Maximal injective subalgebras of tensor products of free group factors

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Abstract

In this article, we prove the following results. Let $L(F(n_i))$ be the free group factor on $n_i$ generators ($n_i \geq 2$) and $\lambda(g_i)$ be one of standard generators of $L(F(n_i))$ for $1 \leq i \leq N$. Let $A_i$ be the abelian von Neumann subalgebra of $L(F(n_i))$ generated by $\lambda(g_i)$. Then the abelian von Neumann subalgebra $\bigotimes_{i=1}^{N} A_i$ is a maximal injective von Neumann subalgebra of $\bigotimes_{i=1}^{N} L(F(n_i))$. When $N$ is equal to infinity, we obtain strongly stable II$_1$ factors (or called McDuff factors) that contain maximal injective abelian von Neumann subalgebras.

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1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. A von Neumann algebra $\mathcal{R}$ is called “injective” if it is the range of a norm one projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{R}$. Since injective von Neumann algebras form a monotone class, it follows that any injective von Neumann algebra is contained in some maximal injective von Neumann algebra.

In his influential list of problems presented at the conference in Baton Rouge in 1967, R. Kadison asked [4, Problem 7] whether each self-adjoint operator in a II$_1$ factor lies in some hyperfinite subfactor. This problem was answered in the negative by S. Popa in his remarkable paper [11].
More specifically, if $L(F(n))$ is the free group factor on $n$ generators, and $\lambda(g)$ is one of the standard generators of $L(F(n))$, then S. Popa showed that the abelian von Neumann subalgebra generated by $\lambda(g)$ is a maximal injective von Neumann subalgebra of $L(F(n))$. It follows that $\lambda(g)$ is not contained in any hyperfinite subfactor of $L(F(n))$, which solves Kadison’s problem as mentioned above. In [3], L. Ge provided more examples of maximal injective von Neumann subalgebras in type $II_1$ factors. Actually he showed that each non-atomic injective finite von Neumann algebra with a separable predual is maximal injective in its free product with any von Neumann algebra associated with a countable discrete group. Note that the type $II_1$ factors listed in [3,11] that contain maximal injective abelian von Neumann subalgebras are all non-$\Gamma$ factors.

In this paper, we provide examples of maximal injective von Neumann subalgebras in strongly stable $II_1$ factors (or McDuff factors). In fact, we consider maximal injective von Neumann subalgebras in tensor products of free group factors. By developing the techniques from [3,11], we are able to prove the following result.

**Theorem.** Suppose $\{n_i\}_{i=1}^N$ is a sequence of integers where $n_i \geq 2$ for each $1 \leq i \leq N$ and $N$ is a finite integer or infinite. Let $F(n_i)$ be the free group with the standard generators $\{g_{i,j}\}_{j=1}^{n_i}$ for $1 \leq i \leq N$. Let the group $G$ be $\bigotimes_{i=1}^N F(n_i)$, the direct sum of all $F(n_i)$’s. And $F(n_i)$ is identified with its canonical image in $G$. Let $\lambda$ be the left regular representation of $G$ and $M = \bigotimes_{i=1}^N L(F(n_i))$ be the group von Neumann algebra associated with $G$. Let $A$ be the abelian von Neumann subalgebra of $M$ generated by the unitary elements $\{\lambda(g_{i,1})\mid 1 \leq i \leq N\}$. Then $A$ is a maximal injective subalgebra of $M$, thus not contained in any hyperfinite subfactor of $M$.

When $N$ is equal to infinity, we obtain examples of strongly stable $II_1$ factors (for example, $\bigotimes_{i=1}^\infty L(F(2))$) that contain maximal injective abelian von Neumann subalgebras.

The organization of the paper is as follows. We introduce some basic knowledge in Section 2. One useful lemma by R. Kadison is quoted. In Section 3, some technical lemmas needed in later section are proved. In Section 4, we prove our main theorem, Theorem 4.1, of the paper.

It was expected by S. Popa that every non-atomic finite injective von Neumann algebra is $*$-isomorphic to a maximal injective subalgebra of each nonhyperfinite type $II_1$ factor. We hope that our work will provide some new insights into S. Popa’s question.

### 2. Preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space, $B(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. (For the general theory of operator algebras, we refer to [6,12,14].) A von Neumann algebra $\mathcal{R}$ is called “injective” if it is the range of a norm one projection from $B(\mathcal{H})$ onto $\mathcal{R}$. Since injective von Neumann subalgebras of a von Neumann algebra $\mathcal{M}$ form a monotone class, it follows that any injective von Neumann subalgebra of $\mathcal{M}$ is contained in some maximal injective von Neumann subalgebra $\mathcal{R}_1$ of $\mathcal{M}$ (see [11]).

Let $\mathcal{M}$ be a finite von Neumann algebra with a tracial state $\tau$. If $\omega$ is a free filter on $\mathbb{N}$ then denote by $\mathcal{M}^\omega$ the quotient of the von Neumann algebra $l^\infty(\mathbb{N},\mathcal{M})$ by the $0$-ideal of the trace $\tau_\omega$, where $\tau_\omega((x_n)_n) = \lim_{n\to \omega} \tau(x_n)$. Then $\mathcal{M}^\omega$ is a finite von Neumann algebra, $\tau_\omega$ is a trace on $\mathcal{M}^\omega$. $\mathcal{M}$ is naturally embedded in $\mathcal{M}^\omega$ as the algebra of constant sequence (see [12]).

A separable type $II_1$ factor $\mathcal{M}$ has the property $\Gamma$ of Murray and von Neumann (see [8]) if for any $x_1, \ldots, x_n \in \mathcal{M}$, $\epsilon > 0$, there exists a unitary element $u \in \mathcal{M}$ such that $\tau(u) = 0$ and $\|ux_i - x_iu\|_2 \leq \epsilon$, $1 \leq i \leq n$. 

It is known in [1] that a separable type II\(_1\) factor \(\mathcal{M}\) has the property \(\Gamma\) of Murray and von Neumann if and only if \(\mathcal{M} \cap \mathcal{M}^\omega \cong I\) and in this case \(\mathcal{M} \cap \mathcal{M}^\omega\) is non-atomic. If \(\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}_0\), then \(\mathcal{M}\) is called a strongly stable II\(_1\) factor (or called a McDuff factor), where \(\mathcal{R}_0\) is the unique hyperfinite type II\(_1\) factor. It is known in [7] that \(\mathcal{M}\) is a strongly stable II\(_1\) factor if and only if \(\mathcal{M} \cap \mathcal{M}^\omega\) is non-commutative. Since \(\mathcal{R}_0 \otimes \mathcal{R}_0 \cong \mathcal{R}_0\), \(\mathcal{R}_0 \cap \mathcal{R}_0^\omega\) is non-commutative.

Let \(\mathcal{R}\) be an injective von Neumann subalgebra of \(\mathcal{M}\). Then \(\mathcal{R}\) can be decomposed as \(\mathcal{R}_1 \oplus \mathcal{R}_2\), where \(\mathcal{R}_1\) is a type I von Neumann subalgebra of \(\mathcal{M}\) and \(\mathcal{R}_2\) is a type II\(_1\) von Neumann subalgebra of \(\mathcal{M}\). From Connes’s celebrated result (see [2]), both of \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are injective. Then we have

**Lemma 2.1.** If \(\mathcal{R}_2 \neq 0\), then \(\mathcal{R}' \cap \mathcal{R}_2^\omega\) (\(\subset \mathcal{R}' \cap \mathcal{M}^\omega\)) is non-commutative.

**Proof.** From [14, Lemma XVI.1.5], it follows that \(\mathcal{R}_2 \cong \mathcal{Z} \otimes \mathcal{R}_0\), where \(\mathcal{Z}\) is the center of \(\mathcal{R}_2\) and \(\mathcal{R}_0\) is the hyperfinite factor of type II\(_1\). From the fact that \(\mathcal{R}_0 \cap \mathcal{R}_0^\omega\) is a non-commutative von Neumann algebra, we obtain that \(\mathcal{R}' \cap \mathcal{R}_2^\omega\) (\(\subset \mathcal{R}' \cap \mathcal{M}^\omega\)) is also non-commutative.

