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Bi-Sobolev mappings with differential matrices in Orlicz Zygmund classes

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A B S T R A C T

Bi-Sobolev mappings $f:\Omega\subset\mathbb{R}^2 \xrightarrow{\text{onto}} \Omega'\subset\mathbb{R}^2$ have been defined as those homeomorphisms such that f and f^{-1} belong to $W^{1,1}_{\mathrm{loc}}$. We deduce regularity properties of the distortion of f from the regularity of the differential matrix Df^{-1} and conversely.

1. Introduction

The theory of mappings of finite distortion has received considerable attention in recent years thanks to its connection with several topics such as elliptic partial differential equations, differential geometry and calculus of variations. The earliest papers on the argument are [4,16,14,15,18] but we also refer to the monograph [17] for a scrupulous treatment of it.

The present paper deals with some properties of homeomorphisms with finite distortion.

Recall that a homeomorphism $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ has finite outer distortion if its Jacobian J_f is strictly positive on the set where $|Df|\neq 0$ and we define its outer distortion function as

$$K_f(x)=\begin{cases}\frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x)>0\\ 1 & \text{for } J_f(x)=0\end{cases}\tag{1.1}$$

Similarly, we say that a homeomorphism $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ has finite inner distortion if its Jacobian is strictly positive on the set where the norm of the adjugate $\operatorname{adj} Df$ of the differential matrix does not vanish. We define its inner distortion function as

$$K_{I,f}(x)=\begin{cases}\frac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1}} & \text{for } J_f(x)>0\\ 1 & \text{for } J_f(x)=0\end{cases}$$

Obviously these two notions coincide in the planar case to which the paper is devoted.

In particular we shall also concern with Bi-Sobolev mappings, which have been introduced in [12] as homeomorphisms $f:\Omega\to\Omega'$ between open subsets of $\mathbb{R}^n$ such that $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\Omega')$ and $f^{-1}\in W^{1,1}_{\mathrm{loc}}(\Omega',\Omega)$. Actually, our interest in this class of mappings is motivated by the fact that they have finite (outer) distortion for $n=2$ (see [10]).

As in [12], we are interested in the connection between the regularity properties of the distortion functions and those of f and f^{-1}.

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In [12] it has been proved that for $W^{1,p}$ Bi-Sobolev mappings, i.e. homeomorphisms $f : \Omega \to \Omega'$ such that $f \in W^{1,p}_{\mathrm{loc}}(\Omega;\Omega')$ and $f^{-1} \in W^{1,p}_{\mathrm{loc}}(\Omega';\Omega)$, the case $p = n$ is somehow critical for what concerns the regularity of the distortion. In fact, if $p = n$ then the distortion functions of f and of its inverse f^{-1} both belong to L^1. If $p > n$ then they belong to some L^q, $q > 1$. In case $p < n$, the authors exhibit a counterexample showing that no L^1 integrability of the distortion can be expected.

Here, we interpolate between the above mentioned results studying the regularity of the distortion functions when the mappings have the differential in an Orlicz space not too far from the critical Lebesgue space L^2.

More precisely, we will deal with planar Bi-Sobolev mappings whose differential matrices belong to an Orlicz class generated by functions of the type

$$\phi(t) = t^2 \log^\alpha(e+t), \qquad \psi(t) = \frac{t^2}{\log^\alpha(e+t)}$$

for some $\alpha > 0$. In particular, we shall obtain the following

Theorem 1.1. *Let $f : \Omega \to \Omega' = f(\Omega)$ be a $W^{1,2}$ Bi-Sobolev map such that*

$$\left|Df^{-1}\right| \in L^2 \log^\alpha L\left(\Omega'\right)$$

for some $\alpha \geqslant 0$. Then

$$K_f \in L \log^\alpha L(\Omega)$$

and there exists a nonnegative constant c, depending only on α, such that

$$\int_\Omega K_f(x) \log^\alpha\left(e + K_f(x)\right) dx \leqslant c \int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^\alpha\left(1 + \left|Df^{-1}(y)\right|\right) dy + \int_\Omega \left|Df(x)\right|^2 dx \tag{1.2}$$

On the contrary, we will exhibit a counterexample (see Section 5) in which we show that assuming that f is a $W^{1,\psi}$ Bi-Sobolev mapping is not enough to have both the distortions in $L \log^{-\alpha} L$.

Recall that the inverse of a homeomorphism of finite distortion $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\Omega')$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega',\Omega)$ while the inverse of a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\Omega')$ belongs to BV_{loc} only (see [11] and [3]).

Motivated by [3] and [10], we are also interested in the study of the integrability properties of the differential matrix of a homeomorphism assuming suitable regularity on the distortion function of the inverse.

In [10], it has been proved that, in the planar case, the L^1-integrability of the distortion K_f of a homeomorphism of the Sobolev class $W^{1,1}$ is a sufficient condition to the L^2-integrability of the differential matrix of the inverse mapping.

We shall examine the case in which the distortion is assumed to belong to an Orlicz class of functions not too far from L^1. More precisely, we shall obtain results in case the distortion belongs to a class smaller than L^1 and in case it has a degree of summability less than L^1, i.e.

