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# Global existence of weak and classical solutions for the Navier–Stokes–Vlasov–Fokker–Planck equations

Myeongju Chae<sup>a,\*</sup>, Kyungkeun Kang<sup>b</sup>, Jihoon Lee<sup>c</sup>

<sup>a</sup> Department of Applied Mathematics, Hankyong National University, Ansung, Republic of Korea

<sup>b</sup> Department of Mathematics, Yonsei University, Seoul, Republic of Korea

<sup>c</sup> Department of Mathematics, Sungkyunkwan University, Suwon, Republic of Korea

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## ABSTRACT

We consider a system coupling the incompressible Navier–Stokes equations to the Vlasov–Fokker–Planck equation. The coupling arises from a drag force exerted by each other. We establish existence of global weak solutions for the system in two and three dimensions. Furthermore, we obtain the existence and uniqueness result of global smooth solutions for dimension two. In case of three dimensions, we also prove that strong solutions exist globally in time for the Vlasov–Stokes system.

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## 1. Introduction

In this paper we consider the motion of particles dispersed in incompressible viscous flows in  $\mathbb{R}^d$ , with  $d = 2, 3$ . Such a model was first introduced by Williams in the context of combustion theory [22], and also found in Cafisch and Papanicolaou [4]. The particles are described by a probability density function  $f(t, x, v) \geq 0$  governed by a kinetic transport equation with a friction force  $F$ ,

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Ff - \sigma \nabla_v f) = 0,$$

\* Corresponding author.

E-mail addresses: mchae@hknu.ac.kr (M. Chae), kkang@yonsei.ac.kr (K. Kang), jihoonlee@skku.edu (J. Lee).

where  $\sigma$  is a diffusive coefficient, which is a nonnegative constant. Here a diffusion effect is taken into account while a collision effect of particles is ignored. The particles are dispersed in a fluid described by its velocity field  $u(t, x)$  satisfying the incompressible Navier–Stokes equations,

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = - \int_{\mathbb{R}^d} F f \, dv, \quad \operatorname{div} u = 0.$$

The coupling arises from the friction force  $F(t, x)$  acting on particles exerted by the fluid. The force  $F(t, x)$  under our consideration is based on a thin spray model [20]; the volume fraction of particle is not considered as a fluid-kinetic coupling, and the force is reduced to be friction force proportional to the relative velocity with some friction constant  $F_0 > 0$ , i.e.  $F = F_0(u - v)$ . Therefore, the external force term in the fluid equation is given by

$$- \int_{\mathbb{R}^d} F f \, dv = F_0 \int_{\mathbb{R}^d} f(v - u) \, dv.$$

If constants  $\sigma, \nu$ , and  $F_0$  are, for simplicity, assumed to be 1, we then have the following Navier–Stokes–Vlasov–Fokker–Planck equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p - \int_{\mathbb{R}^d} (v - u) f \, dv = 0, & \operatorname{div} u = 0, \\ \partial_t f + (v \cdot \nabla_x) f + \nabla_v \cdot ((u - v) f - \nabla_v f) = 0. \end{cases} \tag{1.1}$$

Here  $p$  is the scalar pressure and initial data satisfy the compatibility condition, i.e.  $\operatorname{div} u_0 = 0$ .

We review some known results related to our concerns. In [14], Hamdache studied the Vlasov–Stokes system in a bounded domain and constructed a weak solution with specular reflection boundary conditions. Boudin et al. [2] considered the three dimensional incompressible Navier–Stokes–Vlasov equations in a torus to construct a global weak solution. In [17], Mellet and Vasseur proved the existence of global weak solution to compressible Navier–Stokes–Vlasov–Fokker–Planck equations in a bounded domain with Dirichlet or reflection boundary conditions. When the fluid is inviscid, the local existence of the compressible Vlasov–Euler equations was studied by Baranger and Desvillettes [1]. The local existence in the case of colliding particles was proved by Mathiaud [16]. Stability of solution near Maxwellian, which is equilibrium solution of the form  $(u = 0, f = Me^{-\frac{|v|^2}{2}})$ , was established by Goudon et al. [13] in case that domain is a three dimensional torus (see also [6] for the Vlasov–Euler–Fokker–Planck system). In two dimensions, He [15] showed that a perturbation of the steady state of the system is globally stable for arbitrary initial data converging toward steady state with the exponential rate under specific assumptions. We also mention that there are known results for hydrodynamic limit of the global weak solution of the system (1.1) (see e.g. [11,12] and references therein for other previous results in this direction).

In this paper our main objective is to establish the global existence in time of weak solutions and to study regularity of such solutions for the Navier–Stokes–Vlasov–Fokker–Planck equations in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . The appropriate notion of weak solution is specified in Section 2 (see Definition 6 for details). In two dimensional case, it turns out that weak solutions become strong and unique, provided that initial data are sufficiently regular and decay adequately fast at infinity for phase variables. We also show the global in time existence of the strong solution for the three dimensional Vlasov–Stokes system and Vlasov–Fokker–Planck–Stokes system. Before stating main results, we introduce some function spaces defined as follows:

$$\begin{aligned} \mathcal{V}(\mathbb{R}^d) &= \{u = (u_1, \dots, u_d) \mid u_i \in H_0^1(\mathbb{R}^d)\}, & \mathcal{V}_\sigma(\mathbb{R}^d) &= \{u \in \mathcal{V}(\mathbb{R}^d) \mid \operatorname{div} u = 0\}, \\ \mathcal{H}(\mathbb{R}^d) &= \text{the closure of } \mathcal{V}_\sigma(\mathbb{R}^d) \text{ in } (L^2(\mathbb{R}^d))^d, & \mathcal{V}'(\mathbb{R}^d) &= \{u = (u_1, \dots, u_d) \mid u_i \in H^{-1}(\mathbb{R}^d)\}, \end{aligned}$$

where  $H^{-1}(\mathbb{R}^d)$  is the dual space of  $H_0^1(\mathbb{R}^d)$ , and  $H_0^1(\mathbb{R}^d)$  is the closure of compactly supported smooth functions in  $H^1(\mathbb{R}^d)$ . Now we are ready to state our main results.

**Theorem 1.** *Let  $d = 2$  or  $3$ . Suppose  $(f_0, u_0)$  satisfies*

$$f_0 \geq 0, \quad f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d), \quad \int_{\mathbb{R}^d} (|x|^2 + |v|^2 + |\log f_0|) f_0 \, dv \in L^1(\mathbb{R}^d), \quad u_0 \in \mathcal{H}(\mathbb{R}^d). \tag{1.2}$$

*Then there exists a global weak solution  $(f, u)$  of (1.1) with initials  $(f_0, u_0)$  such that*

$$u \in L^\infty(0, T; \mathcal{H}(\mathbb{R}^d)) \cap L^2(0, T; \mathcal{V}_\sigma(\mathbb{R}^d)) \cap C^0(0, T; \mathcal{V}'(\mathbb{R}^d)), \quad f(t, x, v) \geq 0, \\ f \in L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)) \cap C(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d)), \quad f|v|^2 \in L^\infty(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

Next, we are concerned on the global in time existence of “strong” solutions for the incompressible Navier–Stokes–Vlasov–Fokker–Planck equations in two dimensions.

**Theorem 2.** *Let  $d = 2$ . Suppose  $(f_0, u_0)$  satisfies (1.2) in Theorem 1. Assume further that*

$$\langle v \rangle^k f_0, \langle v \rangle^k \nabla_x f_0 \in L^p(\mathbb{R}^2 \times \mathbb{R}^2), \quad \nabla u_0 \in L^p(\mathbb{R}^2), \tag{1.3}$$

*with  $p \in (2, \infty)$ ,  $k > 3 - \frac{2}{p}$  and  $\langle v \rangle = (1 + |v|^2)^{1/2}$ . Then, there exists a strong solution  $(f, u)$  to (1.1) with initials  $(f_0, u_0)$  such that*

$$\nabla u \in L^\infty(0, T; L^p(\mathbb{R}^2)), \quad |\nabla u|^{\frac{p}{2}} \in L^2(0, T; H^1(\mathbb{R}^2)), \\ \langle v \rangle^k \nabla_x f \in L^\infty(0, T; L^p(\mathbb{R}^2 \times \mathbb{R}^2)), \quad \langle v \rangle^{pk/2} |\nabla_x f|^{\frac{p-2}{2}} \nabla_v \nabla_x f \in L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2)). \tag{1.4}$$

We can also establish the higher regularity with respect to  $v$  and  $x$ , provided that initial data are sufficiently smooth. For notational convenience, let the multi-indices  $\alpha = [\alpha_1, \alpha_2]$  and  $\beta = [\beta_1, \beta_2]$  for nonnegative integers  $\alpha_i, \beta_i$ . For notational convenience, we denote

$$\partial_\beta^\alpha f = \partial_x^\alpha \partial_v^\beta f, \quad \|f\|_{W_x^{m,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}, \\ \|f\|_{W_k^{N,p}(\mathbb{R}^2 \times \mathbb{R}^2)} = \sum_{|\alpha| + |\beta| \leq N} \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}.$$

**Theorem 3.** *Let  $d = 2$ . Suppose  $(f_0, u_0)$  satisfies (1.2) in Theorem 1 and (1.3) in Theorem 2. Assume further that*

$$u_0 \in W^{N,p}(\mathbb{R}^2), \quad f_0 \in W_k^{N,p}(\mathbb{R}^2 \times \mathbb{R}^2) \tag{1.5}$$

*for any nonnegative integer  $N$  with  $p \in (2, \infty)$ ,  $k > 3 - \frac{2}{p}$ . Then, there exists a classical solution  $(f, u)$  to (1.1) on  $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$  satisfying the following integrability conditions:*

$$\begin{aligned}
 & f \in L^\infty(0, T; W_k^{N,p}(\mathbb{R}^2 \times \mathbb{R}^2)), \quad u \in L^\infty(0, T; W^{N,p}(\mathbb{R}^2)), \\
 & \langle v \rangle^{\frac{pk}{2}} \left| \partial_\beta^\alpha f \right|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f \in L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2)) \quad \text{for } |\alpha| + |\beta| \leq N, \\
 & \left| \partial^\alpha u \right|^{\frac{p-2}{2}} \nabla_x \partial^\alpha u \in L^2(0, T; L^2(\mathbb{R}^2)) \quad \text{for } |\alpha| \leq N.
 \end{aligned} \tag{1.6}$$

Furthermore, we prove a uniqueness result for two dimensional Navier–Stokes–Vlasov–Fokker–Planck equations.

