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Loop quantum cosmology and slow roll inflation

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ABSTRACT

In loop quantum cosmology (LQC) the big bang is replaced by a quantum bounce which is followed by a robust phase of super-inflation. Rather than growing unboundedly in the past, the Hubble parameter *vanishes* at the bounce and attains a *finite universal maximum* at the end of super-inflation. These novel features lead to an unforeseen implication: in presence of suitable potentials all LQC dynamical trajectories are funneled to conditions which virtually guarantee slow roll inflation with more than 68 e-foldings, without any input from the pre-big bang regime. This is in striking contrast to certain results in general relativity, where it is argued that the a priori probability of obtaining a slow roll with 68 or more e-foldings is suppressed by a factor e^{-204}.

1. Introduction

Inflationary models have had striking successes, especially in providing a natural explanation of structure formation. These successes bring to fore-front an old question: Does a sufficiently long, slow roll inflation require fine tuning of initial conditions or does it occur generically in a given theoretical paradigm? (See e.g. [1–4]). Such a slow roll requires that initially the inflaton must be correspondingly high-up in the potential. How did it get there? Is it essential to invoke some rare quantum fluctuations to account for the required initial conditions because the a priori probability for their occurrence is low? Or, is a sufficiently long, slow roll inflation robust in the sense that it is realized in 'almost all' dynamical trajectories of the given theory?

To make these questions precise, one needs a stream-lined framework to calculate probabilities of various occurrences *within a given theory*. A mathematically natural framework to carry out this analysis was introduced over two decades ago (see e.g. [5–7]). It invokes Laplace's principle of indifference [8] to calculate the *a priori* probabilities for various occurrences. More precisely, the idea is to use (a flat probability distribution $P(s) = 1$ and) the canonical Liouville measure $d\mu_L$ on the space $\mathbb{S}$ of solutions s of the theory under consideration to calculate the *relative volumes* in $\mathbb{S}$ occupied by solutions with desired properties [5]. In our case, then, the a priori probability is given by the *fractional* Liouville volume occupied by the sub-space of solutions in which a sufficiently long, slow roll inflation occurs. Further physical input can provide a sharper probability distribution $P(s)$ and a more reliable likelihood than the 'bare' a priori probability. However, a priori probabilities can be directly useful if they are very low or very high. In these cases, it would be an especially heavy burden on the fundamental theory to come up with the physical input that significantly alters them.

There is however a conceptual obstacle in this calculation because of the initial singularity in general relativity, where the matter density and curvature both diverge: there is no clean starting point to begin one's counting of e-foldings. For definiteness consider the standard $m^2\phi^2$ potential. If we allow arbitrarily high energy densities at the onset of inflation, then we have to allow initial configurations in which the potential energy is arbitrarily large, i.e., initially the inflaton is arbitrarily high-up in the potential. Then it is easy to achieve a long slow roll. However, one cannot really trust general relativity at arbitrarily high densities and curvatures. Therefore it is not clear that this conclusion is physically reliable. Thus, because of the initial singularity, we know we cannot trust general relativity in certain regimes but the theory itself does not provide clear guidance to restrict the possible initial conditions; it does not have an in-built mechanism to determine its domain of validity.

In addition, calculation of the a priori probability can be subtle because the total Liouville measure of the space of all solutions is often infinite [7]. However, sometimes it is possible to overcome this difficulty by introducing physically motivated regularization schemes and show that the desired probability is insensitive to the details of the scheme. Recently, this strategy was used by Gibbons and Turok [4] to argue that the probability of $\mathcal{N}$ e-folds of a slow roll, single field inflation is suppressed by a factor of $e^{-3\mathcal{N}}$ in general relativity. They concluded that, even if a cosmological model in general relativity allows inflation, one must invoke an extremely

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sharp probability distribution $P(s)$ order to explain *why inflation actually occurred*; "the question of why and how inflation started remains a deep mystery and a challenge for the fundamental theory."