Theorem 1.2. *Let $f : \Omega \to \Omega' = f(\Omega)$ be a homeomorphism in $W^{1,1}$ with finite distortion such that*

$$K_f \in L \log^\alpha L(\Omega) \tag{1.3}$$

for some $\alpha \geqslant 0$. Then

$$\left|Df^{-1}\right| \in L^2 \log^\alpha \log_{\mathrm{loc}} L\left(\Omega'\right) \tag{1.4}$$

Theorem 1.3. *Let $f : \Omega \to \Omega'$ be a $W^{1,1}$ homeomorphism with finite distortion such that $|Df|$ belongs to the space $L \log L(\Omega)$ and that*

$$\frac{K_f}{\log(e + \log(e + |K_f|))} \in L^1(\Omega)$$

Then

$$\left|Df^{-1}\right| \in \frac{L^2}{\log L}\left(\Omega'\right)$$

and hence f^{-1} satisfies condition N.

As in previous papers, our proofs strongly rely on the validity of the area formula for homeomorphisms (see [5]) but here the main difficulty is due the presence of logarithmic factors involved in the summability of our functions.

It is worth pointing out that, assuming the differential matrix in a class of functions smaller than L^2, we have that the distortion of the inverse enjoys a degree of integrability better than L^1 and conversely. On the contrary, when we assume the distortion of the function in a class larger than L^1, we prove that the differential of the inverse is in a space larger than L^2, but the converse is false (see Section 5).

2. Preliminaries

Let Ω and Ω' be bounded domains in $\mathbb{R}^2$. We shall denote by $\mathrm{Hom}(\Omega;\Omega')$ the set of all homeomorphisms $f:\Omega\to\Omega'=f(\Omega)$, by $|Df|$ the operator norm of the differential matrix and by $\operatorname{adj} Df$ the adjugate of Df which is defined by the formula

$$Df\cdot \operatorname{adj} Df=\mathbf{I}\cdot J_f \tag{2.1}$$

where, as usual, $J_f=\det Df$ and $\mathbf{I}$ is the identity matrix.

We will use the well-known area formula for homeomorphisms in $W^{1,1}_{\mathrm{loc}}(\Omega)$, that is

$$\int_B \eta\bigl(f(x)\bigr)\bigl|J_f(x)\bigr|\,dx\leqslant\int_{f(B)}\eta(y)\,dy \tag{2.2}$$

where η is a nonnegative Borel measurable function on $\mathbb{R}^2$ and $B\subset\Omega$ is a Borel set (for more details we refer to [5]). The equality

$$\int_B \eta\bigl(f(x)\bigr)\bigl|J_f(x)\bigr|\,dx=\int_{f(B)}\eta(y)\,dy \tag{2.3}$$

is verified if f is a homeomorphism that satisfies the *Lusin condition N*, i.e. the implication $|E|=0\Longrightarrow|f(E)|=0$ holds for any measurable set $E\subset\Omega$.

Note that the function defined in (1.1) satisfies the so-called distortion inequality

$$\bigl|Df(x)\bigr|^2\leqslant K_f(x)J_f(x)$$

Moreover, by virtue of (2.2), we have that $J_f\in L^1(B)$. Hence, the definition of homeomorphism with finite distortion coincides with the usual one, given for mappings which are not homeomorphisms (see [17]).

In [8] the authors proved that mappings $f\in W^{1,2}(\Omega,\mathbb{R}^2)$ of finite distortion satisfy the Lusin condition N. Here we are interested in mappings of finite distortion whose differential matrices belong to spaces slightly different from L^2. For this reason let us recall the definitions and some basic properties of these spaces.

Let P be an increasing function from $P(0)=0$ to $\lim_{t\to\infty}P(t)=\infty$ and continuously differentiable on $(0,\infty)$. The Orlicz space generated by the function $P(t)$ will be denoted by $L^P(\Omega)$ and it consists of the functions h for which there exists a constant $\lambda=\lambda(h)>0$ such that $\mathcal{P}(\frac{|h|}{\lambda})\in L^1(\Omega)$.

In particular we shall work with the Orlicz–Zygmund spaces $L^s\log^\alpha L$, $1\leqslant s<\infty$, $\alpha\in\mathbb{R}$, which are Orlicz spaces generated by the function $P(t)=t^s\log^\alpha(e+t)$.

For $\alpha>0$, the dual Orlicz space to $L\log^\alpha L(\Omega)$ is the space $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$, generated by the function $Q(t)=\exp(t^{\frac{1}{\alpha}})-1$. The well-known Young's inequality reads as

$$st\leqslant s\log^\alpha(e+s)+\exp\bigl(t^{\frac{1}{\alpha}}\bigr)-1\quad\forall s,t\geqslant 0 \tag{2.4}$$

Observe that if $|g|^\beta\in L^1(\Omega)$ for some $\beta>0$, then

$$\log^\alpha\bigl(e+|g|\bigr)\in\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega) \tag{2.5}$$

for all $\alpha>0$. In fact, we have that

$$\int_\Omega \exp\biggl(\frac{\log^\alpha(e+|g|)}{\lambda}\biggr)^{\frac{1}{\alpha}}dx=\int_\Omega\exp\biggl(\frac{\log(e+|g|)}{\lambda^{\frac{1}{\alpha}}}\biggr)dx=\int_\Omega\bigl(e+|g|\bigr)^{\lambda^{-\frac{1}{\alpha}}}dx$$

Note that the last integral in previous equality is finite for every positive constant λ verifying the inequality $\lambda>\frac{1}{\beta^\alpha}$. Hence, by the definition, the function $\log^\alpha(e+|g|)$ belongs to the space $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$. For more details on Orlicz spaces we refer to [19].

