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Geometry of conformal vector fields

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Abstract. It is well known that the Euclidean space $(R^n, \langle,\rangle)$, the n-sphere $S^n(c)$ of constant curvature c and Euclidean complex space form $(C^n, J, \langle,\rangle)$ are examples of spaces admitting conformal vector fields and therefore conformal vector fields are used in obtaining characterizations of these spaces. In this article, we study the conformal vector fields on a Riemannian manifold and present the existing results as well as some new results on the characterization of these spaces. Taking clue from the analytic vector fields on a complex manifold, we define φ-analytic conformal vector fields on a Riemannian manifold and study their properties as well as obtain characterizations of the Euclidean space $(R^n, \langle,\rangle)$ and the n-sphere $S^n(c)$ of constant curvature c using these vector fields.

Keywords: Conformal vector fields; Ricci curvature; Scalar curvature; Obata's theorem; Laplacian; φ-analytic vector fields

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1. INTRODUCTION

Characterizations of important spaces such as the Euclidean space R^n, the Euclidean sphere S^n, and the complex projective space CP^n, is an important problem in differential geometry and was taken up by several authors (cf. [1–8,10–12,14,13,9,15–23], [26,24,27–32]). In most of these characterizations conformal vector fields play a notable role. Conformal vector fields are important objects for studying the geometry of several kinds of manifolds.

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its flow consists of conformal transformations or, equivalently, if there exists a smooth function f on M (called the potential function of the conformal vector field ξ) that

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satisfies $\pounds_\xi g = 2fg$, where $\pounds_\xi g$ is the Lie derivative of g with respect ξ. We say that ξ is a nontrivial conformal vector field if ξ is a non-Killing vector field (ξ is a Killing vector field if the potential function $f = 0$ or, equivalently, the flow of ξ consists of isometries of the Riemannian manifold). If, in addition, ξ is a closed vector field (or is a gradient of a smooth function), then ξ is said to be a closed conformal vector field (or a gradient conformal vector field). If ξ is a gradient conformal vector field with $\xi = \nabla\rho$ for a smooth function ρ on the Riemannian manifold (M, g), then we get the Poisson equation $\Delta\rho = nf$. Thus the geometry of gradient conformal vector fields on a Riemannian manifold is related to the Poisson equation on the Riemannian manifold. The role of differential equations in studying the geometry of a Riemannian manifold was initiated by the pioneering work of Obata (cf. [21–23]). The work of Obata is about characterizing specific Riemannian manifolds by second order differential equations. His main result is: a necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $S^n(c)$ is that there exists a non constant smooth function f on M that satisfies the differential equation $H_f = -cfg$, where H_f is the Hessian of the smooth function f. Then Tashiro [30] has shown that the Euclidean spaces R^n are characterized by the differential equation $H_f = cg$, and Tanno [28] obtained a similar characterization of spheres.

Recently Garcia-Rio et al. [15,16] have introduced the Laplace operator Δ acting on smooth vector fields on a Riemannian manifold (M, g) and generalized the result of Obata using the differential equation satisfied by a vector field to characterize the n-sphere $S^n(c)$ (cf. Theorem 3.5 in [16]). These authors have also proved that the differential equation

$$\Delta Z = -cZ, c = \frac{S}{n(n-1)},$$

where Z is a smooth vector field on an n-dimensional compact Einstein manifold (M, g) of constant scalar curvature $S > 0$, (that is Z is the eigenvector of the Laplace operator Δ), is a necessary and sufficient condition for M to be isometric to the n-sphere $S^n(c)$ (cf. Theorem 6 in [15]).

The geometry of conformal vector fields is naturally divided in two classes, the geometry of gradient conformal vector fields and the geometry of conformal vector fields which are not closed. The initial work on the subject of conformal vector fields was originated with the geometry of closed or gradient conformal vector fields. Riemannian manifolds admitting closed conformal vector fields or gradient conformal vector fields, have been investigated in [6,10,17–19,21–23,29–32] and it has been observed that there is a close relationship between the potential functions of gradient conformal vector fields and Obata's differential equation.

There are many examples of gradient conformal vector fields, on the n-dimensional sphere $S^n(c)$. If N is the unit normal vector field on $S^n(c)$, in the Euclidean space R^{n+1} with Euclidean metric $\langle,\rangle$, then for any constant vector field Z on the Euclidean space R^{n+1} its restriction to $S^n(c)$ can be expressed as $Z = \xi + fN$, where $f = \langle Z, N\rangle$ is a smooth function and ξ is a vector field on $S^n(c)$. Then it is straightforward to show that ξ is a gradient conformal vector field on $S^n(c)$ with potential function $-\sqrt{c}f$. Other classes of conformal vector gradient vector fields are provided by warped product spaces. For instance, consider an $(n-1)$-dimensional Riemannian manifold (M, g) and an open interval $I \subset R$ and set $\overline{M} = I \times M$ with projections $\pi_1 : \overline{M} \to I$ and $\pi_2 \to M$. Then for a positive function $f : I \to R$, we get the Riemannian manifold $(\overline{M}, \overline{g})$ called the warped product manifold, where the warped product metric $\overline{g}$ is given by

$$\overline{g}(X,Y)=(d\pi_1 X.d\pi_1 Y)+(f\circ\pi_1)^2 g(d\pi_2 X,d\pi_2 Y).$$

If ∂_t is the unit vector field on I, then $\xi=(f\circ\pi_1)\partial_t$ is a gradient conformal vector field on the warped product space $(\overline{M},\overline{g})$ with potential function $f^{'}\circ\pi_1$ (cf. [24], Proposition 35, p. 206). The Euclidean space $(R^n,\langle,\rangle)$ provides many examples of conformal vector fields, a trivial example being the position vector field ξ, which is a gradient conformal vector field. On a complex Euclidean space $(C^n,\langle,\rangle)$ (Euclidean complex space form) with standard complex structure J, the vector field $\xi=\psi+J\psi$ is a conformal vector field that is not closed. Similarly, on the Euclidean space $(R^n,\langle,\rangle)$ with Euclidean coordinates $x^1,\dots,x^n$, the vector field

$$\xi=\psi-\left\langle\psi,\frac{\partial}{\partial x^i}\right\rangle\frac{\partial}{\partial x^j}+\left\langle\psi,\frac{\partial}{\partial x^j}\right\rangle\frac{\partial}{\partial x^i},$$

where i,j are two fixed indices with $i\neq j$, is a conformal vector field that is not closed. On the odd-dimensional unit sphere S^{2n-1} in C^n there are many conformal vector fields which are not closed. Each constant vector field Z on C^n with tangential component Z^T to S^{2n-1} gives a conformal vector field $u=\xi+Z^T$ on S^{2n-1} that is not a closed conformal vector field, where ξ is the Reeb vector field given by the Sasakian structure on the odd-dimensional sphere S^{2n-1}.

The above set of examples suggests that on the spaces $S^n(c)$, the Euclidean space $(R^n,\langle,\rangle)$ and the complex space form $(C^n,\langle,\rangle)$ there are many conformal vector fields. Therefore a natural question is raised: 'could there be characterizations of these spaces using the conformal vector fields?' The aim of this paper is to report on the progress of the geometry of conformal vector fields in answering this question and to introduce some new results in this subject.

2. Preliminaries

Let (M,g) be an n-dimensional Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. A vector field $\xi\in\mathfrak{X}(M)$ is said to be a conformal vector field if

$$\pounds_\xi\, g=2fg \tag{2.1}$$

for a smooth function $f\in C^\infty(M)$, called the potential function, where $\pounds_\xi$ is the Lie derivative with respect to ξ. Using Koszul's formula (cf. [1,5]), we immediately obtain the following for a vector field ξ on M

$$2g(\nabla_X\xi,Y)=(\pounds_\xi\, g)(X,Y)+d\eta(X,Y),\qquad X,Y\in\mathfrak{X}(M), \tag{2.2}$$

where η is the 1-form dual to ξ, that is, $\eta(X)=g(X,\xi)$, $X\in\mathfrak{X}(M)$. Define a skew symmetric tensor field φ of type $(1,1)$ on M by

$$d\eta(X,Y)=2g(\varphi X,Y),\quad X,Y\in\mathfrak{X}(M). \tag{2.3}$$

Then using Eqs. (2.1)–(2.3), we immediately get

$$\nabla_X\xi=fX+\varphi X,\quad X\in\mathfrak{X}(M). \tag{2.4}$$

For a conformal vector field ξ, the skew symmetric tensor field φ in the above equation is called the *associate tensor field* of the conformal vector field.

We shall denote by Δ the Laplace operator acting on smooth functions on M and by λ_1 the first nonzero eigenvalue of the Laplace operator Δ. For a smooth function $h \in C^\infty(M)$ on the Riemannian manifold (M, g), we denote by ∇h the gradient of h and by A_h the Hessian operator $A_h : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $A_h(X) = \nabla_X \nabla h$. On an n-dimensional compact Riemannian manifold (M, g) that admits a conformal vector field ξ, using the skew symmetry of the tensor field φ, Eq. (2.4) gives $\operatorname{div}\xi = nf$ and consequently we have

$$\int_M f = 0, \tag{2.5}$$

which gives

$$\int_M \|\nabla f\|^2 \geq \lambda_1 \int_M f^2. \tag{2.6}$$

Note that the smooth 2-form given by $g(\varphi X, Y)$ is closed, and therefore we have

$$g\left((\nabla\varphi)(X,Y), Z\right) + g\left((\nabla\varphi)(Y,Z), X\right) + g\left((\nabla\varphi)(Z,X), Y\right) = 0, \tag{2.7}$$

where the covariant derivative is $(\nabla\varphi)(X,Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$. Moreover, we compute the curvature tensor field $R(X,Y)\xi$, using Eq. (2.4), to arrive at

$$R(X,Y)\xi = X(f)Y - Y(f)X + (\nabla\varphi)(X,Y) - (\nabla\varphi)(Y,X).$$

Using the above equation in Eq. (2.7) and the skew-symmetry of the tensor field φ, we get

$$g\left(R(X,Y)\xi + Y(f)X - X(f)Y, Z\right) + g\left((\nabla\varphi)(Z,X), Y\right) = 0,$$

that is,

$$(\nabla\varphi)(X,Y) = R(X,\xi)Y + Y(f)X - g(X,Y)\nabla f, \quad X, Y \in \mathfrak{X}(M). \tag{2.8}$$

