c - D\), we can write

\[
G(c) - G(s) \geq L(c^2 - s^2). \tag{23}
\]
Then, for \( c \) as above and recalling (16), we can write

\[
\tau(c) \leq \sqrt{2} \int_0^r \frac{d\xi}{\sqrt{G(c) - G(R)}} + \sqrt{2} \int_{c-D}^{c} \frac{d\xi}{\sqrt{L(c^2 - \xi^2)}}
\]

\[
+ \sqrt{2} \int_{c-D}^{c} \frac{d\xi}{\sqrt{G(c) - G(\xi)}}
\]

\[
\leq \frac{\sqrt{2} R}{\sqrt{G(c) - G(R)}} + \frac{2}{\sqrt{L}} \left( \arcsin \frac{c-D}{c} - \arcsin \frac{R}{c} \right)
\]

\[
+ 2 \sqrt{\min_{c-D < s < c} g(s)}
\]

Hence, using \((g_0)\), we easily obtain that

\[
\limsup_{c \to +\infty} \tau(c) \leq \pi/\sqrt{2L},
\]

so that, letting \( L \) tend to \( +\infty \), we can conclude that

\[
\lim_{c \to +\infty} \tau(c) = 0.
\]

With a completely symmetric argument, one can deduce that \( \lim_{c \to -\infty} \tau(c) = 0 \) and thus the result is proved (recalling a previous remark based on (19)). Q.E.D.

As a first consequence of Lemma 5, we can easily obtain a straightforward proof that \("(21) \Rightarrow (\tau_0)\)" (indeed, if (21) is assumed, for any \( L > 0 \), we can find \( R > d_0 \), such that \( g(x) \geq Lx, \forall x \geq R \) and \( g(x) \leq -Lx, \forall x \leq -R \). Then, by an integration, we have that (23) is satisfied for \( s \in [-|c|, -R] \cup [R, |c|] \), so that \((G_0)\) follows with \( D = 0 = \text{constant with respect to } L \).

Hence, it is clear that Theorem 2 improves the theorem of W. Ding in [14], where the existence of a sequence of infinitely many \( T \)-periodic solutions (with arbitrarily large norm in \( C^1[0, T] \)) for

\[
x'' + g(x) = p(t),
\]

is proved under the more restrictive condition (21).

On the other hand, we easily can produce examples which fit into our result, while [14] does not apply. To this end, it can be convenient to find other more explicit conditions on \( G(x) \), as given next in Corollary 2.
Corollary 2. Assume \((g_0), (p_0), and\)

\[
\exists A > 0, \quad \lim_{|x| \to +\infty} \frac{G(x + A) - G(x)}{x^2} = +\infty. \tag{G_1}
\]

Then, the same conclusion of Theorem 2 holds.

Proof. For any given \(L > 0\), we can find \(R \geq d_0\), such that the inequalities

\[
G(c) - G(s) \geq G(c) - G(c - A) \geq 2L(c - A)^2 \geq Lc^2 \geq L(c^2 - s^2),
\]

hold, for all \(R \leq s \leq c - A\), and all \(-R \geq s \geq c + A\). Then \((G_0)\) follows with \(D = A\) (using (23)) and Theorem 2 can be applied by Lemma 5. Q.E.D.

Assumption \((G_1)\) roughly means that \(G(x)\) grows faster than \(|x|^\eta\) at infinity, for some \(\eta > 3\) and therefore it does not employ the full power of condition \((p_0)\) of Theorem 2. Nevertheless, such hypothesis seems to be worthy of interest as it involves only the consideration of the potential \(G\) of \(g\). For instance, \((G_1)\) clearly holds whenever we can decompose \(G(x)\) as

\[
G(x) = a |x|^\eta + G_1(x), \quad \text{with } a > 0, \eta > 3 \quad \text{and} \quad \lim_{|x| \to +\infty} G_1(x)/|x|^\eta - 1 = 0.
\]

In this connection, we consider the following example.

Example 1. For any constant \(\alpha > 1\), we set

\[
\{x\}^\alpha := x |x|^\alpha - 1,
\]

(according to the notation in \([40]\)). Let \(f: \mathbb{R} \to \mathbb{R}\) be a continuous function such that

\[
\lim_{|x| \to +\infty} f(x) \operatorname{sign}(x) = +\infty, \quad \limsup_{|x| \to +\infty} f(x)/x < +\infty.
\]

By \(F(x)\) we denote a primitive of \(f(x)\). Let \(b > 0\) and \(\alpha > 2\), be fixed constants.

Then it can easily be checked that the function

\[
g(x) := f(x) + b \{x\}^\alpha (1 + \sin(|x|^\alpha + 1)),
\]

with its primitive

\[
G(x) = (b/(\alpha + 1)) |x|^\alpha + F(x) - (b/(\alpha + 1)) \cos(|x|^\alpha + 1) + c,
\]

\(c = \text{constant},\)

satisfies \((g_0)\) and \((G_1)\), but not \((21)\).
A possible function $g$, having this form, is, for instance,

$$g(x) = x + x^3(1 + \sin x^4).$$  \hfill (25)

For such $g$, Corollary 2 implies the existence of infinitely many $T$-periodic solutions (with arbitrarily large norm in $C^1[0, T]$) for the Duffing's equation (24), for any possible continuous and $T$-periodic forcing term $p(t)$.

To the best of our knowledge, no one of the previously known existence theorems for Eq. (24) can be applied for a nonlinearity like $g$ in (25). We also note that in (25), we have that $xg'(x)/g(x)$ is unbounded in any neighbourhood of $-\infty$ or $+\infty$, so that the recent result in [12] is not applicable as well. Q.E.D.

In [36], Z. Opial investigated various relationships among the asymptotic behaviours of $g(x)/x$, $\tau(x)$, and $G(x)/x^2$. In particular, he found an example of a piecewise continuous function $g$ which suggests the possibility that (22) holds, but $(\tau_0)$ is not satisfied (see [36, Sect. 6]). However, from Opial's example, it follows that $\lim \inf_{x \to +\infty} g(x) = 0$, so that our condition $(g_0)$ is not satisfied. The next Example 2 shows that $(g_0) \land (22) \not\Rightarrow (\tau_0)$, even for $C^\infty$-functions.

**Example 2.** Denote by $\Phi(x)$ a $C^\infty$-function with the property that

$$\begin{align*}
\Phi(x) &= 0 \quad \text{for } x \leq 0, \quad \Phi(x) = 1 \quad \text{for } x \geq 1, \\
\Phi'(x) &> 0 \quad \text{for } 0 < x < 1
\end{align*}$$

and let, for each $n \in \mathbb{Z}_+$,

$$a_n := \frac{4}{3}(4^n - 1), \quad b_n := 4a_n + 1, \quad c_n := b_n + 1, \quad d_n = c_n + 1.$$

Note that, by construction,

$$a_{n+1} = d_n + 1, \quad b_n > 4a_n \quad (n = 0, 1, 2, \ldots).$$

Further, denote by $h(x)$, an even $C^\infty$-function defined by

$$h(x) := \begin{cases} 
K, & \text{if } x \in [a_n, b_n[; \\
K + a_{n+2}^2 \Phi(x - b_n), & \text{if } x \in [b_n, c_n[; \\
K + a_{n+2}^2, & \text{if } x \in [c_n, d_n[; \\
K + a_{n+2}^2(1 - \Phi(x - d_n)), & \text{if } x \in [d_n, a_{n+1}[, 
\end{cases}$$

