Periodic surface automorphisms and algebraic independence of Morita–Mumford classes

Toshiyuki Akita\textsuperscript{a}, Nariya Kawazumi\textsuperscript{b,∗}, Takeshi Uemura\textsuperscript{c}

\textsuperscript{a}Faculty of Sciences, Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
\textsuperscript{b}Department of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan
\textsuperscript{c}Hokurei High School, Shin-ei, Kiyota-ku, Sapporo 004-0839, Japan

Received 24 May 1999; received in revised form 12 January 2000
Communicated by E.M. Friedlander

Abstract


MSC: Primary 57R20; secondary 14H15; 20J06; 32G15; 55R40; 57M20; 57S17

0. Introduction

Let $\Sigma_g$ be a closed oriented surface of genus $g$. We denote by $\text{Diff}_+(\Sigma_g)$ the topological group consisting of all orientation-preserving diffeomorphisms of $\Sigma_g$ with the $C^\infty$-topology. It is one of the most important objects in algebraic geometry, as well as in differential topology. In fact, if $g \geq 2$, then a theorem of Earle and Eells [5] tells us the classifying functor of oriented surface bundles $[\cdot, B\text{Diff}_+(\Sigma_g)]$ is homotopy equivalent to the moduli functor of nonsingular complex algebraic curves of genus $g$. Moreover, it implies the homotopy equivalence between $\text{Diff}_+(\Sigma_g)$ and the mapping class group $\pi_0(\text{Diff}_+(\Sigma_g))$.
In the cohomological study of the classifying space $B\text{Diff}_+(\Sigma_g)$ of the diffeomorphism group one often makes use of the Chern classes of the homology representation $\text{Diff}_+(\Sigma_g) \to \text{Aut}(H_1(\Sigma_g; \mathbb{Z}))$. It must be noted that Glover, Grieder, Mislin, Xia and others have studied the Chern classes on periodic surface automorphisms to obtain many remarkable results on the integral cohomology of the classifying space $B\text{Diff}_+(\Sigma_g)$ \cite{6–8,16,23,24}.

The Morita–Mumford classes, introduced by Morita \cite{17} and Mumford \cite{19} independently, play essential roles in the integral cohomology of the classifying space $B\text{Diff}_+(\Sigma_g)$ as well as the Chern classes. Their significance has been revealed by Arbarello, Cornalba, Harer, Miller, Morita and others \cite{3,9,11,13,15,17,18}. In the context of surface bundles the Morita–Mumford classes are defined as follows. Let $\pi: X \to B$ be an oriented fiber bundle whose fiber is a closed oriented surface. The relative tangent bundle $T_{X/B}$ is the oriented real two-dimensional vector bundle over the total space $X$ consisting of all the tangent vectors along the fibers. The $n$th Morita–Mumford class $e_n$ is, by definition, the Gysin image of the $(n+1)$th power of its Euler class $e := e(T_{X/B}) \in H^2(X; \mathbb{Z})$:

$$e_n = e_n(X) := \pi_!(e^{n+1}) \in H^{2n}(B; \mathbb{Z}).$$

It can be regarded as an integral cohomology class defined on the classifying space $B\text{Diff}_+(\Sigma_g)$ of the diffeomorphism group $\text{Diff}_+(\Sigma_g)$ in a natural way.

The purpose of the present paper is to study the Morita–Mumford classes by making use of periodic surface automorphisms. If $G$ is a finite subgroup of $\text{Diff}_+(\Sigma_g)$, then the universal principal $G$-bundle $E_G \to B_G$ induces the homotopy quotient (or the Borel construction) $\pi: (\Sigma_g)_B \to B_G$. The space $(\Sigma_g)_B$ is, by definition, the quotient of $E_G \times \Sigma_g$ by the diagonal action of $G$. The map $\pi$ induced by the first projection provides an oriented fiber bundle with fiber $\Sigma_g$:

$$(\Sigma_g)_B \to (\Sigma_g)_G \xrightarrow{\pi} B_G.$$

Its Morita–Mumford class $e_n((\Sigma_g)_G) \in H^*(B_G; \mathbb{Z}) = H^*(G; \mathbb{Z})$ is equal to the restriction of $e_n$ to the subgroup $G$. Kawazumi and Uemura \cite{14} obtained a fixed-point formula for the Morita–Mumford class $e_n((\Sigma_g)_G)$ on any finite subgroup $G$ of $\text{Diff}_+(\Sigma_g)$, which enables us to compute it in an explicit manner.