As a corollary, we have

**Corollary 2.1.** Let \(\mathcal{M}\) be a finite von Neumann algebra with a tracial state \(\tau\). Let \(\mathcal{R}\) is an injective von Neumann subalgebra of \(\mathcal{M}\). Let \(\mathcal{A}\) be an abelian von Neumann subalgebra of \(\mathcal{R}\). Suppose \(\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2\), where \(\mathcal{R}_1\) is a type I von Neumann subalgebra of \(\mathcal{M}\) and \(\mathcal{R}_2\) is a type II\(_1\) von Neumann subalgebra of \(\mathcal{M}\). If \(\mathcal{R}_2 \neq 0\), then there exists an element \(x \in \mathcal{R}' \cap \mathcal{R}_2^\omega\) not contained in \(\mathcal{A}^\omega\).

Let \(\mathcal{M}\) be a finite von Neumann algebra with a tracial state \(\tau\). If \(\mathcal{N} \subset \mathcal{M}\) is a von Neumann subalgebra then \(E_{\mathcal{N}}\) denotes the normal \(\tau\)-preserving conditional expectation of \(\mathcal{M}\) onto \(\mathcal{N}\). Two von Neumann subalgebras \(\mathcal{N}_1, \mathcal{N}_2\) are mutually orthogonal if \(\tau(x_1 x_2) = \tau(x_1)\tau(x_2)\) for all \(x_1 \in \mathcal{N}_1, x_2 \in \mathcal{N}_2\). An element \(w\) in \(\mathcal{M}\) is called to be orthogonal to the von Neumann subalgebra \(\mathcal{N}\) of \(\mathcal{M}\) if \(\tau(wx) = 0\) for all \(x \in \mathcal{N}\) (this implies \(\tau(w) = 0\)).

Let \(\mathcal{M}\) be a finite von Neumann algebra with a tracial state \(\tau\). Let \(\mathcal{A}\) be a von Neumann subalgebra of \(\mathcal{M}\). Then the normalizer, \(\mathcal{N}(\mathcal{A})\), is defined as the set consisting of all unitary elements \(u\) in \(\mathcal{M}\) such that \(uAu^* = A\). If \(\mathcal{M} = \{\mathcal{N}(\mathcal{A})\}''\), then \(\mathcal{A}\) is called to be regular in \(\mathcal{M}\). If \(\mathcal{A} = \{\mathcal{N}(\mathcal{A})\}'\), then \(\mathcal{A}\) is called to be singular in \(\mathcal{M}\).

Here, we quote a useful lemma from [5].

**Lemma 2.2.** [5] If \(\mathcal{N}_0\) is a countably decomposable von Neumann algebra, \(\mathcal{N}\) is the von Neumann algebra of \(n \times n\) matrices with entries in \(\mathcal{N}_0\), and \(\mathcal{S}\) is an abelian self-adjoint subset of \(\mathcal{N}\), then there is a unitary element (matrix) \(u\) in \(\mathcal{N}\) such that \(uau^{-1}\) has all its non-zero entries on the diagonal for each \(a\) in \(\mathcal{S}\).

The following lemma tells us that a maximal abelian von Neumann subalgebra \(\mathcal{A}\) in a finite type I von Neumann algebra \(\mathcal{M}\) has to be regular in \(\mathcal{M}\).

**Lemma 2.3.** Suppose \(\mathcal{M}\) is a finite type I von Neumann algebra and \(\mathcal{A}\) is a maximal abelian von Neumann subalgebra of \(\mathcal{M}\). Then \(\mathcal{A}\) is regular in \(\mathcal{M}\), i.e. \(\mathcal{M}\) is generated by the normalizer of \(\mathcal{A}\).
Lemma 2.4. We have the following statements.

(1) Suppose $\mathcal{R}_2$ is a type II$_1$ injective von Neumann algebra with a tracial state $\tau$. Suppose $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{R}_2$. Then there exists a unitary element $w$ in $\mathcal{R}_2$ such that $w$ is orthogonal to $\mathcal{A}$ in $\mathcal{R}_2$ with respect to $\tau$, i.e. $\tau(wx) = 0$ for all $x \in \mathcal{A}$.

(2) Let $\mathcal{R}$ be $\mathcal{R}_1 \oplus \mathcal{R}_2$ with a tracial state $\tau$, where $\mathcal{R}_1$ is a type I von Neumann algebra and $\mathcal{R}_2$ is a non-zero type II$_1$ injective von Neumann algebra. Let $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{R}$. Let $P_1$, $P_2$ be the central supports of $\mathcal{R}_1$, $\mathcal{R}_2$, respectively. Then there exists a unitary $w$ in $\mathcal{R}_2$, a partial isometry in $\mathcal{R}$ with $ww^* = w^*w = P_2$, such that $w$ is orthogonal to $\mathcal{A}$ in $\mathcal{R}$ with respect to $\tau$.

Proof. We need only to show (1), since (2) follows from (1) directly. Note that $\mathcal{R}_2$ can be decomposed as $\mathcal{Z} \otimes \mathcal{R}_0$ where $\mathcal{Z}$ is the center of $\mathcal{R}_2$ and $\mathcal{R}_0$ is the unique hyperfinite II$_1$ factor. It is easy to see that $\mathcal{R}_0 \cong \mathcal{R}_0 \otimes M_2(\mathbb{C})$. Thus $\mathcal{R}_2$ can also be viewed as $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$. By Lemma 2.2, there exists a unitary element $u$ in $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$ such that $uau^*$ has all its non-zero entries on the diagonal for each $a$ in $\mathcal{A}$. Suppose $\{e_{ij}\}_{1 \leq i, j \leq 2}$ is the canonical system of matrix units of $M_2(\mathbb{C})$ in $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$. Let $v = I_{\mathcal{Z} \otimes \mathcal{R}_0} \otimes e_{12} + I_{\mathcal{Z} \otimes \mathcal{R}_0} \otimes e_{21}$ and $w = u^*vu$. By direct computation, we get that $w$ is a unitary element in $\mathcal{R}_2$ and orthogonal to $\mathcal{A}$ in $\mathcal{R}_2$ with respect to $\tau$.  

3. Some technical lemmas

Let $\{n_i\}_{i=1}^N$ be a sequence of integers where each $n_i \geq 2$ and $N$ is a finite integer or $\infty$. Let $F(n_i)$ be the free group with the standard generators $\{g_{i,j}\}_{1 \leq j \leq n_i}$. Let

$$G = \bigoplus_{i=1}^N F(n_i),$$

the direct sum of groups $F(n_1), \ldots, F(n_N)$,

$$H_i = \text{subgroup of } F(n_i) \text{ generated by } g_{i,1}, \quad \text{for } 1 \leq i \leq N,$$

$$H = H_1 \times H_2 \times \cdots \times H_N,$$

$$G_i = \left( \bigoplus_{k=1}^{i-1} F(n_k) \right) \times H_i \times \left( \bigoplus_{k=i+1}^N F(n_k) \right), \quad \text{for } 1 \leq i \leq N.$$

Here, we identify $F(n_i)$ with its canonical image in $G$. 

Proof. Decompose $\mathcal{M}$ as $\bigoplus_i \mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$, where $\mathcal{Z}_i$ is abelian von Neumann subalgebra of $\mathcal{M}$. It is sufficient to show the following statement: suppose $\mathcal{A}_i$ is a maximal abelian von Neumann subalgebra in $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$, then $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$ is generated by the normalizer of $\mathcal{A}_i$. Since $\mathcal{A}_i$ is an abelian von Neumann subalgebra, $\mathcal{A}_i$ is generated by a self-adjoint element $x$ in $\mathcal{A}_i$. By Lemma 2.2 we know that there exists a unitary element $u$ in $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$ such that $uxu^*$ is a diagonal matrix, i.e. $uxu^* = \text{diag}(x_1, x_2, \ldots, x_n)$ where $x_j$ is in $\mathcal{Z}_i$. Since $uA_iu^*$ is also a maximal abelian von Neumann subalgebra in $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$ and generated by $uxu^* = \text{diag}(x_1, x_2, \ldots, x_n)$, we easily have that $uA_iu^*$ is generated by the normalizer of $uA_iu^*$; consequently by the normalizer of $\mathcal{A}_i$.  

Lemma 2.4. It is sufficient to show the following statement: suppose $\mathcal{A}_i$ is a maximal abelian von Neumann subalgebra in $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$, then $\mathcal{Z}_i \otimes M_{n_i}(\mathbb{C})$ is generated by the normalizer of $\mathcal{A}_i$.  

