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Area ranking of fuzzy numbers based on positive and negative ideal points[☆]

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ABSTRACT

The maximizing set and minimizing set method is a popular ranking approach for fuzzy numbers, which ranks them based on their left, right and total utilities. This paper presents an alternative ranking approach for fuzzy numbers called area ranking based on positive and negative ideal points, which defines two new alternative indices for the purpose of ranking. The two new indices are defined in terms of a decision maker (DM)'s attitude towards risks and the left and the right areas between fuzzy numbers and the two ideal points. It is shown that the area ranking approach has strong discrimination power and can rank fuzzy numbers that are unable to be discriminated by the maximizing set and minimizing set method. It is also shown that the DM's attitude towards risks may have a significant impact on the ranking of fuzzy numbers. As a side product, a new defuzzification formula is also developed and discussed.

1. Introduction

Ranking fuzzy numbers is a very important issue in fuzzy sets theory and applications, and has been extensively researched in the past. A significant number of ranking approaches have been suggested in the literature [1–28]. Some of them have been reviewed and compared by Bortolan and Degani [29], Chen and Hwang [30], and Wang and Kerre [31,32]. Among the ranking approaches, the maximizing set and minimizing set method proposed by Chen [4] is a very popular approach that has received high citations and wide applications [33–37]. By introducing a maximizing set and a minimizing set at the same time, the method compares and ranks fuzzy numbers in terms of their left, right and total utilities, which are computed based on the introduced maximizing set and minimizing set. In real applications, however, it is found that when the fuzzy numbers to be compared have the same left, right and total utilities, they cannot be ranked by the maximizing set and minimizing set method. To overcome this drawback, the current paper proposes an alternative ranking approach for fuzzy numbers called area ranking based on positive and negative ideal points. The area ranking approach defines two new alternative indices for comparing and ranking fuzzy numbers, which are defined in terms of a decision maker (DM)'s attitude towards risks and the left and the right areas between the fuzzy numbers and the two ideal points. The proposed area ranking approach proves to have very strong discrimination power and to be able to rank fuzzy numbers that cannot be discriminated by the maximizing set and minimizing set method due to their equal total utilities.

The paper is organized as follows. Section 2 briefly reviews the maximizing set and minimizing set method. In Section 3, we propose the area ranking approach and define two new alternative indices for comparing and ranking fuzzy numbers. The proposed area ranking approach is examined with five numerical examples in Section 4. The paper concludes in Section 5.

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2. The maximizing set and minimizing set method

Let $\tilde{A}$ be a normal fuzzy number, whose membership function $\mu_{\tilde{A}}$ is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^L(x), & a \le x \le b, \\ 1, & b \le x \le c \\ f_{\tilde{A}}^R(x), & c \le x \le d, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $f_{\tilde{A}}^L : [a, b] \to [0, 1]$ and $f_{\tilde{A}}^R : [c, d] \to [0, 1]$ are two continuous mappings from the real line R to the closed interval [0, 1]. The former is a strictly increasing function called left membership function and the latter is a monotonically decreasing function called right membership function. If $b \ne c$, $\tilde{A}$ is referred to as a fuzzy interval or a flat fuzzy number. If $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ are both linear, then $\tilde{A}$ is referred to as a trapezoidal fuzzy number and is usually denoted by $\tilde{A} = (a, b, c, d)$, which is plotted in Fig. 1. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number, denoted by $\tilde{A} = (a, b, d)$. So, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.

Alternatively, a fuzzy number can also be generally expressed as

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & -\infty < x < m, \\ 1, & m \le x \le n, \\ R\left(\dfrac{x-n}{\beta}\right), & m < x < +\infty, \end{cases} \tag{2}$$

which is referred to as L–R fuzzy number, denoted by $\tilde{A} = (m, n, \alpha, \beta)_{LR}$, where $m \le n$, $\alpha \ge 0$ and $\beta \ge 0$ are respectively the left-hand and the right-hand spreads, and $L\left(\frac{m-x}{\alpha}\right)$ and $R\left(\frac{x-n}{\beta}\right)$ are continuous and non-increasing functions satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. When L and R are both linear functions, the L–R fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ becomes a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ with $a = m - \alpha$, $b = m$, $c = n$ and $d = n + \beta$.

Suppose there are N fuzzy numbers $\tilde{A}_1, \ldots, \tilde{A}_N$ to be compared or ranked, whose membership functions are denoted by $\mu_{\tilde{A}_i}(x)$, $i = 1, \ldots, N$. The maximizing set and minimizing set method first defines a maximizing set $\tilde{M}$ and a minimizing set $\tilde{G}$, whose membership functions are respectively defined as [4]

$$\mu_{\tilde{M}}(x) = \begin{cases} \left[(x - x_{\min})/(x_{\max} - x_{\min})\right]^k, & x_{\min} \le x \le x_{\max} \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} \left[(x_{\max} - x)/(x_{\max} - x_{\min})\right]^k, & x_{\min} \le x \le x_{\max} \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $x_{\min} = \inf X$, $x_{\max} = \operatorname{Sup} X$, $X = \bigcup_{i=1}^{N} X_i$, $X_i = \{x \mid \mu_{\tilde{A}_i}(x) > 0\}$, and k is a constant reflecting the DM's attitude towards risks with $k > 1$ representing risk-seeking, $k < 1$ risk-aversion and $k = 1$ risk-neutral [36,37]. Usually, k is set as one. Fig. 2 shows the graphical representations of the maximizing set and minimizing set.