Loop quantum cosmology (LQC) provides a new arena to analyze this issue because the big bang singularity is naturally resolved and replaced by a big bounce due to quantum geometry effects [9–13]. Now the question can be posed in an unambiguous way because all solutions are regular. Therefore one can start counting the number of e-foldings from the bounce. The matter density operator has a *finite, universal upper-bound* [11], whence there is an absolute upper bound on how high the inflaton can be up the potential. An unambiguous question now is: Can there still be sufficiently large number of slow roll e-foldings?

The purpose of this communication is to report the main result of a detailed analysis of these questions: in presence of suitable potentials, every solution enjoys an inflationary phase and the a priori probability of obtaining at least 68 e-foldings, desired from phenomenological considerations, is extremely close to 1. Thus, the conclusion reached by Gibbons and Turok in general relativity is reversed in LQC. Away from the Planck regime, LQC is virtually indistinguishable from general relativity. However, in the Planck regime, there are huge differences and these are crucial to our analysis. In particular, since the big-bang is replaced by a non-singular big bounce, initial conditions can be specified at the bounce in a fully controlled fashion. There is a robust phase of *super-inflation* immediately after the bounce [14,15]. Somewhat surprisingly, it shepherds most of the LQC solutions to phase space regions from which a long, slow roll, expansion is almost inevitable. Although several phenomenological consequences of the distinguishing features of LQC have been studied (see, e.g., [16–19]), implications on slow roll inflation have not received as much attention. To our knowledge there have been only two investigations along these lines. The first [20] is aimed at calculating a priori probabilities, as in this Letter, but the bounce and super-inflation were ignored. These were considered in [21] but systematic calculation of probabilities, e.g. through the use of a measure, was not carried out. In terms of the natural Liouville measure, the solutions considered there correspond only to a *very* small region of $\mathbb{S}$.

The material is organized as follows. Section 2 recalls the salient features of LQC that are used in our analysis. Section 3 summarizes the technical results on the super-inflation and inflation phases of LQC dynamics. This discussion of dynamics in the Planck era and during the subsequent slow roll will have applications well beyond the main conclusions of this Letter, e.g., in the analysis of how perturbations evolve during the bounce. Section 4 uses the Liouville measure to show that, in presence of a suitable potential, the a priori probability of inflation is *very* close to 1. Section 5 compares and contrasts our methods and results with those in the literature.

2. Loop quantum cosmology: relevant results

In the LQC treatment of simple cosmological models, the big bang and big crunch singularities are naturally resolved [22]. The origin of this resolution lies in the *quantum geometry effects* that are at the heart of loop quantum gravity [23–25]. Exotic matter is not needed; indeed matter fields can satisfy all the standard energy conditions. Detailed analysis has been carried out in a variety of models: the $k = 0$, ± 1 Friedmann–Lemaître–Robertson–Walker (FLRW) space–times with or without a cosmological constant [10,12,13,26]; Bianchi models [27–29] which admit anisotropies and gravitational waves; and Gowdy models [30] which admit inhomogeneities, and therefore an infinite number of degrees of freedom. The FLRW models have been studied most extensively, using both analytical and numerical methods to solve the exact quantum equations of LQC [10–13]. In these models, the big bang and the big-crunch are replaced by a quantum bounce, which is followed by a robust phase of super-inflation. Interestingly, full quantum dynamics, including the bounce, is well-approximated by certain effective equations. (For a recent review, see [31].)

In this Letter we restrict ourselves to the phenomenologically more interesting case of the $k = 0$ FLRW model (although the method is applicable also to the $k = 1$ case). The matter source will be a scalar field with positive kinetic energy and a suitable potential. Since all the prior discussion of probabilities is based on general relativity, to facilitate comparison we use effective equations rather than the full quantum theory. Finally, we will use the natural Planck units $c = \hbar = G = 1$ (rather than $8\pi G = 1$, often employed in cosmology). The fundamental time unit, $\sqrt{G\hbar/c^5}$, will be referred to as *a Planck second*.