For a measurable function f, we say that $f \in L^{2)}(\Omega)$ if

$$\|f\|_{2)} = \sup_{0<\varepsilon\leqslant 1}\left(\varepsilon\int_\Omega \left|f(x)\right|^{2-\varepsilon}\,dx\right)^{1/(2-\varepsilon)} < \infty$$

Note that $L^{2)}(\Omega)$, endowed with the norm $\|\cdot\|_{2)}$, is a Banach space and that

$$L^{2)}_0(\Omega) = \left\{ f\in L^{2)}(\Omega)\colon\ \lim_{\varepsilon\to 0^+}\varepsilon\int_\Omega \left|f(x)\right|^{2-\varepsilon}\,dx = 0\right\}$$

is the closure of bounded functions in $L^{2)}(\Omega)$, see [2]. Let us recall that these spaces have been introduced by Iwaniec and Sbordone in [13].

In [9], Greco showed the following inclusions

$$\frac{L^2}{\log L}(\Omega)\subset L^{2)}_0(\Omega)\subset L^{2)}(\Omega)\subset \bigcap_{p<2} L^p(\Omega)$$

In [18], the authors proved that $|Df|\in L^{2)}_0$ is a sufficient condition to have that a mapping with finite distortion satisfies the N-property, while N-property may not be verified if $|Df|\in L^{2)}$ only.

We shall use the following result on the higher integrability of the Jacobian of an orientation preserving mapping that is a map such that $J_f\geqslant 0$ a.e. (see for example [21,6]).

Theorem 2.1. *Let $f\in W^{1,1}(\Omega,\mathbb{R}^2)$ be an orientation preserving mapping with finite distortion. Suppose that $|Df|\in \frac{L^2}{\log L}(\Omega)$ and $K_f\in \mathrm{Exp}_\gamma(\Omega)$, for some $\gamma>1$. Then*

$$J_f\in L\log^\alpha L_{\mathrm{loc}}(\Omega)$$

for every $\alpha>0$.

Let us recall that a homeomorphism $f:\Omega\xrightarrow{\text{onto}}\Omega'$ is said to be a Bi-Sobolev map if f belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega;\Omega')$ and its inverse f^{-1} belongs to $W^{1,1}_{\mathrm{loc}}(\Omega';\Omega)$. More specifically, if $f\in W^{1,p}_{\mathrm{loc}}(\Omega;\Omega')$ and $f^{-1}\in W^{1,p}_{\mathrm{loc}}(\Omega';\Omega)$, $1\leqslant p<\infty$, then we say that f is $W^{1,p}$ Bi-Sobolev.

The connection between Bi-Sobolev mappings and mappings with finite distortion is given by the following

Theorem 2.2. *(See [12].) Let $f:A\subset\mathbb{R}^n\to\mathbb{R}^n$ be a Bi-Sobolev map. Suppose that for a measurable set $E\subset A$ we have $J_f=0$ a.e. on E. Then $|\mathrm{adj}\,Df|=0$ a.e. on E. If we moreover assume that $J_f\geqslant 0$ it follows that f has finite inner distortion.*

3. Distortion of $W^{1,2}$ Bi-Sobolev mappings

Let Ω and Ω' be bounded domains in $\mathbb{R}^2$. In Theorem 5 of [12], it has been proved that, given f a $W^{1,2}(\Omega;\Omega')$ Bi-Sobolev mapping, the distortion functions of f and of its inverse f^{-1} both belong to L^1. Moreover in [12], the authors showed that if f is a $W^{1,p}$ Bi-Sobolev mapping for some $p>2$ then K_f and $K_{f^{-1}}$ both belong to a Lebesgue space L^q, some $q>1$.

Here we interpolate between these two results, showing that if the additional regularity of the mapping is due to a logarithmic factor, the improved regularity for the distortion of the inverse mapping is of the same type.

Since every homeomorphism $f\in W^{1,1}(\Omega,\mathbb{R}^2)$ is classically differentiable a.e., it is $J_f\geqslant 0$ or $J_f\leqslant 0$ a.e. in Ω (see Theorem 3.3.4 in [1]). Without loss of generality, from now on we shall assume that $J_f\geqslant 0$ almost everywhere.

Moreover observe that at each point of differentiability of f such that $J_{f^{-1}}(y)>0$ we have

$$Df^{-1}(y) = \left(Df\left(f^{-1}(y)\right)\right)^{-1} \tag{3.1}$$

Now we are ready to embark in the core of the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all we observe that, under our assumption we may apply Theorem 5 in [12] to find that $K_f\in L^1(\Omega)$ and $K_{f^{-1}}\in L^1(\Omega')$. Let us show that the degree of integrability of K_f is actually better than L^1.