(with $n \in \mathbb{Z}_+$), where $K$ is a positive constant. Finally, we set

$$g(x) := xh(x), \quad \text{and} \quad G(x) = \int_0^x g(x) \, dx.$$
By the above definition, it follows that $g$ is a $C^\infty$ odd function, satisfying

$$g(x)/x \geq K > 0, \quad \text{for each } x \neq 0.$$  

Moreover, for every $x \in [a_{n+1}, a_{n+2})$, we have

$$\frac{G(x)}{x^2} > \left( \int_{c_n}^{d_n} g(s) \, ds \right) \left( \frac{a_{n+2}^2}{a_{n+1}^2} \right) = \frac{1}{2} (K + a_{n+2}^2)(d_n^2 - c_n^2)/a_{n+2}^2 \geq \frac{1}{2} \left( 1 + \frac{K}{a_{n+2}^2} \right) (d_n + c_n) > \frac{1}{2} (c_n + d_n),$$

from which we easily obtain (22), using the fact that $G$ is even and that $a_n \to +\infty$, as $n \to +\infty$. It can also be checked that

$$\lim \inf_{c \to +\infty} \tau_0(c) = 0.$$

On the other hand, for $\tau$ defined as in (20), we have

$$\frac{1}{\sqrt{2}} \tau(b_n) = \int_0^{b_n} \frac{d\xi}{\sqrt{G(b_n) - G(\xi)}} = b_n \int_0^1 \frac{dt}{\sqrt{G(b_n) - G(b_n t)}} > b_n \int_{1/4}^1 \frac{dt}{\sqrt{G(b_n) - G(b_n t)}}
= b_n \int_{1/4}^1 \frac{dt}{\sqrt{(b_n - b_n t) \int_0^1 g(b_n \chi(t, \lambda)) \, d\lambda}}
= \int_{1/4}^1 \frac{dt}{\sqrt{(1 - t) \int_0^1 h(b_n \chi(t, \lambda)) \, d\lambda}}
= \int_{1/4}^1 \frac{dt}{\sqrt{K(1 - t)}} = \frac{3}{\sqrt{K}}.$$  

(Note that in the above inequalities we have used the fact that $a_n < b_n \chi(t, \lambda) < b_n$, for $\frac{1}{4} < t < 1$ and $0 \leq \lambda \leq 1$). In this manner, we have proved that

$$\lim \sup_{c \to +\infty} \tau(c) \geq \frac{\sqrt{6}}{\sqrt{K}} > 0.$$  

Therefore, the assumptions $(\tau_0)$ and (22) are not sufficient to ensure the validity of $(\tau_0)$.  

Q.E.D.
The argument developed in the above example suggests the possibility of finding other conditions, sufficient for the validity of \((\tau_0)\), in the case when

\[ \exists D, K > 0, \quad g(x)/x \geq K, \quad \text{for} \quad |x| \geq D. \quad (26) \]

With this respect, the following result can be proved.

**Lemma 6.** Assume \((22)\) and \((26)\) and suppose that there are a constant \(R > 0\) and a continuous function \(\gamma: [0, 1] \to [0, 1]\), such that

\[ G(tx) \leq \gamma(t) G(x), \quad \text{for all} \quad |x| \geq R \quad \text{and} \quad 0 \leq t < 1. \quad (G2) \]

Then, \((\tau_0)\) holds.

**Proof.** Without restriction, we can suppose \(R = D \geq d_0\) (taking, if necessary, the maximum of the three). Let \(\varepsilon: 0 < \varepsilon < 1\) be a fixed (small) constant. For \(c > D/(1 - \varepsilon)\), arguing like in Example 2, we can write

\[ \frac{1}{\sqrt{2}} \tau(c) = \frac{c}{\sqrt{G(c)}} \int_0^{1 - \varepsilon} \frac{dt}{\sqrt{1 - G(ct)/G(c)}} \]

\[ + \int_1^{1 - \varepsilon} \frac{dt}{\sqrt{(1 - \varepsilon) \int_0^1 h(c \chi(t, \lambda)) \chi(t, \lambda) d\lambda}}, \]

where, in order to simplify the notation, we have set

\[ h(x) = g(x)/x, \quad \text{for} \quad |x| \geq D. \]

Then, using \((22)\), \((26)\), and \((G2)\), we easily obtain

\[ \limsup_{c \to +\infty} \tau(c) \leq \frac{8\varepsilon}{\sqrt{K}}, \]

from which we obtain that \(\lim_{c \to +\infty} \tau(c) = 0\), as \(\varepsilon \to 0\). The proof that \(\tau(c) \to 0\), as \(c \to -\infty\), is exactly the same. Thus, \((\tau_0)\) is proved. Q.E.D.

Hypothesis \((G2)\) is satisfied if, for instance, \(G(x)x/|x|^3\) is nondecreasing for \(|x|\) large (a condition considered by Opial in [36, Th. 13], together with \((22)\), in order to ensure the validity of \((\tau_0)\)). For completeness, it should be observed that such condition from [36], actually implies \((21)\), when \((22)\) is assumed.

**Remark 1.** Consider Eq. \((1)\) with the assumptions \((g_0)\) and \((p_0)\). If hypothesis \((\tau_0)\) is not satisfied, that is

\[ \limsup_{c \to +\infty} \tau_0(c) > 0, \]

then...
we can use the above mentioned theorem of T. Ding in [8] (see also [11]) and still obtain the solvability of the $T$-periodic problem for (1), under a nonresonance condition expressed by the fact that the (possibly degenerate) interval

$$[\liminf_{c \to +\infty} \tau_0(c), \limsup_{c \to +\infty} \tau_0(c)]$$

does not contain any number of the form $T/n$, with $n \in \mathbb{N}$. However, T. Ding’s theorem requires the fulfillment of some extra assumptions on $g$ ($g$: globally lipschitzian and semilinear). On the other hand, in [17], A. Fonda and F. Zanolin obtained the existence of $T$-periodic solutions for (1) (without extra assumptions on $g$), if

$$\limsup_{s \to -\infty} \tau(s) > T, \quad \text{or} \quad \limsup_{s \to +\infty} \tau(s) > T.$$ 

Such a result, in particular, can be applied if

$$\limsup_{c \to +\infty} \tau_0(c) > 2T.$$ 

All this discussion seems to indicate the possibility of obtaining a new proof of T. Ding’s theorem in [8] avoiding some of the extra assumptions on $g$ which are required therein.

Finally, we refer to [30] for a general treatment of the periodic problem for Hamiltonian systems (including, as particular examples, the case of the Duffing’s equation), in the light of variational methods. However, we point out that it seems not to be clear how to reach our results through critical point theory techniques.

**Remark 2.** We note that, by means of some standard technical arguments, it is possible to extend Theorems 1 and 2, as well as their corollaries, to the case in which $p: [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory conditions: i.e., $p(\cdot, x, y)$ is measurable for all $(x, y) \in \mathbb{R}^2$ and $p(t, \cdot, \cdot): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$ (see [22, p. 28]).

In this case, a $T$-periodic solution $x(\cdot)$ of (1) is a function defined on $[0, T]$, with $x'$ absolutely continuous, satisfying (1) for a.e. $t \in [0, T]$ and such that

$$x(T) - x(0) = x'(T) - x'(0) = 0.$$ 

Moreover, assumption (p$_0$) has to be substituted by

$$\exists M_0 > 0: |p(t, x, y)| \leq M_0, \quad \text{for a.e.} \quad t \in [0, T] \text{ and all } x, y \in \mathbb{R}$$

and the set $P$ has to be modified accordingly.
Another remark concerning the Duffing's equation (24) with \( p = p(t) \) a \( L^1_{loc} \)-function, is considered at the end of Section 6.

5. PROOF OF THEOREM 2

We maintain all the notations introduced in the preceding sections. We apply Corollary 1. To this end, we consider an arbitrary solution

\[ z(t) = (x(t), y(t)) \]

of system

\[ \begin{align*}
x' &= y, \\
y' &= -g(x) + q(t, x, y),
\end{align*} \]

(3)

for some \( q \in \mathcal{P} \) (\( \mathcal{P} \) being the set of functions defined in Section 2), with

\[ |z(0)| = r \geq d, \]

where \( d > 0 \) is the constant coming from Lemma 2. Let

\[ \theta(t) := \Theta(t; z) \quad \text{and} \quad \rho(t) := \varrho(t; z) \]

be the polar coordinates of \( z(t) \).

As a first step, we find a lower estimate for

\[ |\theta(T) - \theta(0)|/2\pi, \]

if \( r \) is large.

Step 1. Let \( \varepsilon \) be a positive, sufficiently small but fixed constant (for the moment, we just suppose that

\[ 0 < \varepsilon < 1; \]

further conditions on \( \varepsilon \) will be imposed, if necessary, along the proof).