In LQC, spatial geometry is encoded in the volume of a fixed, fiducial cell, rather than the scale factor a; $v = (\text{const}) \times a^3$. The conjugate momentum is denoted by b. On solutions to Einstein's equations, $b = \gamma H$ [11]. (Here $H = \dot{a}/a$ is the standard Hubble parameter and γ is the Barbero–Immirzi parameter of LQC whose value, $\gamma \approx 0.24$, is fixed by the black hole entropy calculation.) However, LQC modifies Einstein dynamics and on solutions to the LQC effective equations we have

$$H = \frac{1}{2\gamma\lambda} \sin 2\lambda b \approx \frac{0.93}{\ell_{\text{Pl}}} \sin 2\lambda b \tag{1}$$

where $\lambda^2 \approx 5.2\ell_{\text{Pl}}^2$ is the 'area-gap', the smallest non-zero eigenvalue of the area operator. In LQC, b ranges over $(0, \pi/\lambda)$ and general relativity is recovered in the limit $\lambda \to 0$. Quantum geometry effects modify the geometric, left side of Einstein's equations. In particular, the Friedmann equation becomes

$$\frac{\sin^2 \lambda b}{\gamma^2 \lambda^2} = \frac{8\pi}{3} \rho \equiv \frac{8\pi}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right). \tag{2}$$

To compare with the standard Friedmann equation $H^2 = (8\pi/3)\rho$, it is often convenient to use (1) to write (2) as

$$\frac{1}{9} \left(\frac{\dot{v}}{v} \right)^2 \equiv H^2 = \frac{8\pi}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}} \right) \tag{3}$$

where $\rho_{\text{crit}} = \sqrt{3}/32\pi^2\gamma^3 \approx 0.41\rho_{\text{Pl}}$. By inspection it is clear from Eqs. (1)–(3) that away from the Planck regime — i.e., when $\lambda b \ll 1$, or, $\rho \ll \rho_{\text{crit}}$ — we recover classical general relativity. However, modifications in the Planck regime are drastic. The main features of this new physics can be summarized as follows.

- In general relativity, the Friedmann equation implies that if the matter density is positive, $\dot{a}$ cannot vanish. Therefore every solution represents *either* a contracting universe *or* an expanding one. By contrast, the LQC modified Friedmann equation (3) implies that $\dot{v}$ vanishes at $\rho = \rho_{\text{crit}}$. This is a quantum bounce. To its past, the solution represents a contracting universe with $\dot{v} < 0$ and to its future, an expanding one with $\dot{v} > 0$.
- As is customary in the literature on probabilities, let us ignore the exceptional de Sitter solutions with eternal inflation. On all other solutions b decreases monotonically from $b = \pi/\lambda$ to 0. Eqs. (2) and (3) imply that $b = \pi/2\lambda$ at the bounce. Thus, each solution undergoes precisely one bounce. The Hubble parameter $H = \dot{v}/3v$ *vanishes* at the bounce and Eq. (1) implies that it is bounded on the full solution space $\mathbb{S}$; $|H| \lesssim 0.93/\ell_{\text{Pl}}$. By contrast, in general relativity, H is large in the entire Planck regime and diverges at the singularity.

- If the potential $V(\phi)$ is bounded below, say $V \geqslant V_o$, then it follows from (2) that $\dot{\phi}^2$ is bounded by $2\rho_{\rm crit} - 2V_o$. If V grows unboundedly for large $|\phi|$, then $|\phi|$ is also bounded. For example, for $V = m^2\phi^2/2$, we have $m|\phi|_{\rm max} = 0.90$.
- When the potential is bounded below, $|\dot{H}|$ is bounded above by $10.29/\ell_{\rm Pl}^2$. The Ricci scalar — the only non-trivial curvature scalar in these models — is bounded above by $31/\ell_{\rm Pl}^2$. Thus, physical quantities which diverge at the big bang of general relativity cannot exceed certain finite, maximum values in LQC. One can also show that if $v \neq 0$ initially, it cannot vanish in finite proper time along any solution. Thus, *the LQC solutions are everywhere regular irrespective of whether one focuses on matter density, curvature or the scale factor.*

Next, the full set of space–time equations of motion can be written in terms of $v(t), \phi(t)$. These variables are subject to the constraint (3) and evolve via:

$$\ddot{v} = \frac{24\pi v}{\rho_{\rm crit}}\big[\big(\rho - V(\phi)\big)^2 + V(\phi)\big(\rho_{\rm crit} - V(\phi)\big)\big],$$
$$\ddot{\phi} + \frac{\dot{v}}{v}\dot{\phi} + V_{,\phi} = 0. \qquad (4)$$

Our task is to obtain the Liouville measure on the space $\mathbb{S}$ of solutions to these equations.