**Theorem 4.** *Let  $d = 2$ . Suppose that  $(u_i, f_i)$  ( $i = 1, 2$ ) are weak solutions with the same initial data,  $(u_1, f_1)|_{t=0} = (u_2, f_2)|_{t=0}$  satisfying (1.2) in Theorem 1. If  $f_2$  satisfies the following integrability condition*

$$\langle v \rangle^{k+\alpha} f_2 \in L^p(0, T; L^q(\mathbb{R}^2 \times \mathbb{R}^2)),$$

where

$$\frac{2}{p} + \frac{2}{q} = 1, \quad 2 < q < \infty, \quad k > 2, \quad \alpha p > 2,$$

then  $u_1 = u_2$  and  $f_1 = f_2$ .

**Remark 1.** The immediate consequence of Theorem 4 is that any weak solution with a little bit high moments estimate for  $f$  should be unique in two dimensions. Furthermore, combining the result of Theorem 2, if initial data are sufficiently regular and decay sufficiently fast at infinity for phase variables, weak solutions become strong and therefore, unique. Another application of the uniqueness result is that weak solutions for the system (1.1) become strong and unique on a half-space with slip boundary condition for  $u$  and specular reflection boundary conditions for  $f$ . More details are found at the end of Section 4.

We also consider the Vlasov–Stokes system ( $\sigma = 0$ ) and Vlasov–Fokker–Planck–Stokes system ( $\sigma > 0$ ) in  $\mathbb{R}^3$ :

$$\begin{cases}
 \partial_t u - \Delta u + \nabla p = \int_{\mathbb{R}^3} (v - u) f \, dv, & \operatorname{div} u = 0, \\
 \partial_t f + (v \cdot \nabla_x) f + \nabla_v \cdot ((u - v) f - \sigma \nabla_v f) = 0, & \sigma \geq 0.
 \end{cases} \tag{1.7}$$

As mentioned earlier, the case  $\sigma = 0$  (Vlasov–Stokes system) on bounded domain  $\Omega$  was considered by Hamdache [14] in two or three dimensions. Among other things, in three dimensions, Hamdache [14] proved the global existence of the solution  $(f, u)$  satisfying

$$\begin{aligned}
 & u \in L^2(0, T; W^{2, \frac{3}{2}}(\Omega)) \cap H^1(0, T; L^{\frac{3}{2}}(\Omega)), \\
 & f \in L^\infty(0, T; L^\infty \cap L^1), \quad |v|^3 f \in L^\infty(0, T; L^1).
 \end{aligned} \tag{1.8}$$

It seems, however, not clear whether a higher regularity of constructed solutions in [14] is available. We consider the system (1.7) in the absence of boundary, i.e.  $\Omega = \mathbb{R}^3$  and in this case we obtain the global in time existence of the strong solution, whose regularity is higher than (1.8). To be more precise, our result reads as follows:

**Theorem 5.** Let  $d = 3$ . Suppose  $(f_0, u_0)$  satisfies (1.2) in Theorem 1. Assume further that

$$\langle v \rangle^k f_0, \langle v \rangle^k \nabla_x f_0, \langle v \rangle^k \nabla_v f_0 \in L^p(\mathbb{R}^3 \times \mathbb{R}^3), \quad \text{and} \quad u_0 \in W^{1,p}(\mathbb{R}^3), \tag{1.9}$$

where  $p \in (3, \infty)$  and  $k > 4 - \frac{3}{p}$ . Then, there exists a strong solution  $(f, u)$  to (1.7) on  $(0, T) \times \mathbb{R}^3 \times \mathbb{R}^3$  satisfying the following integrability conditions for all  $q < p$  and  $r \in (1, \infty)$ :

$$u \in L^r(0, T; W^{2,q}(\mathbb{R}^3)) \cap H^1(0, T; L^q(\mathbb{R}^3)), \quad \langle v \rangle^k \nabla_x f \in L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)),$$

and

$$\langle v \rangle^k \nabla_v f \in L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)).$$

Furthermore, if  $\sigma > 0$ ,  $f$  also satisfies

$$\langle v \rangle^{pk/2} |\nabla_x f|^{\frac{p-2}{2}} \nabla_v \nabla_x f \in L^2(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3)),$$

and

$$\langle v \rangle^{pk/2} |\nabla_v f|^{\frac{p-2}{2}} \nabla_v^2 f \in L^2(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3)).$$

**Remark 2.** The initial condition for  $u_0$  in Theorem 5 could be relaxed. To be more precise, as in [10], we set

$$D_p^{1-\frac{1}{l}, l} := \left\{ w \in L^p_\sigma(\Omega); \|w\|_{D_p^{1-\frac{1}{l}, l}} = \|w\|_{L^p} + \left( \int_0^\infty \|t^{\frac{1}{l}} A_p e^{-tA_p} w\|_{L^p}^l \frac{dt}{t} \right)^{\frac{1}{l}} < \infty \right\},$$

where  $A_p$  is the Stokes operator and  $L^p_\sigma(\Omega)$  is the closure of  $\{u \in C^\infty_0: \operatorname{div} u = 0 \text{ in } \Omega\}$  in  $L^p(\Omega)$  (see [10] for the details). Due to the result of [10],  $\|u_0\|_{W^{1,p}}$  can be replaced by  $\|u_0\|_{D_p^{1-\frac{1}{l}, l}}$ .

This paper is organized as follows: In Section 2, we show the global existence of the weak solutions for the Navier–Stokes–Vlasov–Fokker–Planck system in two or three dimensions using the method of approximation by regularized solutions. Section 3 is devoted to the proof of the global existence of the strong solutions for two dimensional Navier–Stokes–Vlasov–Fokker–Planck equations using the Brezis–Wainger inequality and various energy estimates in  $L^p$ . In Section 4, we prove the higher regularity and uniqueness for the two dimensional Navier–Stokes–Vlasov–Fokker–Planck system and one application of the uniqueness result is provided. In Section 5, we consider three dimensional Vlasov–Stokes and Vlasov–Fokker–Planck–Stokes system and we prove the global in time existence of the strong solution.

## 2. Weak solutions

In this section we will show the global existence of the weak solutions for the Navier–Stokes–Vlasov–Fokker–Planck system in dimension two or three. We start with notations.  $H^1_0(\mathbb{R}^d)$  is used to indicate the closure of compactly supported smooth functions in  $H^1(\mathbb{R}^d)$  and  $H^{-1}(\mathbb{R}^d)$  means the dual space of  $H^1_0(\mathbb{R}^d)$ . As introduced earlier, we also use the function spaces  $\mathcal{V}(\mathbb{R}^d)$ ,  $\mathcal{V}_\sigma(\mathbb{R}^d)$ ,  $\mathcal{H}(\mathbb{R}^d)$  and  $\mathcal{V}'(\mathbb{R}^d)$ . The duality  $\langle w, v \rangle$  for  $w \in \mathcal{V}'(\mathbb{R}^d)$ ,  $v \in \mathcal{V}(\mathbb{R}^d)$  is, as usual, given as  $\langle w, v \rangle = \sum_{i=1}^d \langle w_i, v_i \rangle_{H^{-1} \times H^1_0}$  and we denote  $\mathcal{V}^\circ_\sigma(\mathbb{R}^d) = \{w \in \mathcal{V}'(\mathbb{R}^d) \mid \langle w, v \rangle = 0 \text{ for all } v \in \mathcal{V}_\sigma(\mathbb{R}^d)\}$ .

Next we define the notion of a weak solution for the system (1.1).

**Definition 6.** Let  $I = [0, T]$ . We say a pair  $(f, u)$  is a weak solution of the Navier–Stokes–Vlasov–Fokker–Planck equations (1.1) if the following conditions are satisfied:

(a) The functions  $u$  and  $f$  satisfy

$$\begin{aligned} u &\in L^\infty(I; \mathcal{H}(\mathbb{R}^d)) \cap L^2(I; \mathcal{V}_\sigma(\mathbb{R}^d)) \cap C^0(I; \mathcal{V}'(\mathbb{R}^d)), \quad f(t, x, v) \geq 0, \\ f &\in L^\infty(I; L^\infty \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)) \cap C(I; L^1(\mathbb{R}^d \times \mathbb{R}^d)), \\ &(|x|^2 + |v|^2)f \in L^\infty(I; L^1(\mathbb{R}^d \times \mathbb{R}^d)). \end{aligned}$$

(b) The functions  $u$  and  $f$  solve the Navier–Stokes–Vlasov–Fokker–Planck equations (1.1) in the sense of distributions;

$$\begin{aligned} &\int_{\mathbb{R}^d} (u \cdot \Psi)(T, x) dx + \int_0^T \int_{\mathbb{R}^d} (\nabla u : \nabla \Psi - u \otimes u : \nabla \Psi - u \cdot \partial_t \Psi)(s, x) dx ds \\ &= \int_{\mathbb{R}^d} u_0(x) \Psi(0, x) dx + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v - u) f dv \Psi(s, x) dx ds \end{aligned}$$

with  $\nabla u : \nabla \Psi = \sum_{j,k=1}^n \partial_j u^k \partial_j \Psi^k$  and  $u \otimes u : \nabla \Psi = \sum_{j,k=1}^n u^j u^k \partial_j \Psi^k$ ,

$$\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} f(\partial_t \varphi + v \cdot \nabla_x \varphi + (u - v) \cdot \nabla_v \varphi + \Delta_v \varphi) dx dv dt = \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0 \varphi(0, x, v) dx dv,$$

for any  $\Psi \in C^1(I; (C_c^\infty(\mathbb{R}^d))^d)$  with  $\nabla \cdot \Psi = 0$  and  $\varphi \in C^1(I; C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d))$  with  $\varphi(T, \cdot, \cdot) = 0$ .