The Ricci operator Q is a symmetric $(1,1)$-tensor field that is defined by

$$g(QX, Y) = Ric(X,Y), X, Y \in \mathfrak{X}(M),$$

where Ric is the Ricci tensor of the Riemannian manifold. Choosing a local orthonormal frame $\{e_1, ..e_n\}$ on M, and using

$$Q(X) = \sum R(X, e_i)e_i$$

in Eq. (2.8), we compute

$$\sum (\nabla\varphi)(e_i, e_i) = -Q(\xi) - (n-1)\nabla f. \tag{2.9}$$

A Riemannian manifold (M, g) is said to be an Einstein manifold if $Q = \lambda I$, where λ is a constant called the Einstein constant. The smooth function $S = TrQ$ is called the scalar curvature of the Riemannian manifold, which is a constant on an Einstein manifold. We have the following for the gradient of the scalar curvature S of the Riemannian manifold

$$\frac{1}{2}\nabla S = \sum (\nabla Q)(e_i, e_i). \tag{2.10}$$

Let (M, g) be an n-dimensional Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. Garcia-Rio et. al [16] have initiated the study of the Laplace operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right),$$

where ∇ is the Riemannian connection and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. This operator is a self adjoint elliptic operator with respect to the inner product $\langle , \rangle$ on $\mathfrak{X}^C(M)$, the set of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X, Y \rangle = \int_M g(X, Y), \quad X, Y \in \mathfrak{X}^C(M).$$

A vector field X is said to be an eigenvector of the Laplace operator Δ if there is a constant μ such that $\Delta X = -\mu X$. On a compact Riemannian manifold (M, g), using the properties of Δ with respect to the inner product $\langle , \rangle$, it is easy to conclude that the eigenvalue $\mu \geq 0$. For example consider the n-sphere $S^n(c)$ of constant curvature c as a hypersurface of the Euclidean space R^{n+1} with unit normal vector field N. Take a constant vector field Z on R^{n+1}, which can be expressed as $Z = \xi + fN$, where ξ is the tangential component of Z to $S^n(c)$ and $f = \langle Z, N \rangle$ is the smooth function on $S^n(c)$, $\langle , \rangle$ being the Euclidean metric on R^{n+1}. Then it is easy to show that $\Delta\xi = -c\xi$.

Let $(M; J; g)$ be a $2n$-dimensional Kaehler manifold with complex structure J and Hermitian metric g. We denote by ∇ the Levi-Civita connection on M. Then we have

$$\nabla_X JY = J\nabla_X Y, \quad g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.11}$$

The Riemannian curvature tensor field R and the Ricci tensor field Ric of a Kaehler manifold $(M; J; g)$ satisfy

$$R(JX, JY; JZ, JW) = R(X, Y; Z, W), \quad Ric(JX, JY) = Ric(X, Y). \tag{2.12}$$

A Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form and it is denoted by $M(c)$. The curvature tensor field of a complex space form $M(c)$ has the expression

$$\begin{aligned} R(X, Y)Z = \frac{c}{4} \{ & g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY + 2g(X, JY)JZ \}. \end{aligned} \tag{2.13}$$

Now we state the following useful lemmas.

Lemma 2.1 *([13]). Let ξ be a conformal vector field on an n-dimensional compact Riemannian manifold (M, g) with potential function f. Then,*

$$\int_M g(\nabla f, \xi) = -n \int_M f^2,$$

where ∇f is the gradient of the function f.

Lemma 2.2 ([7,12]). *Let ξ be a conformal vector field on a compact Riemannian manifold (M,g) with potential function f. Then,*

$$\int_M g(\nabla S,\xi) = 2\int_M fS \text{ and } \int_M Ric(\nabla f,\xi) = \int_M \left(\frac{n-2}{2}Sf^2 + \frac{S}{2}g(\nabla f,\xi)\right),$$

where S is the scalar curvature.

Lemma 2.3 ([13]). *Let ξ be a conformal vector field on a compact Riemannian manifold (M,g) with potential function f. Then,*

$$\int_M \left(Ric(\xi,\xi) - n(n-1)f^2 - \|\varphi\|^2\right) = 0.$$

Lemma 2.4 ([11]). *Let ξ be a conformal vector field on a compact Riemannian manifold (M,g) with potential function f. Then,*

$$\int_M \left((n-1)\|\nabla f\|^2 + \frac{n-2}{2}Sf^2 + \frac{S}{2}g(\nabla f,\xi)\right) = 0,$$

where S is the scalar curvature.

Lemma 2.5 ([9]). *Let (M,g) be a Riemannian manifold and f be a smooth function defined on M. Then the Hessian operator A_f satisfies*

$$\sum (\nabla A_f)(e_i,e_i) = Q(\nabla f) + \nabla(\Delta f),$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame, Δ is the Laplace operator on M and $(\nabla A_f)(X,Y) = \nabla_X A_f(Y) - A_f(\nabla_X Y)$, $X,Y \in \mathfrak{X}(M)$.

Lemma 2.6 (*Bochner's Formula [1,9]*). *Let (M,g) be a compact Riemannian manifold and f be a smooth function defined on M. Then*

$$\int_M \left(Ric(\nabla f,\nabla f) + \|A_f\|^2 - (\Delta f)^2\right) = 0.$$

Recall that, given a $(1,1)$ tensor field A on a Riemannian manifold (M,g), we define the divergence of A, $\mathrm{div}A \in \mathfrak{X}(M)$, by

$$g(\mathrm{div}A, X) = \sum g\left((\nabla A)(e_i,X),e_i\right), \quad X \in \mathfrak{X}(M).$$

Lemma 2.7 ([14]). *Let ξ be a conformal vector field on a $2n$-dimensional Kaehler manifold (M,J,g) with potential function f and associated tensor field φ. Then,*

$$2(n-1)\nabla f = \mathrm{div}\varphi - \mathrm{div}(J\varphi J) - J\nabla(TrJ\varphi).$$

Lemma 2.8 ([14]). *Let ρ be a smooth function on a Kaehler manifold (M,J,g). Then $\mathrm{div}(J\nabla\rho) = 0$.*

3. Closed conformal vector fields

In this section, we study the geometry of a Riemannian manifold (M, g) that admits a gradient conformal vector field ξ. Note that if the Riemannian manifold is simply connected, then a closed conformal vector field is a gradient conformal vector field. As the unit sphere $S^n(c)$ admits many gradient conformal vector fields one naturally expects to use gradient conformal vector fields to characterize spheres among compact Riemannian manifolds. We shall give the sketches of proofs of some results of the author for the sake of convenience of the readers. A classical result in this direction is proved by Tanno and Weber (cf. [28]), and is the following:

Theorem 3.1 *([29]). Let (M, g) be a connected compact Riemannian manifold with constant scalar curvature $S > 0$. Then M is isometric to a sphere if it admits a closed conformal vector field ξ that vanishes at some point of M.*

Now we prove the following result:

Theorem 3.2 *([10]). Let (M, g) be an n-dimensional compact and connected Riemannian manifold whose Ricci curvature satisfies*

$$0 < Ric \leq (n-1)\left(2 - \frac{nc}{\lambda_1}\right)c$$

for a constant c and the first nonzero eigenvalue λ_1 of the Laplace operator. Then the Riemannian manifold (M, g) admits a nonzero gradient conformal vector field if and only if it is isometric to $S^n(c)$.

Proof. Suppose the compact connected Riemannian manifold (M, g) admits a nonzero gradient conformal vector field ξ with potential function f. Then in Eq. (2.4) the tensor field $\varphi = 0$ and we have for the constant c that

$$Ric(\nabla f + c\xi, \nabla f + c\xi) = Ric(\nabla f, \nabla f) + 2cRic(\nabla f, \xi) + c^2 Ric(\xi, \xi).$$

Integrating the above equation and using Eq. (2.9) as $Ric(\nabla f, \xi) = -(n-1)\|\nabla f\|^2$, the inequality (2.6) and Lemma 2.4, we arrive at

$$\int_M Ric(\nabla f + c\xi, \nabla f + c\xi) \leq \int_M \left\{Ric(\nabla f, \nabla f) - (n-1)\left(2 - \frac{nc}{\lambda_1}\right)c\|\nabla f\|^2\right\}.$$

Since M has positive Ricci curvature, the condition in the statement of the theorem, together with above inequality, implies that

$$\nabla f = -c\xi, \tag{3.1}$$

which, together with Eq. (2.4), gives

$$\nabla_X \nabla f = -cfX, \quad X \in \mathfrak{X}(M). \tag{3.2}$$

If f is a constant function, Eq. (2.5) would imply $f = 0$, which together with the fact that $\xi = \nabla\rho$ is a gradient conformal vector field, that is, $\Delta\rho = \operatorname{div}\xi = nf = 0$. Then on compact

M, we get ρ is a constant function, that is $\xi = 0$, which is a contradiction. Hence, f is a nonconstant function that satisfies Obata's differential equation (3.2) and therefore M is isometric to $S^n(c)$ (cf. [22,23]).

The converse is trivial as the sphere $S^n(c)$ has $\lambda_1 = nc$ and admits a nonzero gradient conformal vector field and satisfies the inequality in the hypothesis.

Theorem 3.3 ([10]). *Let (M, g) be an n-dimensional compact and connected Einstein manifold with Einstein constant $\lambda = (n-1)c$. Then (M, g) admits a nonzero gradient conformal vector field if and only if $c > 0$ and it is isometric to $S^n(c)$.*

Proof. Suppose the compact connected Einstein manifold (M, g) admits a nonzero gradient conformal vector field ξ with potential function f. Then taking $\varphi = 0$ and $Q(\xi) = (n-1)c\xi$ in Eq. (2.4), we get

$$\nabla f = -c\xi,$$

which is the same as Eq. (3.1), except that in this case we need to verify $c > 0$. The above equation gives $\Delta f = -ncf$, that is f is an eigenfunction of the Laplace operator which is nonconstant (as seen in the proof of Theorem 3.1), and this proves $c > 0$. Then as in the proof of Theorem 3.2, we see that M is isometric to $S^n(c)$. The converse is trivial.