For this, we first construct the phase space Γ. It consists of quadruplets $(v, b; \phi, p_{(\phi)})$, with $\lambda b \in [0, \pi/2]$. The Liouville measure on Γ is simply $\mathrm{d}\mu_{\rm L} = \mathrm{d}v\,\mathrm{d}b\,\mathrm{d}\phi\,\mathrm{d}p_{(\phi)}$. The LQC Friedmann equation implies that these variables must lie on a constraint surface $\bar{\Gamma}$ defined by

$$\frac{3\pi}{2\lambda^2}\sin^2\lambda b = \frac{p_{(\phi)}^2}{2v^2} + 4\pi^2\gamma^2 V(\phi). \qquad (5)$$

They evolve via

$$\dot{v} = \frac{3v}{2\gamma}\frac{\sin 2\lambda b}{\lambda}, \qquad \dot{b} = -\frac{p_{(\phi)}^2}{\pi\gamma v^2},$$
$$\dot{\phi} = \frac{p_{(\phi)}}{2\pi\gamma v}, \qquad \dot{p}_{(\phi)} = -2\pi\gamma|v|V_{,\phi}. \qquad (6)$$

As is well-known, the space of solutions $\mathbb{S}$ is naturally isomorphic to a gauge fixed surface, i.e., a 2-dimensional surface $\hat{\Gamma}$ of $\bar{\Gamma}$ which is intersected by each dynamical trajectory once and only once. Since b is monotonic in each solution, an obvious strategy is to choose for $\hat{\Gamma}$ a 2-dimensional surface $b = b_o$ (a fixed constant) within $\bar{\Gamma}$. Symplectic geometry considerations unambiguously equip $\hat{\Gamma}$ — and hence the solution space $\mathbb{S}$ — with an induced Liouville measure $\mathrm{d}\hat{\mu}_{\rm L}$. Since the dynamical flow preserves the Liouville measure, $\mathrm{d}\hat{\mu}_{\rm L}$ on $\mathbb{S}$ is *independent of the choice of* b_o. The most natural choice in LQC is to set $b_o = \pi/2\lambda$ so that $\hat{\Gamma}$ is just the 'bounce surface'. We will make this choice because it also turns out to be convenient for calculations.

Then $\hat{\Gamma}$ is naturally coordinatized by $(\phi_{\rm B}, v_{\rm B})$, the scalar field and the volume at the bounce. Since $b = \pi/2\lambda$, the constraint (5) determines $p_{(\phi)}$ (or, equivalently, $\dot{\phi}$) up to sign which, without loss of generality, will be taken to be non-negative. The induced measure on $\mathbb{S}$ can be written explicitly as:

$$\mathrm{d}\hat{\mu}_{\rm L} = \frac{\sqrt{3\pi}}{\lambda}[1 - F_{\rm B}]^{\frac{1}{2}}\,\mathrm{d}\phi_{\rm B}\,\mathrm{d}v_{\rm B} \qquad (7)$$

where $F_{\rm B} = V(\phi_{\rm B})/\rho_{\rm crit}$ is the fraction of the total density that is in the potential energy at the bounce. The total Liouville volume of $\hat{\Gamma} \equiv \mathbb{S}$ is infinite because, although $\phi_{\rm B}$ is bounded for suitable potentials such as $m^2\phi^2$, $v_{\rm B}$ is not. However, this non-compact direction represents gauge on the space of solutions $\mathbb{S}$: If $(\phi(t), v(t))$ is a solution to (2) and (4), so is $(\phi(t), \alpha v(t))$ and this rescaling by a constant α simply corresponds to a rescaling of spatial coordinates (or of the fiducial cell) under which physics does not change. Therefore, as discussed in Section 4, there is a natural prescription to calculate fractional volumes of physically relevant sub-regions of $\hat{\Gamma}$ by factoring out the gauge orbits.