(c) The functions  $u$  and  $f$  satisfy the energy inequality,

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f + f \log f dv dx + \int_{\mathbb{R}^d} \frac{|u|^2}{2} dx + \int_0^T D(f, u)(t) dt + \int_0^T \int_{\mathbb{R}^d} |\nabla_x u|^2 dx ds \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_0 + f_0 \log f_0 dv dx + \int_{\mathbb{R}^d} \frac{|u_0|^2}{2} dx, \end{aligned}$$

where  $D(f, u)(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(u - v) f - \nabla_v f|^2 \frac{1}{2} dv dx$ .

We remark that formal computations yield the following equalities for smooth solutions with sufficient integrability:

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f + f \log f dv dx \right) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(u - v) f - \nabla_v f|^2}{f} dv dx &= - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(v - u) f dv dx, \\ \frac{d}{dt} \left( \int_{\mathbb{R}^d} \frac{1}{2} |u|^2 dx \right) + \int_{\mathbb{R}^d} |\nabla_x u|^2 dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u \cdot (v - u) f dv dx. \end{aligned}$$

Thus we deduce an entropy equality for the system as

$$\frac{d}{dt} \mathcal{E}(f, u) + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|(u - v)f - \nabla_v f|^2}{f} dv dx + \int_{\mathbb{R}^d} |\nabla_x u|^2 dx = 0,$$

where

$$\mathcal{E}(f, u) = \int_{\mathbb{R}^d} \frac{|u|^2}{2} dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left( \frac{|v|^2}{2} f + f \log f \right) dv dx.$$

The term  $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \log f dv dx$  has an indefinite sign, however, it can be shown in the following lemma that  $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f (\log f)^- dv dx$  is controlled in terms of initial data.

**Lemma 7.** Assume that  $(f, u)$  is a smooth solution of the system such that  $\mathcal{E}(f_0, u_0) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^2 f_0 dv dx$  is finite. Then it holds that

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f dv dx &\leq C \left( t, \mathcal{E}(f_0, u_0), \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_0 dv dx \right), \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} f (\log f)^- dv dx &\leq C \left( t, \mathcal{E}(f_0, u_0), \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_0 dv dx \right). \end{aligned}$$

Since the proof of Lemma 7 is, in principle, due to the arguments in [18], the details are omitted (see also [12, Proposition 1]).

Lemma 7 implies the entropy inequality

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \left( \frac{|x|^2}{2} + \frac{|v|^2}{2} + |\log f| \right) dv dx + \int_{\mathbb{R}^d} \frac{|u|^2}{2} dx + \int_0^t \int_{\mathbb{R}^d} |\nabla_x u|^2 dx ds \\ &+ \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|(u - v)f - \nabla_v f|^2}{f} dv dx ds \\ &\leq C \left( t, \mathcal{E}(f_0, u_0), \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_0 dv dx \right). \end{aligned} \tag{2.1}$$

The following a priori  $L^p$  estimate for smooth solution is obtained by multiplying  $f^{p-1}$  with  $1 \leq p < \infty$  to the second equation of (1.1) and taking integration by parts,

$$\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^p dv dx + \frac{4(p-1)}{p} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v f^{\frac{p}{2}}|^2 dv dx = d(p-1) \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^p dv dx. \tag{2.2}$$

The above identity implies  $\frac{d}{dt} \|f\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq \frac{d(p-1)}{p} \|f\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}$ , from which we have

$$\|f(t)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq C(t, \|f_0\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}).$$

To prove Theorem 1, we take the usual steps of constructing global weak solutions for the Navier–Stokes equations:

- regularizing the system for which we prove the existence of smooth solutions,
- finding uniform estimates for the solutions of the regularized system,
- passing to the limit on the regularized parameters.

The method of our paper is quite motivated by the work of Mellet and Vasseur [17], which is concerned about the compressible Navier–Stokes–Vlasov–Fokker–Planck equations.

2.1. Regularization

In this subsection, we intend to construct approximate solutions of the system.

For the incompressible Navier–Stokes equations defined on a general unbounded domain  $\Omega$ , Chemin et al. [8, Chapter 2] constructed global weak solutions, using the spectral projection operators  $(P_k)_{k \in \mathbb{Z}}$ , associated to the inhomogeneous Stokes operator. A number of useful properties of the family  $(P_k)_{k \in \mathbb{Z}}$  are listed as follows: For any  $u \in \mathcal{H}(\Omega)$ ,

$$P_k P_{k'} u = P_{\min(k,k')} u, \quad \lim_{k \rightarrow \infty} \|P_k u - u\|_{\mathcal{H}(\Omega)} = 0, \tag{2.3}$$

$$\|\nabla P_k u\|_{L^2(\Omega)} \leq \sqrt{k} \|u\|_{L^2(\Omega)}, \quad \|\Delta P_k u\|_{L^2(\Omega)} \leq k \|u\|_{L^2(\Omega)}. \tag{2.4}$$

In particular (2.4) implies  $P_k u \in L^\infty(\Omega)$  for  $u \in L^2(\Omega)$  in two and three dimensions. In what follows, we adapt notations and several theorems in [8].

**Definition 8.** The bilinear map  $Q$  is defined by

$$Q : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}',$$

$$(u, v) \mapsto -\operatorname{div}(u \otimes v).$$

Apart from the frequency cutoff, we need to modify the Vlasov–Fokker–Planck equation by adding  $\frac{1}{k} \Delta_x f_k$ . From now on we denote by  $\mathcal{H}_k(\mathbb{R}^d)$  the space  $P_k \mathcal{H}(\mathbb{R}^d)$ , unless any confusion is to be expected. Next we introduce the approximating system of (1.1):

$$\begin{cases} \partial_t f_k + v \cdot \nabla_x f_k + \nabla_v \cdot [(u_k - v) f_k] = \Delta_v f_k + \frac{1}{k} \Delta_x f_k, \\ \partial_t u_k(t) = P_k \Delta u_k + P_k Q(u_k, u_k) + P_k (j_k - n_k u_k), \\ f_{k,0} = f_0, \quad u_{k,0} = P_k u_0. \end{cases} \tag{2.5}$$

We assume the initial data  $(f_0, u_0)$  satisfy

$$f_0 \geq 0, \quad f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d), \quad u_0 \in \mathcal{H}(\mathbb{R}^d),$$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^2 + |v|^2) f_0 + f_0 |\log f_0| \, dv \, dx < \infty, \tag{2.6}$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^m + |v|^m) f_0 \, dv \, dx < \infty \quad \text{for all } m \in [0, m_0], \quad m_0 > 2d. \tag{2.7}$$



In the following, we denote

$$j_k(t, x) := \int_{\mathbb{R}^d} f_k v \, dv, \quad n_k(t, x) := \int_{\mathbb{R}^d} f_k \, dv.$$

The next lemma shows that  $L^p$  norms of  $n_k(t, x)$ ,  $j_k(t, x)$  can be bounded, provided that  $f$  and moment of  $f$  are controlled. Since the proof of the next lemma is similar to that of Lemma 3.2 in [17], we just state it without giving the details.

**Lemma 9.** *Let  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function and  $m_0 > 0$ . Assume further that  $f$  satisfies*

$$\|f\|_{L^\infty([0, T] \times \mathbb{R}^d \times \mathbb{R}^d)} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^m f(x, v, t) \, dx \, dv \leq M$$

for any integer  $m \in [0, m_0]$ . Then there exists a constant  $C$ , depending on  $M$ , such that

$$\begin{aligned} \|n(t)\|_{L^p(\mathbb{R}^d)} &\leq C \quad \text{for } p \in [1, (m_0 + d)/d], \\ \|j(t)\|_{L^q(\mathbb{R}^d)} &\leq C \quad \text{for } q \in [1, (m_0 + d)/(d + 1)], \end{aligned}$$

where  $n(t, x) := \int_{\mathbb{R}^d} f \, dv$  and  $j(t, x) := \int_{\mathbb{R}^d} f v \, dv$ .

Now we are ready to prove the existence of weak solutions for the system (2.5).