In [20] it has been proved that f and f^{-1} verify the Lusin N-property, therefore (2.3) holds and then we easily deduce that f and f^{-1}, belonging to $W^{1,2}$, have positive Jacobians almost everywhere:

$$J_f(x)>0\quad \text{a.e. } x\in\Omega\quad\text{and}\quad J_{f^{-1}}(y)>0\quad\text{a.e. } y\in\Omega' \tag{3.2}$$

Using (3.1), (3.2) and (2.3), we get

$$\int_{\Omega} \frac{|Df(x)|^2}{J_f(x)} \log^{\alpha}\left(e + \frac{|Df(x)|^2}{J_f(x)}\right) dx = \int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^{\alpha}\left(e + \frac{|Df^{-1}(y)|^2}{J_{f^{-1}}(y)}\right) dy$$

It follows that

$$\int_{\Omega} K_f(x) \log^{\alpha}\left(e + K_f(x)\right) dx \leqslant c\left[\int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^{\alpha}\left(e + \frac{1}{J_{f^{-1}}(y)}\right) dy + \int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^{\alpha}\left(1 + \left|Df^{-1}(y)\right|\right) dy\right] \tag{3.3}$$

where we have used the elementary inequality

$$\log^{\alpha}(e + xy) \leqslant c\left(\log^{\alpha}(e + x) + \log^{\alpha}(1 + y)\right) \tag{3.4}$$

Note that the second integral in the right-hand side of (3.3) is finite thanks to the assumption $|Df^{-1}| \in L^2 \log^{\alpha} L(\Omega')$. In order to estimate the first integral in the right-hand side of (3.3), we observe that

$$\int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^{\alpha}\left(e + \frac{1}{J_{f^{-1}}(y)}\right) dy \leqslant c\left[\int_{\Omega'} \frac{|Df^{-1}(y)|^2}{J_{f^{-1}}(y)} dy + \int_{\Omega'} \left|Df^{-1}(y)\right|^2 dy\right] \tag{3.5}$$

where we used that $\log^{\alpha}(e + x) \leqslant c(e + x)$ for $c = c(\alpha)$ and for every $x \in \mathbb{R}^+$.

Since the integral in the right-hand side of (3.5) is finite as $K_{f^{-1}} \in L^1(\Omega')$ and $|Df^{-1}| \in L^2 \log L(\Omega')$, we deduce that $K_f \in L \log^{\alpha} L(\Omega)$ combining (3.3) and (3.5). Moreover we obtain that the estimate (1.2) holds. □

As a consequence of previous theorem we show that the distortion function of a $W^{1,2}$ Bi-Sobolev mapping belongs to a weighted L^1-space. Since the weight function is greater than 1, we will establish that the distortion enjoys a degree of integrability higher than L^1. More precisely, we have the following:

Theorem 3.1. *Let $f : \Omega \subset \mathbb{R}^2 \to \Omega' = f(\Omega) \subset \mathbb{R}^2$ be a $W^{1,2}$ Bi-Sobolev map such that*

$$\left|Df^{-1}\right| \in L^2 \log^{\alpha} L\left(\Omega'\right) \tag{3.6}$$

for some $\alpha > 0$. Then

$$\int_{\Omega} K_f(x) \log^{\alpha}\left(e + \left|Df(x)\right|\right) dx < +\infty \tag{3.7}$$

Proof. Under our assumptions we can apply Theorem 1.1 and obtain that

$$\int_{\Omega} K_f(x) \log^{\alpha}\left(e + K_f(x)\right) dx < +\infty \tag{3.8}$$

Therefore, since by Hadamard's inequality $K_f(x) = \frac{|Df(x)|^2}{J_f} \geqslant 1$ a.e. in Ω, we get

$$0 \leqslant \int_{\Omega} K_f(x) \log^{\alpha} K_f(x)\, dx < +\infty \tag{3.9}$$

On the other hand, by the area formula (2.3) and equality (3.1) we also have that

$$\int_{\Omega} K_f(x) \log^{\alpha}\left(e + \frac{|Df(x)|}{J_f}\right) dx = \int_{\Omega} \frac{|Df(x)|^2}{J_f} \log^{\alpha}\left(e + \frac{|Df(x)|}{J_f}\right) dx = \int_{\Omega'} \left|Df^{-1}(y)\right|^2 \log^{\alpha}\left(e + \left|Df^{-1}(y)\right|\right) dy$$

which is finite since $|Df^{-1}| \in L^2 \log^{\alpha} L(\Omega')$.

It follows that

$$0 \leqslant \int_{E \cap \{|Df(x)| \geqslant J_f\}} K_f(x) \log^{\alpha}\left(\frac{|Df(x)|}{J_f}\right) dx < +\infty \tag{3.10}$$

for every measurable set $E \subset \Omega$.

Moreover, if we observe that the function $|x\log^{\alpha} x| \leqslant \frac{1}{e^{\alpha}}$ for every $x \in (0,1)$ and that $f \in W^{1,1}(\Omega;\Omega')$, we get

$$\int\limits_{E\cap\{|Df(x)|<J_f\}} \left| K_f(x)\log^{\alpha}\left(\frac{|Df(x)|}{J_f}\right)\right| dx = \int\limits_{E\cap\{|Df(x)|<J_f\}} \left|Df(x)\right| \left|\frac{|Df(x)|}{J_f}\log^{\alpha}\left(\frac{|Df(x)|}{J_f}\right)\right| dx \leqslant c\int\limits_{\Omega}\left|Df(x)\right| dx < +\infty \tag{3.11}$$