Remark. On a compact Riemannian manifold (M, g) of constant scalar curvature S, if there exists a nonzero gradient conformal vector field ξ with potential function f, then as $\varphi = 0$, taking divergence on both sides of Eq. (2.9), we get

$$\Delta f = -Sf,$$

where we have used Eq. (2.10) with S a constant. From the above equation, we conclude that $S > 0$ (as for nonzero gradient conformal vector field ξ, the potential function is nonconstant). As a consequence, we arrive at the following result:

Corollary. *On a compact and connected Riemannian manifold of constant nonpositive scalar curvature there does not exist a nonzero gradient conformal vector field.*

Theorem 3.4 ([13]). *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature S with $S \leq (n-1)\lambda_1$, λ_1 being the first nonzero eigenvalue of the Laplace operator on M. Then the Riemannian manifold (M, g) admits a nonzero closed conformal vector field ξ with potential function f which satisfies*

$$Ric\left(\nabla f + \frac{S}{n(n-1)}\xi, \nabla f + \frac{S}{n(n-1)}\xi\right) \geq 0,$$

if and only if it is isometric to $S^n(c)$ for a constant c.

Proof. Suppose the compact connected Riemannian manifold (M, g) of constant scalar curvature S admits a nonzero closed conformal vector field ξ with potential function f. Then as $\varphi = 0$, Eq. (2.9) gives

$$Ric(\nabla f, \xi) = -(n-1)\left\|\nabla f\right\|^2. \tag{3.3}$$

Using this equation and Lemmas 2.1, 2.2 and the fact that S is a constant, we get

$$(n-1)\int_M \|\nabla f\|^2 = S\int_M f^2. \tag{3.4}$$

Eq. (2.6) together with the above equation gives the inequality $\lambda_1(n-1) \leq S$ and the equality holds if and only if the equality in (2.6) holds, and the equality in (2.6) holds if and only if $\Delta f = -\lambda_1 f$ (cf. [4]). However, by the hypothesis and $\lambda_1(n-1) \leq S$, we get the equality $\lambda_1(n-1) = S$ and consequently $\Delta f = -\lambda_1 f$. We have

$$Ric\left(\nabla f + \frac{\lambda_1}{n}\xi, \nabla f + \frac{\lambda_1}{n}\xi\right) = Ric(\nabla f, \nabla f) + 2\frac{\lambda_1}{n}Ric(\nabla f, \xi) + \frac{\lambda_1^2}{n^2}Ric(\xi, \xi). \tag{3.5}$$

Using Lemma 2.6, $\Delta f = -\lambda_1 f$ and the Schwartz inequality $\|A_f\|^2 \geq \frac{1}{n}(\Delta f)^2 = \frac{\lambda_1^2}{n}$, we get

$$\int_M Ric(\nabla f, \nabla f) \leq \frac{n-1}{n}\lambda_1^2\int_M f^2, \tag{3.6}$$

and Eqs. (3.3), (3.4) and $S = (n-1)\lambda_1$ imply

$$\int_M Ric(\nabla f, \xi) = -\lambda_1(n-1)\int_M f^2. \tag{3.7}$$

Integrating Eq. (3.5) and using Eqs. (3.6), (3.7) together with Lemma 2.3, we get

$$\int_M Ric\left(\nabla f + \frac{\lambda_1}{n}\xi, \nabla f + \frac{\lambda_1}{n}\xi\right) \leq 0.$$

This, together with the hypothesis, gives

$$Ric\left(\nabla f + \frac{\lambda_1}{n}\xi, \nabla f + \frac{\lambda_1}{n}\xi\right) = 0.$$

Thus, using Eqs. (3.5), (3.7) and Lemma 2.3, we get

$$\int_M \left(Ric(\nabla f, \nabla f) - \frac{n-1}{n}\lambda_1^2 f^2\right) = 0,$$

which together with Lemma 2.6 and equation $\Delta f = -\lambda_1 f$ gives

$$\int_M \left(\|A_f\|^2 - \frac{1}{n}(\Delta f)^2\right) = 0.$$

Hence the Schwartz inequality gives the equality $A_f = \frac{\Delta f}{n} I = -\frac{\lambda_1}{n} f I$, that is,

$$\nabla_X \nabla f = -\frac{\lambda_1}{n} f X, \quad X \in \mathfrak{X}(M). \tag{3.8}$$

The closed conformal vector field ξ being nonzero, the potential function f is a nonconstant function which satisfies Obata's differential equation (3.8). Therefore M is isometric to $S^n(c)$, where $nc = \lambda_1$.

Conversely, choosing a nonzero constant vector field Z on the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ and the vector field $\xi = Z^T$, the tangential component of Z to the sphere $S^n(c)$, we get $Z = \xi + fN$, where N is the unit normal vector field to $S^n(c)$ and $f = \langle Z, N\rangle$. It then follows that

$$\nabla_X \xi = -\sqrt{c} f X, \quad \nabla f = \sqrt{c}\xi, \quad X \in \mathfrak{X}(S^n(c)),$$

that is, ξ is a nonzero closed conformal vector field on $S^n(c)$ with $\lambda_1 = nc$, which satisfies the hypothesis of the theorem.

Recall that a smooth vector field ξ is said to be parallel on the Riemannian manifold (M, g) if $\nabla_X \xi = 0$, $X \in \mathfrak{X}(M)$ and that the Hessian H_f of a smooth function f is defined by

$$H_f(X, Y) = g\left(A_f X, Y\right), \quad X, Y \in \mathfrak{X}(M)\,.$$

To conclude this section, we obtain the following characterization for the Euclidean space using a closed conformal vector field.

Theorem 3.5. *Let (M, g) be an n-dimensional complete and connected Riemannian manifold. Then M admits a nonparallel closed conformal vector field that annihilates the Ricci operator if and only if it is isometric to the Euclidean space $(R^n, \langle,\rangle)$.*

Proof. Suppose the complete and connected Riemannian manifold (M, g) admits a nonparallel closed conformal vector field ξ with potential function f that annihilates the Ricci operator Q. Then as $\varphi = 0$, Eq. (2.9) gives $\nabla f = 0$, that is, the potential function is a constant. Moreover as ξ is nonparallel by Eq. (2.4), we get the constant $f \neq 0$ and $\nabla_X \xi = fX$, $X \in \mathfrak{X}(M)$. Define a smooth function

$$h = \frac{1}{2}\left\|\xi\right\|^2.$$

Then it follows that $X(h) = fg(X, \xi)$, $X \in \mathfrak{X}(M)$, which gives $\nabla h = f\xi$. Since f is a constant,

$$A_h X = f \nabla_X \xi = f^2 X, \quad X \in \mathfrak{X}(M).$$

Thus $H_f(X, Y) = f^2 g(X, Y)$ with positive constant f^2, and therefore (M, g) is isometric to the Euclidean space $(R^n, \langle,\rangle)$ (cf. [23]).

Conversely, the vector field $\xi = \psi$ the position vector field on R^n is a closed conformal vector field on the Euclidean space $(R^n, \langle,\rangle)$ which annihilates the Ricci operator $Q = 0$ of the Euclidean space. Moreover the vector field ξ satisfies $\nabla_X \xi = X$, which is therefore nonparallel.

Remark. We have seen the results using a closed conformal vector field on Riemannian manifolds of constant scalar curvature characterizing the spheres. We know that the n-sphere $S^n(c)$ admits a closed conformal vector field ξ induced by a nonzero constant vector field Z on the Euclidean space $(R^{n+1}, \langle,\rangle)$ that satisfies $\Delta\xi = -c\xi$. This raises a question:

Assuming (M, g) is a compact and connected n-dimensional Riemannian manifold that admits a nontrivial closed conformal vector field ξ satisfying $\Delta\xi = -\lambda\xi$, for a constant $\lambda > 0$ with sectional curvatures of plane sections containing ξ bounded below by λ, is M necessarily isometric to the sphere $S^n(\lambda)$?

4. Energy and Laplacian of conformal vector fields

In this section, we study the geometry of a Riemannian manifold (M, g) that admits a conformal vector field which need not be closed. On a compact Riemannian manifold (M, g), the energy $e(X)$ of a smooth vector field X is defined by

$$e(X) = \frac{1}{2}\int_M \|X\|^2 .$$

For the sphere $S^n(c)$ of constant curvature c in Euclidean space R^{n+1}, if we denote by ξ the tangential component of a nonzero constant vector field Z on R^{n+1} and by N the unit normal vector field on $S^n(c)$, then we have $\nabla_X\xi = -\sqrt{c}\rho X$ and $\nabla\rho = \sqrt{c}\xi$, that is ξ is a conformal vector field with potential function $f = -\sqrt{c}\rho$, where ρ is the normal component of the constant vector field Z. Moreover, we have

$$e(\xi) = c^{-2}e(\nabla f).$$

This example motivates the following question: Is a compact Riemannian manifold (M, g) that admits a nontrivial conformal vector field ξ with potential function f satisfying $e(\xi) = c^{-2}e(\nabla f)$ for a positive constant c necessarily isometric to $S^n(c)$? The following result shows that the answer to this question is in the affirmative for compact Riemannian manifolds of constant scalar curvature.

Theorem 4.1 ([7]). *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature $S = n(n-1)c$. Then the Riemannian manifold (M, g) admits a nonzero closed conformal vector field ξ with potential function f satisfying*

$$e(\xi) \leq c^{-2}e(\nabla f),$$

if and only if it is isometric to $S^n(c)$.

Proof. Suppose the compact connected Riemannian manifold (M, g) of constant scalar curvature S admits a nontrivial conformal vector field ξ with potential function f. Then Lemmas 2.1 and 2.4 imply that

$$e(\nabla f) = \frac{S}{2(n-1)}\int_M f^2, \tag{4.1}$$

which proves that $S = n(n-1)c > 0$ as ξ is a nontrivial conformal vector field, that is, the potential function is nonconstant. By virtue of Lemma 2.1,

$$\int_M \|\nabla f + c\xi\|^2 = \int_M \left(c^2\|\xi\|^2 + \|\nabla f\|^2 - 2ncf^2\right).$$

Using Eq. (4.1) in the above equation, we arrive at

$$\int_M \|\nabla f + c\xi\|^2 = 2c^2\left(e(\xi) - c^{-2}e(\nabla f)\right),$$

which together with the condition in the hypothesis gives

$$\nabla f = -c\xi.$$

The above equation and Eq. (2.4) imply

$$\nabla_X \nabla f = -cfX - c\varphi X, \quad X \in \mathfrak{X}(M).$$

By taking the inner product with X, and as φ is skew symmetric, this yields

$$g\left(\nabla_X \nabla f, X\right) = -cfg(X, X), \quad X \in \mathfrak{X}(M).$$

On polarization, the above equation gives

$$g\left(\nabla_X \nabla f, Y\right) = -cfg(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

which is Obata's differential equation with nonconstant f and constant $c > 0$. Thus M is isometric to $S^n(c)$.