In Fig. 2, the maximizing set $\tilde{M}$ and the minimizing set $\tilde{G}$ intersect the right and left membership functions of fuzzy number $\tilde{A}_i$ respectively at points M_i and G_i. In the case that $\tilde{A}_i$ is a trapezoidal fuzzy number, i.e. $\tilde{A}_i = (a_i, b_i, c_i, d_i)$, the coordinates of M_i and G_i can be determined by the following equations:

$$x_{M_i} = \frac{d_i x_{\max} - c_i x_{\min}}{(d_i - c_i) + (x_{\max} - x_{\min})}, \tag{5}$$

$$u_{M_i} = \frac{d_i - x_{M_i}}{d_i - c_i} = \frac{d_i - x_{\min}}{(d_i - c_i) + (x_{\max} - x_{\min})}, \tag{6}$$

$$x_{G_i} = \frac{b_i x_{\max} - a_i x_{\min}}{(b_i - a_i) + (x_{\max} - x_{\min})}, \tag{7}$$

$$u_{G_i} = \frac{x_{G_i} - a_i}{b_i - a_i} = \frac{x_{\max} - a_i}{(b_i - a_i) + (x_{\max} - x_{\min})}, \tag{8}$$

where $u_{M_i} = \sup_x \left(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{A}_i}(x)\right)$ is referred to as the right utility value of the fuzzy number $\tilde{A}_i$ and $u_{G_i} = \sup_x \left(\mu_{\tilde{G}}(x) \wedge \mu_{\tilde{A}_i}(x)\right)$ as the left utility value. It is obvious that the farther the $\tilde{A}_i$ away from the minimizing set $\tilde{G}$, the smaller the u_{G_i}, and the closer the $\tilde{A}_i$ to the maximizing set $\tilde{M}$, the larger the u_{M_i}. So, the final total utility value of the $\tilde{A}_i$ is defined as [4]

$$u_T(i) = [u_{M_i} + 1 - u_{G_i}]/2, \quad i = 1, \ldots, N. \tag{9}$$

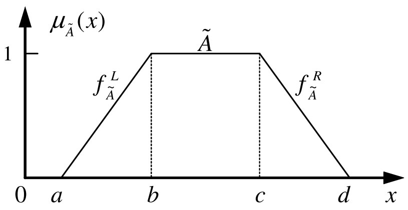

Fig. 1. Membership functions of trapezoidal fuzzy numbers.

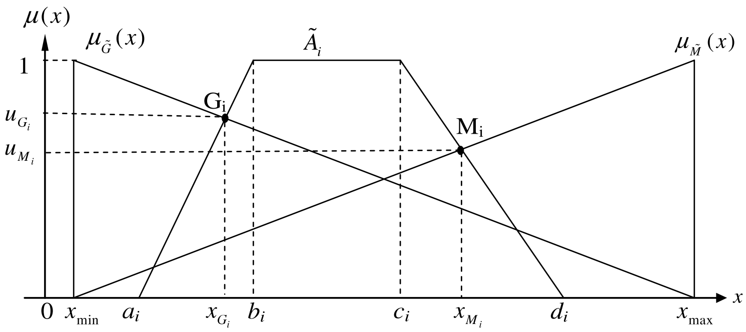

Fig. 2. Graphical representations of maximizing set and minimizing set.

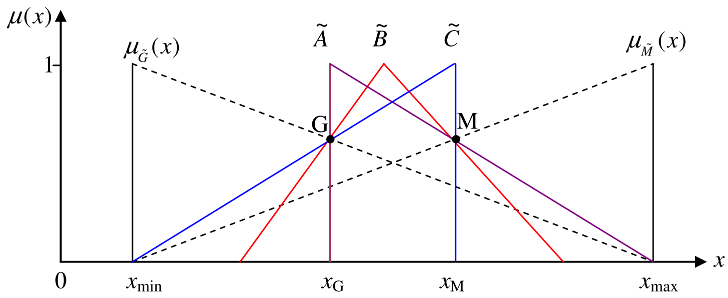

Fig. 3. The maximizing and minimizing sets for fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$.

If the N fuzzy numbers to be compared are trapezoidal fuzzy numbers, i.e. $\tilde{A}_i = (a_i, b_i, c_i, d_i)$, $i = 1, \ldots, N$, then (11) can be further written as

$$u_T(i) = \frac{1}{2}\left(\frac{d_i - x_{\min}}{d_i - c_i + x_{\max} - x_{\min}} + \frac{b_i - x_{\min}}{b_i - a_i + x_{\max} - x_{\min}}\right), \quad i = 1, \ldots, N. \tag{10}$$

The greater the $u_T(i)$, the bigger the fuzzy number $\tilde{A}_i$ and the higher its ranking order. The N fuzzy numbers $\tilde{A}_1, \ldots, \tilde{A}_N$ can therefore be ranked according to their total utilities. Such a ranking approach is referred to as the maximizing set and minimizing set method in the literature.