We confine ourselves to finite cyclic subgroups of the diffeomorphism group $\text{Diff}_+(\Sigma_g)$. Let $\gamma$ be a periodic automorphism of $\Sigma_g$ of period $m$. In other words, $\gamma$ is an element of $\text{Diff}_+(\Sigma_g)$ of finite order $m$. We denote by $\langle \gamma \rangle$ the cyclic subgroup of $\text{Diff}_+(\Sigma_g)$ generated by $\gamma$. Consider the Morita–Mumford classes of the homotopy quotient $(\Sigma_g)_{\langle \gamma \rangle}$ of the surface $\Sigma_g$ by the action of $\gamma$. We denote by $\phi(m)$ the number of the units of the ring $\mathbb{Z}/(m)$, i.e., the Euler function of $m$. As was already shown by Uemura \cite[Theorem 2.1]{22} and partially by the first author, the Morita–Mumford classes admit the periodicity with period $\phi(m)$,

$$e_{n+\phi(m)}((\Sigma_g)_{\langle \gamma \rangle}) = e_n((\Sigma_g)_{\langle \gamma \rangle})(u_\gamma)^{\phi(m)} \in H^{2n+2\phi(m)}(\langle \gamma \rangle; \mathbb{Z})$$
for any $n \geq 0$. Here $u_j$ is a generator of $H^2(\langle \gamma \rangle; \mathbb{Z}) \cong \mathbb{Z}/m$ introduced in Section 1. Moreover, Uemura has computed several interesting examples in [22].

The present paper consists of the following three theorems. In Section 2 we prove

**Theorem A.** If $n \equiv -1 \pmod{\phi(m)}$, we have

$$e_n(\langle \Sigma_\gamma \rangle_{\langle \gamma \rangle}) = 0 \in H^{2n}(\langle \gamma \rangle; \mathbb{Z}).$$

It is deduced from a description of the Morita–Mumford classes on periodic surface automorphisms in terms of the fixed-point data (Proposition 1.1.) By making use of Theorem A, the first author [1,2] has obtained several results concerning of the mod $p$ Morita–Mumford classes of $\text{BDiff}_+(\Sigma_\gamma)$ ($p$ a prime).

Even though we deal only with periodic surface automorphisms, we believe that it would be useful for studying the rational cohomology of the classifying space $\text{BDiff}_+(\Sigma_\gamma)$ of the total diffeomorphism group. Grieder [8] produced strong evidence for our belief. He constructed sufficiently many finite cyclic subgroups of $\text{Diff}_+(\Sigma_\gamma)$ to give an alternative proof of the rational non-triviality of the Chern classes of the homology representation $\text{Diff}_+(\Sigma_\gamma) \to \text{Aut}(H_1(\Sigma_\gamma; \mathbb{Z}))$, which he called the symplectic classes.

In the present paper we strengthen Grieder’s evidence. More precisely, in Section 3, we make use of periodic surface automorphisms to give an alternative and elementary proof of the stable algebraic independence of the Morita–Mumford classes, which was originally proved by Miller [15] and Morita [17] independently.

**Theorem B** (Miller [15, Theorem 1.1], Morita [17, Theorem 6.1]). The ring homomorphism

$$\mathbb{Q}[e_1, e_2, \ldots, e_n, \ldots] \to H^*(\text{BDiff}_+(\Sigma_\gamma); \mathbb{Q})$$

defined by the Morita–Mumford classes is an injection in the stable range $* < 2g/3$.

Our proof is based on Harer’s stability theorem [10,12], the stability of the Morita–Mumford classes [15,17] and the following.

**Theorem C.** For any nonzero polynomial $f = f(X_1, X_2, \ldots) \in \mathbb{Z}[X_1, X_2, \ldots, X_n, \ldots]$, there exists a surface $\Sigma_{\gamma(f)}$ and a periodic automorphism $\gamma(f) \in \text{Diff}_+(\Sigma_{\gamma(f)})$ such that

$$f(e_1, e_2, \ldots)(\langle \Sigma_{\gamma(f)} \rangle_{\langle \gamma(f) \rangle}) \neq 0 \in H^*(\langle \gamma(f) \rangle; \mathbb{Z}).$$

Theorem C will be proved in Section 4. We believe that our proof is more elementary than the original ones. Miller’s proof is based on general loop space theory, and Morita’s is involved with rather elaborate computations, while all we need is a plain observation related to polynomials (Lemmas 4.3 and 4.4.)
1. Fixed-point data

Nielsen [20] classified the periodic surface automorphisms by the fixed-point data. In this section we recall it and describe the Morita–Mumford classes on periodic automorphisms in terms of fixed-point data. We adopt a formulation given by Symonds [21], Glover and Mislin [6] under a slight modification.

