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Optimal strategies for some team games

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Abstract

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Consider a game between teams A and B, consisting of a sequence of matches, where each match takes place between one player i from A and one player j from B. Given the probability that player i wins over player j, we investigate optimal strategies on how to choose a player for the next match, for the following two types of team games. The first type assumes that after each match, the loser is eliminated from the list of remaining players, while the winner remains in the list. The team from which all players are eliminated loses the game. Assuming the Bradley-Terry model as the probability model, we first show that the winning probability does not depend on the strategy chosen. It is also shown that the Bradley-Terry model is essentially the only model for which this strategy independence holds. The second type of game assumes that both players are eliminated after each match. In this case, it is shown that choosing a player with equal probability is an optimal strategy in the sense of maximizing the expected number of wins of matches, provided that information about the order of players in the other teams is not available. The case in which a team knows the ordering of the other team is also studied.

1. Introduction

This paper considers the following team game played by teams A and B, each consisting of a specified number of players. Each team selects one member at each stage to play a match. The win or loss of the game is then determined as a result of such a sequence of matches. A strategy that each team can take is how to choose a player for the next match. Assuming that the winning probability of player i of team A against player j of team B is p_{ij}, this paper investigates properties of optimal strategies.

Among many team games in the above category, this paper is concerned with the following two types. In the first type, the loser of a match is eliminated from the list of players, while the winner remains in the list. The team from which all players are eliminated loses the game. This type of team game is often adopted in the games of "*judo*", "*kendo*" (Japanese fencing), and "*go*". Assume that a positive value representing his or her strength is associated with each player, and that when player i with strength a plays a match against player j with strength b, the probability p_{ij} of i to win the match is generally given by $p_{ij}=p(a,b)$. In particular, if $p(a,b)=a/(a+b)$, it is called the *Bradley–Terry model* (BT model for short) after Bradley and Terry [5] and Bradley [2,3,4], who introduced this to model paired comparisons of sensory test to rank samples (e.g., foods) on the basis of the preference of panelists. Other applications of the BT model have been discussed by Luce [11] for the probabilistic choice theory, Fararo [7] for mathematical sociology and Takeuchi and Fujino [15,16] for a match without a tie in sports. This paper first shows that, under the BT model, any strategy is optimal, i.e., the probability for a team to win the game does not depend on the strategy chosen. This is a generalization of the result of Katoh and Adachi [10] who considered the case in which the player ordering of each team is determined in advance and the rule of choosing the next player from the list is also specified (e.g., in "*judo*" or "*kendo*", a player who won a match must play the next match). It is also shown that the BT model is essentially the only model for which this strategy independence holds.

In the second type of team game, both players have to leave after each match, irrespective of the outcome. If no team has any information on the strategy of the other team, it is shown that choosing with equal probability a player for the next match maximizes the expected number of wins of matches. This result is a restatement of Gale's theorem [8] proved in a different setting. We shall also study the case in which a team knows the player ordering of the other team, and derive some properties of an optimal strategy under the BT model.

2. Team game in which only loser leaves the team

Let $A=\{1,2,\ldots,M\}$ and $B=\{1,2,\ldots,N\}$ be the index sets of the players at the initial stage. At every stage, each team dynamically selects a player from the list of remaining players, and only the loser of the match leaves the team. This is a two-person constant-sum sequential game formulated in the following manner (see [9,17]).

State space:

$$W=\{(S_A,S_B): S_A\subset A,\ S_B\subset B\}-\{(\emptyset,\emptyset)\},$$

where S_A and S_B denote the index sets of remaining players of teams A and B respectively. States $w=(S_A,S_B)$ with $S_A=\emptyset$ or $S_B=\emptyset$ are absorbing states, implying the end of game.

Action sets: The action set of team A (team B) at state (S_A,S_B) is S_A (S_B), i.e., an action (i.e., an index) is selected for the next match from the action set of each team.

Transition law: Assume that the strength of $i\in A$ ($j\in B$) is denoted by a_i (b_j), and the probability of i to defeat j is given by $p(a_i,b_j)$. If actions $i\in S_A$ and $j\in S_B$ are selected at state $w=(S_A,S_B)$, then state w moves to state w' with probability $P(w,i,j,w')$, where

$$P(w,i,j,w')=\begin{cases} p(a_i,b_j), & \text{if } w'=(S_A,S_B-\{j\}),\\ 1-p(a_i,b_j), & \text{if } w'=(S_A-\{i\},S_B),\\ 0, & \text{otherwise.}\end{cases}$$

Payoff function: If a state (S_A,S_B) ($|S_A|\geq 1$, $|S_B|=1$) moves to $(S_A,\emptyset)$, then team A receives 1, while B receives 0. Conversely, if a state (S_A,S_B) ($|S_A|=1$, $|S_B|\geq 1$) moves to $(\emptyset,S_B)$, A receives 0, while B receives 1. Thus it is easy to see that the expected total payoff that team A receives is equal to the winning probability of team A, while the expected total payoff of team B is $1-$(the winning probability of A).

History: A history h^t at the tth stage is a sequence consisting of all states and actions by then:

$$h^t=(w^1,x^1,y^1,\ldots,w^{t-1},x^{t-1},y^{t-1},w^t),$$

where $w^s=(S_A^s,S_B^s)$, x^s and y^s are the state, the action of A and the action of B, respectively, at the sth stage. Let H^t denote the set of all possible h^ts.