Using (\( \tau_0 \)) and (\( g_0 \)), we can find a constant

\[ R_0 := R_0(\varepsilon) > d_0 \]

(with \( d_0 \) coming from (17) in Section 4), such that

\[ \tau(s) = \sqrt{2} \left| \int_0^s \frac{ds}{\sqrt{G(s) - G(\xi)}} \right| < \varepsilon \sqrt{2}, \quad \text{for} \quad |s| \geq R_0 \]

(27)

and

\[ g(s) \text{ sign}(s) > 2M > 0, \quad \text{for} \quad |s| \geq R_0, \]

(28)

are both satisfied.
Next, we fix
\[ R_1 := R_1(\varepsilon) > \max \left\{ d, \frac{R_0 \sqrt{4 + \varepsilon^2}}{\varepsilon} \right\}. \]

Furthermore, let
\[ R_2 := R_2(\varepsilon) > R_1 > R_0, \]
be a (fixed) constant such that, according to Lemma 1,
\[ |z(t)| \geq R_1, \quad \text{for all } t \in [0, T], \quad \text{if } |z(0)| \geq R_2. \]

Now we assume that
\[ |z(0)| = r \geq R_2. \]  \hfill (29)

Our aim is to find an estimate from below of the number of rotations of \( z(\cdot) \) around the origin, during the time \( T \), knowing, by the first inference in Lemma 2, that \( z(t) \) turns clockwise as \( t \) increases.

To this end, we decompose the set
\[ D := \{(a, b) \in \mathbb{R}^2 : |(a, b)| = \sqrt{a^2 + b^2} \geq R_1\}, \]
into six regions \( D_i \) \( (i = 1, 2, \ldots, 6) \) having only boundary points as possible intersections and we study the behaviour of \( z(\cdot) \) in each of these sets. Let us define
\[ D_1 := \{(a, b) \in D : |a| \leq R_0, b > 0\}, \]
\[ D_2 := \{(a, b) \in D : a \geq R_0, b \geq 0\}, \]
\[ D_3 := \{(a, b) \in D : a \geq R_0, b \leq 0\}, \]
\[ D_4 := \{(a, b) \in D : |a| \leq R_0, b < 0\}, \]
\[ D_5 := \{(a, b) \in D : a \leq -R_0, b \leq 0\}, \]
\[ D_6 := \{(a, b) \in D : a \leq -R_0, b \geq 0\}. \]

First, we observe that
\[ z(t) \in D_i, \quad \text{for all } t \in [0, T] \]
(by Lemma 2 and the choice of \( r \geq R_2 \) according to (20)). Moreover, for the validity of the subsequent arguments, it is crucial to observe that \( z(\cdot) \) moves from \( D_i \) to \( D_j \), as \( t \) increases, following the cycle
\[ D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6 \to D_1, \]  \hfill (30)
Actually, this is a consequence of Lemma 2 which ensures that $\theta'(t) < 0$ on $[0, T]$ and that
\[
x' > 0 \quad \text{on } D_1, \quad x' < 0 \quad \text{on } D_4,
\]
\[
y' < 0 \quad \text{on } D_2 \cap D_3, \quad y' > 0 \quad \text{on } D_5 \cap D_6.
\]

We note that at this point of the proof, we cannot guarantee that $z(t)$ complete any rotation and we have not yet proved that any transition from $D_i$ to $D_j$ really occurs (this will be the task in the second "step" of the proof). However, we know that any possible transition must obey the "cyclic rule" (30) described above.

Second, we observe (as a consequence of, respectively, the cyclic rule, the preceding list of signs of $x'$ and $y'$ and the second inference in Lemma 2) that, for each $i = 1, 2, \ldots, 6$, the set
\[
I_i := \{t \in [0, T]: z(t) \in D_i\}
\]
is the union of a finite (possibly empty) number of closed disjoint intervals (possibly degenerate at a point). Accordingly, we define, for $i = 1, 2, \ldots, 6$,
\[
n_i^* := \text{number of nondegenerate closed disjoint intervals contained in } I_i,
\]
so that we can write, for each $i = 1, 2, \ldots, 6$,
\[
I_i = \left( \bigcup_{j=1}^{n_i^*} J_{j}^{(i)} \right) \cup P_i,
\]  
(31)

where $P_i$ is a finite (possibly empty) set and, for every $j: 1 \leq j \leq n_i^*$, $J_{j}^{(i)}$ is a nondegenerate closed interval, contained in $[0, T]$ and maximal with respect to the property that
\[
z(t) \in D_i, \quad \text{for all } t \in J_{j}^{(i)}.
\]

We note that the set $P_i$ contains at most two points, and that $\bigcup_{j=1}^{n_i^*} J_{j}^{(i)} = \emptyset$ if $n_i^* = 0$. In any case, as remarked above, the decomposition of $I_i$ in (31) is given by a disjoint union of a finite number of points and intervals.

A further property of the intervals $J_{j}^{(i)}$ is that any possible intersection of the form
\[
J_{j}^{(i)} \cap J_{k}^{(l)}, \quad \text{for each } i, j, k, l
\]
either is empty, or it consists of exactly one point.
So, by the above discussion, it is clear that

\[
\sum_{i=1}^{6} \left( \sum_{j=1}^{n_i^\#} \text{meas}(J_j^{(i)}) \right) = T.
\]  

(32)

Last, we observe that, by the cyclic rule (30), we obtain

\[
|n_i^\# - n_k^\#| \leq 1, \quad \text{for every } i \neq k
\]  

(33)

and hence,

\[
\frac{|\theta(T) - \theta(0)|}{2\pi} \geq \min \{n_i^\# : 1 \leq i \leq 6\} - 1 \geq n_k^\# - 2,
\]

(34)

where \( k \in \{1, 2, ..., 6\} \) is any fixed index.

In this manner, we have obtained a lower estimate of \(|\theta(T) - \theta(0)|/2\pi\) in terms of the number of times \( n_i^\# \) that \( z(\cdot) \) meets the interior of any of the regions \( D_i \) \((i = 1, 2, ..., 6)\). On the other hand, by (32) and (33), we have

that the integers \( n_i^\# \) \((i = 1, 2, ..., 6)\) may be estimated from below provided that we are able to find (small) upper bounds for the length of the intervals \( J_j^{(i)} \) \((i = 1, 2, ..., 6)\). This goal is achieved in the next step.

**Step 2.** Since \( z(t) \in D = \bigcup_{i=1}^{6} D_i \), for all \( t \in [0, T] \),

then, there is at least one of the nondegenerate intervals \( J_j^{(i)} \).

Now we produce upper bounds for \( \text{meas}(J_j^{(i)}) \), for an arbitrary non-degenerate interval \( J_j^{(i)} \), by examining the six different possibilities that can occur. In order to simplify the notation, we denote by

\[
J = [t_1, t_2] \subseteq [0, T], \quad \text{with } t_1 < t_2.
\]

a generic (but fixed) interval among the \( J_j^{(i)} \).

Suppose that

\[
z(t) \in D_1, \quad \text{for } t \in J.
\]

Then, for all \( t \in J \),

\[
|z(t)| = \sqrt{x(t)^2 + y(t)^2} \geq R_1, \quad |x(t)| \leq R_0.
\]

Hence, from the first equation in (3) and the choice of \( R_1 \), we have

\[
x'(t) = y(t) = |y(t)| \geq \sqrt{R_1^2 - R_0^2} \geq 2R_0/\varepsilon
\]
and thus we obtain, by an integration over $J$,

$$2R_0 \geq x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(s) \, ds \geq 2R_0(t_2 - t_1)/\varepsilon.$$  

So that, we can conclude with

$$\text{meas}(J) = (t_2 - t_1) \leq \varepsilon. \quad (35)$$

With obvious changes in the proof, we can obtain the same estimate also in the case when

$$z(t) \in D_4, \quad \text{for } t \in J.$$ 

Suppose now that

$$z(t) \in D_2, \quad \text{for } t \in J$$

and observe that, from the first equation in (3), we have $x'(t) = y(t) > 0$ and therefore

$$x(t) \leq x(t_2), \quad \text{and } G(x(t)) \leq G(x(t_2)), \quad \text{for } t_1 \leq t < t_2$$

(where, for the inequality in $G$ we have used the fact that $G(\cdot)$ is increasing on $[R_0, +\infty)$, according to (16), (17) and the choice of $R_0$).