Proof. We need only to show (1), since (2) follows from (1) directly. Note that $\mathcal{R}_2$ can be decomposed as $\mathcal{Z} \otimes \mathcal{R}_0$ where $\mathcal{Z}$ is the center of $\mathcal{R}_2$ and $\mathcal{R}_0$ is the unique hyperfinite II$_1$ factor. It is easy to see that $\mathcal{R}_0 \cong \mathcal{R}_0 \otimes M_2(\mathbb{C})$. Thus $\mathcal{R}_2$ can also be viewed as $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$. By Lemma 2.2, there exists a unitary element $u$ in $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$ such that $uau^*$ has all its non-zero entries on the diagonal for each $a$ in $\mathcal{A}$. Suppose $\{e_{ij}\}_{1 \leq i, j \leq 2}$ is the canonical system of matrix units of $M_2(\mathbb{C})$ in $(\mathcal{Z} \otimes \mathcal{R}_0) \otimes M_2(\mathbb{C})$. Let $v = I_{\mathcal{Z} \otimes \mathcal{R}_0} \otimes e_{12} + I_{\mathcal{Z} \otimes \mathcal{R}_0} \otimes e_{21}$ and $w = u^*vu$. By direct computation, we get that $w$ is a unitary element in $\mathcal{R}_2$ and orthogonal to $\mathcal{A}$ in $\mathcal{R}_2$ with respect to $\tau$.  

3. Some technical lemmas

Let $\{n_i\}_{i=1}^N$ be a sequence of integers where each $n_i \geq 2$ and $N$ is a finite integer or $\infty$. Let $F(n_i)$ be the free group with the standard generators $\{g_{i,j}\}_{1 \leq j \leq n_i}$. Let

$$G = \bigoplus_{i=1}^N F(n_i),$$

the direct sum of groups $F(n_1), \ldots, F(n_N)$,

$$H_i = \text{subgroup of } F(n_i) \text{ generated by } g_{i,1}, \quad \text{for } 1 \leq i \leq N,$$

$$H = H_1 \times H_2 \times \cdots \times H_N,$$

$$G_i = \left( \bigoplus_{k=1}^{i-1} F(n_k) \right) \times H_i \times \left( \bigoplus_{k=i+1}^N F(n_k) \right), \quad \text{for } 1 \leq i \leq N.$$
Let $\mathcal{M}_0$ be a finite von Neumann algebra with a tracial state $\tau_0$. Let $G$ acts on $\mathcal{M}_0$ by $\tau_0$-preserving automorphisms. Denote by $\mathcal{M} = \mathcal{M}_0 \rtimes G$ the corresponding crossed product von Neumann algebra. $\mathcal{M}_0$ is identified with its canonical image in $\mathcal{M}$ and denote by $\lambda(g)$, $g \in G$, the unitary elements in $\mathcal{M}$ canonically implementing the action of $G$ on $\mathcal{M}_0$, and by $\tau$ the tracial state on $\mathcal{M}$ that extends $\tau_0$ of $\mathcal{M}_0$.

Note every element $x$ in $L^2(\mathcal{M}, \tau)$ can be uniquely decomposed as $x = \sum_{g \in G} a_g \lambda(g)$, with $a_g \in \mathcal{M}_0$. The set $\{g \in G \mid a_g \neq 0\}$ is called the support of $x$.

If $x = \sum_{g \in F} a_g \lambda(g)$ is in $L^2(\mathcal{M}, \tau)$ and $F \subseteq G$ is a nonempty subset of $G$, we denote by $x_F$ the element $\sum_{g \in F} a_g \lambda(g) \in L^2(\mathcal{M}, \tau)$ and $\|x\|_F = \|x_F\|_2$. For any subsets $F$ and $\tilde{F}$ of $G$, let $F^{-1}$ denote the set $\{g^{-1} \mid g \in F\}$ and $F \tilde{F}$ denote the set $\{gh \mid g \in F, h \in \tilde{F}\}$.

For the subgroup $P \subseteq G$, let $L(P)$ denote the von Neumann subalgebra of $\mathcal{M}$ generated by $\lambda(g)$ with $g \in P$ and $\mathcal{M}_P$ denote the von Neumann subalgebra of $\mathcal{M}$ generated by $\mathcal{M}_0$ and $L(P)$.

The following lemma is essentially from [11, Lemma 2.1].

**Lemma 3.1.** Let $\omega$ be a free ultrafilter on $\mathbb{N}$, and $H$ be a subgroup of $G$. Suppose $x = (x_n)$ is an element in $\mathcal{M}^\omega$ and $y$ is an element in $\mathcal{M}$. Suppose, for every $\epsilon > 0$, there are element $z$ in $\mathcal{M}$ and subsets $S_0$, $S_1$, $S$ of $G$, depending on $\epsilon$, satisfying

(i) $S = G \setminus (S_0 \cup H)$;

(ii) $\|y - zS_1\|_2 \leq \epsilon$, and $\|zS_1\|_2 \leq \|y\|_2$;

(iii) there exists a positive integer $K$ such that $\|(x_n)\|_{S_0} \leq \epsilon$, $\forall n \geq K$;

(iv) $(S_0) \cap (S_1) = \emptyset$; $(S_0) \cap (S_1) = \emptyset$; $(H \cap S_0) \cap (S_1) = \emptyset$;

(v) $(H \cap S_0) \cap (S_1) = \emptyset$; $(S_0) \cap (S_1) = \emptyset$.

Then,

$$\|yx - xy\|_2 \geq \|y(x - E_{\mathcal{M}_H^\omega}(x))\|_2^2 + \|x - E_{\mathcal{M}_H^\omega}(x)\|_2^2.$$ 

**Proof.** Note that the support of $zS_1$, $(x_n)_H$ or $(x_n)_S$ is in $S_1$, $H$ and $S$, respectively. By (iv) and (v), it is easy to check that $zS_1[(x_n)_S]$, $[(x_n)_S]zS_1$ and $zS_1[(x_n)_H] - [(x_n)_H]zS_1$ are mutually orthogonal vectors in $L^2(\mathcal{M}, \tau)$.

Thus, if $\mathcal{H}_\omega$ denotes the ultraproduct Hilbert space obtained as the quotient of $\{(\xi_n)_n \subseteq L^2(\mathcal{M}, \tau) \mid \sup \|\xi_n\|_2 < \infty\}$ by the subspace $\{(\eta_n)_n \subseteq L^2(\mathcal{M}, \tau) \mid \lim_{n \to \omega} \|\xi_n\|_2 = 0\}$, endowed with the norm $\|(\xi_n)_n\|_2 = \lim_{n \to \omega} \|\xi_n\|_2$, then $x' = (zS_1[(x_n)_S]_n)$, $x'' = ([x_n)_S]zS_1)_n$, $x''' = (zS_1[(x_n)_H] - [(x_n)_H]zS_1)_n$ are mutually orthogonal elements in $\mathcal{H}_\omega$. Moreover, $L^2(\mathcal{M}^\omega, \tau)$ is naturally embedded in $\mathcal{H}_\omega$. Note that $E_{\mathcal{M}_H^\omega}(x_n) = (x_n)_H$ for $n \geq 1$. By (i)–(iii) we have:

$$\|y(x - E_{\mathcal{M}_H^\omega}(x)) - x''\|_2 \leq \sup_{n \geq K} \|y(x_n - E_{\mathcal{M}_H^\omega}(x_n)) - zS_1[(x_n)_S]_n\|_2$$

$$\leq \sup_{n \geq K} \|(y - zS_1)(x_n - E_{\mathcal{M}_H^\omega}(x_n))\|_2 + \sup_{n \geq K} \|zS_1[(x_n)_H] - [(x_n)_H]zS_1\|_2$$

$$\leq \sup_{n \geq K} \|(y - zS_1)(x_n - E_{\mathcal{M}_H^\omega}(x_n))\|_2 + \|zS_1\| \left(\sup_{n \geq K} \|(x_n)_H - (x_n)_S\|_2\right)$$
\[ \leq \sup_{n \geq K} \| (y - zS_1)(x_n - E_{M_H^\omega}(x_n)) \|_2 + \| y \| \left( \sup_{n \geq K} \| (x_n)S_0 \|_2 \right) \]
\[ \leq \epsilon \sup(\| x_n \| + \| y \|) , \]
\[ \| (x - E_{M_H^\omega}(x))y - x'' \|_2 \leq \sup_{n \geq K} \| (x_n - E_{M_H^\omega}(x))(y - zS_1) \|_2 \]
\[ \leq \sup_{n \geq K} \| (x_n - E_{M_H^\omega}(x))(y - zS_1) \|_2 + \left( \sup_{n \geq K} \| (x_n)S_0 \|_2 \right) \| y \| \]
\[ \leq \epsilon \sup(\| x_n \| + \| y \|) , \]
\[ \| yE_{M_H^\omega}(x) - E_{M_H^\omega}(x)y - x'' \|_2 \leq \sup_{n \geq K} \| (y - zS_1)E_{M_H^\omega}(x_n) \|_2 + \sup_{n \geq K} \| E_{M_H^\omega}(x_n)(y - zS_1) \|_2 \]
\[ \leq 2\epsilon \sup x_n . \]