Combining (3.10) and (3.11) we obtain

$$\int\limits_{E}\left|K_f(x)\log^{\alpha}\left(\frac{|Df(x)|}{J_f}\right)\right| dx < +\infty \tag{3.12}$$

for every measurable $E \subset \Omega$. Now we note that

$$\begin{aligned}\int\limits_{\Omega}\frac{|Df(x)|^2}{J_f}\log^{\alpha}\left(e+\left|Df(x)\right|\right)dx &\leqslant c\int\limits_{\{|Df(x)|\leqslant e\}} K_f(x)\,dx + c\int\limits_{\{|Df(x)|>e\}} K_f(x)\log^{\alpha}\left(\left|Df(x)\right|\right)dx\\ &= c\int\limits_{\{|Df(x)|\leqslant e\}} K_f(x)\,dx + c\int\limits_{\{|Df(x)|>e\}} K_f(x)\log^{\alpha}\left(K_f(x)\frac{J_f}{|Df(x)|}\right)dx\\ &= c\int\limits_{\{|Df(x)|\leqslant e\}} K_f(x)\,dx + c\int\limits_{\{|Df(x)|>e\}} K_f(x)\left(\log K_f(x)-\log\frac{|Df(x)|}{J_f}\right)^{\alpha}dx\end{aligned}$$

and then

$$\begin{aligned}\int\limits_{\Omega} K_f(x)\log^{\alpha}\left(e+\left|Df(x)\right|\right)dx &\leqslant c\int\limits_{\Omega} K_f(x)\,dx + c\int\limits_{\{|Df(x)|>e\}} K_f(x)\log^{\alpha} K_f(x)\,dx\\ &\quad + c\int\limits_{\{|Df(x)|>e\}}\left|K_f(x)\log^{\alpha}\left(\frac{|Df(x)|}{J_f}\right)\right| dx\end{aligned} \tag{3.13}$$

Recall that $K_f \in L^1(\Omega)$, since f is a $W^{1,2}$ Bi-Sobolev mapping (see [12]). The conclusion follows observing that the last two integrals in the right-hand side of (3.13) are finite thanks to the estimates (3.9) and (3.12). □

Now, we deal with the case of Bi-Sobolev mappings with the differential matrix belonging to a class of functions which is strictly larger than L^2.

Let us recall that there is an example in [10] of a homeomorphism that does not belong to $W^{1,2}$, whose inverse is in $W^{1,2}$ but does not have the distortion in L^1. In the next theorem we will show that, under one more condition, i.e. $K_f \in \mathrm{Exp}_{\gamma}(\Omega)$ for some $\gamma > 1$, the integrability of $K_{f^{-1}}$ is guaranteed. In the case $\gamma = 1$, an analogous sharp result has been obtained in [7] where the integrability of $K_{f^{-1}}$ is guaranteed only if the exponential norm of K_f is sufficiently small. Consistently, assuming only that K_f is exponentially integrable is not sufficient to have $K_{f^{-1}} \in L^1_{\mathrm{loc}}$ (see also the example of Section 5). We'd like to point out that next result is in the same spirit of [7], but takes place on a different scale.

Theorem 3.2. *Let $f : \Omega \to \Omega'$ be a Bi-Sobolev map such that $|Df| \in \frac{L^2}{\log L}(\Omega)$. Suppose that the distortion function $K_f(x)$ belongs to the space $\mathrm{Exp}_{\gamma}(\Omega)$ for some $\gamma > 1$. Then*

$$K_{f^{-1}}(y) \in L^1_{\mathrm{loc}}\left(\Omega'\right)$$

Proof. We have already said that each Bi-Sobolev mapping has finite distortion. Let $A \subset \Omega'$ be any compact set such that for every $y \in A$, f^{-1} is differentiable at y with $J_{f^{-1}}(y) > 0$. Using (2.2) and Young's inequality in (2.4), we easily get

$$\int\limits_{A} K_{f^{-1}}(y)\,dy \leqslant \int\limits_{f^{-1}(A)} K_f(x)J_f(x)\,dx \leqslant \int\limits_{f^{-1}(A)} J_f(x)\log^{\frac{1}{\gamma}}\left(e+J_f(x)\right)dx + \int\limits_{f^{-1}(A)} \exp\left(K_f(x)\right)^{\gamma} dx \tag{3.14}$$

Hence, we conclude using Theorem 2.1. □

4. The regularity of the inverse mapping

In previous section we deduced regularity of the distortion under some regularity assumption on the inverse mapping. Here we want to proceed with the proof of Theorem 1.2 which goes in the opposite direction, i.e. we are going to investigate how the regularity of the distortion influences the integrability of the inverse mapping.

Proof of Theorem 1.2. First of all recall that, since $K_f \in L\log^\alpha L(\Omega)$, we may apply Theorem 6.1 in [10] to find that $J_f(x) > 0$ a.e. in Ω and

$$\log\left(e + \frac{1}{J_f}\right) \in L^1_{\mathrm{loc}}(\Omega) \tag{4.1}$$

Let us consider a compact subset E of Ω. Since f is a homeomorphism with finite distortion we know by Theorem 1.2 in [10] that f^{-1} is a homeomorphism of $W^{1,1}$ with finite distortion. Hence, using area formula stated in (2.2), we get

$$\begin{aligned}
&\int_{f(E)} \left|Df^{-1}(y)\right|^2 \log^\alpha\left(e + \log\left(e + \left|Df^{-1}(y)\right|\right)\right) dy \\
&\quad \leqslant \int_E \frac{|Df(x)|^2}{J_f(x)} \log^\alpha\left(e + \log\left(e + \frac{|Df(x)|}{J_f(x)}\right)\right) dx \\
&\quad = \int_{E\cap\{|Df|\geqslant 1\}} K_f(x)\log^\alpha\left(e + \log\left(e + \frac{|Df(x)|}{J_f}\right)\right) dx + \int_{E\cap\{|Df|<1\}} K_f(x)\log^\alpha\left(e + \log\left(e + \frac{|Df(x)|}{J_f}\right)\right) dx \\
&\quad = I + II
\end{aligned} \tag{4.2}$$