Next, we consider, for $t \in J$, the function

$$w(t) := G(x(t)) - Mx(t) + \frac{1}{2}y(t)^2.$$ 

Differentiating $w(\cdot)$ in $t$ and using Eq. (3), we obtain

$$w'(t) = (g(x(t)) - M) x'(t) + y(t) y'(t)$$

$$= (g(x(t)) - M) y(t) + y(t)(-g(x(t)) + q(t, x(t), y(t)))$$

$$= -y(t)(M - q(t, x(t), y(t))).$$

Since $q \in \mathcal{P}$ (that is $|q(\cdot, \cdot, \cdot)| \leq M$) and $y \geq 0$ in $D_2$, we obtain that the function $w(\cdot)$ is nonincreasing in the interval $J$ and therefore we have, for $t_1 \leq t < t_2$,

$$G(x(t)) - Mx(t) + \frac{1}{2}y(t)^2 \geq G(x(t_2)) - Mx(t_2) + \frac{1}{2}y(t_2)^2$$

$$\geq G(x(t_2)) - Mx(t_2).$$
Then we obtain, for $t_1 \leq t < t_2$,

$$\frac{1}{2} y(t)^2 \geq G(x(t_2)) - G(x(t)) - M(x(t_2) - x(t))$$

$$= \frac{1}{2} (G(x(t_2)) - G(x(t))) + \frac{x(t_2) - x(t)}{2} \left( \frac{G(x(t_2)) - G(x(t))}{x(t_2) - x(t)} - 2M \right)$$

$$= \frac{1}{2} (G(x(t_2)) - G(x(t))) + \frac{x(t_2) - x(t)}{2} (g(\xi) - 2M),$$

where $x(t) < \xi = \xi_{x(t)} < x(t_2)$ comes from the mean value theorem.

Hence, from (28) and recalling that $x \geq R_0$ and $y \geq 0$ in $D_2$, we have (by the first equation in (3))

$$x'(t) = y(t) = |y(t)| \geq \sqrt{G(x(t_2)) - G(x(t))}, \quad \forall t \in [t_1, t_2[, \quad (35)$$

so that, we obtain

$$\frac{x'(t)}{\sqrt{G(x(t_2)) - G(x(t))}} \geq 1, \quad \forall t \in [t_1, t_2[.$$

An integration over the interval $J$, then yields (recalling also (16) and (17))

$$\text{meas}(J) = (t_2 - t_1) = \int_{t_1}^{t_2} 1 \, ds$$

$$\leq \int_{t_1}^{t_2} \frac{x'(s)}{\sqrt{G(x(t_2)) - G(x(s))}} \, ds = \int_{x(t_1)}^{x(t_2)} \frac{d\xi}{\sqrt{G(x(t_2)) - G(\xi)}}$$

$$\leq \int_{0}^{x(t_2)} \frac{d\xi}{\sqrt{G(x(t_2)) - G(\xi)}} = \frac{\tau(x(t_2))}{\sqrt{2}} \leq \varepsilon,$$

where, for the last inequality, we have used (27) and $x(t_2) \geq R_0$ (which comes from $z(t_2) \in D_2$). In this manner, we have established again the validity of (35).

The same argument, with minor changes, can be repeated in order to obtain inequality (35) in the remaining cases

$$z(t) \in D_i, \quad i = 3, 5, 6, \quad \text{for} \quad t \in J.$$

Now we are in position to end the proof.

**Step 3.** According to the estimates given in Step 2, we know that

$$\text{meas}(J_{(i)}^{(i)}) \leq \varepsilon, \quad \forall j = 1, ..., n_i^#, \quad \forall i = 1, 2, ..., 6.$$
Hence, from (32), we obtain

$$T \leq \varepsilon \left( \sum_{i=1}^{6} n_i^* \right),$$

so that, (33) implies

$$T \leq 6\varepsilon (n^* + 1),$$

where we have set

$$n^* := \min\{n_i^*: 1 \leq i \leq 6\}.$$

Finally, from (34), we can conclude that

$$\frac{|\Theta(T; z) - \Theta(0; z)|}{2\pi} \geq \frac{T}{6\varepsilon} - 3,$$

with $z(\cdot)$ satisfying (29). Since the inequality (36) is independent of $z(\cdot)$ and $q \in \mathcal{P}$, recalling the definition of $n^*_\ast(r)$, we realize the validity of the following claim:

**Claim.** For every $\varepsilon: 0 < \varepsilon < 1$, there is $R(\varepsilon) \geq d$, such that

$$n^*_\ast(r) \geq \int \left( \frac{T}{6\varepsilon} - 3 \right), \quad \forall r \geq R(\varepsilon)$$

(where $\text{int}(s) \leq s < \text{int}(s) + 1$, with $\text{int}(s) \in \mathbb{Z}$, is the integer part of $s \in \mathbb{R}$).

To prove the claim it is sufficient to take $R(\varepsilon) = R_2(\varepsilon)$, with $R_2(\varepsilon)$ defined in Step 1.

Then, the hypothesis $(n_1)$ of Corollary 1 is fulfilled as $\varepsilon \to 0^+$ and the result follows from Corollary 1. The proof of Theorem 2 is complete.

Q.E.D.

6. Subharmonic Solutions. Final Remarks and Conclusion

This section is divided into two parts. In the first one, we extend the preceding results to the case of subharmonic solutions. Then, we discuss the possibility of obtaining further generalizations to some related problems.

Accordingly, we start again with equation

$$x'' + g(x) = p(t, x, x'),$$

where $g$ and $p$ are like in Section 2.
Let

\[ m \in \mathbb{N}, \quad m \geq 2, \]

be a fixed integer, our purpose is to prove the existence of periodic solutions to Eq. (1), having minimal period equal to \( mT \). Such solutions will be called subharmonics of order \( m \). To avoid misunderstanding, it should be noted that, at this stage of the discussion, the term "minimal period" is used within the class of periodic functions having period of the form \( kT \), with \( k \in \mathbb{N} \) and therefore, saying that \( x(\cdot) \) is a subharmonic of order \( m \) for Eq. (1), we mean that \( x(\cdot) \) is not a \( kT \)-periodic solution of (1), for each \( k = 1, \ldots, m - 1 \). Indeed, otherwise, we already know (from the discussion developed in Section 4) that for the autonomous equation \( x'' + g(x) = 0 \), with \( g \) satisfying the superlinear growth condition

\[
\lim_{|x| \to +\infty} \frac{g(x)}{x} = +\infty, \tag{21}
\]

the minimal period of the periodic solutions tends to zero (for \( |x(0)| + |x'(0)| \) large) and therefore, in general, we cannot have a periodic solution whose fundamental period is \( mT \). At the end of the section, we see that this is, essentially, the only exceptional case and that, for the forced equation

\[ x'' + g(x) = p(t), \tag{24} \]

with \( p \) nonconstant, we can prove the existence of subharmonics of order \( m \) whose minimal period (in the broader sense) is actually \( mT \).

As a straightforward remark, we also observe that for equations like (1), we cannot expect to have existence of periodic solutions different from those already obtained in the preceding sections. Indeed, for any \( M_0 > 0 \), we can choose a function \( p(t, x, y) := p_0(y) \), with \( p_0: \mathbb{R} \to \mathbb{R} \) continuous and such that

\[ p_0(y) \cdot y < 0, \quad \forall y \neq 0 \quad \text{and} \quad |p_0(y)| < M_0, \quad \forall y \in \mathbb{R}. \]

Then, it can easily be checked that the equation

\[ x'' + g(x) = p_0(x') \]

admits as periodic solutions, only the constant ones which correspond to the zeros of \( g \).

In view of the above remark, from now on, we restrict ourselves to the study of the conservative case, that is, we deal with equation

\[ x'' + g(x) = p(t, x), \tag{37} \]
where \( g: \mathbb{R} \to \mathbb{R} \) and \( p: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions, with \( p \), \( T \)-periodic in the first variable, which satisfy
\[
\lim_{|x| \to +\infty} g(x) \text{ sign}(x) = +\infty, \quad (g_0)
\]
and
\[
\exists M_0 > 0, \quad |p(t, x)| \leq M_0, \quad \forall t, x. \quad (p_0)
\]
For a fixed constant \( M > M_0 \), we also define the set
\[
p := \{ q: |q(t, x)| \leq M, \forall t, x \},
\]
where \( q: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( T \)-periodic in \( t \).