3. Super-inflation and inflation

For our purposes it suffices to focus just on the *post bounce* part of solutions; explicit information from the pre-bounce part is not needed anywhere in the analysis. As explained in Section 1, the key question is: What is the fractional Liouville volume in $\mathbb{S}$ occupied by solutions that exhibit a sufficiently long inflation? To answer it in detail, as is common in literature (see, e.g. [2,4]), we will use $V(\phi) = (1/2)m^2\phi^2$. Then, as we already noted, (5) implies that $m\phi_{\rm B} \in [-0.90, 0.90]$. For definiteness, we will use the phenomenological value [32], $m = 6 \times 10^{-7}\,\mathrm{M_{Pl}}$ (recall that we have set G=1 rather than $8\pi G = 1$). However, as explained in Section 5, the main results are robust even if m were to change by a couple of orders of magnitude.

The idea is to allow all possible initial conditions at the bounce and construct dynamical trajectories by solving (6). The problem can be divided into three parts using the value of the fraction $F_{\rm B}$ at the bounce. In each part, one can introduce suitable approximations to analyze dynamics. Because the evolution equations (4) are invariant under $\phi \to -\phi, \dot{\phi} \to -\dot{\phi}, v \to v, \dot{v} \to \dot{v}$, it suffices to restrict ourselves to initial data with $\dot{\phi}_{\rm B} \geqslant 0$ at the bounce, allowing $\phi_{\rm B}$ to take both positive and negative values. Let us begin with the part $\mathbb{S}^+$ of solutions on which $\phi_{\rm B}$ is non-negative. Then the main results can be summarized as follows. (See also Table 1.)

(i) $F_{\rm B} < 10^{-4}$: *Extreme kinetic energy domination at the bounce.* At the bounce the Hubble parameter H vanishes. However, there is a short phase of super-inflation lasting a fraction of a Planck second during which H increases very rapidly to its maximum value $H_{\rm max} = 0.93$. At this point $\dot{H}$ vanishes and then H starts decreasing and continues to decrease during the rest of the evolution. Since $\dot{\phi} > 0$, the inflaton climbs up the potential during super-inflation and continues to do so after super-inflation ends, till it reaches a turn-around point where $\dot{\phi} = 0$. Then it starts descending. Very soon after that, $\ddot{\phi}$ vanishes. This is the onset of slow roll inflation: during this phase $\dot{H}/H^2$ is in the range 1.6×10^{-2}–3.3×10^{-10} so the slow roll conditions are met. The time required to reach this onset starting from the bounce is in the range of 10^6–10^2 $s_{\rm Pl}$ where $s_{\rm Pl}$ denotes Planck seconds. The number of e-foldings during this slow roll is given approximately by

$$\mathcal{N} \approx 2\pi\left(1 - \frac{\phi_o^2}{\phi_{\rm max}^2}\right)\phi_o^2\ln\phi_o \qquad (8)$$

where ϕ_o is the value of the scalar field at the *onset* of inflation and $\phi_{\rm max} = 1.5 \times 10^6$. Now, ϕ_o increases monotonically with $\phi_{\rm B}$ (and is always larger than $\phi_{\rm B}$). For $\phi_{\rm B} = 0.99$, we have $\phi_o = 3.24$ and $\mathcal{N} = 68$. Thus, *for a kinetic energy dominated bounce, there is a slow roll inflation with over* 68 *e-foldings for all* $\phi_{\rm B} > 1$, i.e., if $F_{\rm B} > 4.4 \times 10^{-13}$.

(ii) $10^{-4} < F_{\rm B} < 0.5$: *Kinetic energy domination at the bounce.* The LQC departures from general relativity are now increasingly significant. The super-inflation era is similar to case (i). However, now the value of $\phi_{\rm B}$ is higher and that of $\dot{\phi}_{\rm B}$ lower while, as before, H is very high at the end of super-inflation. Therefore, the coefficient of friction, H/m^2, is large and one arrives at the slow roll conditions within 10–100 $s_{\rm Pl}$ after the bounce.