The converse is trivial as seen in the above example.

We observe that in the above example of the conformal vector field ξ on the sphere $S^n(c)$ induced by the nonzero constant vector field Z on the Euclidean space R^{n+1} has the potential function $f = -\sqrt{c}\rho$, where ρ is the normal component of the constant vector field Z and that $\nabla f = -c\xi$, $\Delta f = -ncf$ holds. Also the Laplacian of the conformal vector field ξ satisfies $\Delta\xi = -c\xi$, that is the conformal vector field ξ is an eigenvector of the Laplace operator Δ with eigenvalue c. Moreover, as $\Delta f = -ncf$, we have

$$e(\nabla f) = \frac{nc^{-1}}{2}\int_M f^2.$$

This raises a question: is a compact Riemannian manifold (M, g) that admits a nontrivial conformal vector field with $\Delta\xi = -c\xi$, and having energy satisfying the above equality for a constant c, necessarily isometric to the sphere $S^n(c)$? In this direction, we show that the answer is in the affirmative, and prove the following result:

Theorem 4.2 ([11]). *An n-dimensional compact and connected Riemannian manifold (M, g) admits a nontrivial conformal vector field ξ with potential function f that satisfies $\Delta\xi = -\lambda\xi$, $\lambda > 0$ with energy*

$$e(\xi) \leq \frac{n\lambda^{-1}}{2}\int_M f^2,$$

if and only if it is isometric to $S^n(\lambda)$.

Proof. Suppose the compact connected Riemannian manifold (M, g) admits a nontrivial conformal vector field ξ with potential function f satisfying $\Delta\xi = -\lambda\xi$, $\lambda > 0$ and the energy condition in the statement. We use Eq. (2.4) to compute $\Delta\xi$ and arrive at

$$\Delta\xi = \nabla f + \sum(\nabla\varphi)(e_i, e_i), \tag{4.2}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Using $\Delta\xi = -\lambda\xi$ and the above equation, we get

$$Q(\xi) = -(n-2)\nabla f + \lambda\xi,$$

which, together with Lemmas 2.1 and 2.3, gives

$$\int_M \|\varphi\|^2 = 2\lambda\left(e(\xi) - \frac{n\lambda^{-1}}{2}\int_M f^2\right).$$

The above equation with the condition in the statement gives $\varphi = 0$ and, consequently, Eq. (4.2) gives

$$\nabla f = -\lambda\xi.$$

This together with Eq. (2.4) and $\varphi = 0$ leads to Obata's differential equation for nonconstant function f (ξ being a nontrivial conformal vector field) and the constant $\lambda > 0$. Hence M is isometric to $S^n(\lambda)$.

The converse is trivial as $S^n(\lambda)$ admits a nontrivial conformal vector field ξ with potential function f that satisfies the energy condition and $\Delta\xi = -\lambda\xi$.

Theorem 4.3 ([6]). *Let (M, g) be an n-dimensional compact and connected Riemannian manifold (M, g) with constant scalar curvature. Then M admits a nontrivial conformal vector field ξ with potential function f that satisfies $\Delta\xi = -\lambda\xi$ for a constant $\lambda > 0$ and the Ricci curvature in the direction of the vector field ∇f is bounded below by $(n-1)\lambda$ if and only if M is isometric to $S^n(\lambda)$.*

Proof. Suppose the compact connected Riemannian manifold (M, g) of constant scalar curvature admits a nontrivial conformal vector field ξ with potential function f that satisfies $\Delta\xi = -\lambda\xi$ for a constant $\lambda > 0$. Using Eq. (4.2), we get

$$\sum(\nabla\varphi)(e_i, e_i) = -\lambda\xi - \nabla f,$$

which, together with Eq. (2.9), gives

$$Q(\xi) = \lambda\xi - (n-2)\nabla f.$$

Since the scalar curvature S is a constant, we get $\operatorname{div}(Q(\xi)) = fS$. Using the divergence in the above equation, we arrive at

$$fS = n\lambda f - (n-2)\Delta f. \tag{4.3}$$

Recalling the Weitzenbock formula $\delta d\xi + d\delta\xi = -\Delta\xi + Q(\xi)$, where $\delta = -\operatorname{div}$ (cf. [1]), and taking its divergence and using $\operatorname{div}(Q(\xi)) = fS$ and Eq. (2.4), we get

$$n\Delta f = -n\lambda f - fS. \tag{4.4}$$

Then Eqs. (4.3) and (4.4) imply that

$$\Delta f = -n\lambda f \quad \text{and} \quad \int_M \|\nabla f\|^2 = n\lambda \int_M f^2. \tag{4.5}$$

We use the above equation in Lemma 2.6 to get

$$\int_M \left(\left(Ric(\nabla f, \nabla f) - (n-1)\lambda \|\nabla f\|^2 \right) + \left(\|A_f\|^2 - \frac{1}{n}(\Delta f)^2 \right) \right) = 0.$$

Since the Ricci curvature in the direction of the vector field ∇f is bounded below by $(n-1)\lambda$, we can apply the Schwartz inequality in the above equation to obtain

$$A_f = \frac{\Delta f}{n} I = -\lambda f I.$$

The conformal vector field being nontrivial, the potential function f is nonconstant and the above equation gives Obata's differential equation for constant $\lambda > 0$ and therefore M is isometric to $S^n(\lambda)$. The converse is trivial as the conformal vector field ξ on $S^n(c)$ induced by the nonzero constant vector field Z on the Euclidean space R^{n+1} satisfies $\Delta \xi = -c\xi$ and the other conditions in the statement of the theorem.

Note that the scalar curvature of the Riemannian manifold being constant (or the manifold being an Einstein manifold) gives a convenient combination with the presence of a conformal vector field to study the geometry of the manifold, especially in getting the characterizations of spheres using conformal vector field. However, if the scalar curvature of the Riemannian manifold is not a constant, then finding such characterizations becomes a difficult task and we do not find many results in the present literature which address the geometry of Riemannian manifolds of non-constant scalar curvature that admit a conformal vector field. In the next result, we study the geometry of a compact Riemannian manifold of non-constant scalar curvature that admits a nontrivial conformal vector field, under the mild condition that the scalar curvature is constant along the integral curves of the conformal vector field. Such a condition together with an upper bound on the scalar curvature and a lower bound on the Ricci curvature in a certain direction gives a characterization of an n-sphere, as shown in the following theorem:

Theorem 4.4 ([*12*]). *Let (M, g) be an n-dimensional compact and connected Riemannian manifold (M, g) with scalar curvature S and first nonzero eigenvalue of the Laplace operator λ_1 satisfying $S \leq (n-1)\lambda_1$. Then M admits a nontrivial conformal vector field ξ with potential function f that satisfies $\xi(S) = 0$ and the Ricci curvature in the direction of the vector field ∇f is bounded below by $n^{-1}S$ if and only if M is isometric to $S^n(c)$ for a constant c.*

Proof. Suppose the compact connected Riemannian manifold (M, g) admits a nontrivial conformal vector field ξ with potential function f satisfying $\xi(S) = 0$ and $Ric(\nabla f, \nabla f) \geq n^{-1}S$. The condition $\xi(S) = 0$ gives

$$Sg(\nabla f, \xi) = S\xi(f) = \operatorname{div}(fS\xi) - f\operatorname{div}(S\xi) = \operatorname{div}(fS\xi) - nf^2S,$$

which, by Lemma 2.4 implies

$$\int_M \left((n-1) \|\nabla f\|^2 - S f^2 \right) = 0. \tag{4.6}$$

Using the inequality (2.6) in the above equation, we get

$$\int_M ((n-1)\lambda_1 - S) f^2 \le 0.$$

In view of the inequality $S \le (n-1)\lambda_1$ this implies that $((n-1)\lambda_1 - S) f^2 = 0$. Thus, on the connected M, we have either $S = (n-1)\lambda_1$ or $f = 0$. However the conclusion $f = 0$ makes ξ a Killing vector field, which is contrary to our assumption that ξ is a nontrivial conformal vector field. Hence we must have $S = (n-1)\lambda_1$. Inserting in Eq. (4.6), this gives

$$\int_M \|\nabla f\|^2 = \lambda_1 \int_M f^2. \tag{4.7}$$

The above equality gives $\Delta f = -\lambda_1 f$. In view of this conclusion and Eq. (4.7), $S = (n-1)\lambda_1$, Lemma 2.6 takes the form

$$\int_M \left(\left(Ric(\nabla f, \nabla f) - \frac{S}{n} \|\nabla f\|^2 \right) + \left(\|A_f\|^2 - \frac{1}{n}(\Delta f)^2 \right) \right) = 0.$$

Using the lower bound on the Ricci curvature and the Schwartz inequality in the above equation gives

$$A_f = \frac{\Delta f}{n} I = -\frac{\lambda_1}{n} f I. \tag{4.8}$$

Since the potential function f of the nontrivial conformal vector field is nonconstant, Eq. (4.8) gives Obata's differential equation with constant $c = \frac{\lambda_1}{n} > 0$, and therefore M is isometric to the sphere $S^n(c)$. The converse is trivial as the nontrivial conformal vector field on $S^n(c)$ induced by a nonzero constant vector field Z on the Euclidean space R^{n+1} satisfies all the conditions in the statement of the theorem.