Let $\gamma$ be a periodic automorphism of $\Sigma_g$ of period $m$. We denote by $\alpha(x)$ the order of the isotropy group $\langle \gamma \rangle_x$ at a point $x \in \Sigma_g$. It is a divisor of $m$, the order of the subgroup $\langle \gamma \rangle$ of $\text{Diff}_+(\Sigma_g)$ generated by $\gamma$. The exceptional set $S = S_\gamma := \{ x \in \Sigma_g ; \alpha(x) > 1 \}$ of the automorphism $\gamma$ is a $\gamma$-stable finite subset of $\Sigma_g$. Endow the surface $\Sigma_g$ with a $\gamma$-invariant Riemannian metric. If $x \in S$, then the automorphism $\gamma^{m/\alpha(x)}$ acts on the tangent space $T_x \Sigma_g$ by the rotation through the angle $2\pi \beta(x)/\alpha(x)$ for some integer $\beta(x)$ coprime to $\alpha(x)$. The integer $\beta(x)$ is uniquely determined modulo $\alpha(x)$. It is independent of the choice of a $\gamma$-invariant Riemannian metric. Clearly, the functions $\alpha$ and $\beta$ are constant on each $\gamma$-orbit on $S$. Let $\{ x_1, \ldots, x_q \} \subset S$ be a complete system of representatives with respect to the $\gamma$-action. Then the fixed-point data of $\gamma$ is defined by

$$\langle g, m | \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle,$$

where $\alpha_i := \alpha(x_i)$ and $\beta_i := \beta(x_i)$. The rational numbers $\beta_i/\alpha_i \in \mathbb{Q}$ are uniquely determined modulo $\mathbb{Z}$. The fixed-point data satisfies the Riemann–Hurwitz relation

$$1 + \frac{1}{m}(g - 1) - \frac{1}{2} \sum_{i=1}^q \left( 1 - \frac{1}{\alpha_i} \right) \in \mathbb{Z}_{\geq 0}$$

and the relation $\sum_{i=1}^q \beta_i/\alpha_i \equiv 0 \pmod{\mathbb{Z}}$, i.e.,

$$\sum_{i=1}^q \frac{m}{\alpha_i} \beta_i \equiv 0 \pmod{m}. \tag{1.1}$$

As was proved by Nielsen [20], two periodic automorphisms $\gamma_1$ and $\gamma_2$ are conjugate in the diffeomorphism group $\text{Diff}_+(\Sigma_g)$, if and only if their fixed-point data coincide with each other. Moreover, in the case where $m$ is a prime $p$, the fixed-point data $\langle g, p | \beta_1/p, \ldots, \beta_q/p \rangle$, $(\beta_i, p) = 1$, is realized by a periodic automorphism of $\Sigma_g$ if it satisfies the Riemann–Hurwitz relation and relation (1.1) (see, for example, [21, Theorem A], [6, Proposition 2.2]).

Now we describe the Morita–Mumford classes on the automorphism $\gamma$, i.e., those restricted to the subgroup $\langle \gamma \rangle$ it generates

$$e_n|_{\langle \gamma \rangle} = e_n(\langle \Sigma_g \rangle|_{\langle \gamma \rangle}) \in H^{2n}(\langle \gamma \rangle; \mathbb{Z}), \quad n \geq 1,$$

in terms of its fixed-point data. Consider the complex one-dimensional $\langle \gamma \rangle$-module $N_\gamma$ on which $\gamma$ acts by multiplication by $\exp(2\pi \sqrt{-1}/m)$, and denote by $u_\gamma \in H^2(\langle \gamma \rangle; \mathbb{Z})$ the Euler class associated to the $\langle \gamma \rangle$-module $N_\gamma$.