**Proposition 1.** *Let  $k > 0$  be a fixed constant. Suppose that  $f_0$  and  $u_0$  satisfy the conditions (2.6) and (2.7). Then there exists a weak solution  $(f_k, u_k)$  of the system (2.5) in  $[0, T]$  for any  $T > 0$ .*

**Proof.** We shall prove Proposition 1 by a fixed point argument. Motivated by [9], we decouple the system (2.5) replacing  $j_k, n_k$  by  $\tilde{j}_k, \tilde{n}_k$  from a given  $\tilde{f}_k \in L^2(0, T; \mathcal{U})$ , where

$$\mathcal{U} = \left\{ f \in L^2(\mathbb{R}^{2d}) \mid \|f\|_{\mathcal{U}}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\langle x \rangle^m + \langle v \rangle^m) f^2(x, v) \, dv \, dx < \infty, m > d + 2 \right\}.$$

For notational convenience, we denote  $(1 + |x|^2)^{\frac{1}{2}}$  by  $\langle x \rangle$ . Let us consider the equations

$$\partial_t f_k + v \cdot \nabla_x f_k + \nabla_v \cdot (u_k - v) f_k = \Delta_v f_k + \frac{1}{k} \Delta_x f_k, \tag{2.8}$$

$$\partial_t u_k(t) = P_k \Delta u_k + F_k(u_k(t)) + P_k(\tilde{j}_k - \tilde{n}_k u_k), \tag{2.9}$$

for the same initial data as (2.5). We denote  $P_k Q(u_k, u_k)$  by  $F_k(u_k)$ . We note that  $\tilde{j}_k$  and  $\tilde{n}_k$  are bounded in  $L^2([0, T]; L^2(\mathbb{R}^d))$  because in case that  $m > d + 2$ ,

$$\begin{aligned} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \langle v \rangle \tilde{f}_k \, dv \right)^2 dx \right)^{\frac{1}{2}} &\leq C \int_{\mathbb{R}^d} \langle v \rangle^{-\frac{m}{2} + 1} \left( \int_{\mathbb{R}^d} \langle v \rangle^m \tilde{f}_k^2 \, dx \right)^{\frac{1}{2}} dv \\ &\leq C \left( \int_{\mathbb{R}^d} \langle v \rangle^{-m+2} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle v \rangle^m \tilde{f}_k^2 \, dx \, dv \right)^{\frac{1}{2}}. \end{aligned}$$

Due to smoothing properties (2.4), it holds that

$$\begin{aligned} \|F_k(u_k)\|_{L^2(\mathbb{R}^d)} &\leq Ck^{1+\frac{d}{2}}\|u_k\|_{L^2(\mathbb{R}^d)}^2, \\ \|P_k(\tilde{J}_k - \tilde{n}_k u_k)\|_{L^2(\mathbb{R}^d)} &\leq C(\|\tilde{J}_k\|_{L^2(\mathbb{R}^d)} + k\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^2(\mathbb{R}^d)}). \end{aligned}$$

We obtained the second inequality as follows,

$$\begin{aligned} \|P_k\tilde{n}_k u_k\|_{L^2(\mathbb{R}^d)} &\leq \|u_k\|_{L^\infty(\mathbb{R}^d)}\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)} \\ &\leq C(\|\Delta u_k\|_{L^2(\mathbb{R}^d)} + \|u_k\|_{L^2(\mathbb{R}^d)})\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)} \\ &\leq C(k + 1)\|u_k\|_{L^2(\mathbb{R}^d)}\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)} \\ &\leq Ck\|u_k\|_{L^2(\mathbb{R}^d)}\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Hence we have a priori

$$\begin{aligned} \|u_k\|_{L^2(\mathbb{R}^d)} &\leq \|P_k u_0\|_{L^2(\mathbb{R}^d)} + \int_0^t \|\partial_t u_k(s)\|_{L^2(\mathbb{R}^d)} ds \\ &\leq \|P_k u_0\|_{L^2(\mathbb{R}^d)} + C \int_0^t (k^2\|u_k(s)\|_{L^2(\mathbb{R}^d)} + k^{1+\frac{d}{2}}\|u_k(s)\|_{L^2(\mathbb{R}^d)}^2 + k^2\|u_k(s)\|_{L^2(\mathbb{R}^d)}^2) ds + C. \end{aligned}$$

By the usual Picard iteration we can show that there exists the local solution  $u_k$  of (2.9) in  $C(0, T_k; \mathcal{H}_k(\mathbb{R}^d))$  for a short time  $T_k$ . It turns out that such local solutions in fact become global. Indeed,  $u_k$  satisfies the following energy inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|u_k|^2}{2} dx + \int_{\mathbb{R}^d} |\nabla u_k|^2 dx = \int_{\mathbb{R}^d} P_k(\tilde{J}_k - \tilde{n}_k u_k)u_k dx \tag{2.10}$$

$$\leq \|\tilde{J}_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^2(\mathbb{R}^d)} + \|\tilde{n}_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^4(\mathbb{R}^d)}^2 \tag{2.11}$$

$$\leq \|\tilde{J}_k\|_{L^2(\mathbb{R}^d)}\|u_k\|_{L^2(\mathbb{R}^d)} + C\|\tilde{n}_k\|_{L^2(\mathbb{R}^d)}^2\|u_k\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2}\|\nabla u_k\|_{L^2(\mathbb{R}^d)}^2. \tag{2.12}$$

Thus, using the Gagliardo–Nirenberg inequality, the Young inequality and the Gronwall inequality, we obtain

$$\|u_k\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u_k\|_{L^2(0, T; L^2(\mathbb{R}^d))}^2 \leq C,$$

which implies a uniform bound on  $\|u_k(t)\|_{L^2(\mathbb{R}^d)}$  in time, and therefore  $T_k = T$ .

Next we turn to the Vlasov–Fokker–Planck equation (2.8). Since  $u_k \in L^\infty(0, T; L^\infty(\mathbb{R}^d))$ , a nonnegative weak solution to (2.8) can be found similarly as in Carrillo [5, Theorem 3.2] if  $f_0$  satisfies (2.6). The solution  $f_k$  solves (2.8) in the distribution sense, satisfying

$$\begin{aligned} f_k &\in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)), \\ \nabla_v f_k, \nabla_x f_k &\in L^\infty(0, T; L^2(\mathbb{R}^d \times \mathbb{R}^d)), \\ (|x|^2 + |v|^2) f_k &\in L^\infty(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d)), \end{aligned} \tag{2.13}$$

for all  $T > 0$ . The norms are independent with respect to  $\|u_k\|_{L^\infty(0, T; L^\infty(\mathbb{R}^d))}$ . The weak solution satisfies the  $L^p$  bound as (2.2) so that

$$\sup_{t \in [0, T]} \|f_k(t)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq C(T) \|f_0\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \quad \text{for } 1 \leq p \leq \infty. \tag{2.14}$$

Moreover we have the following moment bounds (for its proof, we refer to [5, Lemma 5.4]):

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^m f_k(t, x, v) \, dv \, dx \leq C(m) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^m f_0(x, v) \, dv \, dx. \tag{2.15}$$

For spatial moment we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^m f_k \, dv \, dx = m \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^{m-2} x \cdot v f_k \, dv \, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{m(m+d-2)}{k} |x|^{m-2} f_k \, dv \, dx := I + II.$$

$I$  can be estimated as follows:

$$I \leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^m + |v|^m) f_k \, dv \, dx.$$

If we divide  $|x|^{m-2} f_k = |x|^{m-2} f_k^{\frac{m-2}{m}} f_k^{\frac{2}{m}}$  and use Hölder’s inequality, we have

$$II \leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^m f_k \, dv \, dx \right) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_k \, dv \, dx.$$

Thus we obtain

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^m + |v|^m) f_k(t, x, v) \, dv \, dx \leq C(m, T) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^m + |v|^m + 1) f_0(x, v) \, dv \, dx. \tag{2.16}$$

From the estimates (2.13) and (2.16), it follows that  $f_k \in L^\infty(0, T; \mathcal{U})$ .

Now we can define the operator

$$\mathcal{T}_k : L^2(0, T; \mathcal{U}_N) \subset L^2(0, T; L^2(\mathbb{R}^{2d})) \rightarrow L^2(0, T; \mathcal{U}_N) \tag{2.17}$$

by  $T_k(\tilde{f}_k) = f_k$  for any  $T$ . Here  $\mathcal{U}_N$  is the closed subspace of  $L^2(\mathbb{R}^{2d})$  defined as follows:

$$\mathcal{U}_N = \left\{ f \in \mathcal{U} \mid \|f\|_{\mathcal{U}_N}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\langle x \rangle^m + \langle v \rangle^m) f^2(x, v) \, dv \, dx \leq N, \ m > d + 2 \right\},$$

where  $N$  is a large positive constant depending on initial data such that the estimates (2.18)–(2.19) below hold. Then Proposition 1 follows if a fixed point for  $\mathcal{T}_k$  exists. Note that  $L^2(0, T; \mathcal{U}_N)$  is closed in  $L^2(0, T; L^2(\mathbb{R}^{2d}))$  (see e.g. [21, Theorem XIII.64]). Next we set

$$\mathcal{W} = \{f \in \mathcal{U} \mid \|f\|_{\mathcal{W}} := \|f\|_{\mathcal{U}} + \|\nabla f\|_{L^2(\mathbb{R}^{2d})} < \infty\},$$

and let  $\mathcal{W}'$  be the dual space of  $\mathcal{W}$ . Due to (2.13) it holds that

$$\|\mathcal{T}_k \tilde{f}_k\|_{L^2(0, T; \mathcal{W})} \leq N. \tag{2.18}$$

Moreover, the dual argument shows that

$$\|\partial_t(\mathcal{T}_k \tilde{f}_k)\|_{L^2(0, T; \mathcal{W}')} \leq N. \tag{2.19}$$

Indeed, for  $w \in L^2(0, T; \mathcal{W})$ , we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} \partial_t f_k w &= \int_0^T \int_{\mathbb{R}^{2d}} \left[ v f_k \nabla_x w + (u_k - v) f_k \nabla_v w - \left( \nabla_v f_k \nabla_v w + \frac{1}{k} \nabla_x f_k \nabla_x w \right) \right] \\ &\leq C \|\nabla w\|_{L^2(0, T; L^2(\mathbb{R}^{2d}))} \| |v|^2 f_k \|_{L^1(0, T; L^1(\mathbb{R}^{2d}))} \\ &\quad + \|u_k\|_{L^\infty(0, T; L^\infty(\mathbb{R}^d))} \|f_k\|_{L^2(0, T; L^2(\mathbb{R}^{2d}))} \|\nabla w\|_{L^2(0, T; L^2(\mathbb{R}^{2d}))} \\ &\quad + \left(1 + \frac{1}{k}\right) \|\nabla f_k\|_{L^2(0, T; L^2(\mathbb{R}^{2d}))} \|\nabla w\|_{L^2(0, T; L^2(\mathbb{R}^{2d}))}. \end{aligned}$$

Since  $\mathcal{W}$  is compactly embedded in  $L^2(\mathbb{R}^{2d})$  (see e.g. [21, Theorem XIII.65]), the estimates (2.18) and (2.19) imply that  $\mathcal{T}_k$  is a compact operator on  $L^2(0, T; \mathcal{U})$  by Aubin–Lions’ compactness lemma. Then by the Schauder fixed point theorem  $\mathcal{T}_k$  has in  $L^2(0, T; \mathcal{U})$  a fixed point, which also satisfies (2.13). This deduces the proposition.  $\square$

The weak solution  $(f_k, u_k)$  of the approximated system (2.5) satisfies an entropy equality with dissipation.