In the rest of this section we give examples of conformal vector fields on a Riemannian manifold (M, g), which are eigenvectors of the Laplace operator. They are in abundance on Einstein manifolds and we therefore start with the following lemma:

Lemma 4.1. *On an Einstein manifold (M, g) of constant scalar curvature S, an eigenfunction f, $\Delta f = -\lambda f$, gives an eigenvector of the Laplace operator Δ acting on smooth vector fields, namely $\Delta \nabla f = -\mu \nabla f$ with eigenvalue $\mu = \lambda - n^{-1} S$.*

Proof. Suppose (M, g) is an Einstein manifold of constant scalar curvature S and $\Delta f = -\lambda f$ for a smooth function f and constant $\lambda > 0$. Then Lemma 2.5 gives

$$\sum (\nabla A_f)(e_i, e_i) = \left(\frac{S}{n} - \lambda \right) \nabla f.$$

Using this equation, by a straightforward computation we get

$$\begin{aligned}\Delta\nabla f &= \sum\left(\nabla_{e_i}\nabla_{e_i}\nabla f - \nabla_{\nabla_{e_i}e_i}\nabla f\right) = \sum\left(\nabla A_f\right)(e_i, e_i)\\ &= \left(\frac{S}{n} - \lambda\right)\nabla f = -\mu\nabla f,\end{aligned}$$

which proves the Lemma.

We have already seen that the conformal vector field ξ on the n-sphere $S^n(c)$ induced by a nonzero constant vector field Z on the Euclidean space R^{n+1} satisfies $\Delta\xi = -c\xi$, that is the conformal vector field ξ is an eigenvector of the Laplace operator Δ acting on the smooth vector fields on the sphere $S^n(c)$.

Example 4.1. Let (M, g) be an n-dimensional connected Einstein manifold of constant scalar curvature S. Let $\xi = \nabla\rho$, for a smooth function ρ on M, be the gradient conformal vector field on M with potential function f. Since ξ is a closed vector field we have $\varphi = 0$ and Eq. (2.4) gives $\Delta\rho = nf$. Moreover, the Hessian operation A_ρ of the function ρ satisfies $A_\rho X = fX$ and consequently, for a local orthonormal frame $\{e_1, \ldots, e_n\}$, we have

$$\sum\left(\nabla A_\rho\right)(e_i, e_i) = \nabla f.$$

Combining this equation with Lemma 2.5 for the Einstein manifold and $\Delta\rho = nf$, we get

$$-n(n-1)\nabla f = S\nabla\rho, \tag{4.9}$$

which, together with $\Delta\rho = nf$ gives

$$\Delta f = -\frac{S}{(n-1)}f. \tag{4.10}$$

Using Eqs. (2.4) and (4.9), we get $\nabla_X\nabla f = hX,\ X \in \mathfrak{X}(M)$, where $h = -\frac{S}{n(n-1)}f$ is a smooth function, that is $u = \nabla f$ is a gradient conformal vector field. By Lemma 4.1 and Eq. (4.10), we have $\Delta u = -\frac{S}{n(n-1)}u$. Thus on the Einstein manifold (M, g) the conformal vector field u is an eigenvector of the Laplace operator Δ acting on the smooth vector fields on M.

Example 4.2. Consider the Euclidean space (R^n, g), where $g = \langle,\rangle$ is the Euclidean metric and define a metric $\overline{g}$ on R^n by

$$\overline{g}_u = \left(\frac{2}{1 + \|u\|^2}\right)^2 g_u, \quad u \in R^n.$$

Then the Riemannian connection $\overline{\nabla}$ on the Riemannian manifold $(R^n, \overline{g})$ and the Euclidean connection ∇ on (R^n, g) are related by

$$\overline{\nabla}_X Y = \nabla_X Y - X(f)Y - Y(f)X + g(X, Y)\nabla f, \tag{4.11}$$

where $f = \log(1 + \|u\|^2) - \log 2$. Let Ψ be the position vector field on R^n. Then, using (4.11), we get

$$\overline{\nabla}_X \Psi = \left(\frac{1 - \|u\|^2}{1 + \|u\|^2} \right) X, \quad X \in \mathfrak{X}(R^n) \tag{4.12}$$

which proves that Ψ is a conformal vector field on the Riemannian manifold $(R^n, \overline{g})$. Using (4.12), we obtain

$$\overline{\nabla}_X \overline{\nabla}_X \Psi - \overline{\nabla}_{\overline{\nabla}_X X} \Psi = X(h) X, \tag{4.13}$$

where

$$h = \left(\frac{1 - \|u\|^2}{1 + \|u\|^2} \right).$$

For a local orthonormal frame $\{e_1, \ldots, e_n\}$ on (R^n, g), we get the local orthonormal frame $\{e^f e_1, \ldots, e^f e_n\}$ on $(R^n, \overline{g})$. Using the fact that the gradient ∇h of the function h on the Euclidean space (R^n, g) is given by

$$\nabla h = \frac{-4}{(1 + \|u\|^2)^2} \Psi,$$

and consequently by Eq. (4.13), we conclude that $\overline{\Delta} \Psi = -\Psi$, that is the conformal vector field Ψ on the Riemannian manifold $(R^n, \overline{g})$ is an eigenvector of the Laplace operator $\overline{\Delta}$ acting on the smooth vector fields on R^n.

A smooth vector field u on a Riemannian manifold (M, g) is said to be a harmonic vector field if $\Delta u = 0$. In what follows we give an example of a harmonic conformal vector field.

Example 4.3. Consider the Euclidean space $(R^n, \langle, \rangle)$, and the vector field

$$\xi = \psi + x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i},$$

where ψ is the position vector field $x^1, \ldots, x^n$ are the Euclidean coordinates, and i, j are fixed indices with $i \neq j$. Then it is straightforward to see that ξ is a conformal vector field with potential function 1 and the skew symmetric operator φ in Eq. (2.4) is given by

$$\varphi X = X(x^i) \frac{\partial}{\partial x^j} - X(x^j) \frac{\partial}{\partial x^i}, \quad X \in \mathfrak{X}(R^n)$$

and that

$$(\nabla \varphi)(X, Y) = H_{x^i}(X, Y) \frac{\partial}{\partial x^j} - H_{x^j}(X, Y) \frac{\partial}{\partial x^i} = 0, \quad X, Y \in \mathfrak{X}(R^n).$$

With $f = 1$ Eq. (4.2) and the above equation imply $\Delta \xi = 0$, that is, ξ is a harmonic conformal vector field on the Euclidean space $(R^n, \langle, \rangle)$.

5. Characterizing Euclidean spaces by conformal vector fields

One of the interesting questions in differential geometry is to characterize Euclidean spaces among complete and connected Riemannian manifolds. In this section we study the role of conformal vector fields in characterizing Euclidean spaces. Given a conformal vector field ξ on a Riemannian manifold (M, g) with potential function f, we shall call the skew symmetric tensor field φ appearing in Eq. (2.4) *the associated tensor field* of the conformal vector field ξ.

We have seen that the vector field $\xi = \psi$, where ψ is the position vector field on the Euclidean space $(R^n, \langle,\rangle)$ is a nontrivial conformal vector field which satisfies $\varphi(\xi) = 0$, $Q(\xi) = 0$ as $Q = 0$ for the Euclidean space, the vector field being a gradient conformal vector field $\varphi = 0$. Moreover, the conformal vector field ξ is also harmonic, that is, $\varDelta\xi = 0$. We utilize this information on the Euclidean space to arrive at the following characterizations.

Theorem 5.1. *Let (M, g) be an n- dimensional ($n > 2$) complete and connected Riemannian manifold. Then it admits a nontrivial harmonic conformal vector field ξ with potential function f and associated tensor field φ, that annihilates the Ricci operator and satisfies*

$$R(X, \xi; \xi, X) = \|\varphi X\|^2, \quad X \in \mathfrak{X}(M)$$

if and only if M is isometric to the Euclidean space $(R^n, \langle,\rangle)$.

Proof. Suppose (M, g) is a complete and connected Riemannian manifold $\dim M > 2$ that admits a nontrivial conformal vector field ξ with potential function f, satisfying $\varDelta\xi = 0$, $Q(\xi) = 0$ and the curvature condition given in the statement. Then, using Eq. (2.4), it is straightforward to get

$$\varDelta\xi = \nabla f + \sum(\nabla\varphi)(e_i, e_i) = 0. \tag{5.1}$$

Now, using Eq. (2.9), we have

$$Q(\xi) = -(n-1)\nabla f - \sum(\nabla\varphi)(e_i, e_i) = 0. \tag{5.2}$$

These two equation give

$$(n-2)\nabla f = 0,$$

which in view of $n > 2$, means the potential function f is a constant. This constant cannot be zero, for if it were zero that would imply ξ is a Killing vector field, which is not permissible as ξ is a nontrivial conformal vector field. Next define a smooth function $h = \frac{1}{2}\|\xi\|^2$, which by virtue of Eq. (2.4) satisfies

$$\nabla h = f\xi - \varphi(\xi). \tag{5.3}$$

Note that this equation stipulates that the smooth function h is a nonconstant function, since otherwise that would imply $\varphi(\xi) = f\xi$, that is, $f\|\xi\|^2 = 0$ and $\xi = 0$ which contradicts the assumption that ξ is a nontrivial conformal vector field. Taking the covariant derivative with respect to $X \in \mathfrak{X}(M)$ in Eq. (5.3), and using Eqs. (2.4) and (2.8), we get

$$\nabla_X\nabla h = f^2X - R(X,\xi)\xi - \varphi^2X, \quad X \in \mathfrak{X}(M),$$

which, in view of the curvature condition in the hypothesis, gives

$$H_h(X,Y) = cg(X,Y), \quad X, Y \in \mathfrak{X}(M), \quad c = f^2 > 0.$$

Hence, M is isometric to the Euclidean space $(R^n, \langle , \rangle)$ (cf. [23]). The converse is trivial as the conformal vector field $\xi = \psi$ on the Euclidean space $(R^n, \langle , \rangle)$ meets all the requirements in the statement.