$$u_\gamma := e(N_\gamma) = e((E_{\langle \gamma \rangle} \times N_\gamma)/\langle \gamma \rangle) \in H^2(\langle \gamma \rangle; \mathbb{Z}) = H^2(B_{\langle \gamma \rangle}; \mathbb{Z}).$$
Throughout this paper we call it the cohomology class associated to the automorphism \( \gamma \). As is known, we have

\[ H^*(\langle \gamma \rangle; \mathbb{Z}) = \mathbb{Z}[u_\gamma]/(mu_\gamma). \]

**Proposition 1.1.** For any \( n \geq 1 \) we have

\[ e_n((\Sigma_\alpha)_{\langle \gamma \rangle}) = \sum_{i=1}^{q} \frac{m}{\alpha_i} (\beta_i^*)^n u_i^n \in H^{2n}(\langle \gamma \rangle; \mathbb{Z}), \]

where \( \beta_i^* \) is an integer satisfying \( \beta_i \beta_i^* \equiv 1 \pmod{\alpha_i} \).

**Proof.** The isotropy group \( \langle \gamma \rangle_{x_i} \) at \( x_i \) is generated by \( \gamma^{m/\alpha_i} \). We denote by \( v_i \in H^2(\langle \gamma \rangle_{x_i}; \mathbb{Z}) \) the cohomology class associated to the automorphism \( \gamma^{m/\alpha_i} \). We have

\[ \text{cor}_{\langle \gamma \rangle_{x_i}} (v_i)^n = \text{cor}_{\langle \gamma \rangle_{x_i}} (e(T_{x_i} \Sigma_\alpha))^n = \frac{m}{\alpha_i} (u_i^n), \]

(see, for example, [4, III, Section 9, p. 80]). From the definition of the fixed-point data it follows that

\[ e(T_{x_i} \Sigma_\alpha) = \beta_i^* v_i \in H^2(\langle \gamma \rangle_{x_i}; \mathbb{Z}). \]

Hence Theorem B in [14] implies

\[ e_n((\Sigma_\alpha)_{\langle \gamma \rangle}) = \sum_{i=1}^{q} \frac{m}{\alpha_i} (\beta_i^*)^n u_i^n, \]

as was to be shown. \( \square \)

For simplicity, we introduce an integer \( e_n(\langle \gamma \rangle) \in \mathbb{Z} \) satisfying

\[ e_n((\Sigma_\alpha)_{\langle \gamma \rangle}) = e_n(\langle \gamma \rangle) u_i^n \in H^{2n}(\langle \gamma \rangle; \mathbb{Z}). \]

If \( n \geq 1 \), then \( e_n(\langle \gamma \rangle) \) is uniquely determined modulo the period \( m \), while \( e_0(\langle \gamma \rangle) = 2 - 2g \).

For any \( n \geq 0 \), Proposition 1.1 together with the Riemann–Hurwitz formula implies

\[ e_n(\langle \gamma \rangle) \equiv \sum_{i=1}^{q} \frac{m}{\alpha_i} (\beta_i^*)^n (\bmod{m}). \]

**2. Proof of Theorem A**

Now we prove Theorem A stated in Introduction. As in the preceding section, let \( \gamma \in \text{Diff}_+(\Sigma_\alpha) \) be a periodic automorphism of \( \Sigma_\alpha \), whose fixed-point data is given by \( \langle g, m| \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle \).

Suppose \( n \equiv -1 (\bmod{\phi(m)}) \). Since \( \alpha_i \) is a divisor of \( m \), \( \phi(\alpha_i) \) is a divisor of \( \phi(m) \). Hence we have \( n \equiv -1 (\bmod{\phi(\alpha_i)}) \), and so \( (\beta_i^*)^n \equiv (\beta_i^*)^{\phi(\alpha_i) - 1} \equiv \beta_i \pmod{\alpha_i} \).

Equivalently,

\[ \frac{m}{\alpha_i} (\beta_i^*)^n \equiv \frac{m}{\alpha_i} \beta_i \pmod{m}. \]
Therefore, we obtain
\[ e_n((\Sigma_g)_{(\gamma)}) = \sum_{i=1}^{q} m_{\alpha_i} u_i^{n} = 0 \in H^{2n}(\langle \gamma \rangle; \mathbb{Z}) \cong \mathbb{Z}/m \]
by Proposition 1.1 and (1.1). This completes the proof of Theorem A. \( \square \)

If \( m = 2, 3, 4 \) or 6, we have \( \phi(m) \leq 2 \). Hence Theorem A, together with Theorem 2.1 [22] quoted in the introduction, implies

**Corollary 2.1.** If \( \gamma \) is of order 2, 3, 4 or 6, then
\[ e_n((\Sigma_g)_{(\gamma)}) = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
(2 - 2g) u_i^n & \text{if } n \text{ is even}.
\end{cases} \]

3. Algebraic independence of Morita–Mumford classes

Now, we deduce the stable algebraic independence of the Morita–Mumford classes (Theorem B) from Theorem C, Harer’s stability theorem [10,12] and the stability of the Morita–Mumford classes [15,17]. Theorem C will be proved in the next section.