Table 1
Values of the proper time, the Hubble parameter, the scalar field and their time derivatives at onset of slow roll (where $\ddot{\phi} = 0$). $F_B = V(\phi_B)/\rho_{crit}$ is the ratio of the potential energy density to the total energy density at the bounce. If the value ϕ_B of the scalar field is positive, the inflaton rises up the potential after the bounce while if ϕ_B is negative it descends down the potential (because $\dot{\phi}_B$ is assumed to be positive). For $\phi_B > 0$, there are 68 e-foldings if $F_B = 4.4 \times 10^{-13}$. The bounce is taken to occur at $t = 0$.

$F_B = V(\phi_B)/\rho_{crit}$	Sign$[\phi_B]$	t	ϕ	$\dot{\phi}$	H	$\dot{H}$
0	+/−	1.6×10^6	2.3	-9.7×10^{-8}	2.8×10^{-6}	-1.2×10^{-13}
4.4×10^{-13}	+	1.2×10^6	3.2	-9.7×10^{-8}	4.0×10^{-6}	-1.2×10^{-13}
	−	2.1×10^6	1.3	9.6×10^{-8}	1.6×10^{-6}	-1.2×10^{-13}
1×10^{-4}	+	7.6×10^2	1.5×10^4	-9.8×10^{-8}	1.9×10^{-2}	-1.2×10^{-13}
	−	6.6×10^2	-1.5×10^4	9.8×10^{-8}	1.9×10^{-2}	-1.2×10^{-13}
0.5	+	1.6×10^1	1.1×10^6	-1.4×10^{-7}	9.3×10^{-1}	1.4×10^{-19}
	−	1.5×10^1	-1.1×10^6	1.4×10^{-7}	9.3×10^{-1}	-1.4×10^{-19}
0.8	+	2.0×10^1	1.3×10^6	-2.2×10^{-7}	7.4×10^{-1}	3.6×10^{-13}
	−	1.8×10^1	-1.3×10^6	2.2×10^{-7}	7.4×10^{-1}	3.6×10^{-13}

Consequently, now the change $(\phi_o - \phi_B)$ is negligible, a key feature not shared by regime (i). At the onset of slow roll inflation, the Hubble parameter is now given to an excellent approximation by

$$H_o \approx \left[\frac{8\pi}{3}\rho_{crit}F_B(1-F_B)\right]^{1/2} \approx 1.9\left[F_B(1-F_B)\right]^{1/2} \tag{9}$$

and decreases *very* slowly with $\dot{H}/H^2 < 3.5 \times 10^{-10}$. Thus, the Hubble parameter is essentially frozen to the value (9) and the slow roll condition is met even more easily. This value of H is very high, in the range $1.9 \times 10^{-2}\ s_{Pl}^{-1}$ to $9.3 \times 10^{-1}\ s_{Pl}^{-1}$. The Hubble freezing is an LQC phenomenon: It relies on the fact that H acquires its largest value $H_{max} = 0.93\ s_{Pl}^{-1}$ at the end of super-inflation (and, in the case under consideration, $\dot{\phi}_B$ is not large enough to decrease H more than two orders of magnitude). Eq. (8) implies that throughout this range of F_B there are more than 68 e-foldings.

(iii) $0.5 < F_B < 1$: *Potential energy domination at the bounce.* Now the LQC effects dominate. Again, because $\dot{\phi} > 0$, the inflaton climbs up the potential but now the turn around ($\dot{\phi} = 0$) occurs *during super-inflation*. The change $(\phi_o - \phi_B)$ is even more negligible because the kinetic energy at the bounce is lower than that in case (ii). The Hubble parameter again freezes at the onset of inflation to the value given in (9). The slow roll conditions are easily met as $\dot{H}/H^2$ is less than 1×10^{-11} when $\ddot{\phi} = 0$ (or very soon thereafter). A difference from the slow roll inflation of (i) and (ii) above is that H continues to grow during the slow roll because we are in the super-inflation phase. There are many more than 68 e-foldings already in the super inflation phase. The inflaton exits the super-inflation phase with H at its maximum value, $H_{max} = 0.93$ and little kinetic energy. Therefore, the friction term is large and the inflaton enters a long slow roll inflationary phase. There are many more than 68 e-foldings also in this phase.