3. Area ranking of fuzzy numbers

The maximizing set and minimizing set method proves to be very successful in ranking fuzzy numbers with different left, right and total utilities. However, if the fuzzy numbers to be compared have the same left, right and/or total utilities as shown in Fig. 3, the maximizing set and minimizing set method is found ineffective and the fuzzy numbers cannot be ranked by the method. Consider for example three triangular fuzzy numbers: $\tilde{A} = (5.06, 5.06, 10)$, $\tilde{B} = (3.53, 6, 8.47)$ and $\tilde{C} = (2, 6.94, 6.94)$. For these three fuzzy numbers, we have $x_{\max} = 10$ and $x_{\min} = 2$. It is easy to verify that $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ have the same left, right and total utilities which are 0.618, 0.618 and 0.5, respectively, and cannot be distinguished by the maximizing set and minimizing set method.

To overcome this difficulty, we consider introducing a positive ideal point and a negative ideal point, respectively, rather than the maximizing set and the minimizing set. The positive and negative ideal points are respectively defined

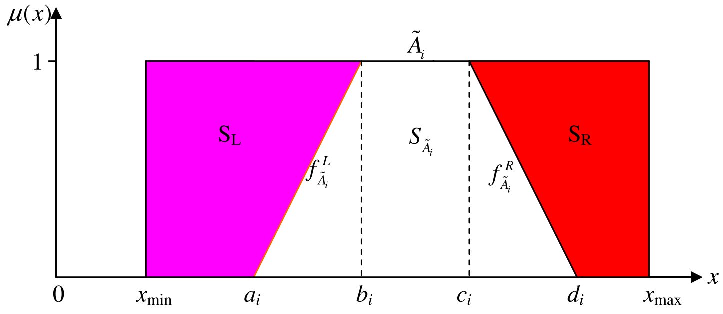

Fig. 4. Area ranking based on the positive and negative ideal points.

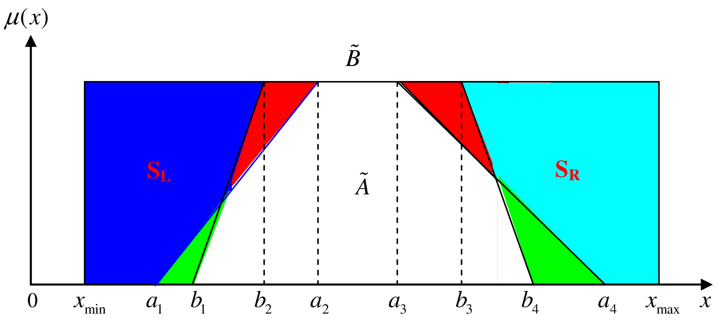

Fig. 5. Fuzzy numbers with equal left and right areas.

as $x_{\max} = \text{Sup}\, X$ and $x_{\min} = \inf X$, where $X = \bigcup_{i=1}^{N} X_i$ and $X_i = \{x | \mu_{\tilde{A}_i}(x) > 0\}$. Note that the positive and negative ideal points are crisp numbers, but can be seen as special cases of fuzzy numbers.

Let $\tilde{A}_i$ be one of the fuzzy numbers to be compared or ranked, whose membership function is defined by Eq. (1). The gaps between $\tilde{A}_i$ and the negative ideal point as well as the positive ideal point form two areas as shown in Fig. 4, which are referred to as the left and right areas, respectively. The two areas are defined by the following equations:

$$S_L(i) = \int_{x_{\min}}^{a_i} \mathrm{d}x + \int_{a_i}^{b_i} \left(1 - f^L_{\tilde{A}_i}(x)\right) \mathrm{d}x = (b_i - x_{\min}) - \int_{a_i}^{b_i} f^L_{\tilde{A}_i}(x)\mathrm{d}x, \tag{11}$$

$$S_R(i) = \int_{c_i}^{d_i} \left(1 - f^R_{\tilde{A}_i}(x)\right) \mathrm{d}x + \int_{d_i}^{x_{\max}} \mathrm{d}x = (x_{\max} - c_i) - \int_{c_i}^{d_i} f^R_{\tilde{A}_i}(x)\mathrm{d}x. \tag{12}$$

In particular, for trapezoidal fuzzy numbers the two areas are computed as

$$S_L(i) = (b_i - x_{\min}) - \int_{a_i}^{b_i} \left(\frac{x - a_i}{b_i - a_i}\right) \mathrm{d}x = \frac{a_i + b_i}{2} - x_{\min}, \tag{13}$$

$$S_R(i) = (x_{\max} - c_i) - \int_{c_i}^{d_i} \left(\frac{d_i - x}{d_i - c_i}\right) \mathrm{d}x = x_{\max} - \frac{c_i + d_i}{2}. \tag{14}$$

It is evident that the farther the $\tilde{A}_i$ away from the negative ideal point $x_{\min}$, the bigger the area $S_L(i)$, and the closer the $\tilde{A}_i$ to the positive ideal point $x_{\max}$, the smaller the area $S_R(i)$. In other words, larger S_L and smaller S_R mean bigger fuzzy numbers.