**Proposition 2.** *The weak solution  $(f_k, u_k)$  of (2.5) given by Proposition 1 satisfies the following equality:*

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_k + f_k \log f_k \, dv \, dx + \int_{\mathbb{R}^d} \frac{|u_k|^2}{2} \, dx + \int_0^T D_k(f_k, u_k)(t) \, dt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} |\nabla_x u_k|^2 \, dx \, ds + \frac{1}{k} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla_x f_k|^2}{f_k} \, dv \, dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_0 + f_0 \log f_0 \, dv \, dx + \int_{\mathbb{R}^d} \frac{|u_0|^2}{2} \, dx, \end{aligned}$$

where

$$D_k(f_k, u_k)(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(u_k - v) f_k - \nabla_v f_k|^2 \frac{1}{f_k} dv dx.$$

**Proof.** As in [5], the weak solution  $f_k$  of (2.8) satisfies

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_k + f_k \log f_k dv dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\chi_\lambda(u_k) - v) f_k - \nabla_v f_k|^2 \frac{1}{f_k} dv dx + \frac{1}{k} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla_x f_k|^2}{f_k} dv dx \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_k(u_k - v) f_k dv dx. \end{aligned}$$

We add the energy inequality (2.10) to the above equality replacing  $\tilde{j}_k, \tilde{n}_k$  with  $j_k, n_k$ . Using

$$\begin{aligned} \int_{\mathbb{R}^d} P_k(j_k - n_k u_k) u_k dx &= \int_{\mathbb{R}^d} (j_k - n_k u_k) u_k dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} v f_k dv \right) u_k dx - \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f_k dv \right) u_k u_k dx \\ &= - \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_k(u_k - v) f_k dv dx \right), \end{aligned}$$

we get the desired result.  $\square$

### 2.2. Proof of Theorem 1

In this subsection we construct a weak solution  $(f, u)$  of (1.1) with the initial data  $(f_0, u_0)$  satisfying the condition (1.2). Before giving the proof, we recall a useful lemma regarding the matter on weak convergence for product of two weakly convergent functions (see [19, Lemma 5.1]).

**Lemma 10.** Let  $\Omega$  be  $\mathbb{R}^d$  or a bounded open domain with smooth boundary. Suppose  $g^n, h^n$  converge weakly to  $g, h$  respectively in  $L^{p_1}(0, T; L^{p_2}(\Omega)), L^{q_1}(0, T; L^{q_2}(\Omega))$  where  $(p_1, p_2), (q_1, q_2)$  are conjugate pairs, and  $1 \leq p_i, q_i \leq \infty$ . We assume that for some  $m \geq 0$  which is independent of  $n$ ,

$$\begin{aligned} & \partial_t g^n \text{ bounded in } L^1(0, T; W^{-m,1}(\Omega)), \\ & \|h^n - h^n(\cdot, \cdot + \xi)\|_{L^{q_1}(0,T;L^{q_2}(\Omega))} \rightarrow 0 \text{ as } |\xi| \rightarrow 0. \end{aligned}$$

Then  $g^n h^n \rightharpoonup gh$  in the sense of distribution uniformly in  $n$ .

Now we present the proof of Theorem 1.

**Proof of Theorem 1.** Consider an approximating sequence  $f_0^n$  to  $f_0$  satisfying (2.7), and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|v|^2 + |x|^2) |f_0^n - f_0| dx dv + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f_0^n| \log f_0^n - f_0 \log f_0 | dx dv \rightarrow 0.$$

We denote by  $(f^n, u^n)$  the weak solution constructed in Proposition 1 for the system (2.5) with initial data  $f^n(0, \cdot, \cdot) = f_0^n(\cdot, \cdot)$  and  $u^n(0, \cdot) = P_n u_0(\cdot)$ .

There are several uniform estimates on  $(f^n, u^n)$ . The  $L^p$  estimate (2.14) yields the existence of a constant  $C$  independent of  $\lambda, n$  such that

$$\|f^n\|_{L^\infty(0,T;L^p(\mathbb{R}^d \times \mathbb{R}^d))} \leq C, \quad 1 \leq p \leq \infty. \tag{2.20}$$

Proceeding similarly with Lemma 7, the approximated entropy inequality (Proposition 2) gives

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f^n + f^n |\log f^n| \, dv \, dx + \int_{\mathbb{R}^d} \frac{|u^n|^2}{2} \, dx \\ & + \int_0^T D_n(f^n, u^n)(t) \, dt + \int_0^T \int_{\mathbb{R}^d} |\nabla_x u^n|^2 \, dx \, dt \\ & \leq C \left( T, \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x|^2 + |v|^2) f_0^n \, dv \, dx, \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_0^n \log f_0^n \, dv \, dx, \int_{\mathbb{R}^d} \frac{|P_n u_0|^2}{2} \, dx \right). \end{aligned}$$

Hence we deduce the existence of a uniform constant  $C$  such that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |x|^2 + |v|^2) f^n(t, x, v) \, dv \, dx \leq C, \tag{2.21}$$

$$\|u^n\|_{L^\infty(0,T;\mathcal{H}(\mathbb{R}^d))} + \|u^n\|_{L^2(0,T;\mathcal{V}_\sigma(\mathbb{R}^d))} \leq C. \tag{2.22}$$

In light of Lemma 9 ( $m_0 = 2$ ), (2.20) and (2.21) yield a constant  $C$  such that

$$\begin{aligned} \|n^n\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} & \leq C, \quad 1 \leq p < \frac{d+2}{d}, \\ \|j^n\|_{L^\infty(0,T;L^q(\mathbb{R}^d))} & \leq C, \quad 1 \leq q < \frac{d+2}{d+1}. \end{aligned} \tag{2.23}$$

Note that the above constant  $C$  is uniform since  $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^2 f^n \, dx \, dv$  is uniformly bounded in (2.21). We now take  $n$  to infinity. First of all, by (2.20)–(2.22), there exist

$$f \in L^\infty(0, T; L^p(\mathbb{R}^d \times \mathbb{R}^d)) \quad \text{for } 1 < p < \infty, \quad u \in L^2(0, T; \mathcal{V}_\sigma(\mathbb{R}^d))$$

such that

$$\begin{aligned} f^n & \rightharpoonup f \quad L^\infty(0, T; L^p(\mathbb{R}^d \times \mathbb{R}^d))\text{-weak}^* \quad \text{for } p \in (1, \infty), \\ u^n & \rightarrow u \quad L^2(0, T; \mathcal{V}_\sigma) \text{ weakly.} \end{aligned}$$

By the same arguments in [17], we have

$$\begin{aligned} n^n & \rightharpoonup n \quad L^\infty(0, T; L^p(\mathbb{R}^d))\text{-weak}^* \quad \text{for } p \in \left(1, \frac{d+2}{d}\right), \\ j^n & \rightharpoonup j \quad L^\infty(0, T; L^q(\mathbb{R}^d))\text{-weak}^* \quad \text{for } q \in \left(1, \frac{d+2}{d+1}\right), \end{aligned} \tag{2.24}$$

with  $j = \int_{\mathbb{R}^d} v f \, dv$  and  $n = \int_{\mathbb{R}^d} f \, dv$ . Note that (2.22), (2.23) imply the source term of Navier–Stokes part,  $(j^n - n^n u^n)$ , is in  $L^2([0, T]; \mathcal{V}'_\sigma(\mathbb{R}^d))$  uniformly with respect to  $n$ ; for any  $w \in L^2([0, T]; \mathcal{V}_\sigma(\mathbb{R}^d))$ , it holds that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} P_n[(j^n - n^n u^n)] w \, dx \, dt &\leq \int_0^T \|j^n\|_{L^{\frac{6}{5}}} \|w\|_{L^6} + \|n^n\|_{L^{\frac{3}{2}}} \|u^n\|_{L^2} \|w\|_{L^6} \, dt \\ &\leq C \int_0^T \|\nabla w\|_{L^2} (1 + \|\nabla u^n\|_{L^2}) \, dt. \end{aligned}$$