Finally, let us consider the part $\mathbb{S}^-$ of the solution space on which $\phi_B < 0$. The main difference now is that the inflaton starts rolling down the potential immediately after the bounce. As before, in case (ii) the Hubble freezing occurs soon after the end of super-inflation and in (iii) during super-inflation. The value of H_o is again given by (9). In case (i), differences can arise from the part $\mathbb{S}^+$ of the solution space because now the kinetic energy is very large at the bounce point so the inflaton can transit from a negative to a positive value before the onset of inflation. But after the onset, the situation is the same as in (ii). In this case, there are more than 68 e-foldings if $F_B > 1.4 \times 10^{-11}$ or $\phi_B \notin [-5.7, 0]$.

These general features of LQC dynamics emerge from analytical calculations based on approximations that are tailored to the three cases considered above. They were confirmed by detailed numerical simulations performed in MATLAB using a Runge–Kutta (4, 5) algorithm (ode45) to solve the set of coupled ODEs. Both relative and absolute tolerances were set at 3×10^{-14} and the preservation of the Hamiltonian constraint (5) to this order was verified on each solution. To ensure numerical accuracy, the natural logarithm of volume was treated as fundamental in the simulations. As noted above, the Barbero–Immirzi parameter was set at 0.24 and inflaton mass 6×10^{-7} (in units $c = \hbar = G = 1$). A large number of simulations were performed. Table 1 summarizes a few illustrative results.

4. Measure and probabilities

As explained in section 2, the space $\mathbb{S}$ of solutions can be coordinatized by pairs (ϕ_B, ν_B). However, physics does not change under $(\phi_B, \nu_B) \to (\phi_B, \alpha\nu_B)$, where α is a constant. In particular, the number of slow-roll e-foldings is insensitive to this rescaling of ν_B. Therefore, only those regions $\mathcal{R}$ in $\mathbb{S}$ that contain complete gauge orbits are physically relevant. These are of the type $\mathcal{R} = I \times \mathbb{R}^+$ where I is a closed interval in $[-\phi_{max}, \phi_{max}]$ and $\mathbb{R}^+$ denotes the ν_B axis. To calculate fractional volumes $P_{\mathcal{R}}$ of such regions it is natural to factor out by the 'volume of the gauge orbits'. This suggests an obvious strategy, commonly used in the physics literature:

$$P_{\mathcal{R}} = \lim_{\nu_0 \to 0} \frac{\text{Liouville Volume of } [I \times I_{\nu_0}]}{\text{Liouville Volume of } [I_{total} \times I_{\nu_0}]} = \frac{\int_I d\phi_B\,[1-F_B]^{\frac{1}{2}}}{\int_{-\phi_{max}}^{\phi_{max}} d\phi_B\,[1-F_B]^{\frac{1}{2}}} \tag{10}$$

where we have set $I_{\nu_0} = [\nu_0, 1/\nu_0]$ (with $\nu_0 > 0$). This physical idea can be mathematically justified by the 'group averaging technique' [33] to obtain a physical measure on $\mathbb{S}$ by averaging $d\hat{\mu}_L$ over the orbit of the 'gauge group.'

Let us now apply this strategy to calculate the probability that, prior to re-heating, there are at least 68 slow roll e-foldings in LQC. Since F_B ranges over $[0, 1]$ and there are requisite number of e-foldings if $F_B > 1.4 \times 10^{-11}$, it follows from (10) that the required probability is greater than 0.99999. Moreover, numerical simulations show that even when $F_B \leqslant 1.4 \times 10^{-11}$ there are at least 6.1 e-foldings in LQC. Thus the probability of obtaining at least 6.1 e-foldings is 1. By contrast, the Gibbons and Turok result implies that in general relativity even this probability is suppressed by a factor of $e^{-18.3} \approx 1.1 \times 10^{-8}$ [4].