This shows that the vectors \( y(x - E_{M_H^\omega}(x)), (x - E_{M_H^\omega}(x))y, yE_{M_H^\omega}(x) - E_{M_H^\omega}(x)y \) can be approximated arbitrarily well in \( H_\omega \) by some mutually orthogonal vectors and hence they are mutually orthogonal in \( L^2(M_\omega, \tau) \). Since their sum is equal to \( yx - xy \) we get

\[ \| yx - xy \|_2^2 = \| y(x - E_{M_H^\omega}(x)) \|_2^2 + \| (x - E_{M_H^\omega}(x))y \|_2^2 + \| yE_{M_H^\omega}(x) - E_{M_H^\omega}(x)y \|_2^2 \]
\[ \geq \| y(x - E_{M_H^\omega}(x)) \|_2^2 + \| (x - E_{M_H^\omega}(x))y \|_2^2 . \]  \[ \square \]

Recall that, for the subgroup \( P \subset G \), let \( L(P) \) denote the von Neumann subalgebra of \( M \) generated by \( \lambda(g) \) with \( g \in P \), and \( M_P \) denote the von Neumann subalgebra of \( M \) generated by \( M_0 \) and \( L(P) \). Following the preceding notations, we let

\[ A = M_H, \quad A_i = M_{H_i}, \quad N_i = M_{G_i}, \quad \text{for } 1 \leq i \leq N. \]

We have the following lemma, which is the extension of [11, Lemma 2.1].

**Lemma 3.2.** Suppose that \( N \) is finite. Let \( \omega \) be a free ultrafilter on \( \mathbb{N} \). Suppose \( x \) is an element in \( M_\omega \) \( (= (M_0 \rtimes G)_\omega) \) that commutes \( A \) and

\[ E_{N_1}(x) = \cdots = E_{N_N}(x) = E_A(x). \]
Then for any \( y \in M \) with \( E_{N_1}(y) = \cdots = E_{N_N}(y) = E_A(y) = 0 \), we have

\[ \|yx - xy\|_2^2 \geq \| y(x - E_A(x)) \|_2^2 + \| (x - E_A(x))y \|_2^2 . \]

**Proof.** We are going to use Popa’s trick [11] to prove the lemma. Let \( (x_n)_n \) be a sequence of elements in \( M \) representing \( x \in M_\omega \). We might assume that

\[ E_{N_1}(x_n) = \cdots = E_{N_N}(x_n) = E_A(x_n), \quad \forall n \in \mathbb{N} . \]
Let $F = \text{span}\{a_g \lambda(g) \mid a_g \in M_0, \ g \in G\}$, a weakly dense $*$-subalgebra in $M$. For any $y \in M$ with $E_{N_1}(y) = \cdots = E_{N_N}(y) = E_A(y) = 0$, and any $\epsilon > 0$, by Kaplansky density theorem there exists $z \in F$ such that

$$\|y - z\|_2 < \epsilon, \quad \|z\| \leq \|y\|, \quad E_{N_1}(z) = \cdots = E_{N_N}(z) = E_A(z) = 0.$$ 

Let $S_1$ be the support of $z$. Since $E_{N_i}(z) = \cdots = E_{N_N}(z) = E_A(z) = 0$, we have that $S_1 \cap (\bigcup_i G_i \cup H) = \emptyset$ and $z_{S_1} = z$.

Note that every element (or word) $w$ in $G = F(n_1) \times \cdots \times F(n_N)$ can be uniquely written as

$$w = (g_{1,1})^{m_1} \cdots (g_{N,1})^{m_N} w_1 \cdots w_N (g_{1,1})^{p_1} \cdots (g_{N,1})^{p_N},$$

where $(g_{i,1})^{m_i} w_i (g_{i,1})^{p_i}$ is a reduced word in $F(n_i)$ for $1 \leq i \leq N$.

Note $S_1$ is a finite subset of $G$. Let $N_0 - 1$ be the maximal length of the words $g$ in $S_1$. For every $i$, let

$$S_i^0 = \{ g = (g_{1,1})^{m_1} \cdots (g_{N,1})^{m_N} w_1 \cdots w_N (g_{1,1})^{p_1} \cdots (g_{N,1})^{p_N} \in G \mid$$

$$w_i \text{ starts with a non-zero power of } g_{i,j} \text{ for some } j \geq 2; \text{ and } 0 \leq |m_i| \leq 2N_0 - 1\},$$

$$S_0 = \left( \bigcup_i S_i^0 \cup \left( \bigcup_i S_i^0 \right)^{-1} \cup \left( \bigcup_i G_i \right) \right) \setminus H,$$

$$S = G \setminus (S_0 \cup H).$$

**Claim.** For such $x, y, z, H, S_0, S_1, S$, we have:

(i) $S = G \setminus (S_0 \cup H)$;
(ii) $\|z_{S_1}\| \leq \|y\|, \quad \|y - z_{S_1}\|_2 \leq \epsilon$;
(iii) there exists some positive integer $K$ such that $\|(x_n)\|_{S_0} \leq \epsilon, \forall n \geq K$;
(iv) $(SS_1) \cap (S_1 S) = \emptyset, (S_1 H) \cap (S_1 S) = \emptyset, (H S_1) \cap (S_1 S) = \emptyset;
(v) (H S_1) \cap (S S_1) = \emptyset, (S_1 H) \cap (S S_1) = \emptyset.$

**Proof of the Claim.** (i) and (ii) follow directly from the choices of $x, y, z, H, S_0, S_1, S$.

(iii) We want to show that $\|(x_n)_{S_0}\|_2$ is small for $n$ large. Note that $(x_n)_{G_i} = E_{N_i}(x_n) = E_A(x_n) = (x_n)_H$, for $1 \leq i \leq N$. It follows that

$$\|(x_n)_{S_0}\|_2 \leq \sum_i \|(x_n)_{S_i^0}\|_2 + \sum_i \|(x_n)_{(S_i^0)^{-1}}\|_2.$$ 

It will be sufficient to control the norms in the right-hand side. Let $N_1$ be an integer multiple of $4N_0$ such that $N_1 \geq 32N_0N^3 \|x\|^2 \epsilon^{-2}$. By hypothesis, there exists $K = K(\epsilon, N_1)$ such that if $n \geq K$, then

$$\|\lambda(g_{i,1})^{k_i} x_n \lambda(g_{i,1})^{-k_i} - x_n\|_2 < (2N)^{-2} \epsilon$$

for all $1 \leq i \leq N, |k_1|, \ldots, |k_N| \leq N_1$. So if $1 \leq i \leq N, 0 < 4N_0 |k_i| < N_1$ and $n \geq K$, then we have
Using the parallelogram identity in the Hilbert space $L^2(\mathcal{M}, \tau)$, we get the inequalities
\[
\| (x_n)_{S_i} \|_2^2 = \| \lambda(g_{i,1}) S_{g_{i,1}}^0 \lambda(g_{i,1})^{-4N_0k_i} \|_2^2 \\
\leq 2 \| \lambda(g_{i,1}) S_{g_{i,1}}^0 \lambda(g_{i,1})^{-4N_0k_i} - (x_n)_{g_{i,1}} S_{g_{i,1}}^0 \lambda(g_{i,1})^{-4N_0k_i} \|_2^2 \\
+ 2 \| (x_n)_{g_{i,1}} S_{g_{i,1}}^0 \lambda(g_{i,1})^{-4N_0k_i} \|_2^2 \\
\leq (2N)^{-3} \epsilon^2 + 2 \| (x_n)_{g_{i,1}} S_{g_{i,1}}^0 \lambda(g_{i,1})^{-4N_0k_i} \|_2^2.
\]