Dynamic strategy: A dynamic strategy π_A for team A is a sequence $(\pi_A^1,\pi_A^2,\ldots,\pi_A^t,\ldots)$, where π_A^t is a conditional probability distribution on S_A^t for a given history h^t. In other words, $\pi_A^t(i\mid h^t)$ is the probability that, given h^t, team A selects player i ($\in S_A^t$) at the tth match. π_B^t is defined in a similar manner. Let Π_A (Π_B) denote the set of all possible dynamic strategies for team A (B).

Let $V_{\pi_A,\pi_B}(h^t)$ be the expected total payoff that, for a given history h^t (up to the tth stage), team A receives under strategies π_A and π_B. Team A (B) tries to find a strategy π_A (π_B) that maximizes (minimizes) $V_{\pi_A,\pi_B}(h^t)$. It is known in the theory of finite stage two-person constant-sum sequential game (see [9,17]), that the secure levels

$$\max_{\pi_A\in\Pi_A}\Big[\min_{\pi_B\in\Pi_B} V_{\pi_A,\pi_B}(h^t)\Big] \tag{1}$$

of team A and

$$\min_{\pi_B\in\Pi_B}\Big[\max_{\pi_A\in\Pi_A} V_{\pi_A,\pi_B}(h^t)\Big] \tag{2}$$

of team B coincide, and depend only on the current state $w^t=(S_A,S_B)$ of history h^t. Furthermore, any maximizer π_A^* of (1) and any minimizer π_B^* of (2) are optimal for both teams A and B in the sense that

$$V_{\pi_A,\pi_B^*}(h^t)\le V_{\pi_A^*,\pi_B^*}(h^t)\le V_{\pi_A^*,\pi_B}(h^t)$$

holds for any $\pi_A\in\Pi_A$ and $\pi_B\in\Pi_B$. For a history h^t with $w^t=(S_A,S_B)$, denote the value of (1) and (2) by

$$V(S_A,S_B)=\max_{\pi_A\in\Pi_A}\ \min_{\pi_B\in\Pi_B} V_{\pi_A,\pi_B}(h^t)=\min_{\pi_B\in\Pi_B}\ \max_{\pi_A\in\Pi_A} V_{\pi_A,\pi_B}(h^t)$$

and call it the *value of game* with initial state (S_A,S_B). It is also known that these $V(S_A,S_B)$ satisfy the following optimality equations:

$$V(S_A,\emptyset)=1,$$

$$V(\emptyset,S_B)=0,$$

$$V(S_A,S_B)=\max_{\alpha\in P(S_A)}\ \min_{\beta\in P(S_B)}\left\{\sum_{i\in S_A}\sum_{j\in S_B}\alpha(i)\beta(j)[p(a_i,b_j)V(S_A,S_B-\{j\}) + (1-p(a_i,b_j))V(S_A-\{i\},S_B)]\right\} \quad (3)$$

for $S_A\ne\emptyset$ and $S_B\ne\emptyset$,

where $P(X)$ is the set of all probability distributions over set X. Based on a solution to (3), an optimal strategy π_A^* can be constructed by a maximizer α in the right-hand side of (3) at each state $w^t=(S_A,S_B)$. This means that π_A^* depends only on the current state (not the whole history). An optimal strategy π_B^* can be similarly constructed by selecting a minimizer β.

2.1. *Strategy independence under the BT model*

Here we assume the BT model, i.e.,

$$p(a_i,b_j)=\frac{a_i}{a_i+b_j},$$

and show that the winning probability is independent of the strategy chosen, i.e., any dynamic strategy is optimal. Of course, this result depends on the probability model of the game, and we shall show in the next subsection that the strategy independence holds only for the BT model.

Lemma 2.1. *For the team game of this section, the optimality equations* (3) *reduce to*

$$V(S_A,S_B)=\frac{a_i}{a_i+b_j}V(S_A,S_B-\{j\})+\frac{b_j}{a_i+b_j}V(S_A-\{i\},S_B) \quad (4)$$

for all $i\in S_A$ and $j\in S_B$,

where S_A and S_B satisfy $S_A\ne\emptyset$ and $S_B\ne\emptyset$ (i.e., (3) *is independent of α and β).*

Proof. This is proved by induction on $(m,n)=(|S_A|,|S_B|)$. For notational convenience, let

$$U_{ij}=\frac{a_i}{a_i+b_j}V(S_A,S_B-\{j\})+\frac{b_j}{a_i+b_j}V(S_A-\{i\},S_B). \tag{5}$$