5. Discussion

In this Letter we reported the results of a systematic analysis of LQC dynamics in the context on inflation. In LQC, all solutions of the quantum as well as effective equations are regular and, for the FLRW models under consideration, effective equations provide an excellent approximation to the full quantum dynamics. To facilitate comparison with the earlier work in general relativity, we focused on effective equations. Every solution of these equations is determined by its initial data at the bounce. We divided the space of these initial data into three classes and used approximation schemes to extract the behavior of the dynamical trajectories they lead to. These analytical results were then confirmed by detailed and high precision numerical calculations where both relative and absolute tolerances were set at 3×10^{-14}. By examining *all* the dynamical trajectories (not just 'generic ones') we were able to conclude that for the $m^2\phi^2$ potential with m chosen to satisfy phenomenological constraints, the a priori probability of obtaining at least 68 e-foldings is greater than 0.99999. By contrast, the Gibbons and Turok [4] argument says that, in general relativity, this a priori probability is suppressed by a factor $e^{-204} \sim 2.5 \times 10^{-89}$!

Thus, the situation in LQC is dramatically different from that implied by the Gibbons–Turok analysis in general relativity. Note that we used the same potential as in the detailed calculations of Gibbons and Turok [4], as well as Kofman, Linde and Mukhanov [2]. Authors of [2] have argued that a sufficiently long, slow roll inflation will occur generically within general relativity. Thus, the thrust of their conclusion is opposite to that of [4]. However, they used a measure which is not preserved under dynamics and requires an additional structure. Therefore there has been some debate [1,2,4] about the appropriateness of the procedure employed to arrive at their conclusion. In LQC one does not have to take a stand on this issue: As in [4] we use the natural Liouville measure which is preserved by dynamics and yet the conclusions of [4] are reversed. Finally, the procedure we used to handle the fact that the total Liouville volume of $\mathbb{S}$ is infinite is physically and mathematically well motivated and it also constituted the basis of the regularization scheme used in [4].

Our detailed analysis made a crucial use of the salient differences between LQC and general relativity in the Planck regime. Since LQC has its basis in LQG, a candidate fundamental theory of quantum gravity, it has precise predictions in the Planck regime of the simple cosmological models that have been traditionally used [1–7] in probability considerations. Consequently, we do not have to worry about setting judicious initial conditions at the singular big bang. The bounce is regular and we considered *all possible* initial data there. The LQC dynamics are such that if F_B, the fraction of total energy that is in the potential at the bounce, satisfies $F_B > 1.4 \times 10^{-11}$ a slow roll with 68 e-foldings is inevitable. Thus, in LQC a sufficiently long slow roll inflation may not result *only if* $F_B < 1.4 \times 10^{-11}$. Since by definition $F_B \in [0, 1]$ for all initial conditions, (10) implies that the probability of a sufficiently long slow roll inflation is extremely close to 1.

These main results are quite robust. For example, we could change the value of the mass used in the main calculations. The probability of obtaining at least 68 e-foldings in fact grows slightly if m is decreased (so long as it is non-zero). What if we increase the mass? To check robustness, let us be generous with phenomenological constraints and increase it by *two orders of magnitude*, i.e., require only $m < 6 \times 10^{-5} M_{Pl}$. Even then the a priori probability of *not* obtaining at least 68 e-foldings is *less than* 2.7×10^{-4}. Thus, we do not have to fine tune the mass. The situation is similar with respect to adding a quartic term to the potential with a phenomenologically permissible coupling constant. Finally LQC provides a neat separation between the regime in which the quantum geometry effects dominate and the regime in which general relativity serves as an excellent approximation. Therefore it is possible to separate the two types of effects. These issues will be discussed in the detailed paper.

To conclude, we emphasize that we have discussed prediction of LQC *only* in presence of a scalar field with suitable potentials. If there is no potential at all, there is still a period of accelerated expansion due to super inflation but, unfortunately, it does not yield a sufficient number of e-foldings. So the issue of the origin of the required potential — and of the inflaton itself — still remains. Although there have been some tantalizing suggestions [34] that promoting the Barbero–Immirzi parameter γ to a field could provide a natural avenue to address these issues, these ideas have not been analyzed in sufficient detail.

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