Now we use the fact that
\[
\{ g_{i,1}^{4N_0k_i} S_{g_{i,1}}^0, g_{i,1}^{-4N_0k_i} \}_{k_i \in \mathbb{Z}}
\]
are disjoint subsets of $G$, so that summing up the above inequalities for all $k_i, 0 < 4N_0|k_i| \leq N_1$, we have
\[
\left( \frac{N_1}{2N_0} \right) \| (x_n)_{S_i} \|_2^2 < \left( \frac{N_1}{2N_0} \right) (2N)^{-3} \epsilon^2 + 2 \| x_n \|_2^2
\]
so that
\[
\| (x_n)_{S_i} \|_2^2 < (2N)^{-3} \epsilon^2 + 2 \| x_n \|_2^2 \left( \frac{N_1}{2N_0} \right)^{-1} \leq (2N)^{-2} \epsilon^2.
\]

Similarly, we get $\| (x_n)_{S_i}^{-1} \|_2 < (2N)^{-1} \epsilon$ and thus $\| (x_n)_{S_i} \|_2 < \epsilon$ for all $n \geq K$.

(iv) We want to show that $(SS_i) \cap (S_i S) = \emptyset$, $(S_i H_i) \cap (S_i S) = \emptyset$, $(H_i S_i) \cap (S_i S) = \emptyset$. Denote by $l(w)$ the length of the reduced word $w$ in $G$. Since every element $w$ in $G$ can be uniquely expressed as,
\[
w = (g_{1,1})^{m_1} \ldots (g_{N,1})^{m_N} w_1 \ldots w_N (g_{1,1})^{p_1} \ldots (g_{N,1})^{p_N} \in G,
\]
where $g_{i,1}^{m_i} w_i g_{i,1}^{p_i}$ is a reduced word in $L(F(n_i))$ and $w_i$ does neither start nor end with any power of $g_{i,1}$. Then we can define the following functions as
\[
s_i(w) = |m_i|, \quad e_i(w) = |p_i|, \quad l_i(w) = l(w_i), \quad t(w) = l(w_1 \ldots w_N),
\]
where $| \cdot |$ denotes the absolute value function.
Fact. By the definition of $S$, we know that, for all $1 \leq i \leq N$, every reduced word $w$ in $S$ contains a non-zero power of $g_{i,j}$ for some $j \geq 2$; and begins, also ends, with the absolute value of the power of $g_{i,1}$ greater than in $2N_0 - 1$, i.e.

$$
\min_{1 \leq i \leq N} l_i(w) > 0, \quad \min_{1 \leq i \leq N} s_i(w) > 2N_0 - 1, \quad \min_{1 \leq i \leq N} e_i(w) > 2N_0 - 1. \quad (**) 
$$

Let $g_1$ be any element in $S_1$, the support of $z$. From $S_1 \cap (\bigcup_{1 \leq i \leq N} G_i \cup H) = \emptyset$, it follows that, for every $i$, $g_1$ contains a non-zero power of $g_{i,j}$ for some $j \geq 2$, i.e.

$$
\min_{1 \leq i \leq N} l_i(g_1) > 0 \quad \text{and} \quad l(g_1) \leq N_0 - 1. \quad (a) 
$$

By the fact (**), we know that, for all $w$ in $S$,

$$
\min_{1 \leq i \leq N} l_i(w) > 0, \quad \min_{1 \leq i \leq N} s_i(w) > 2N_0 - 1, \quad \min_{1 \leq i \leq N} e_i(w) > 2N_0 - 1. \quad (b) 
$$

From (a), (b) and definition (*), it follows that

$$
\min_{1 \leq i \leq N} s_i(wg_1) > 2N_0 - 1, \quad \min_{1 \leq i \leq N} e_i(wg_1) \leq N_0, \quad \forall wg_1 \in SS_1, 
$$

$$
\min_{1 \leq i \leq N} s_i(g_1w) \leq N_0, \quad \min_{1 \leq i \leq N} e_i(g_1w) > 2N_0 - 1, \quad \forall g_1w \in S_1S. 
$$

Hence, $(SS_1) \cap (S_1S) = \emptyset$.

Let $g_2$ be another element in $S_1$ (so, not in $\bigcup_{1 \leq i \leq N} G_i \cup H$). Similar as (a), we have that $\min_{1 \leq i \leq N} l_i(g_2) > 0$ and $l(g_2) \leq N_0 - 1$. Combining with the fact (**), we know that $\min_{1 \leq i \leq N} s_i(w) > 2N_0 - 1$ and $\min_{1 \leq i \leq N} l_i(w) > 0$ for every $w \in S$, we obtain that $l_i(g_2w) \geq N_0 + 1$. Therefore, $\tau(g_2w) \geq l_i(g_2w) \geq N_0 + 1$. On the other hand, it is easy to see that $\tau(g_1h) = \tau(g_1) \leq N_0$ for all $g_1 \in S_1$ and $h \in H$. Thus $g_1H \cap g_2S = \emptyset$ whence $(S_1H) \cap (S_1S) = \emptyset$. Similarly, we also have $(H_1S_1) \cap (S_1S) = \emptyset$.

(v) The proof of $(H_1S_1) \cap (SS_1) = \emptyset$ and $(S_1H) \cap (SS_1) = \emptyset$ is similar as the one of $(S_1H) \cap (S_1S) = \emptyset$. We skip it here. This ends the proof of the Claim. \(\square\)

**Proof of Lemma 3.2** (continued). As a summary, for such $x$, $y$, $z$, $H$, $S_0$, $S_1$, $S$, we have that:

(i) $S = G \setminus (S_0 \cup H)$;

(ii) $\|y - zS_1\| \leq \epsilon$, $\|zS_1\| \leq \|y\|$;

(iii) there exists some positive integer $K$ such that $\|x_n\|_{S_0} \leq \epsilon, \forall n \geq K$;

(iv) $(SS_1) \cap (S_1S) = \emptyset$, $(S_1H) \cap (S_1S) = \emptyset$, $(H_1S_1) \cap (S_1S) = \emptyset$;

(v) $(H_1S_1) \cap (SS_1) = \emptyset$, $(S_1H) \cap (SS_1) = \emptyset$.

Applying Lemma 3.1, we get

$$
\|yx - xy\|^2 \geq \|y(x - E_{A_w}(x))\|^2 + \|(x - E_{A_w}(x))y\|^2. \quad \square 
$$

Note when $N = 1$, $G_1 = H_1 = H$, as an application of the preceding lemma, we have the following corollary.
Corollary 3.1. Let \( \mathcal{M}_0 \) be \( L(\mathbb{Z}) \) or \( CI \), and the group \( G = F(n_1) \) acts trivially on \( \mathcal{M}_0 \). Then

1. \( \mathcal{M} = \mathcal{M}_0 \times G = \mathcal{M}_0 \otimes L(G) \), and \( \mathcal{A} = \mathcal{M}_0 \times H = \mathcal{M}_0 \otimes L(H) \).
2. Let \( \omega \) be a free ultrafilter on \( \mathbb{N} \). Suppose \( x \) is an element in \( \mathcal{M}^\omega \) \((= (\mathcal{M}_0 \times G)^\omega)\) that commutes \( \mathcal{A} \). Then for any \( y \in \mathcal{M} \) with \( E_{\mathcal{A}}(y) = 0 \), we have

\[
\|yx - xy\|_2^2 \geq \|y(x - E_{\mathcal{A}}(x))\|_2^2 + \|y - E_{\mathcal{A}}(x)y\|_2^2.
\]

Generally, when \( N \) is finite, we have the following corollary.