Then we fix
\[
T_0 := mT,
\]
and thus we can easily rewrite Lemmas 1 and 2 of Section 1, for the planar system
\[
x' = y, \quad y' = -g(x) + q(t, x). \quad (38)
\]
In particular, we can find the constants \( r_0 \) and \( d \), with \( 0 < r_0 < d \), coming from (6) and Lemma 2 and we can represent any solution \( z(\cdot) = (x(\cdot), y(\cdot)) \) of (38), in the polar form
\[
z(t) = (q(t; z) \cos \Theta(t; z), q(t; z) \sin \Theta(t; z)), \quad \forall t \in [0, mT],
\]
provided that \( |z(0)| \geq r_0 \).

Now we define the nonnegative integers
\[
n_*(r)_m := \max \left\{ n \in \mathbb{Z}_+: n \leq \inf \left\{ \frac{|\Theta(mT; z) - \Theta(0; z)|}{2\pi} : |z(0)| = r \right\} \right\},
\]
where the \( \inf \{ \ldots \} \) is taken over all \( q \in \mathbb{P} \) and all solutions \( z(\cdot) \) of (38);
\[
n^{*}(r)_m := \min \left\{ n \in \mathbb{Z}_+: n \geq \sup \left\{ \frac{|\Theta(mT; z) - \Theta(0; z)|}{2\pi} : |z(0)| = r \right\} \right\},
\]
where the \( \sup \{ \ldots \} \) is taken over all \( q \in \mathbb{P} \) and all solutions \( z(\cdot) \) of (38), which are the "\( m \)-rotation numbers" analogous to the integers \( n_*(r), n^{*}(r) \), introduced in Section 3. Then we can state the following:
THEOREM 3. Suppose that there are two constants \( r_1, r_2 \in [d, +\infty) \), with \( r_1 \neq r_2 \), and there is a prime number \( \zeta > m \), such that
\[
\mathbf{n}(r_2)_m > \zeta > \mathbf{n}(r_1)_m.
\] (m_0)

Then, Eq. (37) has at least one \( mT \)-periodic solution \( x(\cdot) \), with
\[
\min\{r_1, r_2\} \leq \sqrt{x(0)^2 + x'(0)^2} \leq \max\{r_1, r_2\}.
\]

Moreover, \( x(\cdot) \) is not \( kT \)-periodic, for any \( 1 \leq k < m \).

Proof. We consider only the situation when
\[
r_1 < r_2,
\]
which is the significant one for our applications (we do not give the proof for the case \( r_1 > r_2 \), since it can be performed by similar arguments).

As a first step, we observe that there is a sequence \( \{q_k\} \) of continuous functions, with \( q_k \in \mathcal{F}, \ q_k \to p \) (as \( k \to +\infty \)), uniformly on the compact subsets of \( \mathbb{R}^2 \) and there is an index \( k_2 \), such that, for every \( k \geq k_2 \) and each \( z_0 = (x_0, y_0) \in B[r_2] \), there exists a unique solution \( z_k(\cdot; z_0) = (x_k(\cdot; z_0), y_k(\cdot; z_0)) \) on \( [0, mT] \), to equation
\[
x' = y, \quad y' = -g(x) + q_k(t, x),
\] (38_k)
satisfying the initial condition \( z_k(0; z_0) = z_0 \).

(The proof of the above remark is exactly the same as that of the corresponding Lemma 3 in Section 3 and therefore it is omitted).

Let \( q_m \in \{q_k: k \geq k_2\} \) be fixed and denote by
\[
\varphi_m: B[r_2] \to \mathbb{R}^2, \quad \varphi_m(w) = z(mT; w),
\]
the \( m \)th iterate of the Poincaré map \( \varphi \) associated to the system
\[
x' = y, \quad y' = -g(x) + q_m(t, x),
\] (38_m)
where
\[
z(t; w) = (x(t; w), y(t; w))
\]
is the unique solution of (38_m) defined on \( [0, mT] \), with \( z(0; w) = w \in B[r_2] \). Clearly,
\[
\varphi_m: B[r_2] \to \varphi_m(B[r_2]) \subset \mathbb{R}^2
\]
is an area-preserving homeomorphism and
\[
\varphi_m: \mathcal{A} \to \varphi_m(\mathcal{A}) \subset \mathbb{R}^2 \setminus \{O\},
\]
where \( A \) is the annulus: \( A := B[r_2]\setminus B(r_1) \). We also note that

\[
\varphi_m^{-1}(O) \in B(r_1).
\]

Arguing as in the proof of Theorem 1, we consider the polar coordinates \( \Theta(t; w) \) and \( \varrho(t; w) \) corresponding to \( z(t; w) \), defined by

\[
\Theta(t; w) := \Theta(t; z(\cdot; w)), \quad \varrho(t; w) := \varrho(t; z(\cdot; w)) = |z(t; w)|,
\]

\( \forall t \in [0, mT] \), so that, for \( w \in B[r_2] \), we obtain

\[
\varphi_m(w) = (\varrho(mT; w) \cos \Theta(mT; w), \varrho(mT; w) \sin \Theta(mT; w)).
\]

Hence, from the definition of \( n_*(r)_m \) and \( n^*(r)_m \), and by hypothesis \((m_0)\), we can find a constant \( \varepsilon_0 \), with

\[
0 < \varepsilon_0 < \frac{1}{2},
\]

such that

\[
-\Theta(mT; w) + \Theta(0; w) < 2\pi(\xi - \varepsilon_0), \quad \text{for all } |w| = r_1,
\]

\[
-\Theta(mT; w) + \Theta(0; w) > 2\pi(\xi + \varepsilon_0), \quad \text{for all } |w| = r_2.
\]

Now, we consider the lifting \( \varphi_m^0 \) of \( \varphi_m \), defined by

\[
\varphi_m^0(\theta, \rho) = (\theta^*, \rho^*), \quad \text{with } \theta^* = \theta + h(\theta, \rho), \quad \rho^* = f(\theta, \rho)
\]

and

\[
h(\theta, \rho) = \Theta(mT; \Pi(\theta, \rho)) - \Theta(0; \Pi(\theta, \rho)) + 2\pi\xi, \quad f(\theta, \rho) = |\varphi_m(\Pi(\theta, \rho))|,
\]

where \( \Pi: \mathbb{H}^+ = \{ (\theta, \rho): \rho > 0 \} \to \mathbb{R}^2 \setminus \{0\} \), is the standard covering projection given by \( \Pi(\theta, \rho) = (\rho \cos \theta, \rho \sin \theta) \). It can easily be checked that \( h \) and \( f \) are continuous functions defined in \( A^0 := \Pi^{-1}(A) \), which are \( 2\pi \)-periodic in the \( \theta \)-variable.

Since the twist condition for inner and outer boundaries of \( A \) is fulfilled by

\[
h(\theta, r_1) > 2\pi\varepsilon_0 > 0 > -2\pi\varepsilon_0 \geq h(\theta, r_2), \quad \forall \theta \in [0, 2\pi],
\]

we conclude that the generalized Poincaré–Birkhoff theorem of W. Ding can be applied (see Theorem A.2 in the Appendix). Thus, we obtain the existence of at least two (geometrically distinct) fixed points \((\theta_i, \rho_i)\) \((i = 1, 2)\), of \( \varphi_m^0 \) in \( A^0 \), satisfying \( h(\theta_i, \rho_i) = 0 \), for \( i = 1, 2 \).
Let $w_\# = \Pi(\theta_1, \rho_1)$, be a fixed point of $\varphi_m$ in $A$. From $h(\theta_1, \rho_1) = 0$, we obtain that $|\Theta(mT; w_\#) - \Theta(0; w_\#)| = 2\pi \zeta$.