However, it is not enough to consider only the two areas S_L and S_R. In some cases, different fuzzy numbers may have equal left and right areas, as shown in Fig. 5, in which the two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ have equal left and right areas, but their risks (or uncertainties) are different due to their different support intervals and shapes. Therefore, the DM's attitude towards risks should be considered.

By considering the DM's risk attitude, we define two new alternative Ranking Indices based on Areas (RIA) for comparing and ranking fuzzy numbers:

$$\text{RIA}_1(i) = \frac{1}{2}\left[\left(\frac{S_L(i)}{x_{\max} - x_{\min}}\right) r_L(i) + \left(1 - \frac{S_R(i)}{x_{\max} - x_{\min}}\right) r_R(i)\right], \tag{15}$$

$$\text{RIA}_2(i) = \frac{S_L(i) r_L(i)}{S_L(i) r_L(i) + S_R(i) r'_R(i)}, \tag{16}$$

where $S_L(i)/(x_{\max} - x_{\min})$ and $S_R(i)/(x_{\max} - x_{\min})$ are respectively the percentages of the left area $S_L(i)$ and the right area $S_R(i)$ in the total area of the rectangle encompassed by the ideal points, as illustrated in Fig. 4, and

$$r_L(i) = 1 + (\alpha - 0.5)\frac{b_i - a_i}{x_{\max} - x_{\min}}, \tag{17}$$

$$r_R(i) = 1 + (\alpha - 0.5)\frac{d_i - c_i}{x_{\max} - x_{\min}}, \tag{18}$$

$$r'_R(i) = 1 - (\alpha - 0.5)\frac{d_i - c_i}{x_{\max} - x_{\min}}, \tag{19}$$

in which $r_L(i)$ is a left risk factor, both $r_R(i)$ and $r'_R(i)$ are right risk factors defined for different purposes, and $0 \le \alpha \le 1$ is the DM's attitude towards risks with $\alpha = 0.5$ representing risk-neutral, $0.5 < \alpha \le 1$ risk-seeking and $0 \le \alpha < 0.5$ risk-aversion. Both RIA_1 and RIA_2 are defined to be positive and dimensionless. Either of them can be used for comparing and ranking purposes. The bigger the two indices, the higher the fuzzy number is ranked.

RIA_1 is similar to Eq. (9) in mathematical form. The only differences between them lie in that the utilities in (9) are replaced by the percentages in (15) and that the DM's attitude towards risks is taken into consideration in (15) while Eq. (9) does not. Since the right area $S_R(i)$ is the smaller the better, it is converted into a benefit type index, i.e. $1 - S_R(i)/(x_{\max} - x_{\min})$ for addition with the percentage of the left area $S_L(i)$. RIA_2 is a relative closeness index taken from TOPSIS, a popular multiple attribute decision making approach known as Technique for Order Preference by Similarity to Ideal Solution [38], which selects the best alternative as the one closest to the positive ideal solution and farthest from the negative ideal solution. The difference between RIA_2 and the relative closeness in TOPSIS lies in that the former uses areas to measure the relative closeness of each fuzzy number relative to the positive ideal point, whereas the latter utilizes Euclidean distances to measure the relative closeness of each decision alternative relative to the positive ideal solution. The concept of relative closeness has recently been utilized for ranking fuzzy numbers by Wang et al. [25], but their relative closeness index was defined in a slightly different way as $d_i = \frac{d_i^L}{1+d_i^R}$, where d_i^L and d_i^R are the left and right deviation degrees of the fuzzy number $\tilde{A}_i$ with respect to the minimal and maximal reference sets, which are two fuzzy sets rather than crisp numbers. RIA_1 and RIA_2 are essentially alternative to each other. Either of them can be used for comparing and ranking fuzzy numbers. We offer the two indices in the text to give the readers an alternative choice and to provide them with more flexibility in deciding on which one to use.

The risk factors in Eqs. (17)–(19) are defined to reflect the impacts of the DM's attitude towards risks on RIA_1 and RIA_2. Intuitively, risk-neutral should have no impact on them, whereas risk-seeking and risk-aversion should respectively have a positive and a negative impact on the two ranking indices. If we utilize $\alpha = 0.5$ to represent risk-neutral, then the risk factors should meet the conditions of $r_L(i) = r_R(i) = 1$ for $\alpha = 0.5$, $r_L(i) > 1$ and $r_R(i) > 1$ if $0.5 < \alpha \le 1$ which stands for risk-seeking, and $r_L(i) < 1$ and $r_R(i) < 1$ if $0 \le \alpha < 0.5$ which indicates risk-aversion. Bigger impact means higher risk. The left risk is intuitively highly correlated with the left-hand spread, namely, the difference between a_i and b_i. The bigger the spread, the higher the left risk. When $a_i = b_i$, there should be no left risk. For simplicity, we define the left risk as a linear function of $b_i - a_i$, as indicated by Eq. (17). As such, we define the right risk to be a linear function of $d_i - c_i$, as shown by Eq. (18).