Theorem 5.2. *An n-dimensional ($n > 2$) complete and connected Riemannian manifold (M, g) of nonnegative Ricci curvature admits a nontrivial harmonic conformal vector field ξ that annihilates the associated tensor field φ if and only if M is isometric to the Euclidean space $(R^n, \langle , \rangle)$.*

Proof. Suppose (M, g) is a complete and connected Riemannian manifold $\dim M > 2$ of nonnegative Ricci curvature that admits a nontrivial conformal vector field ξ with potential function f satisfying $\Delta\xi = 0$ and $\varphi(\xi) = 0$. Using Eqs. (2.9) and (5.1), we get

$$Q(\xi) = -(n-2)\nabla f,$$

that is

$$Ric(\xi, \xi) = -(n-2)\xi(f). \tag{5.4}$$

Next, using Eq. (2.4) and $\varphi(\xi) = 0$, we compute

$$(\nabla\varphi)(X, \xi) = -f\varphi X - \varphi^2 X, \quad X \in \mathfrak{X}(M).$$

For an orthonormal frame $\{e_1, \ldots, e_n\}$, choose $X = e_i$ in the above equation and take the inner product with e_i and sum the resulting equation, to get

$$-g\left(\xi, \sum(\nabla\varphi)(e_i, e_i)\right) = \|\varphi\|^2,$$

which, together with Eq. (5.1), gives

$$\xi(f) = \|\varphi\|^2.$$

Using this last equation in (5.4), we arrive at $Ric(\xi, \xi) + (n-2)\|\varphi\|^2 = 0$. Since the Ricci curvature is nonnegative and $n > 2$ it follows that $\varphi = 0$. By Eq. (5.1), we get $\nabla f = 0$, that is, f is a constant, which cannot be zero by virtue of the fact that ξ is a nontrivial conformal vector field. Finally, the smooth function $h = \frac{1}{2}\|\xi\|^2$ defined on M has gradient $\nabla h = f\xi$ and as in the proof of previous theorem, h is a nonconstant function that satisfies

$$H_h(X,Y) = cg(X,Y)$$

for a constant $c > 0$ and therefore M is isometric to the Euclidean space $(R^n, \langle , \rangle)$. The converse is trivial.

We denote by η the smooth 1-form dual to the conform vector field ξ on a Riemannian manifold (M, g), that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Finally, we arrive at the following characterization of the Euclidean space.

Theorem 5.3 (*[12]*). *An n-dimensional complete and connected Riemannian manifold (M, g), $(n \geq 3)$, admits a nontrivial harmonic conformal field ξ that annihilates the Ricci operator and satisfies $d\eta(X, \xi) = 0$ for smooth vector fields X on M, where η is the 1-form dual to ξ, if and only if M is isometric to the Euclidean space $(R^n, \langle,\rangle)$.*

In the rest of this section we are interested in obtaining characterizations of the Euclidean complex space form $(C^n, J, \langle,\rangle)$ using specific conformal vector fields on a Kaehler manifold. On the Euclidean complex space form there exists a conformal vector field whose expression for its covariant derivative motivates the definition of a specific vector field on the Kaehler manifold, which we call a special conformal vector field. We have seen in the introduction that the vector field $\xi = \psi + J\psi$ is a conformal vector field on the Kaehler manifold $(C^n, J, \langle,\rangle)$, which is not Killing, where ψ is the position vector field, J is the complex structure and $\langle,\rangle$ is the Hermitian metric on C^n. For this conformal vector field ξ on $(C^n, J, \langle,\rangle)$, we have

$$\nabla_X \xi = X + JX, \quad X \in \mathfrak{X}(C^n).$$

Motivated by the above expression, we say a smooth vector field ξ on a Kaehler manifold (M, J, g) is a special conformal vector field if it satisfies

$$\nabla_X \xi = fX + \rho JX, \quad X \in \mathfrak{X}(M), \tag{5.5}$$

for smooth functions f, ρ defined on M. Note that a conformal vector field ξ on a Kaehler manifold (M, J, g) is a special conformal vector field if the associated tensor field φ of ξ satisfies $\varphi = \rho J$, for a smooth function ρ on M. These special conformal vector fields are a particular case of the vector fields considered in [19]. We observe that for a special conformal vector field ξ on a Kaehler manifold (M, J, g), we have $\varphi = \rho J$ and $J\varphi J = -\varphi$ and therefore, using Eqs. (2.4) and (2.11), we have

$$\operatorname{div}\varphi = -J\nabla\rho, \quad \operatorname{div}(J\varphi J) = J\nabla\rho \text{ and } Tr(J\varphi) = -2n\rho. \tag{5.6}$$

Hence, by Lemma 2.7,

$$(n-1)\left(\nabla f - J\nabla\rho\right) = 0. \tag{5.7}$$

Theorem 5.4 (*[14]*). *A $2n$-dimensional $(n > 1)$ complete and simply connected complex space form $M(c)$ admits a nontrivial conformal vector field ξ with potential function f and associated tensor field φ satisfying $\operatorname{div}\varphi = J\nabla r$ and $\operatorname{div}(J\varphi J) = J\nabla s$ for smooth functions r, s on M if and only if M is isometric to the Euclidean complex space form $(C^n, J, \langle,\rangle)$.*

Proof. Suppose $M(c)$ is a complete and simply connected complex space form $\dim M > 2$ that admits a nontrivial conformal vector field ξ with potential function f and associated tensor field φ satisfying $\operatorname{div}\varphi = J\nabla r$ and $\operatorname{div}(J\varphi J) = J\nabla s$ for smooth functions r, s. Using Eq. (2.13), we get

$$Q(\xi) = \frac{c}{2}(n+1)\xi.$$

Also, using $\operatorname{div}\varphi = J\nabla r$ in Eq. (2.9), we get

$$J\nabla r = (2n-1)\nabla f + Q(\xi) = (2n-1)\nabla f + \frac{c}{2}(n+1)\xi.$$

Taking divergence in the above equation and using Lemma 2.8, we arrive at

$$(2n-1)\Delta f + n(n+1)cf = 0. \tag{5.8}$$

By Lemma 2.7 we have

$$2(n-1)\nabla f = J\nabla r - J\nabla s + J\nabla(TrJ\varphi).$$

Taking divergence and using Lemma 2.8, gives $2(n-1)\Delta f = 0$, that is, $\Delta f = 0$. From Eq. (5.8) we have $n(n+1)cf = 0$, hence either $c = 0$ or $f = 0$. But the choice $f = 0$ contradicts the fact that ξ is a nontrivial conformal vector field, and hence $c = 0$. Consequently, M is isometric to the Euclidean complex space form $(C^n, J, \langle,\rangle)$. The converse is trivial as the Euclidean complex space form $(C^n, J, \langle,\rangle)$ admits the nontrivial conformal vector field $\xi = \psi + J\psi$, with potential function $f = 1$ and associated tensor field $\varphi = J$ satisfying $\mathrm{div}\varphi = 0$ and $\mathrm{div}(J\varphi J) = 0$, and thus meets the requirements.

Since a special conformal vector field ξ on a complex space form $M(c)$ satisfies Eq. (5.6), as a direct consequence of Theorem 5.4, we have the following:

Corollary 5.1 ([14]). *A $2n$-dimensional ($n > 1$) complete and simply connected complex space form $M(c)$ admits a nontrivial special conformal vector field ξ, if and only if M is isometric to the Euclidean complex space form $(C^n, J, \langle,\rangle)$.*

Corollary 5.2 ([14]). *On a hyperbolic complex space form there does not exist a nontrivial special conformal vector field.*

Theorem 5.5 ([14]). *A $2n$-dimensional ($n > 1$) complete and simply connected Kaehler manifold (M, J, g) admits a nontrivial harmonic special conformal vector field ξ if and only if it is isometric to the Euclidean complex space form $(C^n, J, \langle,\rangle)$.*

Proof. Suppose that ξ is a nontrivial harmonic special conformal vector field on a $2n$-dimensional Kaehler manifold (M, J, g). Then the associated tensor field φ of ξ satisfies $\varphi = \rho J$ for some smooth function ρ. Using Eq. (5.5), we compute

$$\Delta\xi = \nabla f + J\nabla\rho.$$

Since $\Delta\xi = 0$ (ξ is harmonic), Eq. (5.7) and $n > 1$ imply $\nabla f = 0$ and $\nabla\rho = 0$, that is, both functions f and ρ are constants. Thus Eq. (5.5) takes the form

$$\nabla_X\xi = c_1X + c_2JX, \quad X \in \mathfrak{X}(M). \tag{5.9}$$

Then for the smooth function $h = \frac{1}{2}\|\xi\|^2$ on M, using Eq. (5.9), we compute the gradient ∇h and the Hessian operator as

$$\nabla h = c_1\xi - c_2J\xi, \tag{5.10}$$

and

$$A_h(X) = (c_1^2 + c_2^2)X = cX, \quad X \in \mathfrak{X}(M). \tag{5.11}$$

If the constant $c = 0$, that would give $c_1 = c_2 = 0$, which by Eq. (5.9) would imply that ξ is parallel and hence Killing, thereby contradicting the fact that ξ is a nontrivial conformal vector field. Hence $c > 0$. Moreover, if h is a constant function, Eq. (5.10) would imply that $\xi = 0$ (since constants c_1 and c_2 cannot both be zero), which contradicts the fact that ξ is a nontrivial conformal vector field and hence h is a nonconstant function. By virtue of Eq. (5.11) h satisfies

$$H_h(X,Y) = cg(X,Y), \quad X,Y \in \mathfrak{X}(M).$$

Hence M is isometric to the Euclidean complex space form C^n (cf. [29]).

The converse is trivial as the Euclidean complex space form $(C^n, J, \langle,\rangle)$ admits a nontrivial harmonic special conformal vector field $\xi = \psi + J\psi$.

6. CONFORMAL VECTOR FIELDS ON KAEHLER MANIFOLDS

In this section, we study conformal vector fields on a Kaehler manifold. The model spaces in Kaehler manifolds, the Euclidean space C^n and the complex projective space CP^n, are counterparts of Riemannian manifolds, the Euclidean space R^n, and the unit sphere S^n respectively. However, with respect to the existence of conformal vector fields the analogy is not complete as it is known that on the complex projective space, CP^n, $(n > 1)$ there does not exists a nontrivial conformal vector field, whereas on the unit sphere S^n there are many. Indeed, it is known that a conformal vector field on a $2n$-dimensional compact Kaehler manifold (M, J, g), $(n > 1)$ is Killing (cf. [17]). This result limits the use of nontrivial conformal vector fields in characterizing complex projective spaces beyond the real dimension 2. Therefore the focus in the geometry of conformal vector fields on Kaehler manifolds is to find conditions for a conformal vector field on a noncompact Kaehler manifold to be Killing.