Let \( \Sigma_{g,1} \) be a compact oriented surface of genus \( g \) with 1 boundary component, and \( \text{Diff}_+(\Sigma_{g,1}) \) the topological group consisting of all orientation-preserving diffeomorphism of \( \Sigma_{g,1} \) fixing the boundary pointwise. Harer’s stability theorem [10] tells us that the stabilization homomorphism \( H^* (B\text{Diff}_+(\Sigma_{g+1,1}); \mathbb{Z}) \rightarrow H^* (B\text{Diff}_+(\Sigma_{g,1}); \mathbb{Z}) \) and the forgetful homomorphism \( \sigma^*: H^* (B\text{Diff}_+(\Sigma_g); \mathbb{Z}) \rightarrow H^* (B\text{Diff}_+(\Sigma_{g,1}); \mathbb{Z}) \) are both isomorphisms in the integral stable range \( * < g/3 \). As was shown in [17, Proposition 3.4] or [15, Lemma 1.4], the stabilization homomorphism maps the \( n \)th Morita–Mumford class \( \sigma^* e_n \in H^{2n}(B\text{Diff}_+(\Sigma_{g+1,1}); \mathbb{Z}) \) to \( \sigma^* e_n \in H^{2n}(B\text{Diff}_+(\Sigma_{g,1}); \mathbb{Z}) \). Therefore, if a non-zero homogeneous polynomial \( f(e_1, e_2, \ldots, e_n) \) of the Morita–Mumford classes of degree \( 2d \) does not vanish in \( H^{2d}(B\text{Diff}_+(\Sigma_{g,1}); \mathbb{Z}) \) for some \( g_0 \) with \( 2d < g_0/3 \), then it does not vanish for any \( g \) in the integral stable range \( 2d < g/3 \) either.

Theorem C implies that \( f \) does not vanish in \( H^*(B\text{Diff}_+(\Sigma_{g(f)}); \mathbb{Z}) \). Let \( \langle g(f), m|\beta_1/x_1, \ldots, \beta_q/x_q \rangle \) be the fixed-point data of the automorphism \( \gamma(f) \). Then \( \langle g(f) + vm, m|\beta_1/x_1, \ldots, \beta_q/x_q \rangle \) is realized by a periodic automorphism for any integer \( v \geq 0 \). In fact, it satisfies relation (1.1) and the Riemann–Hurwitz relation. From Proposition 1.1 the polynomial \( f(e_1, e_2, \ldots, e_n) \) of the modified automorphism does not vanish either. Thus, we obtain a surface \( \Sigma_{g_0} \) and a periodic automorphism \( \gamma_0 \in \text{Diff}_+(\Sigma_{g_0}) \) such that \( 2d < g_0/3 \) and \( f(e_1, e_2, \ldots, e_n) \) does not vanish in \( H^{2d}(\langle \gamma_0 \rangle; \mathbb{Z}) \). Hence it does not vanish in the integral stable range.

Consequently, it turns out that the natural ring homomorphism
\[ \mathbb{Z}[e_1, e_2, \ldots, e_n, \ldots] \rightarrow H^*(B\text{Diff}_+(\Sigma_g); \mathbb{Z}) \]
defined by the Morita–Mumford classes is an injection in the integral stable range \( * < g/3 \). Since an injection tensored by \( \mathbb{Q} \) is also an injection, this proves the stable
rational algebraic independence of the Morita–Mumford classes in the integral stable range $* < g/3$. Finally, the improved stability theorem [12] assures us that the stable rational algebraic independence holds also in the rational stable range $* < 2g/3$. This completes the proof of Theorem B.

4. Proof of Theorem C

Finally we give a proof of Theorem C.