Corollary 3.2. Let \( \mathcal{M}_0 \) be \( L(\mathbb{Z}) \) or \( CI \), and the group \( G = F(n_1) \times \cdots \times F(n_N) \) acts trivially on \( \mathcal{M}_0 \). Then

1. \( \mathcal{M} = \mathcal{M}_0 \otimes L(G) = \mathcal{M}_0 \otimes L(H) \), \( \mathcal{A} = \mathcal{M}_0 \otimes L(H) \), \( A_i = \mathcal{M}_0 \otimes L(G_i) \).
2. Suppose that \( N \) is finite. Let \( \omega \) be a free ultrafilter on \( \mathbb{N} \). Suppose \( x \) is an element in \( \mathcal{M}^\omega \) \((= (\mathcal{M}_0 \times G)^\omega)\) that commutes \( \mathcal{A} \) and \( E_{\mathcal{M}_0 \otimes L(G_1)}^\omega(x) = \cdots = E_{\mathcal{M}_0 \otimes L(G_N)}^\omega(x) = E_{\mathcal{A}}(x) \). Then for any \( y \in \mathcal{M} \) with \( E_{\mathcal{M}_0 \otimes L(G_1)}(y) = \cdots = E_{\mathcal{M}_0 \otimes L(G_N)}(y) = E_{\mathcal{A}}(y) = 0 \), we have

\[
\|yx - xy\|_2^2 \geq \|y(x - E_{\mathcal{A}}(x))\|_2^2 + \|x - E_{\mathcal{A}}(x)y\|_2^2.
\]

4. Abelian maximal injective subalgebras of tensor products of free group factors

Let \( \{n_i\}_{i=1}^m \) be a sequence of integers where \( n_i \geq 2 \) for all \( 1 \leq i \leq m \) and \( m \) is a positive integer or infinite. Let \( F(n_i) \) be the free group with the standard generators \( \{g_{i,j}\}_{j=1}^{n_i} \) for all \( 1 \leq i \leq m \). Let

\[
G^{(m)} = \bigtimes_{i=1}^m F(n_i), \quad \text{the direct sum of groups } F(n_1), \ldots, F(n_m);
\]

\[
H_i = \text{subgroup of } F(n_i) \text{ generated by } g_{i,1}, \quad \text{for } 1 \leq i \leq m;
\]

\[
H^{(m)} = H_1 \times H_2 \times \cdots \times H_m;
\]

\[
G_i = \bigtimes_{k=1}^{i-1} F(n_k) \times H_i \times \bigtimes_{k=i+1}^m F(n_k), \quad \text{for } 1 \leq i \leq m;
\]

\[
J_i = \bigtimes_{k=1}^{i-1} F(n_k) \times H_k \times \bigtimes_{k=i}^m F(n_k), \quad \text{for } 1 \leq i \leq m.
\]

Here, \( F(n_i) \) is identified with its canonical image in \( G^{(m)} \).

Let \( \lambda \) be the left regular representation of \( G^{(m)} \) and \( \mathcal{M}_m = L(G^{(m)}) \) the group von Neumann algebra associated with \( G^{(m)} \). Denote by \( \mathcal{A}^{(m)} \) (or \( A_i, N_i, B_i \)) the von Neumann subalgebra \( L(H^{(m)}) \) (or \( L(H_i), L(G_i), L(J_i) \), respectively) of \( \mathcal{M}_m \), for all \( 1 \leq i \leq m \).

From the construction of \( \mathcal{A}^{(m)} \), it is easy to see the following lemma.
Lemma 4.1. \(A^{(m)}\) is a maximal abelian von Neumann subalgebra of \(M_m\).

Moreover, we have that

Lemma 4.2. \(A^{(m)}\) is a singular maximal abelian von Neumann subalgebra of \(M_m\).

Proof. For any group element \(g\) in \(G^{(m)} \setminus (\bigcup G_i)\), we have \(\lambda(g)\) is orthogonal to \(N(A^{(m)})\), where \(N(A^{(m)})\) is the von Neumann subalgebra of \(M_m\) generated by the normalizer of \(A^{(m)}\) in \(M_m\). But the Hilbert space generated in \(L^2(G^{(m)})\) by \(\lambda(g)\) with \(g \in G^{(m)} \setminus (\bigcup G_i)\), coincides with the orthogonal subspace of \(L^2(\bigcup G_i)\) in \(L^2(G^{(m)})\). Therefore, \(N(A^{(m)}) \subset L^2(\bigcup G_i)\). Moreover, for all \(i \neq j\), if the unitary element \(\lambda(g)\) is such that \(g\) is in \(G_i\) and not contained in \(G_j\), then \(\lambda(g)A_j\) is orthogonal to \(N(A^{(m)})\). Therefore, for all \(g\) not contained in \(\bigcap G_i\), \(\lambda(g)\) is orthogonal to \(N(A^{(m)})\). It follows that \(N(A^{(m)}) = A^{(m)}\). Combining with the preceding lemma, \(A^{(m)}\) is a singular maximal abelian von Neumann subalgebra of \(M_m\). \(\square\)

Lemma 4.3. Let \(B \cong L(\mathbb{Z})\). Then we also have that \(B \otimes A^{(m)}\) is a singular maximal abelian von Neumann subalgebra of \(B \otimes M_m\).

Proof. The proof of the lemma is almost identical with the one of Lemma 4.2. So we skipped it here. \(\square\)

The following lemma is Corollary 3.3 in [11]. For the reader’s convenience, we present a proof here.

Lemma 4.4. \(A^{(1)}\) is a maximal injective abelian von Neumann subalgebra in \(M_1\).

Proof. Assume \(R\) is a maximal injective von Neumann subalgebra of \(M_1\) and \(A^{(1)} \subset R \subset M_1\). Let \(M_0 = C\) and the group \(G^{(1)}\) acts trivially on \(M_0\). Then \(M_1 = M_0 \rtimes G^{(1)}\).

Decompose \(R = R_1 \oplus R_2\) where \(R_1\) is a type I von Neumann subalgebra and \(R_2\) is a type II\(_1\) injective von Neumann subalgebra. If \(R_2 \neq 0\), from Corollary 2.1 we can find some \(x\) in \(R' \cap R_2^\omega\) but not contained in \((A^{(1)})^\omega\). By Lemma 2.4 we can find a unitary \(w\) in \(R_2\) such that \(w\) is orthogonal to \(A^{(1)}\) in \(R\), whence \(E_{A^{(1)}}(w) = 0\). By Corollary 3.1 and the fact that \(x \in R_2^\omega\), we have that \(0 = \|xw - wx\| \geq \|(x - E_{A^{(1)}}(x))w\| > 0\), which is a contradiction. Therefore \(R_2 = 0\) and \(R = R_1\). From Lemmas 2.3 and 4.2, it follows that \(A^{(1)} = R\). \(\square\)

Lemma 4.5. Let \(B \cong L(\mathbb{Z})\). \(B \otimes A^{(1)}\) is a maximal injective subalgebra of \(B \otimes M_1\) (= \(B \otimes L(F(n_1))\)).

Proof. Suppose \(R\) is a maximal injective von Neumann subalgebra and \(B \otimes A^{(1)} \subset R \subset B \otimes M_1\). Suppose \(R = R_1 \oplus R_2\), where \(R_1\) is a type I von Neumann subalgebra of \(B \otimes M_1\) and \(R_2\) is a type II\(_1\) injective von Neumann subalgebra of \(B \otimes M_1\). If \(R_2 \neq 0\), from Corollary 2.1 we can find some \(x\) in \(R' \cap R_2^\omega\) but not contained in \((B \otimes A^{(1)})^\omega\). By Lemma 2.4 we can find a unitary \(w\) in \(R_2\) such that \(w\) is orthogonal to \(B \otimes A^{(1)}\) in \(R\). Let \(M_0 = B\) and the group \(G^{(1)}\)
acts trivially on $B$. Then $B \otimes M_1 = M_0 \rtimes G^{(1)}$. By Corollary 3.1 and the fact that $x \in \mathcal{R}_2^\omega$, we have that
\[
0 = \|xw_w - wx\|_2 \geq \|(x - E_{(A(1) \otimes B)^\omega}(x))w\|_2 > 0,
\]
which is a contradiction. Therefore $\mathcal{R}_2 = 0$ and $\mathcal{R} = \mathcal{R}_1$. From Lemmas 2.3 and 4.3, it follows that $B \otimes A^{(1)} = \mathcal{R}$. \hfill $\Box$

A slight modification of [9, Lemma 2.3] shows the following result. For the sake of completeness, we also sketch its proof here.