It is worth pointing out that since the risk factor $r'_R(i)$ appears on the denominator of Eq. (16), its value is thus defined by Eq. (19) to be less than one for risk-seeking, (i.e. $0.5 < \alpha \le 1$) to make a positive impact on RIA_2 and to be greater than one for risk-aversion to maintain a negative impact on RIA_2.

For risk-neutral DMs, i.e. $\alpha = 0.5$, both RIA_1 and RIA_2 have their simplest forms, as shown below:

$$\text{RIA}_1(i) = \frac{1}{2}\left(1 + \frac{S_L(i)}{x_{\max} - x_{\min}} - \frac{S_R(i)}{x_{\max} - x_{\min}}\right), \tag{20}$$

$$\text{RIA}_2(i) = \frac{S_L(i)}{S_L(i) + S_R(i)}. \tag{21}$$

Substituting (11) and (12) into (20), we get

$$\text{RIA}_1(i) = \frac{1}{x_{\max} - x_{\min}}\left[\frac{1}{2}\left(b_i + c_i + \int_{c_i}^{d_i} f_{\tilde{A}_i}^R(x)\mathrm{d}x - \int_{a_i}^{b_i} f_{\tilde{A}_i}^L(x)\mathrm{d}x\right) - x_{\min}\right]. \tag{22}$$

Let

$$x_0 = \frac{1}{2}\left(b_i + c_i + \int_{c_i}^{d_i} f_{\tilde{A}_i}^R(x)\mathrm{d}x - \int_{a_i}^{b_i} f_{\tilde{A}_i}^L(x)\mathrm{d}x\right). \tag{23}$$

It is easy to see that x_0 has nothing to do with the positive and negative ideal points. If we rewrite Eq. (23) as

$$\int_{a_i}^{b_i} f_{\tilde{A}_i}^L(x)\mathrm{d}x + (x_0 - b_i) = (c_i - x_0) + \int_{c_i}^{d_i} f_{\tilde{A}_i}^R(x)\mathrm{d}x, \tag{24}$$

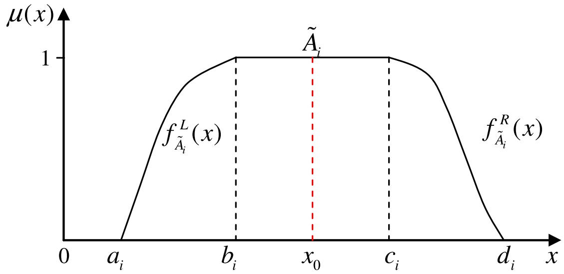

Fig. 6. Defuzzification of symmetrical fuzzy intervals by simple partition.

Table 1
Area ranking of triangular fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in Example 1.

DM's risk attitude (α)	RIA_1			RIA_2			Ranking
	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	
0	0.430	0.423	0.415	0.486	0.423	0.358	$\tilde{A} > \tilde{B} > \tilde{C}$
0.5	0.537	0.500	0.463	0.553	0.500	0.447	$\tilde{A} > \tilde{B} > \tilde{C}$
1	0.644	0.577	0.511	0.642	0.577	0.514	$\tilde{A} > \tilde{B} > \tilde{C}$

then we can further see that for a symmetrical fuzzy interval, the both sides of the equation happen to divide the area of $\tilde{A}_i$ into two equivalent parts, as shown in Fig. 6. The Eq. (23), however, does not make any assumption about the shapes of fuzzy numbers and is thus applicable to normal fuzzy numbers of any shape. For example, if $\tilde{A}_i$ is a triangular fuzzy number, say $\tilde{A}_i = (a_i, b_i, c_i)$, then it is computed from (23) that $x_0 = \frac{a_i+2b_i+c_i}{4}$; if $\tilde{A}_i$ is a trapezoidal fuzzy number, then it is derived from (23) that $x_0 = \frac{a_i+b_i+c_i+d_i}{4}$ no matter whether $\tilde{A}_i = (a_i, b_i, c_i, d_i)$ is symmetrical or non-symmetrical. Since the expression $x_0 = \frac{a_i+b_i+c_i+d_i}{4}$ is consistent with the defuzzification formula developed by Chen [39–42], who defuzzified a trapezoidal fuzzy number by simply dividing it into two equivalent areas, such a consistency and the good property that x_0 has nothing to do with the positive and negative ideal points enable us to have sufficient reason to view x_0 defined by Eq. (23) as a defuzzified value of fuzzy number $\tilde{A}_i$. It provides a more general formula for defuzzifying normal fuzzy numbers than the defuzzification formula $x_0 = \frac{a_i+b_i+c_i+d_i}{4}$ proposed by Chen [39–42]. The former, i.e. Eq. (23), is applicable to normal fuzzy numbers of any shape, whereas the latter is only suitable to the trapezoidal fuzzy numbers satisfying $b_i \le \frac{a_i+b_i+c_i+d_i}{4} \le c_i$, otherwise $x_0 = \frac{a_i+b_i+c_i+d_i}{4}$ cannot be derived by dividing the area of $\tilde{A}_i$ into two equivalent areas.