Then, as $\varphi_m$ is the $m$th iterate of the Poincaré map associated to system \( (38_k) \), for an arbitrary but fixed $k \geq k_2$, we have that for each $k \geq k_2$, there exists a solution $z_k(\cdot) = (x_k(\cdot), y_k(\cdot))$, of \( (38_k) \), defined on $[0, mT]$, such that

$$z_k(0) = z_k(mT) \in A$$

and

$$|\Theta(mT; z_k) - \Theta(0; z_k)| = \left| \int_0^{mT} \frac{y_k'(t) x_k(t) - x_k'(t) y_k(t)}{x_k(t)^2 + y_k(t)^2} \, dt \right| = 2\pi \zeta.$$ 

Finally, by the same argument as in Lemma 4, we obtain that $z_k(\cdot)$ admits a subsequence converging to a solution $z(\cdot) = (x(\cdot), y(\cdot))$ of system

$$x' = y, \quad y' = -g(x) + p(t, x),$$

such that,

$$z(0) \in A, \quad z(t + mT) = z(t), \quad \forall t \in \mathbb{R}$$

and

$$\left| \frac{\Theta(mT; z) - \Theta(0; z)}{2\pi} \right| = \frac{1}{2\pi} \left| \int_0^{mT} \frac{y'(t) x(t) - x'(t) y(t)}{x(t)^2 + y(t)^2} \, dt \right| = \zeta.$$ 

Correspondingly, $x(\cdot)$ is a $mT$-periodic solution of Eq. (37), with

$$r_1 \leq \sqrt{x(0)^2 + x'(0)^2} \leq r_2.$$ 

Our last task is to prove that such solution $x(\cdot)$ is not $kT$-periodic, for any $1 \leq k \leq m - 1$. To this end, assume, by contradiction, that there is $k \in \mathbb{N}$, with

$$1 \leq k < m,$$

such that

$$x(t + kT) = x(t), \quad \forall t \in \mathbb{R}.$$ 

By the $kT$-periodicity of $p(\cdot, x(\cdot))$, we then obtain that $z(t + kT) = z(t)$, for all $t \in \mathbb{R}$ and therefore,

$$l := |\Theta(kT; z) - \Theta(0; z)|/2\pi < \zeta,$$
is a positive integer \( l \) counts the number of rotations of \( z \) around the origin, during the time \( kT < mT \). Using the fact that \( z(\cdot) \) is at the same time \( mT \)-periodic and \( kT \)-periodic, we obtain

\[
2\pi(l \cdot m) = |\Theta(kmT; z) - \Theta(0; z)| = 2\pi(k \cdot \zeta)
\]

and so,

\[
\zeta = l \cdot m/k.
\]

Thus, \( \zeta \) divides \( l \cdot m \) and this is a contradiction to the fact that \( \zeta \) is a prime number larger than \( l \) and \( m \). In this manner, we complete the proof.

Q.E.D.

As a direct consequence of Theorem 3, we have now the following:

**Corollary 3.** Assume (besides \( (g_0) \) and \( (p_0) \))

\[
\lim_{n \to +\infty} n_*(r) = +\infty. \tag{n_1}
\]

Then, Eq. (37) has a sequence \( \{x_j(\cdot)\} \) of \( mT \)-periodic solutions, such that

\[
\lim_{j \to +\infty} \left( \min\{\sqrt{x_j(t)^2 + x'_j(t)^2}; t \in \mathbb{R}\} \right) = +\infty. \tag{13}
\]

Moreover, each of the solutions \( x_j(\cdot) \) is not \( kT \)-periodic, for any \( 1 \leq k < m \).

**Proof.** First, we observe that \( (n_1) \) implies that

\[
\lim_{r \to +\infty} n_*(r)_m = +\infty. \tag{39}
\]

From Lemma 1, Lemma 2, and (39), we can find two increasing sequences \( \{r_1^{(j)}\}, \{r_2^{(j)}\} \) of positive real numbers and an increasing sequence of prime numbers \( \{\zeta_j\} \), with \( \zeta_j > m, \forall j \in \mathbb{N} \), such that the following conditions are satisfied:

\[
d < r_1^{(j)} < r_2^{(j)} < r_1^{(j+1)}, \quad \forall j \in \mathbb{N};
\]

\[
\lim_{j \to +\infty} r_1^{(j)} = \lim_{j \to +\infty} r_2^{(j)} = +\infty;
\]

for every \( j \in \mathbb{N}, j \geq 2 \) and each \( x(\cdot) \) solution of (1),

\[
r_1^{(j)} \leq \sqrt{x(0)^2 + x'(0)^2} \leq r_2^{(j)} \Rightarrow r_2^{(j-1)} \leq \sqrt{x(t)^2 + x'(t)^2} \leq r_1^{(j+1)}, \quad \forall t \in [0, mT];
\]

\[
n_*(r_2^{(j)}) > \zeta_j > n_*(r_1^{(j)}), \quad \forall j \in \mathbb{N}.
\]
Then, we can apply Theorem 3, arguing like in the corresponding part of Corollary 1. In particular, we obtain that the estimate (14) in Corollary 1, holds for all $t \in [0, mT]$. Hence the result easily follows. Q.E.D.

From Corollary 3, it is clear that we can obtain a multiplicity result for the subharmonic solutions of Eq. (37), by producing sufficient conditions for the validity of $(n)$. This goal has been already achieved in Theorem 2 of Section 4. Accordingly, we can state:

**Theorem 4.** Assume $(g_0), (p_0),$ and

$$\lim_{|c| \to +\infty} \tau(c) = 0. \quad \tag{\tau_0}$$

Then, the same conclusion of Corollary 3, holds.

(Recall that, according to (20), $\tau(s) := \sqrt{2 \left| \int_0^s \frac{d\xi}{\sqrt{G(s) - G(\xi)}} \right|}$, for $|s|$ large).

The proof of Theorem 4 is omitted, since it is exactly the same of that of Theorem 2 (see Section 5). The only difference is in the fact that this time, Corollary 3, instead of Corollary 1, is applied. We want only to mention that, according to the remark at the end of Theorem 2, the condition $(\tau_0)$ is reported here in an equivalent but different form (using the map $\tau$ instead of $\tau_0$).

Following Remark 2 it is not difficult to extend our result to the case when $p$ is a Carathéodory function, provided that $|p(t, x, y)|$ is essentially bounded by a constant $M_0$ on $[0, mT] \times \mathbb{R} \times \mathbb{R}$.

Finally, we can put together Theorems 2 and 4 and state a unifying result concerning the periodic boundary value problem for the Duffing's equation

$$x'' + g(x) = p(t), \quad (24)$$

where $g, p: \mathbb{R} \to \mathbb{R}$ are continuous functions, with $p(\cdot), T$-periodic.

**Theorem 5.** Assume $(g_0)$ and $(\tau_0)$. Then, for each $m \geq 1$, Eq. (24) has a sequence $\{x_{(m)}(\cdot)\}$ of $mT$-periodic solutions, such that

$$\lim_{j \to +\infty} (\min_{t \in \mathbb{R}} \{ |x_{(m)}(t)| + |x_{(m)}'(t)| \}) = +\infty.$$ 

Moreover, each of the solutions $x_{(m)}(\cdot)$ is not $kT$-periodic, for any $1 \leq k < m$.

Finally, if $p(\cdot)$ is not constant, then, each of the solutions $x_{(m)}(\cdot)$ has $mT$ as minimal period.

**Proof.** For the first part of the assertion, it is sufficient to observe that the assumption $(p_0)$ is fulfilled just choosing $M_0 \geq |p|_{\infty}$. Then, we can
apply Theorems 2 and 4 and obtain the result. Then, suppose that the forcing term $p(\cdot)$ is nonconstant and let $\sigma \leq T$, be its minimal period. Hence, $\sigma > 0$ and

$$\exists l \in \mathbb{N}, \quad l \cdot \sigma = T.$$  

We also observe that any periodic solution $x(\cdot)$ of Eq. (24) (with $x(\cdot)$ not necessarily $mT$-periodic, for $m \in \mathbb{N}$) must be nonconstant and its minimal period $T_x$ must satisfy the condition

$$\exists n \in \mathbb{N}, \quad n \cdot \sigma = T_x.$$  

(40)

Now, we apply Theorems 2 and 4 to Eq. (24), looking for solutions of period $(m \cdot l)\sigma$, for any $m \in \mathbb{N}$. In this manner, for every $m \in \mathbb{N}$, we can find a sequence of solutions $\{x_{(m)}(\cdot)\}$, satisfying (13) and such that each of the $x_{(m)}(\cdot)$ is $(m \cdot l)\sigma$-periodic and it is not $k \cdot \sigma$-periodic, for any $1 \leq k < m \cdot l$. Hence, recalling (40), we conclude that $(m \cdot l)\sigma = mT$ is actually the minimal period of the solution $x_{(m)}(\cdot)$, for all $j \in \mathbb{N}$. Thus we complete the proof. Q.E.D.