Since $|Df(x)| \leqslant |Df(x)|^2$ on the set $E \cap \{|Df| \geqslant 1\}$, we have

$$I \leqslant \int_{E\cap\{|Df|\geqslant 1\}} K_f(x)\log^\alpha\left(e + \log\left(e + \frac{|Df(x)|^2}{J_f}\right)\right) dx \leqslant \int_E K_f \log^\alpha\left(e + \log(e + K_f)\right) dx < +\infty \tag{4.3}$$

thanks to the assumption (1.3). In order to estimate II, we use Young's inequality in (2.4)

$$\begin{aligned}
II &\leqslant \int_{E\cap\{|Df|<1\}} K_f(x)\log^\alpha\left(e + \log\left(e + \frac{1}{J_f}\right)\right) dx \\
&\leqslant \int_\Omega K_f(x)\log^\alpha\left(e + K_f(x)\right) dx + \int_\Omega \exp\left(\log^\alpha\left(e + \log\left(e + \frac{1}{J_f}\right)\right)\right)^{\frac{1}{\alpha}} dx \\
&\leqslant \int_\Omega K_f(x)\log^\alpha\left(e + K_f(x)\right) dx + \int_\Omega \left(e + \log\left(e + \frac{1}{J_f}\right)\right) dx
\end{aligned} \tag{4.4}$$

that is finite thanks to (1.3) and (4.1). This concludes the proof. □

In [22] and [23] there are examples of homeomorphisms belonging to $W^{1,p}(\Omega,\mathbb{R}^n)$ for every $p < n$ that do not satisfy the condition N. Here we shall prove Theorem 1.3 which states that, under suitable conditions on the distortion, a homeomorphism $f \in W^{1,p}(\Omega,\mathbb{R}^2)$, $p < 2$, still satisfies the condition N. Let us note that our further assumption is not on the integrability of the differential matrix but on the distortion function.

Proof of Theorem 1.3. Since $f \in W^{1,1}$ is a homeomorphism with finite distortion, we have that f^{-1} is a homeomorphism with finite distortion belonging to $W^{1,1}$ (see [10]) and therefore we may apply the area formula in (2.2). Moreover, since f^{-1} has finite distortion, we have that $\{y \in \Omega'\colon\ J_{f^{-1}}(y) = 0\} \subset \{y \in \Omega'\colon\ |Df^{-1}(y)| = 0\}$ and hence

$$\int_{\Omega'} \frac{|Df^{-1}(y)|^2}{\log(e + |Df^{-1}(y)|)} dy = \int_{\{J_{f^{-1}}(y)>0\}} \frac{|Df^{-1}(y)|^2}{\log(e + |Df^{-1}(y)|)} dy$$

By (2.2) and (3.1) we get

$$\int_{\Omega'} \frac{|Df^{-1}(y)|^2}{\log(e+|Df^{-1}(y)|)}\,dy \leqslant \int_{\Omega} \frac{|Df(x)|^2}{J_f(x)\log(e+\frac{|Df(x)|}{J_f(x)})}\,dx = \int_{\Omega} \frac{K_f(x)}{\log(e+\frac{K_f(x)}{|Df(x)|})}\,dx$$

$$= \int_{\{K_f \leqslant |Df|\log(e+K_f)\}} \frac{K_f(x)}{\log(e+\frac{K_f(x)}{|Df(x)|})}\,dx + \int_{\{K_f > |Df|\log(e+K_f(x))\}} \frac{K_f(x)}{\log(e+\frac{K_f(x)}{|Df(x)|})}\,dx$$

$$\leqslant \int_{\Omega} |Df(x)| \log\bigl(e+K_f(x)\bigr)\,dx + \int_{\Omega} \frac{K_f(x)}{\log(e+\log(e+K_f(x)))}\,dx \tag{4.5}$$

Note that the second integral in the right-hand side of (4.5) is finite thanks to the assumption. In order to prove that also the first integral is finite we observe that

$$\int_{\Omega} |Df(x)| \log(e+K_f)\,dx = \int_{\Omega} |Df(x)| \log\left(e+\frac{|Df(x)|^2}{J_f}\right)dx$$

$$\leqslant c\int_{\Omega} |Df(x)| \log\bigl(e+|Df(x)|\bigr)\,dx + c\int_{\Omega} |Df(x)| \log\left(e+\frac{1}{J_f(x)}\right)dx$$

$$\leqslant c\int_{\Omega} |Df(x)| \log\bigl(e+|Df(x)|\bigr)\,dx + c\int_{\Omega} \frac{|Df(x)|}{(J_f(x))^{\frac{1}{2}}}\,dx + c\int_{\Omega} |Df(x)|\,dx$$

$$= c\int_{\Omega} |Df(x)| \log\bigl(e+|Df(x)|\bigr)\,dx + \int_{\Omega} K_f(x)^{\frac{1}{2}}\,dx + \int_{\Omega} |Df(x)|\,dx \tag{4.6}$$

where we used the elementary inequalities

$$\log(e+xy) \leqslant \log(e+x) + \log(1+y) \quad \text{and} \quad \log(e+x) \leqslant (e+x)^{\frac{1}{2}} \quad \forall x > 0$$

Note that the first and the third integrals in the right-hand side of (4.6) are finite by assumption, while the second one can be estimated as follows by Young' inequality

$$\int_{\Omega} K_f(x)^{\frac{1}{2}}\,dx \leqslant \int_{\Omega} \frac{K_f(x)}{\log(e+\log(e+K_f(x)))} + c\bigl(|\Omega|\bigr) \tag{4.7}$$