Due to the fact that the positive and negative ideal points x_{max} and x_{min} are both the same for the fuzzy numbers that are to be compared together, Eqs. (22) and (23) clearly reveal that by RIA_1 the comparison of fuzzy numbers comes down to the comparison of their defuzzified values and the comparison has nothing to do with the defined positive and negative ideal points. Such a conclusion, however, only holds for risk-neutral DMs.

4. Numerical examples

We now turn to verifying the validity of the proposed area ranking approach. Five numerical examples are examined in this section to show good discrimination powers of RIA_1 and RIA_2 in comparing and ranking fuzzy numbers. The results obtained by the maximizing set and minimizing set method are also provided and discussed.

Example 1. Consider three triangular fuzzy numbers $\tilde{A} = (5.06, 5.06, 10)$, $\tilde{B} = (3.53, 6, 8.47)$ and $\tilde{C} = (2, 6.94, 6.94)$, as shown in Fig. 7. These three fuzzy numbers are found to have equal left, right and total utilities and therefore cannot be ranked by the maximizing set and minimizing set method. They have been re-examined using the area ranking approach developed in this paper. The results are shown in Table 1, from which it is clear that the three triangular fuzzy numbers are all ranked by RIA_1 and RIA_2 no matter what an attitude the DM has towards risks. The two ranking indices produce the same ranking: $\tilde{A} > \tilde{B} > \tilde{C}$, which is not affected by the DM's risk attitude.

Example 2. Consider three trapezoidal fuzzy numbers $\tilde{A} = (2, 7, 10, 15)$, $\tilde{B} = (3, 6, 11, 14)$ and $\tilde{C} = (4, 5, 12, 13)$, as shown in Fig. 8. For these three trapezoidal fuzzy numbers, we have $x_{min} = 2$ and $x_{max} = 15$. It is found that the three trapezoidal fuzzy numbers have different left and right utilities, but their total utilities are all the same (see Table 2). Therefore, they cannot be discriminated by the maximizing set and minimizing set method. By computing their left and right areas, it is also found that the three trapezoidal fuzzy numbers have the same left and right areas, which are shown in the last two columns of Table 2. This reveals that they cannot be discriminated by considering only the two areas either. To rank the three fuzzy numbers, DM's attitude towards risks has to be taken into consideration. Table 3 shows area rankings for DMs with different attitudes towards risks. It is observed that risk-neutral DMs view the three trapezoidal fuzzy numbers

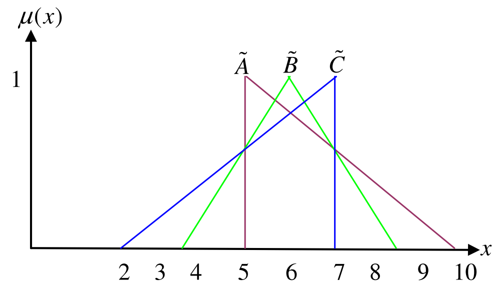

Fig. 7. Fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in Example 1.

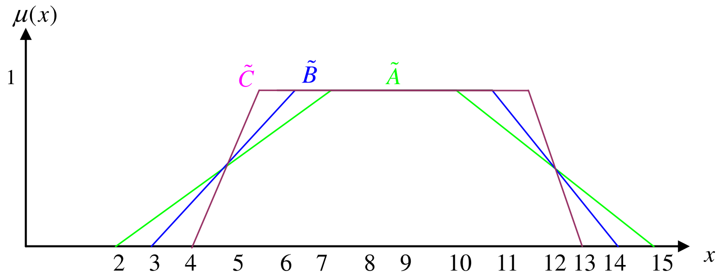

Fig. 8. Fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in Example 2.

Table 2
Utilities and areas of trapezoidal fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in Example 2.

Fuzzy number	Utilities			Areas	
	u_M	u_G	u_T	S_L	S_R
$\tilde{A}$	0.722	0.722	0.5	2.5	2.5
$\tilde{B}$	0.750	0.750	0.5	2.5	2.5
$\tilde{C}$	0.786	0.786	0.5	2.5	2.5

Table 3
Area rankings of the trapezoidal fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in Example 2.

DM's risk attitude (α)	RIA_1			RIA_2			Ranking
	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	
0	0.404	0.442	0.481	0.404	0.442	0.481	$\tilde{C} > \tilde{B} > \tilde{A}$
0.5	0.500	0.500	0.500	0.500	0.500	0.500	$\tilde{A} \sim \tilde{B} \sim \tilde{C}$
1	0.596	0.558	0.519	0.596	0.558	0.519	$\tilde{A} > \tilde{B} > \tilde{C}$

as no difference and risk-seeking DMs prefer $\tilde{A}$ to $\tilde{B}$ to $\tilde{C}$, whereas risk-averse DMs think $\tilde{C}$ is better than $\tilde{B}$, which is better than $\tilde{A}$. Obviously, DMs' attitudes towards risks have significant impacts on the final ranking of the three trapezoidal fuzzy numbers.