Fix an integer $n \geq 2$ and a prime $p$ satisfying the conditions

$$ p \equiv 1 \pmod{k} \quad (2 \leq k \leq n) \quad \text{and} \quad p \geq 5. \quad (4.1) $$

The finite field $\mathbb{F}_p = \mathbb{Z}/(p)$ may be identified with the endomorphism ring of the abelian group $\mathbb{Z}/p$, $\text{End}_{\mathbb{Z}}(\mathbb{Z}/p)$. In fact, $a \in \mathbb{F}_p$ induces an endomorphism

$$ t \mod p \in \mathbb{Z}/p \mapsto \text{at} \mod p \in \mathbb{Z}/p. $$

As in (1.2) we write

$$ e_i((\Sigma_{g})_{(\gamma)}) = e_i(\gamma) \mu^j \in H^2(\gamma; \mathbb{Z}) $$

for any periodic automorphism $\gamma \in \text{Diff}_+(\Sigma_g)$. The main ingredient of the proof of Theorem C is

**Proposition 4.1.** For each $a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_p^n$, there exists a surface $\Sigma_{g(a)}$ and a periodic diffeomorphism $\gamma(a) \in \text{Diff}_+(\Sigma_{g(a)})$ of order $p$ such that

$$ e_i(\gamma(a)) \equiv (2 + ((p - 1)/2) \gamma) a^j + \sum_{2 \leq k \leq i} k \alpha^j_k \pmod p \quad (4.2) $$

for any $i \geq 1$.

**Proof.** From condition (4.1) there exists an integer $\mu_k \in \mathbb{Z}$, $2 \leq k \leq n$, such that $\mu_k \mod p$ is of order $k$ in the multiplicative group $\mathbb{F}_p = \mathbb{Z}/(p)$. Clearly, for any $c \in \mathbb{Z}$, we have

$$ \sum_{j=0}^{k-1} \mu_k^{j} \equiv \begin{cases} 0 & \text{if } k \mid c \\ k & \text{if } k \nmid c \end{cases} \pmod p. \quad (4.3) $$

Now, for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_p^n$, we construct a surface $\Sigma_{g(a)}$ and a periodic diffeomorphism $\gamma(a) \in \text{Diff}_+(\Sigma_{g(a)})$ of order $p$ as follows.

In the case where $a = 0$, we choose a surface $\Sigma_{g(0)}$ which admits a fixed-point free automorphism $\gamma(0)$ of order $p$. For the rest, we assume $a \neq 0$. Let $a_k_1, a_k_2, \ldots, a_k_n$ ($1 \leq k_1 < k_2 < \cdots < k_n \leq n$) be the nonzero components of $a = (a_1, a_2, \ldots, a_n)$. Choose an integer $\alpha_k^x$ such that $a_k^x \alpha_k^x \equiv 1 \pmod p$ for each $x$.

Consider the case where $a_1 = 0$, i.e., $k_1 \geq 2$. Define

$$ g(a) := 1 - p + \frac{1}{2}(p - 1) \sum_{x=1}^{m} k_2, $$
which is nonnegative, since \( k_1 \geq 2 \). Then the fixed-point data
\[
\langle g(a), p | a^*_\alpha r^j_k / p; 1 \leq \alpha \leq m, 0 \leq j < k_x \rangle
\]
are realized by a periodic automorphism \( \gamma(a) \) of \( \Sigma_{g(a)} \). In fact, the Riemann–Hurwitz relation follows from the definition of \( g(a) \), and relation (1.1),
\[
\sum_{z=1}^{m} \sum_{j=0}^{k_z-1} a^*_z r^j_k \equiv 0 \pmod{p}
\]
follows from (4.3). From (1.3) and (4.3) we deduce
\[
e_i(\gamma(a)) \equiv \sum_{z=1}^{m} \sum_{j=0}^{k_z-1} a^*_z r^{-ij}_k \equiv \sum_{k_z \mid i} a^*_z k_x \equiv \sum_{z \leq \frac{k_z}{i}} ka^*_z \pmod{p}.
\]