(i) $m=1$. We consider the case of $S_A=\{1\}$ without loss of generality. We prove

$$V(S_A,S_B)=\prod_{k\in S_B}\frac{a_1}{a_1+b_k} \tag{6}$$

by induction on $n=|S_B|$, because this implies

$$\begin{aligned}V(S_A,S_B)&=\frac{a_1}{a_1+b_j}\left(\prod_{k\in S_B-\{j\}}\frac{a_1}{a_1+b_k}\right)+\frac{b_j}{a_1+b_j}\times 0\\&=\frac{a_1}{a_1+b_j}V(S_A,S_B-\{j\})+\frac{b_j}{a_1+b_j}V(\emptyset,S_B)\\&=U_{1j}\end{aligned}$$

for any $j\in S_B$, i.e., the lemma statement. For $n=1$, (6) is obvious. For general n, (3) becomes

$$\begin{aligned}V(S_A,S_B)&=\min_{\beta\in P(S_B)}\left\{\sum_{j\in S_B}\beta(j)\left(\frac{a_1}{a_1+b_j}V(S_A,S_B-\{j\})+\frac{b_j}{a_1+b_j}V(\emptyset,S_B)\right)\right\}\\&=\min_{\beta\in P(S_B)}\left\{\sum_{j\in S_B}\beta(j)\frac{a_1}{a_1+b_j}\left(\prod_{k\in S_B-\{j\}}\frac{a_1}{a_1+b_k}\right)\right\}\\&=\min_{\beta\in P(S_B)}\left\{\sum_{j\in S_B}\beta(j)\left(\prod_{k\in S_B}\frac{a_1}{a_1+b_k}\right)\right\}\\&=\left(\prod_{k\in S_B}\frac{a_1}{a_1+b_k}\right)\min_{\beta\in P(S_B)}\left\{\sum_{j\in S_B}\beta(j)\right\}\\&=\prod_{k\in S_B}\frac{a_1}{a_1+b_k},\end{aligned} \tag{7}$$

where the second equality follows from the induction hypothesis.

(ii) $n=1$. After proving

$$V(S_A,S_B)=1-\prod_{k\in S_A}\frac{b_1}{a_k+b_1} \tag{8}$$

for $S_B=\{1\}$ by an argument similar to (i), we easily obtain

$$V(S_A,S_B)=U_{i1}$$

for any $i\in S_A$.

(iii) For $m=2$ and $n=2$. Assume $S_A=\{1,2\}$ and $S_B=\{1,2\}$ without loss of generality. From (i) and (ii), U_{11} becomes

$$U_{11}=\frac{a_1}{a_1+b_1}V(\{1,2\},\{2\})+\frac{b_1}{a_1+b_1}V(\{2\},\{1,2\})$$

$$=\frac{a_1}{a_1+b_1}\left(1-\frac{b_2}{a_1+b_2}\cdot\frac{b_2}{a_2+b_2}\right)+\frac{b_1}{a_1+b_1}\left(\frac{a_2}{a_2+b_1}\cdot\frac{a_2}{a_2+b_2}\right)$$

(by (6) and (8))

$$=\frac{a_1(a_2+b_1)(a_1a_2+a_1b_2+a_2b_2)+a_2^2b_1(a_1+b_2)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)}$$

$$=\frac{a_1a_2\{a_1a_2+(a_1+a_2)(b_1+b_2)\}+b_1b_2(a_1^2+a_1a_2+a_2^2)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)}.$$

Since this is invariant under interchanging a_1 and a_2 and/or b_1 and b_2, U_{ij} does not depend on the choice of either $i\in S_A$ or $j\in S_B$. Therefore

$$\begin{aligned}V(S_A,S_B)&=\max_{\alpha\in P(S_A)}\min_{\beta\in P(S_B)}\left\{\sum_{i\in S_A}\sum_{j\in S_B}\alpha(i)\beta(j)U_{ij}\right\}\\&=U_{ij}\max_{\alpha\in P(S_A)}\min_{\beta\in P(S_B)}\left\{\sum_{i\in S_A}\sum_{j\in S_B}\alpha(i)\beta(j)\right\}\\&=U_{ij}\end{aligned}\tag{9}$$

holds for any $i\in S_A$ and $j\in S_B$.

(iv) Assume $m\geq 3$, $n\geq 2$ or $m\geq 2$, $n\geq 3$. First for any $k\in S_A$ with $k\neq i$, and $l\in S_B$ with $l\neq j$,

$$\begin{aligned}U_{ij}&=\frac{a_i}{a_i+b_j}\left(\frac{a_k}{a_k+b_l}V(S_A,S_B-\{j,l\})+\frac{b_l}{a_k+b_l}V(S_A-\{k\},S_B-\{j\})\right)\\&\quad+\frac{b_j}{a_i+b_j}\left(\frac{a_k}{a_k+b_l}V(S_A-\{i\},S_B-\{l\})+\frac{b_l}{a_k+b_l}V(S_A-\{i,k\},S_B)\right)\\&=\frac{a_k}{a_k+b_l}\left(\frac{a_i}{a_i+b_j}V(S_A,S_B-\{j,l\})+\frac{b_j}{a_i+b_j}V(S_A-\{i\},S_B-\{l\})\right)\\&\quad+\frac{b_l}{a_k+b_l}\left(\frac{a_i}{a_i+b_j}V(S_A-\{k\},S_B-\{j\})+\frac{b_j}{a_i+b_j}V(S_A-\{i,k\},S_B)\right)\\&=\frac{a_k}{a_k+b_l}V(S_A,S_B-\{l\})+\frac{b_l}{a_k+b_l}V(S_A-\{k\},S_B)\\&=U_{kl}\end{aligned}\tag{10}$$

follows from the induction hypothesis. Next we shall show below that $U_{ij}=U_{kj}$ and $U_{ij}=U_{il}$. Consider without loss of generality that $m\geq 3$, and choose $g\in S_A$ with $g\neq i,k$, and $h\in S_B$ with $h\neq j$. Applying (10) with k and l replaced by g and h respectively, we have $U_{ij}=U_{gh}$, while $U_{gh}=U_{kj}$ follows from (10) with i, j and l replaced by g, h and j respectively. This proves $U_{ij}=U_{kj}$. Similarly, (10) and $U_{il}=U_{kl}$ (apply the above discussion with j replaced by l) yield $U_{ij} = U_{il}$. Consequently, U_{ij} is constant for all i and j. Similarly to the case of (iii), this proves that (9) holds for any $i \in S_A$ and $j \in S_B$. $\square$