Recall that a smooth vector field ξ on a Kaehler manifold (M, J, g) is said to be an analytic vector field if the flow of ξ consists of holomorphic diffeomorphisms (that is, the differentials of flow diffeomorphisms commute with the complex structure J), which is equivalent to the requirement that

$$\pounds_\xi J = 0,$$

where $\pounds_\xi$ is the Lie-derivative with respect the vector field ξ. A smooth vector field ξ on a Kaehler manifold (M, J, g) is said to be an analytic conformal vector field if it is both an analytic vector field and a conformal vector field. As pointed out earlier, on a compact Kaehler manifold (M, J, g) with $\dim M > 2$ each conformal vector field is Killing. However on noncompact Kaehler manifolds nontrivial conformal vector fields are in abundance, for instance the vector field $\xi = \psi + J\psi$ on the Kaehler manifold $(C^n, J, \langle,\rangle)$ is a nontrivial conformal vector field. Moreover this vector field is an analytic vector field on the Kaehler manifold $(C^n, J, \langle,\rangle)$. For a smooth vector field X on a Kaehler manifold (M, J, g), we set $\overline{X} = JX$, $X \in \mathfrak{X}(M)$ and it is easy see that if ξ is an analytic vector field on the Kaehler manifold (M, J, g), then so is $\overline{\xi}$.

Lemma 6.1 ([8]). *Let ξ be an analytic conformal vector field with potential function f on a Kaehler manifold (M, J, g). Then the associated tensor field φ of the conformal vector field ξ satisfies $J \circ \varphi = \varphi \circ J$ and the covariant derivative of the vector field $\overline{\xi}$ is given by*

$$\nabla_X \overline{\xi} = AX + fJX, \quad X \in \mathfrak{X}(M),$$

where $A = J \circ \varphi$ is a symmetric $(1,1)$ tensor field.

Lemma 6.2 ([8]). *Let ξ be an analytic conformal vector field with potential function f on a Kaehler manifold (M, J, g). Then the tensor field $A = J \circ \varphi$ satisfies*

$$\operatorname{div} A = \Delta\bar{\xi} - J\nabla f.$$

Lemma 6.3 ([8]). *Let ξ be an analytic conformal vector field with potential function f on a Kaehler manifold (M, J, g). Then the tensor field $A = J \circ \varphi$ satisfies*

$$(\nabla A)(X, Y) - (\nabla A)(Y, X) = J[R(X, Y)\xi + Y(f)X - X(f)Y], \quad X, Y \in \mathfrak{X}(M).$$

Lemma 6.4 ([8]). *Let ξ be an analytic conformal vector field with potential function f on a $2n$-dimensional Kaehler manifold (M, J, g). Then the trace $h = TrA$ tensor field $A = J \circ \varphi$ satisfies*

$$\nabla h = \Delta\bar{\xi} - Q(\bar{\xi}) - 2J\nabla f \quad \text{and} \quad \nabla h = -2Q(\bar{\xi}) - 2nJ\nabla f.$$

Note that if ξ is an analytic conformal vector field with potential function f on a $2n$-dimensional Kaehler manifold (M, J, g), then using Eqs. (2.9) and (4.2), we have

$$\Delta\xi + Q(\xi) = -2(n-1)\nabla f. \tag{6.1}$$

Theorem 6.1 ([8]). *Let (M, J, g) be a Kaehler manifold of constant scalar curvature $S \neq 0$ and $\dim M \neq 4$. Then a harmonic analytic conformal vector field on M is Killing.*

Proof. Suppose that ξ is a harmonic analytic conformal vector field on a $2n$-dimensional Kaehler manifold (M, J, g) with potential function f. Then Eq. (6.1) gives $Q(\xi) = -2(n-1)\nabla f$ and as Eq. (2.12) implies that the Ricci operator Q commutes with the complex structure J, we have

$$Q(\bar{\xi}) = -2(n-1)J\nabla f. \tag{6.2}$$

Using Eq. (6.2) and the fact that $\Delta\bar{\xi} = J\Delta\xi = 0$ in Lemma 6.4, we get

$$-J\nabla h = 2(n-2)\nabla f.$$

Taking divergence in the above equation and using Lemma 2.8, we get $(n-2)\Delta f = 0$. Since $\dim M \neq 4$, we have $n \neq 2$ and consequently $\Delta f = 0$, which together with Eq. (6.2) gives $\operatorname{div}(Q(\xi)) = 0$. However, as the scalar curvature S is a constant, we have $\operatorname{div}(Q(\xi)) = fS = 0$ and the scalar curvature $S \neq 0$ implies that $f = 0$. Hence the conformal vector field is Killing.

Theorem 6.2 ([8]). *Let (M, J, g) be a $2n$-dimensional Kaehler Einstein manifold ($n > 1$). If ξ is an analytic conformal vector field on M, then either ξ is a Killing vector field or else (M, J, g) is Ricci flat.*

Proof. Suppose that ξ is an analytic conformal vector field on a $2n$-dimensional Kaehler Einstein manifold (M, J, g) with potential function f. Then using $Q(\xi) = (2n)^{-1}S\xi$ in

Lemma 6.4, we get

$$\nabla h = -\frac{S}{n}\bar{\xi} - 2nJ\nabla f \quad \text{and} \quad J\nabla h = 2n\nabla f + \frac{S}{n}\xi. \tag{6.3}$$

Taking divergence in equations in (6.3) and using Lemma 2.8, we arrive at

$$\Delta h = -\frac{S}{n}h \quad \text{and} \quad \Delta f = -\frac{S}{n}f. \tag{6.4}$$

Now, taking the covariant derivative in the first equation in (6.3) with respect to $X \in \mathfrak{X}(M)$, we get

$$A_h(X) = -\frac{S}{n}AX - \frac{S}{n}fJX - 2nJA_fX, \quad X \in \mathfrak{X}(M),$$

which gives

$$(\nabla A_h)(X,Y) = -\frac{S}{n}(\nabla A)(X,Y) - \frac{S}{n}X(f)JY - 2nJ(\nabla A_f)(X,Y), \tag{6.5}$$

$X, Y \in \mathfrak{X}(M)$. For a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on M, take $X = Y = e_i$ in Eq. (6.5) and sum, while using Eqs. (6.4) and Lemma 2.5, one arrives at

$$\frac{S}{2n}\nabla h - \frac{S}{n}\nabla h = -\frac{S}{n}J\nabla f - 2nJ\left(\frac{S}{2n}\nabla f - \frac{S}{n}\nabla f\right) - \frac{S}{n}\sum(\nabla A)(e_i, e_i). \tag{6.6}$$

Lemma 6.4 together with the property of the curvature tensor of a Kaehler manifold $R(X,Y)JZ = JR(X,Y)Z$, implies

$$X(h) = -Ric(X, \bar{\xi}) - g(X, J\nabla f) + g\left(X, \sum(\nabla A)(e_i, e_i)\right),$$

that is,

$$\sum(\nabla A)(e_i, e_i) = \nabla h + \frac{S}{2n}\bar{\xi} + J\nabla f.$$

Using the above equation in (6.6), we arrive at

$$\frac{S}{2n}\left(\nabla h + \frac{S}{n}\bar{\xi} - 2(n-2)J\nabla f\right) = 0. \tag{6.7}$$

From the second equation in Lemma 6.4 we have

$$\nabla h + \frac{S}{n}\bar{\xi} = -2nJ\nabla f.$$

This, together with Eq. (6.7), gives

$$(n-1)SJ\nabla f = 0.$$

Thus either $S = 0$, that is, the Kaehler manifold is Ricci flat or else $\nabla f = 0$, which gives $\Delta f = 0$. By Eq. (6.4) we have $f = 0$ for $(S \neq 0)$, hence the vector field ξ is Killing.

7. φ-ANALYTIC CONFORMAL VECTOR FIELDS

Given a conformal vector field ξ on a Riemannian manifold (M, g) with potential function f and associated tensor field φ, if the conformal vector field ξ is not closed, then $\varphi \neq 0$ and it is a skew symmetric tensor field. Taking clue from analytic vector fields on a complex manifold, one considers conformal vector fields whose flow leaves the associated tensor field φ invariant. Such conformal vector fields on a Riemannian manifold will be the subject of study in this section.

Definition. A conformal vector field ξ on a Riemannian manifold (M, g) with associated tensor field φ is said to be a φ-analytic conformal vector field if the tensor field φ is invariant under the flow of ξ.

It follows from the above definition that a conformal vector field ξ is φ-analytic if and only if

$$(\pounds_\xi \varphi)(X) = 0, \quad X \in \mathfrak{X}(M). \tag{7.1}$$

We have seen that the vector field $\xi = \psi + J\psi \in \mathfrak{X}(\mathcal{C}^n)$, where ψ is the position vector field and J is the complex structure on the complex Euclidean space $\mathcal{C}^n$ is a conformal vector field with potential function $f = 1$ and associated tensor field $\varphi = J$ and it satisfies Eq. (7.1), hence ξ is a φ-analytic vector field. Note that the conformal vector fields on the n-sphere $S^n(c)$ induced by constant vector fields on R^{n+1} are φ-analytic vector fields. Furthermore, the vector field

$$\xi = \psi + x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$$

on the Euclidean space $(R^n, \langle , \rangle)$, where ψ is the position vector field and $x^1, \ldots, x^n$ are Euclidean coordinates and $i \neq j$ are two fixed indices, is a φ-analytic vector field with the associated tensor field φ given by

$$\varphi(X) = X(x^i)\frac{\partial}{\partial x^j} - X(x^j)\frac{\partial}{\partial x^j}, \quad X \in X(R^n).$$

Note that on a connected Riemannian manifold (M, g) a nontrivial conformal vector field ξ cannot have a constant length. For if h is the square of the length of ξ, then using Eq. (2.4), we would get

$$\nabla h = 2\left(f\xi - \varphi\xi\right),$$

which for a constant h, would give

$$f\left\|\xi\right\|^2 = 0.$$

The above equation on connected M gives either $f = 0$ or $\xi = 0$ and both of these are ruled out for a nontrivial conformal vector field. Indeed, this proves that a conformal vector field of constant length on a connected Riemannian manifold is Killing.