Next consider the case where \( a_1 \neq 0 \), i.e., \( k_1 = 1 \). Define
\[
g(a) := \frac{1}{2} (p - 1) \left( 1 + \sum_{z=2}^{m} k_x \right).
\]
Then the fixed-point data
\[
\langle g(a), p | a^*_\alpha r^j_k / p, a^*_\alpha r^j_k / p, -2a^*_\alpha r^j_k / p; 2 \leq \alpha \leq m, 0 \leq j < k_x \rangle
\]
are realized by a periodic automorphism \( \gamma(a) \) of \( \Sigma_{g(a)} \). In fact, the Riemann–Hurwitz relation follows from the definition of \( g(a) \), and relation (1.1),
\[
a^*_1 (1/p + 1/p + (-2)/p) + \sum_{z=2}^{m} \sum_{j=0}^{k_z-1} a^*_z r^j_k \equiv 0 \pmod{p}
\]
follows from (4.3). From (1.3) and (4.3) we deduce
\[
e_i(\gamma(a)) \equiv (2 + ((p - 1)/2)') a_1^i + \sum_{z \leq \frac{k_z}{i}} ka^*_z \pmod{p}.
\]

Thus, for any \( a \in \mathbb{F}_p^n \), we have constructed a periodic automorphism \( \gamma(a) \) whose \( e_i \) is given by (4.2). This completes the proof. \( \Box \)

In order to prove Theorem C, we introduce some graded commutative algebras over \( \mathbb{F}_p \). We denote by \( \mathbb{F}_p[X_1, \ldots, X_n] \) the polynomial algebra over \( \mathbb{F}_p \) on generators \( X_i, 1 \leq i \leq n \) with \( \deg X_i = 2i \), by \( \mathbb{F}_p[Y_1, \ldots, Y_n] \) the polynomial algebra over \( \mathbb{F}_p \) on generators \( Y_i, 1 \leq i \leq n \) with \( \deg Y_i = 2 \), and by \( A_{p,n} \) the commutative \( \mathbb{F}_p \)-algebra consisting of all \( \mathbb{F}_p \)-valued functions on the \( n \)-dimensional vector space \( \mathbb{F}_p^n \).

Define two homomorphisms of \( \mathbb{F}_p \)-algebras \( \varphi_{p,n} : \mathbb{F}_p[X_1, \ldots, X_n] \to \mathbb{F}_p[Y_1, \ldots, Y_n] \) and \( \psi_{p,n} : \mathbb{F}_p[Y_1, \ldots, Y_n] \to A_{p,n} \) by
\[
\varphi_{p,n}(X_i) := (2 + ((p - 1)/2)'Y_1^i + \sum_{z \leq \frac{k_z}{i}} kY_1^i, \\
\psi_{p,n}(Y_i)(a) := a_i \quad (a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_p^n). \tag{4.4}
\]

Clearly, we have
Lemma 4.2. If \( f = f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n] \) is a homogeneous polynomial of degree \( 2d > 0 \), then
\[
f(e_1, \ldots, e_n)((\Sigma \varphi(a))(\gamma(a))) = \psi_{p,n} \varphi_{p,n}(\tilde{f})(a) u(a)^d \in H^*(\langle \gamma(a) \rangle; \mathbb{Z})
\]
for any \( a \in \mathbb{F}_p^n \). Here \( \Sigma \varphi(a) \) and \( \gamma(a) \) are as in Proposition 4.1, \( u(a) \in H^2(\langle \gamma(a) \rangle; \mathbb{Z}) \) is the cohomology class associated to the automorphism \( \gamma(a) \), and \( \tilde{f} := f \mod (p) \in \mathbb{F}_p[X_1, \ldots, X_n] \).

Moreover, we have

Lemma 4.3. The homomorphism \( \varphi_{p,n} \) is an injection.

Proof. The image of \( \varphi_{p,n} \) is contained in the subalgebra \( \langle Y_1, Y_2^2, \ldots, Y_n^n \rangle \) generated by \( Y_i^i, 1 \leq i \leq n \). From (4.4) it follows that
\[
\varphi_{p,n}(X_1) = \frac{1}{2}(p + 3)Y_1,
\]
\[
\varphi_{p,n}(X_i) = iY_i^i + (\text{terms in } Y_1, Y_2^2, \ldots, Y_{i-1}^{i-1}) \quad \text{if } i \geq 2.
\]

Since \( p \geq 5 \), \( (p + 3)/2 \) is invertible in \( \mathbb{F}_p \). Condition (4.1) implies that each \( i \) is invertible in \( \mathbb{F}_p \). Hence \( \varphi_{p,n} : \mathbb{F}_p[X_1, \ldots, X_n] \to \langle Y_1, Y_2^2, \ldots, Y_n^n \rangle \) has an inverse. Therefore \( \varphi_{p,n} \) is an injection.