We then have the following compactness result for  $(u^n)_{n \in \mathbb{N}}$  (compare to [8, Proposition 2.7]):

$$\lim_{(n, \lambda) \rightarrow \infty} \int_0^T \int_K |u^n(t, x) - u(t, x)|^2 \, dx \, dt = 0, \tag{2.25}$$

for any  $T > 0$  and compact subset  $K$  of  $\mathbb{R}^d$ . In addition, for  $\Psi \in L^2([0, T]; \mathcal{V}(\mathbb{R}^d))$  and  $\Phi \in L^2([0, T] \times \mathbb{R}^d)$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} \nabla u^n(t, x) \nabla \Psi(t, x) \, dx \, dt &= \int_0^T \int_{\mathbb{R}^d} \nabla u(t, x) \nabla \Psi(t, x) \, dx \, dt, \\ \lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} u^n(t, x) \Phi(t, x) \, dx \, dt &= \int_0^T \int_{\mathbb{R}^d} u(t, x) \Phi(t, x) \, dx \, dt. \end{aligned} \tag{2.26}$$

Furthermore, for any  $\psi \in C^1(\mathbb{R}^+; \mathcal{V}_\sigma(\mathbb{R}^d))$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} (u^n(t, x) - u(t, x)) \psi(t, x) \, dx \right| = 0. \tag{2.27}$$

Applying a test function  $\Psi$  in  $C^1([0, T]; \mathcal{V}_\sigma(\mathbb{R}^d))$ , we obtain

$$\begin{aligned} \frac{d}{dt} \langle u^n(t), \Psi(t) \rangle &= \langle P_n \Delta u^n(t), \Psi(t) \rangle + \langle P_n Q(u^n(t), u^n(t)), \Psi(t) \rangle \\ &\quad + \langle P_n(j^n - n^n u^n), \Psi(t) \rangle + \left\langle u^n(t), \frac{d}{dt} \Psi(t) \right\rangle. \end{aligned} \tag{2.28}$$

Following the argument in [8], that is, using (2.25)–(2.27) and the fact

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|P_n \Psi(t) - \Psi(t)\|_{\mathcal{V}(\mathbb{R}^d)} = 0, \tag{2.29}$$

we can pass to the limit with respect to  $n$  so that

$$\begin{aligned} & \int_{\mathbb{R}^d} u \cdot \Psi(T, x) dx + \int_0^T \int_{\mathbb{R}^d} (\nabla u : \nabla \Psi - u \otimes u : \nabla \Psi - u \cdot \partial_t \Psi)(s, x) dx ds \\ &= \int_{\mathbb{R}^d} u_0(x) \Psi(0, x) dx + \lim_{(\lambda, n, k) \rightarrow \infty} \int_0^T \langle P_n(j^n - n^n u^n), \Psi \rangle dt. \end{aligned}$$

For the last term, it suffices to show that  $n^n u^n$  term converges in the sense of distribution for it holds that

$$\lim_{n \rightarrow \infty} \int_0^T \langle n^n u^n, P_n \Psi \rangle dt = \lim_{n \rightarrow \infty} \int_0^T \langle n^n u^n, \Psi \rangle dt$$

by (2.29). Indeed, with the aid of Lemma 10, the parallel arguments in Section 3.3 of [17] lead to

$$n^n u^n \rightharpoonup nu, \quad u^n f^n \rightharpoonup uf$$

in the distribution sense, which implies that  $(f^n, u^n)$  converge to a weak solution  $(f, u)$  of (1.1).

Finally taking the limit in the approximated entropy equality (Proposition 2) and using the convexity of the entropy we deduce

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f + f \log f dv dx + \int_{\mathbb{R}^d} \frac{|u|^2}{2} dx + \int_0^T D(f, u)(t) dt + \int_0^T \int_{\mathbb{R}^d} |\nabla_x u|^2 dx ds \\ & \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f_0 + f_0 \log f_0 dv dx + \int_{\mathbb{R}^d} \frac{|u_0|^2}{2} dx, \end{aligned}$$

where

$$D(f, u)(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(u - v)f - \nabla_v f|^2 \frac{1}{f} dv dx.$$

Thus the entropy inequality holds for the weak solution  $(f, u)$ . The proof is completed.  $\square$

Next we remark on the weak solution of three dimensional Vlasov–Fokker–Planck–Stokes equations, which will be discussed in Section 5.

**Remark 3.** Let  $\sigma \geq 0$ . We consider the Vlasov–Fokker–Planck–Stokes equations,

$$\begin{cases} \partial_t u - \Delta u + \nabla p = \int_{\mathbb{R}^d} (v - u) f dv, & \operatorname{div} u = 0, \\ \partial_t f + (v \cdot \nabla_x) f + \nabla_v \cdot ((u - v) f - \sigma \nabla_v f) = 0. \end{cases} \tag{2.30}$$

A weak solution of the Vlasov–Stokes equations on  $\mathbb{R}^3$  can be constructed in the same way as above in the sense of Definition 6. The case  $\sigma = 0$  on a bounded domain in  $\mathbb{R}^d, d \geq 2$ , with the appropriate boundary condition was considered in [14].



**Remark 4.** In case of 3-dimensional Vlasov–Fokker–Planck–Stokes equations, if the initial data are assumed to satisfy (1.2) and if  $\int_{\mathbb{R}^3} |v|^3 f_0 \, dv \in L^1(\mathbb{R}^3)$ , then the weak solution  $u$  of (2.30) satisfies the following estimate as mentioned in [14, Remark 3.1]:

$$\begin{aligned} & \|u\|_{L^2(0,T;W^{2,3/2}(\mathbb{R}^3))} + \|\partial_t u\|_{L^2(0,T;L^{3/2}(\mathbb{R}^3))} + \|\nabla p\|_{L^2(0,T;L^{3/2}(\mathbb{R}^3))} \\ & \leq C(\|j - nu\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))}, \|u_0\|_{W^{1,\frac{3}{2}}(\mathbb{R}^3)}). \end{aligned} \tag{2.31}$$

Indeed, a priori estimate shows that

$$\|n\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|j\|_{L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))} \leq C(T)(\|f_0\|_{L^\infty} + 1)A^3, \tag{2.32}$$

where  $A = \| |v|^3 f_0 \|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{1}{6}} + C(\|f_0\|_{L^\infty} + 1)\|u\|_{L^2(0,T;V_\sigma(\mathbb{R}^3))}$  and  $C(T)$  is a uniform constant. Then the inequality (2.31) is the direct consequence of the mixed norm estimates for the Stokes system (see e.g. [10]). We note that the a priori estimate (2.32) can be shown in a rigorous manner by following the construction in Section 2. Since its verification is rather straightforward, the details are omitted (compare to [14, Lemma 2.1]).

### 3. 2D strong solutions

In this section, we provide a priori estimates for the proof of Theorem 2. The main ingredients are the high moments estimates in Proposition 3 and the following version of the Brezis–Wainger [3] inequality,

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + \|\nabla u\|_{L^2(\mathbb{R}^2)})(1 + \log^+ \|\nabla u\|_{L^p(\mathbb{R}^2)})^{\frac{1}{2}} + C\|u\|_{L^2(\mathbb{R}^2)}, \tag{3.1}$$

for  $u \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$  with  $p > 2$ . We refer to [7] for the use of (3.1) to the proof of global existence of strong solutions for two dimensional partially viscous Boussinesq equations.

Before stating Proposition 3, we recall the following type of Gronwall’s inequality. Since its verification is straightforward, we state it without presenting its proof.

**Lemma 11.** *Let  $T > 0$  and nonnegative functions  $f, g : [0, T] \rightarrow \mathbb{R}^+$ . Assume  $g$  is integrable;  $C(t) := \int_0^t g(s) \, ds < \infty$  and  $f' \leq C_1 f + C_2 g f^a$  for  $0 \leq a \leq 1$ , where  $C_1$  and  $C_2$  are positive constants. We have the following:*

- (i) *If  $a = 1$ , then  $f(t) \leq f(0)e^{C_1 t + C_2 C(t)}$ .*
- (ii) *If  $0 \leq a < 1$ , then  $f(t) \leq e^{C_1 t}(f(0) + C_2(1 - a)^{\frac{1}{1-a}} C^{\frac{1}{1-a}}(t))$ .*

**Proposition 3.** *Suppose a pair  $(f, u)$  is a weak solution of Eqs. (1.1). Furthermore if the initial datum  $f_0$  satisfies*

$$\| \langle v \rangle^k f_0 \|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)} < \infty, \quad \text{for any } k > 0, p \geq 2,$$

where  $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$ , then  $f$  satisfies

$$\langle v \rangle^k f \in L^\infty(0, T; L^p(\mathbb{R}^2 \times \mathbb{R}^2)) \quad \text{and} \quad \langle v \rangle^{\frac{pk}{2}} \nabla_v |f|^{\frac{p}{2}} \in L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2)).$$

**Proof.** Multiplying  $\langle v \rangle^{kp} f^{p-1}$  on both sides of the equation of  $f$  in (1.1) and integrating over  $\mathbb{R}^2 \times \mathbb{R}^2$ , we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_p \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\ &= -\frac{1}{p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u \cdot \nabla_v) f^p \langle v \rangle^{kp} dv dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (vf) \langle v \rangle^{kp} f^{p-1} dx dv \\ &+ \frac{1}{p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f^p \nabla_v^2 \langle v \rangle^{kp} dv dx := J_{11} + J_{12} + J_{13}. \end{aligned} \tag{3.2}$$

Here easily we have

$$\begin{aligned} J_{11} &= \frac{k}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{kp-2} (u \cdot v) f^p dv dx \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{kp-1} |u| f^p dv dx, \\ J_{12} &= 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{kp} f^p dv dx + \frac{1}{p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v \cdot \nabla_v f^p \langle v \rangle^{kp} dv dx \\ &= \left(2 - \frac{2}{p}\right) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{kp} f^p dv dx - k \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v|^2 \langle v \rangle^{kp-2} f^p dv dx, \end{aligned}$$

and

$$J_{13} \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{kp-2} f^p dv dx.$$