Theorem 2.2 (Strategy independence). *For the team game described in this section, any $\pi_A \in \Pi_A$ (any $\pi_B \in \Pi_B$) is an optimal dynamic strategy for team A (B).*

Proof. Since U_{ij} of (5) does not depend on the choice of either i or j as shown in the proof of Lemma 2.1, it is clear that any pair of distributions $\alpha \in P(S_A)$ and $\beta \in P(S_B)$ respectively attain the max and min of the right-hand side of optimality equations (3). The theorem follows from this observation and the known properties of optimal strategies stated in conjunction with (1) and (2). $\square$

Some team games have additional rules in carrying out matches. In "*judo*", "*kendo*" or "*go*", the following rule is often used. Each team determines the ordering of all players in advance. Let $(1,2,\dots,M)$ and $(1,2,\dots,N)$ be such initial orderings of teams A and B respectively, without loss of generality. Then player 1 of team A and player 1 of team B play the first match. The winner must play the next match in succession. In general, if $(i,i+1,\dots,M)$ is the ordering of team A just before the tth stage, the first player i participates the tth match, and the ordering for the $(t+1)$-st stage is updated as follows.

$$
\begin{array}{ll}
(i,i+1,\dots,M), & \text{if } i \text{ wins,} \\
(i+1,i+2,\dots,M), & \text{if } i \text{ loses.}
\end{array}
$$

The ordering of team B is similarly updated. This type of team game is called "*elimination series*". In "*soft tennis*" (a variation of "*hard*" tennis, which was originated in Japan), the winner of a match does not play the next match, but is placed in the last position of the list. This type of team game is called "*exterminatory series*".

Formally, these additional rules are defined as a sequence $r=(r^1,r^2,\dots,r^t,\dots)$, where $r^t=(R_A^t,R_B^t)$ is a pair of mappings which restricts the index sets of players for the next match to $R_A^t(h^t)$ $(\subset S_A^t)$ and $R_B^t(h^t)$ $(\subset S_B^t)$ respectively. Although a team game with an additional rule r is not a sequential game as originally stated in this section, we can observe that any dynamic strategies taken by teams A and B are still optimal for such a game because the sets of all dynamic strategies Π_A (respectively Π_B) also include those strategies which obey the additional rule r.

Corollary 2.3. *Even if an additional rule r as stated above is imposed, the winning probability of a team is independent of the chosen strategies.*

Corollary 2.3 specialized to elimination and exterminatory series was first shown by [10].

2.2. Condition for strategy independence

In this subsection we consider the converse of Theorem 2.2. That is, the condition on the probability model, under which the strategy independence of Theorem 2.2 holds, is derived, assuming that player $i \in A$ ($j \in B$) has strength a_i (b_j), and the probability that $i \in A$ wins a match against $j \in B$ is $p(a_i, b_j)$. Although, this $p(\cdot, \cdot)$ is originally defined over domain $\{(a_i, b_j) \mid i \in A, j \in B\}$, we extend its domain to the following symmetric set,

$$X = \{(a_i, b_j)\colon i \in A, j \in B\} \cup \{(b_j, a_i)\colon j \in B, i \in A\}, \tag{11}$$

and assume the following for any $(a, b) \in X$:

$$0 < p(a, b) < 1, \tag{12}$$

$$p(a, b) + p(b, a) = 1. \tag{13}$$

Property (13) states that matches are unbiased and that strengths of players in two teams are measured on a common platform.

With this notation, optimality equations (3) become

$$V(S_A, \emptyset) = 1,$$

$$V(\emptyset, S_B) = 0,$$

$$V(S_A, S_B) = \max_{\alpha \in P(S_A)} \min_{\beta \in P(S_B)} \left\{ \sum_{i \in S_A} \sum_{j \in S_B} \alpha(i)\beta(j)[p(a_i, b_j)V(S_A, S_B - \{j\}) + p(b_j, a_i)V(S_A - \{i\}, S_B)] \right\} \tag{14}$$

for $S_A \neq \emptyset$ and $S_B \neq \emptyset$.

Lemma 2.4. *Under the above assumption, the strategy independence of Theorem 2.2 implies that*

$$p(a_i, b_j)p(a_{i'}, b_{j'})p(b_{j'}, a_i)p(b_j, a_{i'}) = p(b_j, a_i)p(b_{j'}, a_{i'})p(a_i, b_{j'})p(a_{i'}, b_j) \tag{15}$$

holds for any $i, i' \in A$ and $j, j' \in B$.