Estimates (4.5)–(4.7) yield $|Df^{-1}| \in \frac{L^2}{\log L}(\Omega')$ and we conclude the proof using Theorem A in [18]. □

Theorem 1.3 also shows that if we assume that K_f is in a class of functions strictly larger than L^1 then the differential matrix of the inverse mapping belongs to a class of functions larger than L^2. In the same spirit we prove the following result:

Theorem 4.1. *Let $f : \Omega \to \Omega'$ be a $W^{1,1}$ homeomorphism with finite distortion such that $|Df|$ belongs to the space $L\log L(\Omega)$ and that*

$$\frac{K_f}{\log(e+\log(e+|Df|))} \in L^1(\Omega)$$

Then

$$\bigl|Df^{-1}\bigr| \in \frac{L^2}{\log L}\bigl(\Omega'\bigr)$$

and hence f^{-1} satisfies condition N.

Proof. Arguing as in previous theorem we have that we may apply the area formula at (2.2) and equality (3.1) to get

$$\int_{\Omega'} \frac{|Df^{-1}|^2}{\log(e+|Df^{-1}(y)|)}\,dy \leqslant \int_{\Omega} \frac{|Df(x)|^2}{J_f(x)\log(e+\frac{|Df(x)|}{J_f(x)})}\,dx = \int_{\Omega} \frac{K_f(x)}{\log(e+\frac{K_f(x)}{|Df(x)|})}\,dx$$

$$= \int\limits_{\{K_f(x)\leqslant|Df|\log(e+|Df|)\}} \frac{K_f(x)}{\log(e+\frac{K_f}{|Df(x)|})}\,dx + \int\limits_{\{K_f>|Df|\log(e+|Df|)\}} \frac{K_f(x)}{\log(e+\frac{K_f(x)}{|Df(x)|})}\,dx$$

$$\leqslant \int\limits_\Omega |Df(x)|\log\bigl(e+|Df(x)|\bigr)\,dx + \int\limits_\Omega \frac{K_f(x)}{\log(e+\log(e+|Df(x)|))}\,dx < +\infty \tag{4.8}$$

i.e. the conclusion. □

Finally, assuming that the distortion function belongs to a slightly larger space than $\frac{L^1}{\log\log L}$, we also have the following

Theorem 4.2. *Let $f:\Omega\to\Omega'$ be a $W^{1,p}$ homeomorphism, $1<p<2$, with finite distortion and assume that*

$$\lim_{\delta\to0^+}\delta\int\limits_\Omega K_f(x)^{1-\delta}\,dx = 0$$

Then f^{-1} satisfies condition N.

Proof. As in Theorem 1.3 we have that $\{y\in\Omega':\ J_{f^{-1}}(y)=0\}\subset\{y\in\Omega':\ |Df^{-1}(y)|=0\}$ and hence applying the area formula in (2.2) and Hölder's inequality, we get

$$\varepsilon\int\limits_{\Omega'}\bigl|Df^{-1}(y)\bigr|^{2-\varepsilon}dy = \varepsilon\int\limits_{\{J_{f^{-1}}>0\}}\frac{|Df^{-1}(y)|^{2-\varepsilon}}{J_{f^{-1}}(y)}J_{f^{-1}}(y)\,dy \leqslant \varepsilon\int\limits_\Omega\frac{|Df(x)|^{2-\varepsilon}}{(J_f(x))^{1-\varepsilon}}\,dx$$

$$= \varepsilon\int\limits_\Omega\bigl|Df(x)\bigr|^\varepsilon K_f^{1-\varepsilon}(x)\,dx \leqslant \|Df\|_p^\varepsilon\left(\varepsilon^{\frac{p}{p-\varepsilon}}\int\limits_\Omega K_f^{\frac{(1-\varepsilon)p}{p-\varepsilon}}(x)\,dx\right)^{\frac{p-\varepsilon}{p}}$$

Observe that for $\delta>0$ such that $\frac{(1-\varepsilon)p}{p-\varepsilon}=1-\delta$ we have that $\varepsilon=\frac{\delta p}{\delta+p-1}\leqslant\delta\frac{p}{p-1}$, hence $\varepsilon^{\frac{p}{p-\varepsilon}}\leqslant\delta(\frac{p}{p-1})^{\frac{p}{p-\varepsilon}}$. As a consequence we have that

$$\varepsilon\int\limits_{\Omega'}\bigl|Df^{-1}(y)\bigr|^{2-\varepsilon}dy \leqslant \|Df\|_{L^p}^\varepsilon\left(\frac{p}{p-1}\right)\left(\delta\int\limits_\Omega K_f^{1-\delta}(x)\,dx\right)^{\frac{1-p}{1-p-\delta}}$$

Letting $\delta\to0^+$ it follows that $\varepsilon\to0^+$ and therefore, thanks to the assumption on the distortion function K_f, we get

$$\lim_{\varepsilon\to0^+}\varepsilon\int\limits_{\Omega'}\bigl|Df^{-1}(y)\bigr|^{2-\varepsilon}dy = 0$$

that is $|Df^{-1}|\in L_0^{2)}$. It follows that f satisfies condition N (see [18]). □

5. Some counterexamples

We consider a radial stretching on the ball $B_{\frac1e}=\{x\in\mathbb{R}^2:\ |x|\leqslant\frac1e\}$

$$f(x)=\rho\bigl(|x|\bigr)\frac{x}{|x|} \tag{5.1}$$

with ρ a strictly increasing function, continuous in $(0,1]$ and locally Lipschitz in $(0,1]$, satisfying $\rho(0^+)=0$ and $\rho(\frac1e)=1$. Notice that f is a homeomorphism of $B_{\frac1e}$ onto B_1.