Having seen many examples of φ-analytic conformal vector fields, to see that this is a stronger notion than conformal vector field, consider the odd dimensional unit sphere (S^{2n-1}, g), $n > 1$. The Reeb vector field $\xi^{'}$ defines the Sasakian structure $(\varphi, \xi^{'}, \eta^{'}, g)$ that satisfies

$$\nabla_X \xi^{'} = -\varphi X, \quad X \in \mathfrak{X}(S^{2n-1})$$

and

$$(\nabla\varphi)(X, Y) = g(X, Y)\xi^{'} - \eta(Y)X\ , \quad X, Y \in \mathfrak{X}(S^{2n-1}).$$

Using the gradient conformal vector field $u = Z^T$ on S^{2n-1} induced by a constant vector field Z on the Euclidean space R^{en}, which satisfies

$$u = \nabla f \text{ and } \nabla_X u = -fX, \quad X \in \mathfrak{X}(S^{2n-1}),$$

where $f = \langle Z, N\rangle$, N being the unit normal vector field to S^{2n-1}, we get the vector field $\xi = -u - \xi^{'}$. This is a conformal vector field with potential function f and associated tensor field φ. We claim that this conformal vector field ξ with potential function f and associated tensor field φ is not a φ-analytic vector field. If it were, then an easy computation shows that

$$(\pounds_\xi\, \varphi)(X) = g(X, \xi^{'})u - g(u, X)\xi^{'} = 0, \quad X \in \mathfrak{X}(S^{2n-1}),$$

and that the vector fields u and $\xi^{'}$ are parallel. Hence there exists a smooth function ρ on S^{2n-1} such that $u = \rho\xi^{'}$. Taking the covariant derivative with respect to $X \in \mathfrak{X}(S^{2n-1})$, we get $\rho\varphi X = X(\rho)\xi^{'} + fX$. Operating φ on this equation, to obtain $f\varphi X = \rho\eta(X)\xi^{'} - \rho X$, then eliminating φX using the last two equations, we get

$$(f^2 + \rho^2)X = \left(\rho^2\eta(X) - fX(\rho)\right)\xi^{'}, \quad X \in \mathfrak{X}(S^{2n-1}).$$

Choose unit vector field X orthogonal to $\xi^{'}$ in the above equation to conclude that $f = \rho = 0$, which is a contradiction as u is a nontrivial gradient conformal vector field. Hence ξ is not a φ analytic vector field.

First, we prove the following result, which gives an important property of a φ-analytic conformal vector field.

Theorem 7.1 ([11]). *A conformal vector field ξ on a Riemannian manifold (M, g) with potential function f is a φ-analytic conformal vector field if and only if there exists a smooth function ρ on M such that $\nabla f = \rho\xi$.*

Proof. Suppose ξ is a φ-analytic vector field with potential function f. Then using Eqs. (2.4) and (7.1), we get

$$(\nabla\varphi)(\xi, X) = 0, \quad X \in \mathfrak{X}(M).$$

In view of Eq. (2.8), this gives

$$g(X, \xi)\nabla f = g(X, \nabla f)\xi, \quad X \in \mathfrak{X}(M).$$

Thus, we get $\nabla f \wedge \xi = 0$ and consequently the vector fields ∇f and ξ are parallel. Hence, there exists a smooth function ρ on M such that $\nabla f = \rho\xi$.

Conversely, assume that $\nabla f = \rho\xi$. Then using Eqs. (2.4) and (2.8), we have

$$\begin{aligned}(\pounds_\xi\, \varphi)(X) &= [\xi, \varphi X] - \varphi[\xi, X] = (\nabla\varphi)(\xi, X)\\ &= g(X, \nabla f)\xi - g(X, \xi)\nabla f = 0\end{aligned}$$

which proves that ξ is a φ-analytic vector field.

A conformal vector field ξ on a Riemannian manifold (M, g) with associated tensor field φ is said to be a *null conformal vector field* if it satisfies $\varphi(\xi) = 0$. The following result shows that on non-positively curved Riemannian manifolds φ-analytic conformal vector fields are in abundance.

Theorem 7.2 ([11]). *A null conformal vector field ξ on a Riemannian manifold (M, g) with potential function f satisfying $R(\nabla f, \xi; \nabla f, \xi) \leq 0$ is a φ-analytic conformal vector field.*

Proof. Using $\varphi(\xi) = 0$ and Eq. (2.8), we get

$$-\varphi\left(f\nabla f + \varphi(\nabla f)\right) = R(\nabla f, \xi)\xi. \tag{7.2}$$

Taking the inner product in the above equation with ∇f,

$$R(\nabla f, \xi; \xi, \nabla f) = \left\|\varphi(\nabla f)\right\|^2.$$

Since the sectional curvature is non-positive, we get $\varphi(\nabla f) = 0$, and consequently, $[\nabla f, \xi] = f\nabla f - \nabla_\xi \nabla f$. On the other hand Eq. (7.2) gives $R(\nabla f, \xi)\xi = 0$. Thus

$$\nabla_{\nabla f} f\xi - \nabla_\xi (f\nabla f) - \nabla_{f\nabla f}\xi + \nabla_{\nabla_\xi \nabla f}\xi = 0,$$

which, on using Eq. (2.4), gives

$$\left\|\nabla f\right\|^2 \xi - \xi(f)\nabla f + \varphi\left(\nabla_\xi \nabla f\right) = 0.$$

Taking the inner product in the above equation with ξ, we get

$$g(\nabla f, \xi)^2 = \left\|\nabla f\right\|^2 \left\|\xi\right\|^2$$

that is, $\nabla f = \rho\xi$ for a smooth function ρ on M. By Theorem 7.1, this proves that ξ is a φ-analytic vector field.

Finally, we use a specific type of φ-analytic vector field to find a characterization for spheres. If the function ρ appearing in the characterization of the φ-analytic conformal vector field ξ in Theorem 7.1 is a constant, then we say that ξ is a constant type φ-analytic conformal vector field.

Theorem 7.3 ([11]). *Let ξ be a nontrivial φ-analytic conformal vector field of constant type on an n-dimensional compact and connected Riemannian manifold (M, g). Then M is isometric to the n-sphere $S^n(c)$.*

Proof. Note that we have $\nabla f = \alpha\xi$, where α is a constant. The constant $\alpha \neq 0$ for otherwise the potential function f will be a constant. By Eq. (2.5) this will imply $f = 0$, which is contrary to our assumption that ξ is a nontrivial conformal vector field. Hence, $\alpha \neq 0$ and the vector field $\xi = \alpha^{-}\nabla f$ is a closed vector field, which by Eq. (2.3) gives $\varphi = 0$. Thus, taking covariant derivative of both sides of equation $\nabla f = \alpha\xi$ with respect to $X \in \mathfrak{X}(M)$ and using Eq. (2.4), we get

$$\nabla_X \nabla f = \alpha f X, \quad X \in \mathfrak{X}(M). \tag{7.3}$$

We claim that α is a negative constant. To see that observe that Eq. (7.3) gives $\Delta f = n\alpha f$, that is, f is an eigenfunction of the Laplace operator Δ. Since this is an elliptic operator on the compact Riemannian manifold its eigenvalue is either $n\alpha = 0$ or $n\alpha < 0$. The first option implies $\Delta f = 0$, that is, f is a constant which is ruled out as seen above. Hence $\alpha < 0$ and we put $\alpha = -c$, $c > 0$ which makes Eq. (7.3) Obata's differential equation. That proves M is isometric to $S^n(c)$.

Finally, we have the following characterization of the Euclidean space $(R^n, \langle,\rangle)$ using φ-analytic conformal vector fields.

Theorem 7.4. *An n-dimensional $(n > 1)$,complete and connected Riemannian manifold (M, g) admits a nontrivial null φ-analytic conformal vector field ξ with potential function f and associated tensor field φ satisfying*

$$\mathrm{div}\varphi = 0 \textit{ and } Ric(\xi, \xi) = 0,$$

if and only if it is isometric to the Euclidean space $(R^n, \langle,\rangle)$.

Proof. Suppose the complete and connected Riemannian manifold (M, g) admits a nontrivial null φ-analytic conformal vector field that satisfies the conditions in the statement. Then we have

$$\varphi(\xi) = 0 \text{ and } \nabla f = \rho\xi \tag{7.4}$$

for a smooth function ρ on M. Since, $\mathrm{div}\varphi = 0$, we have for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M

$$\sum g\left((\nabla\varphi)(e_i X), e_i\right) = 0, \quad X \in \mathfrak{X}(M),$$

which gives

$$\sum(\nabla\varphi)(e_i, e_i) = 0, \tag{7.5}$$

From Eqs. (2.9), (7.4) and $Ric(\xi, \xi) = 0$ we conclude that

$$(n-1)\rho \|\xi\|^2 = 0.$$

As the vector field ξ is a nontrivial conformal vector field and $n > 1$, the above equation gives $\rho = 0$, which by virtue of Eq. (7.4) implies that f is a constant. Moreover, f is a nonzero constant, for $f = 0$ makes ξ Killing, which is contrary to the assumption that ξ is a nontrivial conformal vector field. Also, using Eq. (2.4), we compute

$$\mathrm{div}(\varphi(\xi)) = -\|\varphi\|^2 - g\left(\xi, \sum(\nabla\varphi)(e_i, e_i)\right).$$

Using Eqs. (7.4) and (7.5), we conclude that $\varphi = 0$ and thus Eq. (2.4) for the function $h = \frac{1}{2}\|\xi\|^2$ gives

$$H_h(X, Y) = cg(X, Y),$$

where the constant $c = f^2 > 0$ and hence M is isometric to the Euclidean space $(R^n, \langle , \rangle)$ (cf. [25]). The converse is trivial as the Euclidean space $(R^n, \langle , \rangle)$ admits the nontrivial conformal vector field $\xi = \psi$, where ψ is the position vector field, that satisfies all requirements in the statement.

Remark. Given a φ-analytic vector field ξ on a Riemannian manifold (M, g) with potential function f, it gives rise to another smooth function ρ on M that satisfies $\nabla f = \rho\xi$. It will be worth exploring the properties of this function to utilize these properties in getting different characterizations of the spheres $S^n(c)$ as well as the Euclidean spaces $(R^n, \langle , \rangle)$. For instance in Theorem 7.3, we used $\rho =$ constant for characterizing spheres. Similarly, one could expect the bounds on ρ could be used to get some new characterizations of spheres as well as Euclidean spaces. Moreover, as we have seen all closed conformal vector fields are trivially φ-analytic conformal vector fields as the associated tensor field $\varphi = 0$ for these conformal vector fields. The examples of nontrivial φ-analytic vector fields known so far are on the Euclidean spaces, therefore it will be worth constructing nontrivial φ-analytic conformal vector fields on other spaces.

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