Proof. First note that

$$U_{ij} = p(a_i, b_j)V(S_A, S_B - \{j\}) + p(b_j, a_i)V(S_A - \{i\}, S_B)$$

in (14) gives the expected total payoff (the winning probability) of team A when the game starts from initial state (S_A, S_B), and teams A and B use the strategies that respectively choose the first players $i \in S_A$ and $j \in S_B$ with probability 1 and obey their optimal strategies thereafter. Further note that, as properties (6) and (8) in the proof of Lemma 2.1 (replace $a_i/(a_i + b_j)$ in the proof by $p(a_i, b_j)$),

$$V(S_A, S_B) = \coprod_{j \in S_B} p(a_i, b_j) \tag{16}$$

holds if $S_A=\{i\}$ and

$$V(S_A,S_B)=1-\prod_{i\in S_A} p(a_i,b_j) \tag{17}$$

holds if $S_B=\{j\}$.

Now consider the case of $S_A=\{1,2\}$ and $S_B=\{1,2\}$. We have

$$\begin{aligned}
U_{11}&=p(a_1,b_1)V(S_A,S_B-\{1\})+p(b_1,a_1)V(S_A-\{1\},S_B)\\
&=p(a_1,b_1)\{1-p(b_2,a_1)p(b_2,a_2)\}+p(b_1,a_1)\{p(a_2,b_1)p(a_2,b_2)\}\\
&\qquad \text{(by (16) and (17))}\\
&=p(a_1,b_1)\{(p(a_1,b_2)+p(b_2,a_1))(p(a_2,b_2)+p(b_2,a_2))-p(b_2,a_1)p(b_2,a_2)\}\\
&\qquad +p(b_1,a_1)p(a_2,b_1)p(a_2,b_2) \quad \text{(by (13))}\\
&=p(a_1,b_1)p(a_1,b_2)p(a_2,b_2)+p(a_1,b_1)p(a_1,b_2)p(b_2,a_2)\\
&\qquad +p(a_1,b_1)p(b_2,a_1)p(a_2,b_2)+p(b_1,a_1)p(a_2,b_1)p(a_2,b_2)\\
&=p(a_1,b_1)p(a_1,b_2)p(a_2,b_2)\{p(a_2,b_1)+p(b_1,a_2)\}\\
&\qquad +p(a_1,b_1)p(a_1,b_2)p(b_2,a_2)\{p(a_2,b_1)+p(b_1,a_2)\}\\
&\qquad +p(a_1,b_1)p(b_2,a_1)p(a_2,b_2)\{p(a_2,b_1)+p(b_1,a_2)\}\\
&\qquad +p(b_1,a_1)p(a_2,b_1)p(a_2,b_2)\{p(a_1,b_2)+p(b_2,a_1)\} \quad \text{(by (13))}.
\end{aligned} \tag{18}$$

Similarly, we can obtain U_{12}. From the expressions of U_{11} and U_{12},

$$\begin{aligned}
U_{11}-U_{12}&=p(a_1,b_1)p(b_2,a_1)p(a_2,b_2)p(b_1,a_2)\\
&\qquad -p(a_2,b_1)p(b_2,a_2)p(a_1,b_2)p(b_1,a_1)\\
&=p(a_1,b_1)p(a_2,b_2)p(b_2,a_1)p(b_1,a_2)\\
&\qquad -p(b_1,a_1)p(b_2,a_2)p(a_1,b_2)p(a_2,b_1).
\end{aligned} \tag{19}$$

To prove the assertion, assume without loss of generality that (15) does not hold for $i=1$, $i'=2$, $j=1$, $j'=2$. Then it follows $U_{11}\neq U_{12}$ from (19). This means that the winning probability of team A from state $(\{1,2\},\{1,2\})$ becomes different depending upon whether team B chooses 1 or 2 with probability 1 for the first player. Also, by (12), there are strategies of teams A and B, under which state $(\{1,2\},\{1,2\})$ is reachable from initial state (A,B) with positive probability (e.g., the game of "*elimination series*" with respective initial orderings $(M,M-1,\dots,2,1)$ of team A and $(N,N-1,\dots,2,1)$ of team B). Therefore the winning probability of team A (from initial state (A,B)) becomes different depending upon whether team B chooses 1 or 2 with probability 1 for the next player when state $(\{1,2\},\{1,2\})$ is reached. Accordingly, (15) is necessary for the strategy independence. □

We shall now identify the functional form of $p(a,b)$. The following lemma is a slight generalization of the well-known result (Subsection 5.1.2 in the English translation of Aczél [1]), which derives the stated functional form under the assumption that the domain X of $p(\cdot,\cdot)$ is a Cartesian product of some set X', i.e., $X=X'\times X'$.

Lemma 2.5. *Under* (12) *and* (13), $p(\cdot,\cdot)$ *defined over* X *of* (11) *satisfies* (15) *if and only if*

$$p(a,b)=\frac{f(a)}{f(a)+f(b)} \tag{20}$$

for some function $f(\cdot)$ *with* $f(x)>0$, *defined over* $Y=\{a_i\colon i\in A\}\cup\{b_j\colon j\in B\}$.

Proof. Define the function $r(a,b)$ over X of (11) by

$$r(a,b)=\log\frac{p(a,b)}{1-p(a,b)}. \tag{21}$$

Condition (13) implies

$$r(a,b)+r(b,a)=0 \tag{22}$$

for any $(a,b)\in X$, and (15) becomes

$$r(a_i,b_j)+r(a_{i'},b_{j'})+r(b_{j'},a_i)+r(b_j,a_{i'})=0 \tag{23}$$

for any $i,i'\in A$ and $j,j'\in B$. We shall show below that the general solution of the system of functional equations (22) and (23) is

$$r(a,b)=q(a)-q(b), \tag{24}$$

for some function $q(\cdot)$ on Y.