Example 3. Consider two L–R fuzzy numbers $\tilde{A}_1 = (6, 6, 3, 3)_{LR}$ and $\tilde{A}_2 = (6, 6, 1, 1)_{LR}$ which are taken from Wang et al. [25] and were investigated by Chen and Lu [6]. Fig. 9 shows the pictures of the two L–R fuzzy numbers. According to the computational results in Wang et al. [25], the two L–R fuzzy numbers have the same mode and symmetric spreads, most of existing ranking approaches cannot thus distinguish them. For instance, by using the ranking approaches in [3,10,24,27] and the maximizing set and minimizing set method of Chen [4], the ranking of $\tilde{A}_1$ and $\tilde{A}_2$ is always the same, i.e. $\tilde{A}_1 \sim \tilde{A}_2$. Wang et al. [25] utilized their ranking approach and got the ranking $\tilde{A}_2 > \tilde{A}_1$. They believed that decision makers preferred the result $\tilde{A}_2 > \tilde{A}_1$ intuitionally. Does the DM really prefer $\tilde{A}_2$ to $\tilde{A}_1$? We do not think this is always the case. It can be seen from Fig. 9 that $\tilde{A}_1$ has higher risk than $\tilde{A}_2$. For a risk-seeking DM, he/she would prefer $\tilde{A}_1$ to $\tilde{A}_2$ rather than $\tilde{A}_2$ to $\tilde{A}_1$. If the DM is risk-averse, then he/she would prefer $\tilde{A}_2$ to $\tilde{A}_1$. For a risk-neutral DM, he/she does not care too much about the risks of the two fuzzy numbers and would like each of them equally. Table 4 shows the results and the rankings of the two L–R fuzzy numbers obtained by the area ranking approach, which are fully consistent with the results analyzed above. This shows the strong discrimination power of the proposed area ranking approach and its good advantages.

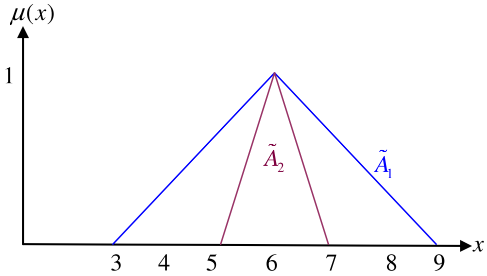

Fig. 9. Fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ in Example 3.

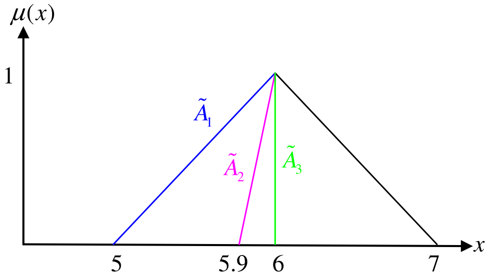

Fig. 10. Fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ in Example 4.

Table 4
Area ranking of *L–R* fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ in Example 3.

DM's risk attitude (α)	RIA_1		RIA_2		Ranking
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_1$	$\tilde{A}_2$	
0	0.375	0.458	0.375	0.458	$\tilde{A}_2 > \tilde{A}_1$
0.5	0.500	0.500	0.500	0.500	$\tilde{A}_1 \sim \tilde{A}_2$
1	0.625	0.542	0.625	0.542	$\tilde{A}_1 > \tilde{A}_2$

Table 5
Area ranking of *L–R* fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ in Example 4.

DM's risk attitude (α)	RIA_1			RIA_2			Ranking
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	
0	0.375	0.513	0.531	0.375	0.597	0.615	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
0.5	0.500	0.613	0.625	0.500	0.655	0.667	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
1	0.625	0.712	0.719	0.625	0.722	0.727	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$

Example 4. Consider the example investigated by Abbasbandy and Asady [1] and Wang et al. [25], which contains three *L–R* fuzzy numbers $\tilde{A}_1 = (6, 6, 1, 1)_{LR}$, $\tilde{A}_2 = (6, 6, 0.1, 1)_{LR}$ and $\tilde{A}_3 = (6, 6, 0, 1)_{LR}$ to be compared and ranked, as shown in Fig. 10. Table 5 gives the results obtained by the area ranking approach, from which it is seen clearly that the three *L–R* fuzzy numbers are always ranked as $\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$ regardless of the DM's attitude towards risks. Such a ranking is not only fully consistent with the ranking obtained by the maximizing set and minimizing set method, which calculates the total utilities of the three *L–R* fuzzy numbers as $u_T(1) = 0.5$, $u_T(2) = 0.571$ and $u_T(3) = 0.583$, but also consistent with the ranking obtained by the sign distance method of Abbasbandy and Asady [1] and the ranking approach of Wang et al. [25]. Table 6 summarizes the ranking results obtained by different methods, where the ranking $\tilde{A}_2 > \tilde{A}_3 > \tilde{A}_1$ by the method of Chu and Tsao [10] and the ranking $\tilde{A}_1 > \tilde{A}_2 > \tilde{A}_3$ by the CV index of Cheng [8] are thought of to be unreasonable and not consistent with human intuition [25].