**Lemma 4.6.** Let $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ be finite von Neumann algebras with a normal faithful tracial state $\tau$. Let $E_{\mathcal{N}}$ be a normal $\tau$-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Suppose $\mathcal{P}' \cap \mathcal{M} = \mathcal{N}$. Then, for every $x$ in $\mathcal{M}$,
\[
E_{\mathcal{N}}(x) \in \mathcal{N} \{vxv^* \mid v \text{ unitary element in } \mathcal{P}\}.
\]

**Proof.** Note $\mathcal{P}' \cap \mathcal{M} = \mathcal{N}$. It suffices to show that, if $x$ is an element in $\mathcal{M}$ such that $E_{\mathcal{N}}(x) = 0$, then $0 \in \mathcal{N} \{vxv^* \mid v \text{ unitary element in } \mathcal{P}\}$. Denote by $K_x = \mathcal{N} \{vxv^* \mid v \text{ unitary element in } \mathcal{P}\}$. Since $K_x$ is a weakly compact convex subset of $\mathcal{M}$, by inferior semi-continuity of the norm $\| \cdot \|_2$, it follows there is an element $y_0 \in K_x$ such that $\|y_0\|_2 = \inf \{\|y\|_2 \mid y \in K_x\}$. Since $\| \cdot \|_2$ is a Hilbert norm and $K_x$ is convex, it follows that $y_0$ is the unique element in $K_x$ with the property. But $vK_xv^* \subset K_x$ for all unitary element $v \in \mathcal{P}$. Therefore, $v_0y_0v^* \in K_x$ and $\|v_0y_0v^*\|_2 = \|y_0\|_2$. From the uniqueness of the element $y_0$, it follows that $v_0y_0v^* = y_0$ for all $v \in \mathcal{P}$, whence $y_0 \in \mathcal{P}' \cap \mathcal{M} = \mathcal{N}$. By the assumption, $x$ is orthogonal to $\mathcal{N}$. Therefore $\tau(v^*xv) = \tau(xvv^*) = 0$ for all $y \in \mathcal{N}, v \in \mathcal{P}$. It follows that $y_0$ is orthogonal to $\mathcal{N}$ and $y_0 \in \mathcal{N}$. Hence $y_0 = 0$, i.e. $0 \in \mathcal{N} \{vxv^* \mid v \text{ unitary element in } \mathcal{P}\}$. \hfill $\Box$

**Lemma 4.7.** Let $\mathcal{B} \cong L(\mathbb{Z})$. Suppose that, when $m < k$, $\mathcal{B} \otimes A^{(m)}$ is a maximal injective von Neumann subalgebra of $\mathcal{B} \otimes \mathcal{M}_m$. Then $\mathcal{B} \otimes A^{(k)}$ is also a maximal injective von Neumann subalgebra of $\mathcal{B} \otimes \mathcal{M}_k$.

**Proof.** Assume $\mathcal{R}$ is a maximal injective von Neumann subalgebra of $\mathcal{B} \otimes \mathcal{M}_k$ and $\mathcal{B} \otimes A^{(k)} \subset \mathcal{R} \subset \mathcal{B} \otimes \mathcal{M}_k$. Let $\mathcal{M}_0 = \mathcal{B}$ and the group $G^{(k)}$ acts trivially on $\mathcal{M}_0$. Then $\mathcal{B} \otimes \mathcal{M}_k = \mathcal{M}_0 \rtimes G^{(k)}$.

Let $E_{\mathcal{B} \otimes L(G_i)}$ be the trace-preserving condition expectation from $\mathcal{B} \otimes \mathcal{M}_k$ onto $\mathcal{B} \otimes L(G_i)$. Actually, if $x$ is expressed as $\sum_{g \in G^{(i)}} a_g \lambda(g)$, then $E_{\mathcal{B} \otimes L(G_i)}(x) = \sum_{g \in G^{(i)}} a_g \lambda(g)$, where $a_g \in \mathcal{B}$.

If $E_{\mathcal{B} \otimes L(G_i)}(\mathcal{R}) \supsetneq \mathcal{B} \otimes A^{(k)}$, there exists some $x$ in $\mathcal{R}$ such that $E_{\mathcal{B} \otimes L(G_i)}(x)$ is not contained in $\mathcal{B} \otimes A^{(k)}$. Let
\[
\mathcal{P} = \mathcal{B} \otimes (\bigotimes_{j=1}^{i-1} \mathbb{C}I) \otimes A_i \otimes (\bigotimes_{j=i+1}^k \mathbb{C}I),
\]
\[
\mathcal{N} = \mathcal{B} \otimes (\bigotimes_{j=1}^{i-1} L(F(n_j))) \otimes A_i \otimes (\bigotimes_{j=i+1}^k L(F(n_j))) = \mathcal{B} \otimes L(G_i),
\]
\[
\mathcal{M} = \mathcal{B} \otimes (\bigotimes_{j=1}^{i-1} L(F(n_j))) \otimes L(F(n_i)) \otimes (\bigotimes_{j=i+1}^k L(F(n_j))) = \mathcal{B} \otimes \mathcal{M}_k.
\]

It is not hard to check that $\mathcal{P}' \cap \mathcal{M} = \mathcal{N}$ (see [13, Corollary 9.11]). From Lemma 4.6 and the fact that both $x$ and $\mathcal{P}$ are contained in $\mathcal{R}$, it follows that $E_{\mathcal{B} \otimes L(G_i)}(x)$ is contained in $\mathcal{R}$. Denote
\[
E_{B \otimes L(G_i)}(x) \text{ by } y. \text{ Therefore } y \text{ is in } R \cap (B \otimes L(G_i)) \text{ but not contained in } B \otimes A(k). \text{ Let } S \text{ be the von Neumann subalgebra generated by } y \text{ and } B \otimes A(k) \text{ in } R \cap (B \otimes L(G_i)). \text{ Since } R \text{ is injective, } S \text{ is also injective and contained in } B \otimes L(G_i). \text{ Note that}
\[
B \otimes A(k) \simeq (B \otimes A_i) \otimes (\bigotimes_{j=1}^{i-1} A_j) \otimes (\bigotimes_{j=i+1}^{k} A_j) \subset S \subset B \otimes L(G_i)
\]
\[
\simeq (B \otimes A_i) \otimes (\bigotimes_{j=1}^{i-1} L(F(n_j))) \otimes (\bigotimes_{j=i+1}^{k} L(F(n_j))).
\]
By the assumption of the lemma, we know \( S = B \otimes A(k) \), contradicting with the fact that \( 0 \neq y \in S \setminus B \otimes A(k) \). Hence we obtain \( E_{B \otimes L(G_i)}(R) = B \otimes A(k) \) for all \( 1 \leq i \leq k \). It follows that we have \( E_{B \otimes L(G_i)}(R) = \cdots = E_{B \otimes L(G_k)}(R) = B \otimes A(k) \).

Therefore, we can assume that \( E_{B \otimes L(G_i)}(R) = \cdots = E_{B \otimes L(G_k)}(R) = B \otimes A(k) \). Again decompose \( R = R_1 \oplus R_2 \) where \( R_1 \) is a type I von Neumann subalgebra and \( R_2 \) is a type II\(_1\) injective von Neumann subalgebra. If \( R_2 \neq 0 \), from Corollary 2.1 we can find some \( x \) in \( R' \cap R_2^{\omega} \) but not contained in \( (B \otimes A(k))^{\omega} \). By Lemma 2.4 we can find a unitary \( w \) in \( R_2 \) such that \( w \) is orthogonal to \( B \otimes A(k) \) in \( R \), whence \( E_{B \otimes L(G_i)}(w) = \cdots = E_{B \otimes L(G_k)}(w) = E_{B \otimes A(k)}(w) = 0 \). By Corollary 3.2 and the fact that \( x \in R_2^{\omega} \), we have that
\[
0 = \|xw - wx\|_2 \geq \| (x - E_{(B \otimes A(k))^{\omega}}(x)) w \|_2 > 0,
\]
which is a contradiction. Therefore \( R_2 = 0 \) and \( R = R_1 \). From Lemmas 2.3 and 4.3, it follows that \( B \otimes A(k) = R \). □

**Lemma 4.8.** Let \( B \cong L(\mathbb{Z}) \). Assume that, when \( m < k \), \( B \otimes A(m) \) is a maximal injective von Neumann subalgebra in \( B \otimes M_m \). Then \( A(k) \) is a maximal injective von Neumann subalgebra of \( M_k \).