(i) If $\{a_i\colon i\in A\}\cap\{b_j\colon j\in B\}\neq\emptyset$, then

$$a_{i^*}=b_{j^*}=c$$

holds for some $i^*\in A$ and $j^*\in B$, and $r(c,c)=0$ holds by (22). Therefore, (22) and (23) imply

$$r(a_i,b_j)=r(a_i,c)-r(b_j,c) \tag{25}$$

for any $i\in A$ and $j\in B$. Thus, define

$$q(x)=r(x,c) \tag{26}$$

for $x\in Y$, and we have (24).

(ii) $\{a_i\colon i\in A\}\cap\{b_j\colon j\in B\}=\emptyset$. Fix $i'=1$ and $j'=1$. Then, from (22) and (23), we have

$$r(a_i,b_j)=r(a_i,b_1)+r(b_1,a_1)-r(b_j,a_1) \tag{27}$$

for any $i\in A$ and $j\in B$. Thus, define a function $q(\cdot)$ by

$$q(x)=\begin{cases} r(a_i,b_1)+r(b_1,a_1), & \text{if } x=a_i \text{ for some } i\in A,\\ r(b_j,a_1), & \text{if } x=b_j \text{ for some } j\in B,\end{cases}$$

and we have (24).

Now, using

$$f(x)=e^{q(x)}, \quad x\in Y, \tag{28}$$

$p(a,b)$ is written by (21) and (24) as follows.

$$p(a,b)=\frac{f(a)}{f(a)+f(b)}, \quad (a,b)\in X. \tag{29}$$

In addition, $f(x)$ is positive valued by (28). This proves the "*only if*" part.

Conversely, it is clear that any $p(a,b)$ of (20) satisfies (15) as well as (12) and (13). □

In view of this lemma, we conclude that, given $p(a,b)$ for which the strategy independence holds, we again obtain the BT model by redefining strength a of a player by $f(a)$. This observation together with Theorem 2.2 yields the following theorem.

Theorem 2.6. *Under* (12) *and* (13), *the strategy independence of Theorem* 2.2 *holds if and only if the winning probability* $p(a_i,b_j)$ *obeys the BT model.*

3. Team game in which both winner and loser leave

In this model, teams A and B both have N players whose index sets are denoted by

$$A=\{1,2,\dots,N\}, \qquad B=\{1,2,\dots,N\}.$$

Each player participates a match exactly once. After each match, two players who participated the match leave the teams irrespective of the outcome. The objective of each team is to maximize the expected number of wins.

3.1. The case in which the strategy of the opponent is unknown

First consider the case in which the strategy of the other team is unknown. Assuming that each team dynamically selects a player for the next match, we derive an optimal strategy that maximizes the expected number of wins. Instead of the BT model, we assume here a more general model that the winning probability of player $i\in A$ over player $j\in B$ is given by p_{ij}. Thus $1-p_{ij}$ is the probability that player $j\in B$ wins over player $i\in A$. This problem can be formulated as the following two-person constant-sum sequential game.

State space:

$$W=\{(S_A,S_B)\colon S_A\subset A,\ S_B\subset B,\ |S_A|=|S_B|\},$$

where S_A and S_B stand for the index sets of remaining players of teams A and B, respectively. State $w=(\emptyset,\emptyset)$ is an absorbing state, implying the end of the game.

Action sets: The action set of team A (team B) at state (S_A,S_B) is S_A (S_B).

Transition law: If actions $i\in S_A$ and $j\in S_B$ are respectively selected at state $w=(S_A,S_B)$, then state w moves to state w' with probability $P(w,i,j,w')$, where

$$P(w,i,j,w')=\begin{cases}1, & \text{if } w'=(S_A-\{i\},S_B-\{j\}),\\ 0, & \text{otherwise.}\end{cases}$$

Payoff function: If $i\in S_A$ wins a match, then team A receives 1, while B receives 0. Otherwise, team B receives 1, while A receives 0. Thus the expected payoff for team A (respectively team B) is p_{ij} (respectively $1-p_{ij}$).

History and *strategy* are defined in the same way as in Section 2.

Let $V_{\pi_A,\pi_B}(h^t)$ be the expected total payoff for team A under strategies π_A and π_B, for a given history $h^t\in H^t$. Similarly to the case of Section 2, it is known (e.g., [9,17]) that there exists a real number $V(S_A,S_B)$ (the *value of game* with initial state (S_A,S_B)) for $w^t=(S_A,S_B)$ of history h^t such that

$$V(S_A,S_B)=\max_{\pi_A\in\Pi_A}\min_{\pi_B\in\Pi_B}V_{\pi_A,\pi_B}(h^t)=\min_{\pi_B\in\Pi_B}\max_{\pi_A\in\Pi_A}V_{\pi_A,\pi_B}(h^t).$$

These $V(S_A,S_B)$ can be obtained by solving the following optimality equations:

$$\begin{aligned}&V(\emptyset,\emptyset)=0,\\ &V(S_A,S_B)=\max_{\alpha\in P(S_A)}\min_{\beta\in P(S_B)}\left\{\sum_{i\in S_A}\sum_{j\in S_B}\alpha(i)\beta(j)[p_{ij}\right.\\ &\qquad\qquad\qquad\qquad\left.+V(S_A-\{i\},S_B-\{j\})]\right\}\\ &\quad\text{for } (S_A,S_B)\neq(\emptyset,\emptyset).\end{aligned}\tag{30}$$

An optimal strategy π_A^* can be constructed by always selecting a maximizer α at each state $w^t=(S_A,S_B)$, while an optimal policy π_B^* can be similarly constructed by selecting a minimizer β.