The estimates of  $J_{12}$  and  $J_{13}$  are direct. Using Hölder’s inequality and Sobolev’s inequality,  $J_{11}$  can be estimated as follows (we decompose  $\langle v \rangle^{kp-1} = \langle v \rangle^{kp-\epsilon} \langle v \rangle^{+\epsilon-1}$  with  $0 < \epsilon < \frac{kp}{kp+2}$  and use Hölder’s exponent  $\frac{kp-\epsilon}{kp} + \frac{\epsilon}{kp} = 1$ ):

$$\begin{aligned} J_{11} &\leq C \int_{\mathbb{R}^2} |u| \left( \int_{\mathbb{R}^2} \langle v \rangle^{kp} f^{p \cdot \frac{kp}{kp-\epsilon}} dv \right)^{\frac{kp-\epsilon}{kp}} \left( \int_{\mathbb{R}^2} \frac{1}{\langle v \rangle^{kp \cdot \frac{(1-\epsilon)}{\epsilon}}} dv \right)^{\frac{\epsilon}{kp}} dx \\ &\leq C \|u\|_{L^{\frac{kp}{\epsilon}}(\mathbb{R}^2)} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{kp-\epsilon}{k}} \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{\epsilon}{k}} \\ &\leq C \|u\|_{L^2(\mathbb{R}^2)}^{\frac{2\epsilon}{kp}} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-\frac{2\epsilon}{kp}} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{kp-\epsilon}{k}} \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{\epsilon}{k}}. \end{aligned}$$

From the estimates  $J_{11}$ ,  $J_{12}$ , and  $J_{13}$ , we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_p \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\ & \leq C \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C \|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-\frac{2\epsilon}{kp}} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{kp-\epsilon}{k}}, \end{aligned}$$

where we used that  $\|u\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))} < C$ . The Gronwall inequality in Lemma 11 deduces Proposition 3.  $\square$

Next we prove Theorem 2 by presenting a priori estimates for  $\omega := \nabla \times u$  and  $\langle v \rangle^k \nabla_x f$ .

**Proof of Theorem 2.** We consider the vorticity equation in two dimensions:

$$\partial_t \omega - \Delta \omega + (u \cdot \nabla) \omega = - \int_{\mathbb{R}^2} (v \times \nabla_x) f \, dv - \nabla_x \times (nu), \tag{3.3}$$

where  $\omega = \partial_1 u_2 - \partial_2 u_1$  and  $n(x, t) = \int_{\mathbb{R}^2} f(t, x, v) \, dv$ . Let  $p > 2$ . Multiplying  $|\omega|^{p-2} \omega$  on both sides of Eq. (3.3) and integrating over  $\mathbb{R}^2$ , we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p(\mathbb{R}^2)}^p + C_p \|\nabla |\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f |\omega|^p \, dv \, dx \\ & \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v| |\nabla_x f| |\omega|^{p-1} \, dv \, dx + C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u| |\nabla_x f| |\omega|^{p-1} \, dv \, dx := J_{22} + J_{23}. \end{aligned}$$

What it follows,  $\epsilon$  will be chosen as a sufficiently small positive constant and  $k$  is a positive number satisfying  $k > 3 - \frac{2}{p}$ . Using the Hölder inequality, the Young inequality, and the Gagliardo–Nirenberg–Sobolev inequality, we have

$$\begin{aligned} J_{22} & \leq C \int_{\mathbb{R}^2} |\omega|^{p-1} \left( \int_{\mathbb{R}^2} \langle v \rangle^k |\nabla_x f| \frac{1}{\langle v \rangle^{k-1}} \, dv \right) dx \\ & \leq C \int_{\mathbb{R}^2} |\omega|^{p-1} \left( \int_{\mathbb{R}^2} \langle v \rangle^{kp} |\nabla_x f|^p \, dv \right)^{\frac{1}{p}} dx \\ & \leq C \|\omega\|_{L^p(\mathbb{R}^2)}^{p-1} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}, \end{aligned}$$

and

$$\begin{aligned} J_{23} & \leq C \int_{\mathbb{R}^2} \|\nabla_x f\|_{L^p(\mathbb{R}^2)} \|\omega\|^{\frac{p}{2}} \|\omega\|^{\frac{2(p-1)}{4}} \|u\|_{L^{\frac{2p}{p-1}}(\mathbb{R}^2)} \, dv \\ & \leq C \|u\|_{L^2(\mathbb{R}^2)}^{\frac{p-1}{p}} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{p}} \|\omega\|^{\frac{p}{2}} \|\omega\|^{\frac{p-1}{2}} \|\nabla |\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{p-1}{p}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)} \\ & \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p+1}} \|\omega\|_{L^p(\mathbb{R}^2)}^{p \cdot \frac{p-1}{p+1}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p \cdot \frac{2}{p+1}} + \epsilon \|\nabla |\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p+1}} \|\omega\|_{L^p(\mathbb{R}^2)}^p + C \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + \epsilon \|\nabla |\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

From the estimates  $J_{22}$  and  $J_{23}$ , we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_p \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f|\omega|^p \, dv \, dx \\ & \leq C(\|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p-1}} + 1) \|\omega\|_{L^p(\mathbb{R}^2)}^p + C \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + \epsilon \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \tag{3.4}$$

Next, we consider the equation of  $\nabla_x f$ :

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \nabla_x f\|_{L^p}^p + C_p \|\langle v \rangle^{\frac{kp}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2}^2 \\ & = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (\nabla_x(uf)) \langle v \rangle^{kp} |\nabla_x f|^{p-2} \nabla_x f \, dv \, dx \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (v \nabla_x f) \langle v \rangle^{kp} |\nabla_x f|^{p-2} \nabla_x f \, dv \, dx \\ & \quad - \frac{1}{p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v (|\nabla_x f|^p) \nabla_v \langle v \rangle^{kp} \, dv \, dx := J_{31} + J_{32} + J_{33}. \end{aligned} \tag{3.5}$$

We first estimate  $J_{31}$ , which is a rather troublesome term to control compared to other two terms. To estimate  $J_{31}$ , we divide  $J_{31}$  into three parts using integration by parts

$$\begin{aligned} |J_{31}| & = \left| -\frac{1}{p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u \cdot \nabla_v) (|\nabla_x f|^p) \langle v \rangle^{kp} \, dv \, dx \right. \\ & \quad \left. - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\nabla_x u \cdot \nabla_v) f |\nabla_x f|^{p-2} \nabla_x f \langle v \rangle^{kp} \, dv \, dx \right| \\ & \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u| \langle v \rangle^{kp-1} |\nabla_x f|^p \, dv \, dx + C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\nabla u| |f| |\nabla_v |\nabla_x f|^{\frac{p}{2}}| |\nabla_x f|^{\frac{p}{2}-1} \langle v \rangle^{kp} \, dv \, dx \\ & \quad + C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\nabla u| |f| |\nabla_x f|^{p-1} \langle v \rangle^{kp-1} \, dv \, dx := \tilde{J}_{31} + \tilde{J}_{32} + \tilde{J}_{33}. \end{aligned}$$

Here  $\tilde{J}_{31}$ ,  $\tilde{J}_{32}$ , and  $\tilde{J}_{33}$  can be estimated as follows. The estimate of  $\tilde{J}_{31}$  is direct as follows,

$$\tilde{J}_{31} \leq \|u\|_{L^\infty(\mathbb{R}^2)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p.$$

Applying Hölder's inequality to  $1/p + (p - 2)/(2p) + 1/2 = 1$ , we have

$$\begin{aligned} \tilde{J}_{32} & \leq C \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} |\nabla u|^p |f|^p \, dx \, dv \right)^{\frac{1}{p}} \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{p-2}{2}} \\ & \leq C \|\nabla u\|_{L^{2p}(\mathbb{R}^2)} \left( \int_{\mathbb{R}^2} \langle v \rangle^{pk} \|f\|_{L^p(\mathbb{R}^2)}^{p-1} \|\nabla_x f\|_{L^p(\mathbb{R}^2)} \, dv \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 & \times \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{p-2}{2}} \\
 & \leq C \|\nabla u\|_{L^{2p}(\mathbb{R}^2)} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{1-\frac{1}{p}} \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{p-2}{2} + \frac{1}{p}} \\
 & \leq C \|\nabla u\|_{L^{2p}(\mathbb{R}^2)}^2 \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{2-\frac{2}{p}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-2+\frac{2}{p}} + \epsilon \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\
 & \leq C \|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p}} \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{4(p-1)}{p^2}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-2+\frac{2}{p}} + \epsilon \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\
 & \leq C \|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{2p}{p^2-2p+2}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + \epsilon \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 + \epsilon \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2,
 \end{aligned}$$

where  $\frac{2p}{p^2-2p+2} < 2$  if  $p > 2$ . We also have

$$\begin{aligned}
 \tilde{J}_{33} & \leq C \|\nabla u\|_{L^{2p}(\mathbb{R}^2)} \|\langle v \rangle^k f\|_{L^{2p}(\mathbb{R}^2)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-1} \\
 & \leq C \|\omega\|_{L^p(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{p}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-1} \\
 & \leq C \|\omega\|_{L^p(\mathbb{R}^2)}^{\frac{p-1}{2p-1}} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2p-1}{2}(p-1)} + \epsilon \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2.
 \end{aligned}$$

The terms  $J_{32}$  and  $J_{33}$  are controlled as follows:

$$J_{32} + J_{33} \leq C \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p.$$

Since its verification is rather straightforward, we skip its details. Collecting all the estimates,