Setting $r=|x|$, it is well known that

$$Df(x)=\frac{\rho(r)}{r}I+\left[\rho'(r)-\frac{\rho(r)}{r}\right]\frac{x\otimes x}{r^2} \tag{5.2}$$

and

$$J_f(x)=\rho'(r)\left[\frac{\rho(r)}{r}\right] \tag{5.3}$$

Hence, the Jacobian J_f is positive and integrable over $B_{\frac1e}$.

For more details, see for example [17].

Taking in (5.1)

$$\rho(|x|) = \frac{1}{\log^{\frac{1}{2}} \frac{1}{|x|}}$$

one can easily check that

$$|Df(x)| \sim \frac{1}{|x| \log^{\frac{1}{2}} \frac{1}{|x|}}, \qquad J_f(x) \sim \frac{1}{2} \frac{1}{|x| \log^2 \frac{1}{|x|}}$$

and

$$K_f(x) \sim 2 \log \frac{1}{|x|}$$

Hence $f \in W^{1,1}(B_{\frac{1}{e}}; B_1)$ is a homeomorphism with exponentially integrable distortion such that $|Df| \in L^2 \log^{-\alpha} L(B_{\frac{1}{e}})$ for every $\alpha > 0$.

Moreover its inverse $f^{-1}(y)$ is given by

$$f^{-1}(y) = \frac{y}{|y|} \exp\left(-\frac{1}{|y|^2}\right)$$

and thus

$$|Df^{-1}(y)| \sim \frac{c}{|y|^3} \exp\left(-\frac{1}{|y|^2}\right), \qquad J_f^{-1}(y) \sim \frac{c}{|y|^4} \exp\left(-\frac{2}{|y|^2}\right)$$

and

$$K_{f^{-1}}(y) \sim \frac{c}{|y|^2}$$

Note that $|Df^{-1}| \in L \log L(B_1)$.

The following assertions hold true:

- *The converse of Theorem* 1.3 *is false, i.e.*

$$|Df^{-1}| \in L \log L(B_1) \quad \text{and} \quad |Df| \in L^2 \log^{-1} L(B_{\frac{1}{e}})$$
$$\not\Downarrow$$
$$\frac{K_{f^{-1}}}{\log(e + \log(e + K_{f^{-1}}))} \in L^1(B_1)$$

In fact

$$\int_{B_1} \frac{K_{f^{-1}}}{\log(e + \log(e + K_{f^{-1}}))} \, dy = c \int_{B_1} \frac{1}{|y|^2 \log(e + \log(e + \frac{1}{|y|^2}))} \, dy \geqslant c \int_0^{\frac{1}{e}} \frac{1}{\rho \log(\frac{1}{\rho})} \, d\rho \to +\infty \tag{5.4}$$

- *The converse of Theorem* 4.1 *is false, i.e.*

$$|Df^{-1}| \in L \log L(B_1) \quad \text{and} \quad |Df| \in L^2 \log^{-1} L(B_{\frac{1}{e}})$$
$$\not\Downarrow$$
$$\frac{K_{f^{-1}}}{\log(e + \log(e + |Df^{-1}|))} \in L^1(B_1)$$

In fact

$$\int_{B_1} \frac{K_{f^{-1}}}{\log(e + \log(e + |Df^{-1}|))} \, dy = c \int_{B_1} \frac{1}{|y|^2 \log(e + \log(e + \frac{e^{\frac{-1}{|y|^2}}}{|y|^3}))} \, dy \geqslant c \int_0^{\frac{1}{e}} \frac{1}{\rho \log(e + \log(e + \frac{e^{\frac{-1}{\rho^2}}}{\rho^3}))} \, d\rho \tag{5.5}$$

Since $\frac{e^{\frac{-1}{\rho^2}}}{\frac{1}{\rho^2}} \to 0$ as $\rho \to 0^+$, we have that $e^{\frac{-1}{\rho^2}} \leqslant \frac{1}{\rho^2}$ for $\rho < \bar{\rho}$. Setting $\rho_0 = \min\{\bar{\rho}, \frac{1}{e}\}$, we have

$$\int_0^{\frac{1}{e}} \frac{1}{\rho \log(e + \log(e + \frac{e^{\frac{-1}{\rho^2}}}{\rho^3}))}\, d\rho \geqslant c \int_0^{\rho_0} \frac{1}{\rho \log(\frac{1}{\rho})}\, d\rho \to +\infty \tag{5.6}$$

We conclude inserting (5.6) in (5.5).

- *Theorem* 3.2 *is false under the weaker assumption* $K_f \in \mathrm{EXP}(B_{\frac{1}{e}})$, *i.e.*

$$|Df| \in L^2 \log^{-\alpha} L(B_{\frac{1}{e}}) \quad \forall \alpha > 0 \quad \text{and} \quad K_f \in \mathrm{EXP}(B_{\frac{1}{e}})$$
$$\not\Downarrow$$
$$K_{f^{-1}} \in L^1(B_1)$$

In fact

$$\int_{B_1} K_{f^{-1}}\, dy = \int_0^1 \frac{1}{\rho}\, d\rho \to +\infty$$

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