Gale [8] discussed a closely related game called "game with finite resource", where "resource" corresponds to players in our case. He considered an asymmetric situation that, in our context, only one team (say team A) is allowed to dynamically choose a player for the next match, while the other team (team B) must determine the ordering of players in advance (i.e., a static strategy). He shows that, if team B chooses a static strategy to select one ordering out of $N!$ possibilities with equal probability, the expected total payoff is independent of the dynamic strategy of team A. Similar properties are also observed in Ross [13,14], and Dror [6] for a card game named "*Goofspiel*".

The following theorem on our dynamic game is also similar to Gale's result. In fact, its proof can be easily done by slightly modifying Gale's proof; hence it is omitted.

Theorem 3.1. *For* $|S_A| = |S_B| = n \geq 1$,

$$V(S_A, S_B) = \frac{1}{n} \sum_{i \in S_A} \sum_{j \in S_B} p_{ij} \tag{31}$$

holds. In addition, the following term in the right-hand side of (30)

$$\sum_{i \in S_A} \sum_{j \in S_B} \alpha(i)\beta(j)[p_{ij} + V(S_A - \{i\}, S_B - \{j\})]$$

becomes constant with respect to $\beta \in P(S_B)$, *if* $\alpha(i) = 1/n$ *for all* $i \in S_A$, *and becomes constant with respect to* $\alpha \in P(S_A)$, *if* $\beta(j) = 1/n$ *for all* $j \in S_B$. *(Therefore the* max min *of the right-hand side of* (30) *is attained if we let* $\alpha(i) = 1/n$ *for* $i \in S_A$, *and* $\beta(j) = 1/n$ *for* $j \in S_B$.)

3.2. The case in which the ordering of the other team is known

Assuming the BT model, consider the case in which both teams determine the orderings of players before the game starts. Furthermore, in determining the ordering of team A, we assume that the ordering of team B is known to team A in advance. In this setting, Theorem 3.1 is no longer true. We shall derive in this subsection some properties of an optimal ordering for team A. Let a_i for $i \in A$ and b_j for $j \in B$ be the strength of players as defined in Section 2, and assume

$$a_1 < a_2 < \cdots < a_N, \qquad b_1 < b_2 < \cdots < b_N \tag{32}$$

for simplicity. For an ordering $(i_1, i_2, \ldots, i_N)$ of team A and $(j_1, j_2, \ldots, j_N)$ of team B, the expected number of wins of team A is

$$\sum_{m=1}^{N} \frac{a_{i_m}}{a_{i_m} + b_{j_m}}. \tag{33}$$

Thus, we can fix the ordering of team B as

$$(1, 2, \ldots, N) \tag{34}$$

without loss of generality. It should be noted that maximizing (33) is a special case of the assignment problem in combinatorial optimization, and an optimal ordering can be efficiently obtained by an appropriate algorithm (e.g., [12]). In this subsection, we are interested in properties of an optimal ordering.

Lemma 3.2. *For an optimal ordering* $(i_1, i_2, \ldots, i_N)$ *of team* A,

$$(a_{i_k} - a_{i_l})(b_k b_l - a_{i_k} a_{i_l}) \geq 0 \tag{35}$$

holds for any k, l *with* $1 \leq k < l \leq N$.

Proof. If we exchange i_k and i_l in an optimal ordering $(i_1, i_2, \ldots, i_N)$, the expected number of wins must not increase, i.e.,

$$\frac{a_{i_l}}{a_{i_l}+b_k}+\frac{a_{i_k}}{a_{i_k}+b_l}-\frac{a_{i_k}}{a_{i_k}+b_k}-\frac{a_{i_l}}{a_{i_l}+b_l}\le 0.$$

From this we have

$$(b_k-b_l)(a_{i_k}-a_{i_l})(b_kb_l-a_{i_k}a_{i_l})\le 0.$$

The lemma now follows from assumption $b_k<b_l$ of (32). □

Lemma 3.3. *For any subsequence $\alpha=(i_{k_1},i_{k_2},\dots,i_{k_n})$ of an optimal ordering $(i_1,i_2,\dots,i_N)$ of team A, such that*

$$i_{k_1}=\max_{1\le m\le n} i_{k_m},$$

indices i_{k_m} for $m=1,2,\dots,n$ are monotone decreasing in m (cf. (32)).