Example 5. Consider the two sets of fuzzy numbers taken from Yao and Wu [27]:

Set 1: $\tilde{A}_1 = (0.5, 0.5, 0.1, 0.5)_{LR}$, $\tilde{A}_2 = (0.7, 0.7, 0.3, 0.2)_{LR}$ and $\tilde{A}_3 = (0.9, 0.9, 0.5, 0.1)_{LR}$;
Set 2: $\tilde{B}_1 = (0.4, 0.7, 0.1, 0.2)_{LR}$, $\tilde{B}_2 = (0.7, 0.7, 0.4, 0.2)_{LR}$ and $\tilde{B}_3 = (0.7, 0.7, 0.2, 0.2)_{LR}$,

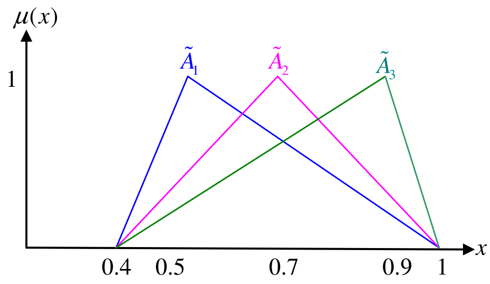

Fig. 11. Fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ in Example 5.

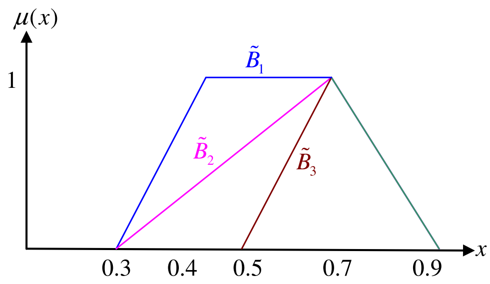

Fig. 12. Fuzzy numbers $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$ in Example 5.

Table 6
Ranking results of the L–R fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ in Example 4 by different approaches [25].

Ranking approach	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	Ranking
Wang et al. [25]	0.25	0.5339	0.5625	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
Sign distance ($p = 1$) [1]	6.12	12.45	12.5	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
Sign distance ($p = 2$) [1]	8.52	8.82	8.85	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
Chu and Tsao [10]	3	3.126	3.085	$\tilde{A}_2 > \tilde{A}_3 > \tilde{A}_1$
Cheng's distance index [8]	6.021	6.349	6.7519	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
Cheng's CV index [8]	0.028	0.0098	0.0089	$\tilde{A}_1 > \tilde{A}_2 > \tilde{A}_3$

Table 7
Area ranking of L–R fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ in Example 5.

DM's risk attitude (α)	RIA_1			RIA_2			Ranking
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	
0	0.208	0.375	0.542	0.115	0.375	0.729	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
0.5	0.333	0.500	0.667	0.167	0.500	0.833	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
1	0.458	0.625	0.792	0.271	0.625	0.885	$\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$

Table 8
Area ranking of L–R fuzzy numbers $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$ in Example 5.

DM's risk attitude (α)	RIA_1			RIA_2			Ranking
	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	
0	0.385	0.458	0.556	0.282	0.533	0.682	$\tilde{B}_3 > \tilde{B}_2 > \tilde{B}_1$
0.5	0.458	0.583	0.667	0.333	0.667	0.750	$\tilde{B}_3 > \tilde{B}_2 > \tilde{B}_1$
1	0.531	0.708	0.778	0.394	0.762	0.808	$\tilde{B}_3 > \tilde{B}_2 > \tilde{B}_1$

which are shown in Figs. 11 and 12, respectively. According to the computational results by Wang et al. [25], most of the ranking approaches inclusive of their approach rank the fuzzy numbers in Set 1 as $\tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$ and those in Set 2 as $\tilde{B}_3 > \tilde{B}_2 > \tilde{B}_1$. These rankings are also achieved by the area ranking approach regardless of the DM's attitude towards risks, as revealed by Tables 7 and 8. It is also found that the maximizing set and minimizing set method calculates the total utilities of $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$ as 0.344, 0.5 and 0.656, respectively, and the total utilities of $\tilde{B}_1, \tilde{B}_2$ and $\tilde{B}_3$ as 0.445, 0.575 and 0.625, respectively, which give the same rankings for the two sets of fuzzy numbers.

5. Conclusions

In this paper we have developed an area ranking approach for fuzzy numbers by introducing a positive and a negative ideal point, respectively. The area ranking approach considers not only the left and right areas between fuzzy numbers and the two ideal points, but also the DM's attitude towards risks, which has rarely been considered in existing fuzzy ranking approaches. Two new alternative indices have been defined for the purpose of ranking and tested with five numerical examples. It has been shown that the proposed area ranking approach has very strong discrimination power and can compare and rank fuzzy numbers that are unable to be ranked by the maximizing set and minimizing set method. It has also been shown that the DM's attitude towards risks may have a significant impact on the ranking of fuzzy numbers. In comparison with those approaches that do not consider the DM's attitude towards risks, the proposed area ranking approach is more flexible and more practical.

We have also come up with a defuzzification formula, which turns out to be more general than that proposed by Chen [39–42]. The new defuzzification formula is not only suitable for trapezoidal and triangular fuzzy numbers, but also suitable for any *L*–*R* fuzzy numbers. It is therefore expected to have more applications in the future.

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