Remark 3. Theorem 5 generalizes completely the precedingly quoted result of W. Ding [14]. Indeed, first, we avoid requiring the unique solvability of the associated Cauchy problems, then, we consider the condition $(\tau_0)$ which is more general than (21), and, finally, we obtain the existence of infinitely many subharmonics of order $m$, for each $m \geq 2$. It is also clear that Theorem 4 contains as particular cases all the results already improved by W. Ding (like, e.g., [7, 24, 32, 31]). With respect to the analogous result of Nakajima [34] (which is proved under the more restrictive condition (21)), we do not assume any symmetry hypothesis on $p(t)$ or $g(x)$.

As a last step, we study the solvability of Eq. (24), assuming $g: \mathbb{R} \rightarrow \mathbb{R}$ continuous and $p: \mathbb{R} \rightarrow \mathbb{R}$, $T$-periodic, with $p \in L^1_{\text{loc}}$. In this case, solutions are intendent in the Caratheodory sense, namely, a $mT$-periodic solution $x(\cdot)$ of (24) (for $m \in \mathbb{N}$), is a $mT$-periodic function $x(\cdot): \mathbb{R} \rightarrow \mathbb{R}$, with $x'(\cdot)$ absolutely continuous and $x(\cdot)$ satisfying (24) for a.e. $t \in \mathbb{R}$.

In this case, the following generalization of Theorem 5 can be obtained.

**Theorem 6.** Assume $(g_0)$ and $(\tau_0)$. Then, the same conclusions of Theorem 5, hold.

Theorem 6 widely generalizes the earlier mentioned theorems of Fučík and Lovicar [20], and (for the scalar case, only) Bahri and Berestycki [2]. Indeed, recall, that in [20] no multiplicity of solutions was obtained and the condition (21) was assumed, while, for what concerns Bahri and
Berestycki's result for the scalar case, we observe that in [2], the authors assumed \( p \in L^2_{\text{loc}} \) and \( g \) satisfying a superlinear growth restriction less general than (21). Moreover, the existence and multiplicity of subharmonics was not considered in such paper.

The proof of Theorem 6 does not come immediately from the above theorems, but, nevertheless, it can be easily achieved by suitably modifying the arguments developed along the preceding pages. Indeed, for the proof of our last result, we need to obtain periodic solutions to the equivalent first order planar system

\[
x' = y + \tilde{p}(t), \quad y' = -g(x) + \bar{p},
\]

where

\[
\tilde{p} := \frac{1}{T} \int_0^T p(t) \, dt \quad \text{and} \quad \bar{p}(t) := \int_0^t (p(s) - \bar{p}) \, ds.
\]

Observe that, from \( p \in L^1_{\text{loc}} \), it follows that \( \tilde{p} : \mathbb{R} \to \mathbb{R} \) is \( T \)-periodic and continuous (even absolutely continuous) and that the solutions of system (41) are in a one-to-one correspondence with the (generalized) solutions of (24).

In order to solve the periodic problem for Eq. (41), we then consider the larger class of planar systems

\[
x' = y + p_1(t, x, y), \quad y' = -g(x) + p_2(t, x, y),
\]

where \( g, p_1, p_2 \) are continuous functions, with \( g \) satisfying condition \( (g_0) \) and with \( p_1, p_2, T \)-periodic in \( t \) and satisfying \( (p_0) \), for a suitable constant \( M_0 > 0 \). Then, following the same lines of Section 2, we fix a constant \( M > M_0 \) and perform a phase-plane analysis for all the systems of the form

\[
x' = y + q_1(t, x, y), \quad y' = -g(x) + q_2(t, x, y),
\]

with \( q_1, q_2 \) continuous functions, \( T \)-periodic in \( t \) and such that \( q_i \in \mathcal{P}, i=1,2 \), that is

\[
|q_i(t, x, y)| \leq M, \quad \forall t, x, y, \quad \forall i = 1, 2.
\]

Now, a first step toward the obtention of Theorem 6, consists in the proof that every solution \( z(t) = (x(t), y(t)) \) of (42), is noncontinuable. To this end, we use a trick from [26, p. 119] and consider the Liapunov-like function

\[
V(a, b) = (G(a) - G_{\text{min}}) + \frac{1}{2}(b + K(a))^2 + L,
\]

where \( K: \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function such that \( K(x) = M \text{ sign}(x) \), for \(|x| \geq 1\) and \( L > 0 \) is a sufficient large constant (depending
only on \( g, G, M \), and \( \max_{|x| \leq 1} |K'(x)| \). Then, we can easily evaluate the derivative of \( V \) along a solution \( z(t) \) of Eq. (42) and obtain, 
\[
\frac{d}{dt} V(z(t)) \leq c_0 V(z(t)),
\]
for all \( t \geq t_0 \), where \( z(\cdot) \) is defined (where \( c_0 > 0 \) is a suitable constant independent of \( z(\cdot) \)). Hence, using the property \( V(a, b) \to +\infty \) as \( a^2 + b^2 \to +\infty \), it follows that all the noncontrollable solutions of system (42) (with \( q_1, q_2 \in \mathbb{P} \)) are defined on \( t \geq t_0 \). A similar argument yields the continuability for \( t \leq t_0 \).

Then, by simple changes in the proofs of the results in Sections 2 and 3, it is easy to obtain a version of Lemmas 1, 2, 3, and 4 for system (42) and thus it is possible to prove results corresponding to Theorem 1, Corollary 1, Theorem 3, and Corollary 3, for Eq. (42,). The checking of this part is omitted, since it is just an elementary repetition of the preceding arguments. The only point that is worthy of mention is that the multiplicity of harmonic and subharmonic solutions for system (42,), can be obtained in the case when the associated Poincaré maps are homeomorphisms preserving the Lebesgue measure. By Liouville theorem, this happens, for instance, if
\[
p_1(t, x, y) = p_1(t, y), \quad p_2(t, x, y) = p_2(t, x)
\]
and, respectively, if \( q_1 = q_1(t, y) \) and \( q_2 = q_2(t, x) \), in system (42).

Finally, we prove the analogue of Theorem 2 for Eq. (42,).

For this last step, we have to modify the proof of Theorem 2 given in Section 5, by introducing two further subsets in the decomposition of the set \( D \). More precisely, for \( D \) defined as in Section 5, we set
\[
D_1 := \{(a, b) \in D : |a| \leq R_0, b > 0\}, \\
D_2 := \{(a, b) \in D : a \geq R_0, b \geq M\}, \\
D_3 := \{(a, b) \in D : a \geq R_0, |b| \leq M\}, \\
D_4 := \{(a, b) \in D : a \geq R_0, b \leq -M\}, \\
D_5 := \{(a, b) \in D : |a| \leq R_0, b < 0\}, \\
D_6 := \{(a, b) \in D : a \leq -R_0, b \leq -M\}, \\
D_7 := \{(a, b) \in D : a \leq -R_0, |b| \leq M\}, \\
D_8 := \{(a, b) \in D : a \leq -R_0, b > M\}.
\]

In this case, the analogue of the "cyclic rule" (30), from the transitions of a solution \( z(t) \) of Eq. (42), with \( |z(0)| \) large, is
\[
D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6 \to D_7 \to D_8 \to D_1.
\]

Now, if
\[
z(t) \in D_3, \quad \text{for } t \in J = [t_1, t_2].
\]
then, from the second Eq. (42), we have

\[ |y'(t)| \geq |g(x(t))| - M \geq 2M/\varepsilon, \]

provided that, according to hypothesis \((g_0)\), we have chosen the constant \(R_0 = R_0(\varepsilon)\) in such a way that \(g(s) \text{ sign}(s) \geq (2 + \varepsilon)M/|s|\), for \(|s| \geq R_0\).