Proof. Because objective function (33) is additive, maximization of the partial sum

$$\sum_{m=1}^{n}\frac{a_{\sigma_m}}{a_{\sigma_m}+b_{k_m}}$$

over all permutations $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_n)$ of $(i_{k_1},i_{k_2},\dots,i_{k_n})$ can be done independently of other components in the whole sequence, where $(k_1,k_2,\dots,k_n)$ is the subsequence of $(1,2,\dots,N)$ of team B corresponding to $(i_{k_1},i_{k_2},\dots,i_{k_n})$. Therefore it suffices to show the following: if $i_1=N$ in an optimal ordering $(i_1,i_2,\dots,i_N)$ of team A, then $(i_1,i_2,\dots,i_N)=(N,N-1,\dots,1)$.

Assume $i_p<i_q$ for some $p<q$. By Lemma 3.2, we have

$$(a_N-a_{i_p})(b_1b_p-a_Na_{i_p})\ge 0,$$

$$(a_{i_p}-a_{i_q})(b_pb_q-a_{i_p}a_{i_q})\ge 0.$$

By $a_{i_p}<a_{i_q}<a_N$ (recall (32)), this implies

$$b_1b_p-a_Na_{i_p}\ge 0,$$

$$b_pb_q-a_{i_p}a_{i_q}\le 0.$$

On the other hand, $b_1b_p<b_pb_q$ and $a_{i_p}a_{i_q}<a_Na_{i_p}$ must hold by $b_1<b_q$ and $a_{i_q}<a_N$. This is a contradiction. □

Lemma 3.4. *For any subsequence $\alpha=(i_{k_1},i_{k_2},\dots,i_{k_n})$ of an optimal ordering $(i_1,i_2,\dots,i_N)$ of team A, at least one of the following four cases occurs.*

$$i_{k_1}=\min_{1\le m\le n} i_{k_m}, \tag{36}$$

$$i_{k_n}=\min_{1\le m\le n} i_{k_m}, \tag{37}$$

$$i_{k_1}=\max_{1\le m\le n} i_{k_m}, \tag{38}$$

$$i_{k_n}=\max_{1\le m\le n} i_{k_m}. \tag{39}$$

In addition, (38) *implies* (37).

Proof. Similarly to the proof of Lemma 3.3, we only consider the case of $n=N$, i.e., $(i_{k_1}, i_{k_2}, \ldots, i_{k_n})=(i_1, i_2, \ldots, i_N)$.

If $N=1,2$, the lemma is obvious. For $N=3$, the optimal ordering not satisfying the condition of the lemma is (3, 1, 2) only (this satisfies (38) but not (37)). However, this does not satisfy the condition of Lemma 3.3, and hence it is not optimal. For $N=4$, an ordering which does not satisfy this lemma but satisfies the condition of Lemma 3.3 is (2, 1, 4, 3) only. From Lemma 3.2, we have

$$b_1b_2-a_2a_1\ge 0 \quad\Rightarrow\quad b_1/a_2\ge a_1/b_2, \tag{40}$$

$$b_1b_4-a_2a_3\le 0 \quad\Rightarrow\quad b_1/a_2\le a_3/b_4, \tag{41}$$

$$b_2b_4-a_1a_3\le 0 \quad\Rightarrow\quad b_2/a_1\le a_3/b_4, \tag{42}$$

$$b_2b_3-a_1a_4\le 0 \quad\Rightarrow\quad b_2/a_1\le a_4/b_3, \tag{43}$$

$$b_3b_4-a_4a_3\ge 0 \quad\Rightarrow\quad b_4/a_3\ge a_4/b_3. \tag{44}$$

Using (44), (43), (40), (41) in this order,

$$b_4/a_3\ge a_4/b_3\ge b_2/a_1\ge a_2/b_1\ge b_4/a_3$$

follows. Thus, consider the case of

$$b_4/a_3=a_4/b_3=b_2/a_1=a_2/b_1=c.$$

Then from (42),

$$c=b_4/a_3\le a_3/b_4=1/c$$

holds. This implies $c\le 1$, and hence $a_1=b_2/c>cb_1=a_2$, which contradicts (32).

For $N>4$, if we consider an ordering which does not satisfy this lemma but satisfies the condition of Lemma 3.3, there exist l, m with $1<l<m<N$, $i_l=1$ and $i_m=N$. Hence, applying a similar argument as in $N=4$ to i_1, i_l, i_m, i_N, we can derive a contradiction.

The last assertion of the lemma directly follows from Lemma 3.3. □

This lemma states that for any subsequence $\alpha=(i_{k_1}, i_{k_2}, \ldots, i_{k_n})$ of an optimal sequence of team A, at least one of the following four matches must take place:

team A		team B
the weakest player in α	vs	the weakest player in β
the weakest player in α	vs	the strongest player in β
the strongest player in α	vs	the weakest player in β
the strongest player in α	vs	the strongest player in β,

where β is the corresponding subsequence $(k_1, k_2, \ldots, k_n)$ of team B.

When $N=3$, ordering $(3,1,2)$ can only be eliminated by Lemma 3.3. However, when $N=4$, the following 11 out of $4!=24$ possible orderings are excluded by Lemmas 3.3 and 3.4:

$$(1,4,2,3),\quad (2,1,4,3),\quad (2,4,1,3),\quad (3,1,2,4),\quad (3,1,4,2),\quad (3,4,1,2),$$
$$(4,1,2,3),\quad (4,1,3,2),\quad (4,2,1,3),\quad (4,2,3,1),\quad (4,3,1,2).$$

The fraction of orderings to be excluded by Lemmas 3.3 and 3.4 becomes significant as N becomes large.

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