The following is easily deduced from the fact that \( \mathbb{F}_p \) is an integral domain:

Lemma 4.4. If \( h \neq 0 \in \mathbb{F}_p[Y_1, \ldots, Y_n] \) and \( \deg(h) < 2p \), then \( \psi_{p,n}(h) \neq 0 \in A_{p,n} \).

Now we can prove Theorem C.

Proof of Theorem C. We endow the polynomial ring \( \mathbb{Z}[X_1, X_2, \ldots, X_n, \ldots] \) with a grading by \( \deg(X_i) = 2i \). It suffices to show the case where \( f = f(X_1, X_2, \ldots) \in \mathbb{Z}[X_1, X_2, \ldots, X_n, \ldots] \) is a nonzero homogeneous polynomial with degree \( 2d \). When \( d = 0 \), then the theorem is trivial. So we assume \( d > 0 \). Choose an integer \( n \geq 2 \) such that \( f \in \mathbb{Z}[X_1, X_2, \ldots, X_n] \).

In view of Dirichlet’s theorem there exists a prime \( p \) satisfying the following:
\[
p \equiv 1 \pmod{k} \quad (2 \leq k \leq n),
\]
\[
f \mod(p) \neq 0 \in \mathbb{F}_p[X_1, \ldots, X_n], \quad (4.5)
\]
\[
\deg(f) = 2d < 2p. \quad (4.6)
\]

Then \( \varphi_{p,n}(\tilde{f}) \neq 0 \in \mathbb{F}_p[Y_1, \ldots, Y_n] \) by (4.5) and Lemma 4.3. Here, \( \tilde{f} := f \mod(p) \in \mathbb{F}_p[X_1, \ldots, X_n] \). From (4.6) and Lemma 4.4 follows \( \psi_{p,n}(\varphi_{p,n}(\tilde{f})) \neq 0 \in A_{p,n} \). This means \( \psi_{p,n}(\varphi_{p,n}(\tilde{f}))(a) \neq 0 \in \mathbb{F}_p \) for some \( a \in \mathbb{F}_p^n \).
The homomorphism $F_p \to H^{2d}(\langle \gamma(a) \rangle; \mathbb{Z})$, $1 \mapsto u(a)^d$ is an isomorphism since $d > 0$. Therefore, from Lemma 4.2, we obtain

$$f(e_1, \ldots, e_n)((\Sigma_{g(a)} \langle \gamma(a) \rangle) = \psi_{p,n} \varphi_{p,n}(\tilde{f})(a)u(a)^d \neq 0 \in H^{2d}(\langle \gamma(a) \rangle; \mathbb{Z}).$$

This completes the proof. □

A similar construction to Proposition 4.1 gives

**Proposition 4.5.** If $n \geq 2$ and $p$ is a prime satisfying $p \equiv 1 \pmod{n}$, then there exists a surface $\Sigma_g$ and a periodic automorphism $\gamma \in \text{Diff}_+(\Sigma_g)$ of period $p$ such that

$$e_k((\Sigma_g \langle \gamma \rangle)) = \begin{cases} 0 & \text{if } n \nmid k, \\ \nu_k \gamma & \text{if } n \mid k. \end{cases}$$

Here $u_\gamma \in H^2(\langle \gamma \rangle; \mathbb{Z})$ is the cohomology class associated to the automorphism $\gamma$.

In the case where $n$ is an even number or a multiple of 3, this $\gamma$ is nothing but a periodic automorphism constructed by the third author [22, Section 4].

It should be remarked that the situation for the first Morita–Mumford class $e_1$ is slightly different. If $p$ is a prime greater than 3, there exists a surface $\Sigma_g$ and a periodic automorphism $\gamma \in \text{Diff}_+(\Sigma_g)$ of period $p$ such that $e_1((\Sigma_g \langle \gamma \rangle)) \neq 0$ (see [14, Section 3]). On the other hand, the Grothendieck–Riemann–Roch theorem implies $e_1 = 0 \in H^2(B\text{Diff}_+(\Sigma_g); \mathbb{Z}/12)$ for any $g \geq 1$ (see [17, p. 555]).

**Acknowledgements**

The authors would like to thank Professor Shigeyuki Morita for valuable comments and encouragement. The second and the third authors would like to express their gratitude to Professors Iku Nakamura, Tomohide Terasoma and Makoto Matsumoto and Doctor Kazuhide Kubo for helpful discussions.

**References**