**Proof.** Assume \( R \) is a maximal injective von Neumann subalgebra of \( M_k \) and \( A(k) \subset R \subset M_k \). Let \( M_0 = \mathbb{C} \) and the group \( G(k) \) acts trivially on \( M_0 \). Then \( M_k = M_0 \rtimes G(k) \).

Let \( E_{L(G_i)} \) be the trace-preserving condition expectation from \( M_k \) onto \( L(G_i) \). Actually, if \( x \) is expressed as \( \sum_{g \in G_i} a_g \lambda(g) \), then \( E_{L(G_i)}(x) = \sum_{g \in G_i} a_g \lambda(g) \), where \( a_g \in M_0 = \mathbb{C} I \).

If \( E_{L(G_i)}(R) \nsubseteq A(k) \), there exists some \( x \) in \( R \) such that \( E_{L(G_i)}(x) \) is not contained in \( A(k) \). Let
\[
\mathcal{P} = (\bigotimes_{j=1}^{i-1} CI) \otimes A_i \otimes (\bigotimes_{j=i+1}^k CI),
\]
\[
\mathcal{N} = (\bigotimes_{j=1}^{i-1} L(F(n_j))) \otimes A_i \otimes (\bigotimes_{j=i+1}^k L(F(n_j))) = L(G_i),
\]
\[
\mathcal{M} = (\bigotimes_{j=1}^{i-1} L(F(n_j))) \otimes L(F(n_i)) \otimes (\bigotimes_{j=i+1}^k L(F(n_j))) = M_k.
\]
It is not hard to check that \( \mathcal{P}' \cap M = \mathcal{N} \) (see [13, Corollary 9.11]). From Lemma 4.6 and the fact that both \( x \) and \( \mathcal{P} \) are contained in \( R \), it follows that \( E_{L(G_i)}(x) \) is contained in \( R \). Denote \( E_{L(G_i)}(x) \) by \( y \). Therefore \( y \) is in \( R \cap L(G_i) \) but not contained in \( A(k) \). Let \( S \) be the von Neumann subalgebra generated by \( y \) and \( A(k) \) in \( R \cap L(G_i) \). Since \( R \) is injective, \( S \) is also injective and contained in \( L(G_i) \). Note that
Theorem 4.1. Following the notations as above. Suppose \( \{n_i\}_{i=1}^N \) is a sequence of integers where \( n_i \geq 1 \) for all \( 1 \leq i \leq N \) and \( N \) is finite or infinite. Let \( F(n_i) \) be the free group with the standard generators \( \{g_{i,j}\}_{j=1}^{n_i} \) for all \( 1 \leq i \leq N \). Let the group \( G \) be \( \bigoplus_{i=1}^N F(n_i) \), the direct sum of \( F(n_1), \ldots, F(n_N) \). And \( F(n_i) \) is identified with its canonical image in \( G \). Let \( \lambda \) be the left regular representation of \( G \) and \( \mathcal{M} = L(G) \cong \bigotimes_{i=1}^N L(F(n_i)) \) be the group von Neumann algebra associated with \( G \). Let \( \mathcal{A} \) be the abelian von Neumann subalgebra of \( \mathcal{M} \) generated by the unitary elements \( \{\lambda(g_{i,j})\} \) \( 1 \leq i \leq N \). Then \( \mathcal{A} \) is a maximal injective subalgebra of \( \mathcal{M} \) and not contained in any hyperfinite subfactor of \( \mathcal{M} \).

The proof of Theorem 4.1 is divided into two different cases: (i) \( N \) is finite, (ii) \( N \) is infinite. Therefore the theorem will follow easily from the following two propositions.

Proposition 4.1. When \( N \) is finite, \( \mathcal{A} \) is a maximal injective subalgebra of \( \mathcal{M} \). Moreover, if \( \mathcal{B} \cong L(\mathbb{Z}) \), then \( \mathcal{B} \otimes \mathcal{A} \) is a maximal injective abelian von Neumann subalgebra of \( \mathcal{B} \otimes \mathcal{M} \).

Proposition 4.2. When \( N \) is infinite, \( \mathcal{A} \) is a maximal injective subalgebra of \( \mathcal{M} \).

Proof of Proposition 4.1. The proposition follows easily from Lemmas 4.4, 4.5, 4.7 and 4.8. □

Proof of Proposition 4.2. Recall

\[
A^{(k)} \simeq A_i \otimes \left( \bigotimes_{j=1}^{i-1} A_j \right) \otimes \left( \bigotimes_{j=i+1}^k A_j \right) \subset S \subset L(G_i)
\]

\[
\simeq A_i \otimes \left( \bigotimes_{j=1}^{i-1} L(F(n_j)) \right) \otimes \left( \bigotimes_{j=i+1}^k L(F(n_j)) \right).
\]

By the assumption of the lemma, we know \( S = A^{(k)} \), contradicting with the fact that \( 0 \neq y \in S \setminus A^{(k)} \). Hence we obtain \( E_{L(G_i)}(R) = A^{(k)} \) for all \( 1 \leq i \leq k \). It follows that we have \( E_{L(G_1)}(R) = \cdots = E_{L(G_k)}(R) = \mathcal{A}^{(k)} \).

Therefore, we can assume that \( E_{L(G_1)}(R) = \cdots = E_{L(G_k)}(R) = \mathcal{A}^{(k)} \). Again decompose \( R = R_1 \oplus R_2 \) where \( R_1 \) is a type I von Neumann subalgebra and \( R_2 \) is a type \( II_1 \) injective von Neumann subalgebra. If \( R_2 \neq 0 \), from Corollary 2.1 we can find some \( x \in R' \cap R_2^\omega \) but not contained in \( \mathcal{A}^{(k)} \). By Lemma 2.4 we can find a unitary \( w \) in \( R_2 \) such that \( w \) is orthogonal to \( A^{(k)} \) in \( R \), whence \( E_{L(G_1)}(w) = \cdots = E_{L(G_k)}(w) = E_{A^{(k)}}(w) = 0 \). By Corollary 3.2 and the fact that \( x \in R_2^\omega \), we have that \( 0 = ||xw - wx||_2 \geq ||(x - E_{A^{(k)}}(w))w||_2 > 0 \), which is a contradiction. Therefore \( R_2 = 0 \) and \( R = R_1 \). From Lemmas 2.3 and 4.2, it follows that \( \mathcal{A}^{(k)} = R \). □
It is easy to see that $B_i ↗ M$. Assume that $R$ is an injective von Neumann subalgebra of $M$ that contains $A$ properly. Hence there exists some $x$ in $R$ but not in $A$. There exists some positive number $a$ such that $∥x∥_2 > a > ∥E_A(x)∥_2$. Note that $E_{B_i}(x) → x$ when $i$ goes to infinity. There is some $k \in \mathbb{N}$ such that $∥E_{B_k}(x)∥_2 > a$. Since $B_k = L(J_k) \cong (\bigotimes_{j=k}^{∞} A_j) \otimes L(F(n_1)) \otimes \cdots \otimes L(F(n_{k-1}))$,

by similar arguments as in Lemma 4.8 it follows $E_{B_k}(x) \in R \cap B_k$. Denote $E_{B_k}(x)$ by $y$. Note that $∥y∥_2 > a > ∥E_A(y)∥_2$. It follows that $y$ is not contained in $A$. Let $S$ be the von Neumann subalgebra of $R \cap B_k$ generated by $y$ and $A$. Since $R$ is injective, $S$ is also injective. By Proposition 4.1, $A (\cong \bigotimes_{i=1}^{∞} A_i)$ is maximal injective in $B_k (\cong (\bigotimes_{j=k}^{∞} A_j) \otimes L(F(n_1)) \otimes \cdots \otimes L(F(n_{k-1})))$. It contradicts with the fact that $0 \neq y \in S \setminus A$ and $S$ is injective. Hence $A$ is a maximal injective von Neumann subalgebra of $M$.

Remark. When $N$ is infinite, we obtain examples of strongly stable II$_1$ factors (or McDuff factors), infinite tensor products of free group factors, that contain abelian von Neumann subalgebras as the maximal injective abelian von Neumann subalgebras. These McDuff factors have self-adjoint operators that are not contained in any hyperfinite subfactor, which also answers Kadison’s problem #7 in the negative.

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References