Then, after an integration over the interval \(J\) and taking into account of the sign of \(y'\) on \(D_1^3\), we obtain

\[ 2M \geq y(t_1) - y(t_2) = \int_{t_1}^{t_2} |y'(s)| \, ds \geq 2M(t_2 - t_1)/\varepsilon. \]

So that, we can conclude with

\[ \text{meas}(J) = (t_2 - t_1) \leq \varepsilon, \]

that is, (35) of Section 5 is achieved.

The same estimate can be found if \(z(t) \in D_i^1\), \(\forall t \in J\).

If \(z(t) \in D_i\), \(\forall t \in J\), with \(i = 1, 5\) or if \(i = 2, 4, 6, 8\), we have only to repeat (with easy changes), the corresponding computations developed in "Step 2" of Section 5 (see also the analogous estimates in [16, 17]). Of course, during this process, we need to impose further conditions for the choice of the constant \(R_0\) (this part of the proof is omitted). Thus, we can re-establish (35) in all the situations. Then, like in the claim of Section 5, we finally obtain

for every \(\varepsilon: 0 < \varepsilon < 1\), there is \(R(\varepsilon) \geq d\), such that

\[ n_+(r) \geq \text{int}(T/8\varepsilon - 3), \forall r \geq R(\varepsilon). \]

In this manner, we can prove an analogue of Theorem 2 for Eq. (42,).

As a last step, working as in the first part of Section 6, we can restate all the results for the subharmonic solutions of system (42,), when \(p_1 = p_1(t, y)\) and \(p_2 = p_2(t, x)\) and so, the proof of Theorem 6 is accomplished, arguing like in the proof of Theorem 5.

Thus all the gaps are filled and our final result is proved. Q.E.D.

Remark 4. All the results contained in this paper, can be suitably extended to equations of the form

\[ x'' + f(t, x) = p(t, x, x'), \]

provided that we can work in a class of nonlinearities for which the global existence of the solutions of the Cauchy problems is ensured. We note that this problem arises even for the simpler equation

\[ x'' + f(t, x) = 0, \quad \text{(43)} \]
with \( f(\cdot, x)/x \to +\infty \), as \(|x| \to +\infty\), and, indeed, the global existence of the solutions is not always guaranteed, because of some counterexamples (see the pertinent discussion in [23]).

Another strong connection with the results discussed in this paper, comes from the Littlewood problem about the boundedness of all the solutions of the Duffing's equation (24), with \( g \) sublinear or superlinear at infinity.

We recall that, recently, Y. Long [29] constructed an example of a superlinear \( C^\infty \)-function \( g \) such that Eq. (24) possesses unbounded solutions for some piecewise constant periodic forcing term \( p \). On the other hand, in [9], T. Ding solved the Littlewood conjecture in the affirmative, assuming some supplementary conditions of symmetry on \( g \) or \( p \). Further results of boundedness for the solutions have been recently obtained also by Liu Bin (see [27]), using the Moser "twist" theorem [33], for a particular equation of the form (43). (Incidentally, we have to note that the example by Y. Long shows that the Littlewood conjecture cannot be true in its full generality and therefore, there is a gap in the main result of [10]).

It could be interesting to check whether some of these results of boundedness can be obtained by assuming the relaxed condition (\( \tau_0 \)) which generalizes the more classical superlinear growth restrictions.

7. Appendix

In this final part of the paper, we recall the main tools which are used for the proof of the fixed points for the Poincaré's operators associated to planar Duffing's systems considered in the preceding sections.

First, we state the fixed point theorem of Fučík and Lovíšek in [20] in a form which is useful for our purposes.

**Theorem A.1.** Let \( \Phi: \mathbb{R}^2 \to B[R] \to \mathbb{R}^2 \) be a continuous mapping, \( M \subseteq B[R] \) a compact set such that 0 is contained in a bounded component of \( \mathbb{R}^2 \setminus M \) and such that

\[
\forall w \in M, \quad \exists \lambda \in \mathbb{R}, \quad \Phi(w) = \lambda w.
\]

Then \( \Phi \) has a fixed point.

In [20, 19] it is assumed that \( \Phi \) is defined on \( \mathbb{R}^2 \) and the existence of a fixed point \( w^* \) of \( \Phi \), with \( w^* \in \text{co}(M) \), follows (see also the corresponding remark in [40, p. 294]). However, it is clear that Theorem A.1 is a straightforward consequence of Fučık and Lovíšek's fixed point theorem.
Indeed, for \( \Phi \) defined on the closed disc \( B[R] \), as above, it is sufficient to consider any continuous extension \( \Phi_0 \) of \( \Phi \) to \( \mathbb{R}^2 \) and observe that there is (according Fucik and Lovicar result) a fixed point \( w^* \) of \( \Phi_0 \) with \( w^* \in \text{co}(M) \subseteq B[R] \). Thus we also have \( w^* = \Phi(w^*) \) and Theorem A.1 is proved. Q.E.D.

Second, we recall the generalized version of Poincaré–Birkhoff "twist" theorem due to W. Ding in [14]. The proof of such result is based on a preceding work of H. Jacobowitz (see [25]) extending the Poincare–Birkhoff theorem. For a detailed proof of the Poincaré–Birkhoff theorem according to its original formulation we refer to [4] (for another extension, see also [18]).

For a better understanding of the next part (taken from [14]) we recall that any continuous map \( \Phi: \mathbb{R}^2 \supseteq \text{dom } \Phi \to \mathbb{R}^2 \setminus \{O\} \) can be lifted to a continuous map \( \Phi^0: H^+ \supseteq \text{dom } \Phi^0 \to H^+ \), where \( H^+ = \{ (\theta, \rho): \rho > 0 \} \) is the universal covering space of \( \mathbb{R}^2 \setminus \{O\} \), with covering projection \( \Pi: H^+ \to \mathbb{R}^2 \setminus \{O\} \), \( \Pi(\theta, \rho) = (\rho \cos \theta, \rho \sin \theta) \) and \( \Pi \circ \Phi^0 = \Phi \circ \Pi \). (See, e.g., [21].)

In the proofs of Theorem 1 in Section 2 and Theorem 3 in Section 6, we use the generalized Poincaré–Birkhoff theorem of W. Ding according to the following statement.

Let \( 0 < r_1 < r_2 \) and consider the sets \( A = \{ w \in \mathbb{R}^2: r_1 \leq |w| \leq r_2 \} = B[r_2] \setminus B(r_1) \) and \( A^0 = \{ (\theta, \rho): r_1 \leq \rho \leq r_2 \} \subseteq H^+ \). Observe that \( A^0 = \Pi^{-1}(A) \). Then we have:

**Theorem A.2.** Let \( \varphi: B[r_2] \to \varphi(B[r_2]) \subseteq \mathbb{R}^2 \), be an area-preserving homeomorphism such that \( \varphi(A) \subseteq \mathbb{R}^2 \setminus \{O\} \). Let \( \varphi^0: A^0 \to H^+ \) be a lifting of \( \varphi|_A \) (i.e., \( \Pi \circ \varphi^0 = \varphi \circ \Pi \) in \( A^0 \)) of the form

\[
\theta^* = \theta + h(\theta, \rho), \quad \rho^* = f(\theta, \rho),
\]

where \( h \) and \( f \) are continuous in \( A^0 \) and \( 2\pi \)-periodic in \( \theta \). Assume that \( \varphi \) is twist on \( A \), i.e.,

\[
h(\theta, r_1) \cdot h(\theta, r_2) < 0, \quad \forall \theta \in [0, 2\pi]
\]

and

\[
O \in \varphi(B(r_1)).
\]

Then, \( \varphi \) has at least two fixed points in \( A \). Moreover, such fixed points of \( \varphi \) correspond to two (geometrically distinct) fixed points \( (\theta_i, \rho_i) \) \( (i = 1, 2) \), of \( \varphi^0 \) in \( A^0 \), satisfying \( h(\theta_i, \rho_i) = 0 \), for \( i = 1, 2 \).
We note that for the obtention of the results of our paper, we always met the twist condition

$$h(\theta, r_1) > 0 > h(\theta, r_2), \quad \forall \theta \in [0, 2\pi]$$

(see Corollaries 1 and 3). It seems to be worthy to mention that, in this case, Theorem A.2 is a straightforward consequence of [15, Th. 1] as well.

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