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \nabla_x f\|_{L^p}^p + C_p \|\langle v \rangle^{\frac{kp}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2}^2 \\
 & \leq C (\|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{2p}{p^2-2p+2}} + \|u\|_{L^\infty(\mathbb{R}^2)} + 1) \|\langle v \rangle^k \nabla_x f\|_{L^p}^p + C \|\omega\|_{L^p(\mathbb{R}^2)}^p \\
 & \quad + \epsilon \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 + \epsilon \|\langle v \rangle^{\frac{pk}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2.
 \end{aligned} \tag{3.6}$$

Adding (3.4) and (3.6), we obtain

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} (\|\omega\|_{L^p(\mathbb{R}^2)}^p + \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p) \\
 & \quad + C_p \|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f |\omega|^p dv dx + C_p \|\langle v \rangle^{\frac{kp}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2}^2 \\
 & \leq C (\|\omega\|_{L^p(\mathbb{R}^2)}^p + \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p) (1 + \|u\|_{L^\infty(\mathbb{R}^2)} + \|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{2p}{p^2-2p+2}}) \\
 & \leq C (\|\omega\|_{L^p(\mathbb{R}^2)}^p + \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p) (1 + \|\nabla u\|_{L^2(\mathbb{R}^2)} (1 + \log^+ \|\omega\|_{L^p}) + \|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{2p}{p^2-2p+2}}),
 \end{aligned}$$

where we used the Brezis–Wainger inequality (3.1). Using Gronwall’s inequality, we have

$$\sup_{0 \leq t \leq T} (\|\omega\|_{L^p(\mathbb{R}^2)}^p + \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p) \leq C_0 e^{e^{C_1 T}}.$$

This completes the proof of Theorem 2.  $\square$

**4. 2D higher regularity and uniqueness**

In this section we obtain the higher regularity (Theorem 3) for the strong solution  $(f, u)$  constructed in Section 3. The uniqueness assertion (Theorem 4) can be proved with assumptions on the integrability condition for Theorem 2. Theorem 3 is the consequence of the following a priori estimate for the strong solution  $(f, u)$  to (1.1).

Suppose  $(f_0, u_0)$  satisfies the conditions in Theorem 3, and

$$u_0 \in W^{N,p}(\mathbb{R}^2), \quad \sum_{|\alpha|+|\beta| \leq N, |\beta| \leq m'} \|\langle v \rangle^k \partial_\beta^\alpha f_0\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)} < C \tag{4.1}$$

for any nonnegative integer  $0 \leq m' \leq N$  with  $p \in (2, \infty), k > 3 - \frac{2}{p}$ . Then a strong solution  $(f, u)$  to (1.1) on  $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$  satisfies the following a priori inequality:

$$\begin{aligned} & \sum_{|\alpha|+|\beta|=N, |\beta| \leq m'} \left( \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p(t) + \int_0^t \|\langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \partial_\beta^\alpha \nabla_v f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2(t) dt \right) \\ & + \|u\|_{W^{N,p}(\mathbb{R}^2)}^p + \sum_{|\alpha'| \leq N} \int_0^t \|\partial^{\alpha'} u\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2}} \|\nabla_x \partial^{\alpha'} u\|_{L^2(\mathbb{R}^2)}^2(t) dt \leq C. \end{aligned} \tag{4.2}$$

The above constant  $C$  depends only on  $m', \|u_0\|_{W^{N,p}(\mathbb{R}^2)}, \sum_{|\alpha|+|\beta|=N, |\beta| \leq m'} \|\langle v \rangle^k \partial_\beta^\alpha f_0\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}$ .

The case that  $m' = 0$  will be treated in the lemma below.

**Lemma 12.** Suppose  $(f_0, u_0)$  satisfies the conditions in Theorem 3, and

$$u_0 \in W^{m,p}(\mathbb{R}^2), \quad \|\langle v \rangle^k f_0\|_{W_x^{m,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)} < C$$

for any given  $0 \leq m \leq N$  with  $p \in (2, \infty), k > 3 - \frac{2}{p}$ . Then there exists a constant  $C_m$  depending on  $m, \|u_0\|_{W^{m,p}(\mathbb{R}^2)}$ , and  $\|\langle v \rangle^k f_0\|_{W_x^{m,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}$  such that

$$\begin{aligned} & \|\langle v \rangle^k f\|_{W_x^{m,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p(t) + \sum_{|\alpha| \leq m} \int_0^t \|\langle v \rangle^{\frac{pk}{2}} |\partial_x^\alpha f|^{\frac{p-2}{2}} \partial_x^\alpha \nabla_v f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2(t) dt \\ & + \|u\|_{W^{m,p}(\mathbb{R}^2)}^p + \sum_{|\alpha| \leq m} \int_0^t \|\partial^\alpha u\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2}} \|\nabla_x \partial^\alpha u\|_{L^2(\mathbb{R}^2)}^2(t) dt \leq C_m. \end{aligned} \tag{4.3}$$

Let us postpone the proof of Lemma 12 for a moment. Assuming the lemma, we will obtain the inequality (1.6) as follows. This proves Theorem 3.

**Proof of Theorem 3.** We use induction on  $m'$ . Assume (1.6) is valid for  $m'$ . For  $|\beta| = m' + 1$ , taking  $\partial_\beta^\alpha$  derivatives to the  $f$  equation of (1.1) and inner product with  $\langle v \rangle^{pk} |\partial_\beta^\alpha f|^{p-2} \partial_\beta^\alpha f$ , we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2)}^p + \left\| \langle v \rangle^{\frac{kp}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \partial_\beta^\alpha \nabla v f \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\ & \leq C \sum_{|\beta| \leq m'+1} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p \\ & \quad + C \sum_{|\alpha|+|\beta|=N, |\beta| \leq m'} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)} \sum_{|\alpha|+|\beta|=N, |\beta| \leq m'+1} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-1} \\ & \quad + \sum_{\gamma \leq \alpha} C_\gamma \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \langle v \rangle^{pk} \partial^\gamma u \cdot \nabla v \partial_\beta^{\alpha-\gamma} f |\partial_\beta^\alpha f|^{p-2} \partial_\beta^\alpha f \, dv \, dx. \end{aligned}$$

By integration by parts we bound the last line by

$$\begin{aligned} & C \sum_{\gamma \leq \alpha} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \langle v \rangle^{pk-1} |\partial^\gamma u| |\partial_\beta^{\alpha-\gamma} f| |\partial_\beta^\alpha f|^{p-1} \, dx \, dv \\ & \quad + C \sum_{\gamma \leq \alpha} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \langle v \rangle^{pk} |\partial^\gamma u| |\partial_\beta^{\alpha-\gamma} f| |\partial_\beta^\alpha f|^{p-2} |\nabla v \partial_\beta^\alpha f| \, dx \, dv := \mathcal{I}_{41} + \mathcal{I}_{42}. \end{aligned}$$

We estimate  $\mathcal{I}_{41}$  by

$$\begin{aligned} \mathcal{I}_{41} & \leq C \left( \|u\|_{L^\infty(\mathbb{R}^2)} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p \right. \\ & \quad \left. + \sum_{0 < \gamma \leq \alpha} \int_{\mathbb{R}^2} \langle v \rangle^{pk-1} \|\partial^\gamma u\|_{L^p(\mathbb{R}^2)} \|\partial_\beta^{\alpha-\gamma} f\|_{L_x^\infty(\mathbb{R}^2)} \|\partial_\beta^\alpha f\|_{L_x^{p-1}(\mathbb{R}^2)}^{p-1} \, dv \right) \\ & \leq C \|u\|_{L^\infty(\mathbb{R}^2)} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p \\ & \quad + C \sum_{0 < \gamma \leq \alpha} \|\partial^\gamma u\|_{L^p(\mathbb{R}^2)} \int_{\mathbb{R}^2} \langle v \rangle^{pk-1} \|\partial_\beta^{\alpha-\gamma} f\|_{L_x^{\frac{p-2}{p}}(\mathbb{R}^2)} \|\partial_\beta^{\alpha-\gamma} \nabla_x f\|_{L_x^{\frac{2}{p}}(\mathbb{R}^2)} \|\partial_\beta^\alpha f\|_{L_x^{p-1}(\mathbb{R}^2)}^{p-1} \, dv \\ & \leq C \|u\|_{L^\infty(\mathbb{R}^2)} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_{N-(m'+1)} \sum_{|\alpha|+|\beta|=N, |\beta| \leq m'+1} \left\| \langle v \rangle^k \partial_\beta^\alpha f \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p, \end{aligned} \tag{4.4}$$

using the Sobolev interpolation inequality such that for  $p \in (2, \infty)$

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C \|u\|_{L^p(\mathbb{R}^2)}^{\frac{p-2}{p}} \|\nabla u\|_{L^p(\mathbb{R}^2)}^{\frac{2}{p}}, \tag{4.5}$$

and Lemma 12.

We estimate  $\mathcal{I}_{42}$  by

$$\begin{aligned}
 \mathcal{I}_{42} &\leq C \left( \|u\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} |\partial_\beta^\alpha f|^{p-1} |\nabla_v \partial_\beta^\alpha f| dx dv \right. \\
 &\quad \left. + \sum_{0 < \gamma \leq \alpha} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} |\partial^\gamma u| |\partial_\beta^{\alpha-\gamma} f| |\partial_\beta^\alpha f|^{p-2} |\partial_\beta^\alpha \nabla_v f| dx dv \right) \\
 &\leq C \|u\|_{L^\infty(\mathbb{R}^2)} \|\langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{p}{2}} \\
 &\quad + C \sum_{0 < \gamma \leq \alpha} \|\langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} |\partial^\gamma u|^2 |\partial_\beta^{\alpha-\gamma} f|^2 |\partial_\beta^\alpha f|^{p-2} dx dv \right)^{\frac{1}{2}} \\
 &\leq C \epsilon (\|u\|_{L^\infty(\mathbb{R}^2)} + 1) (\|\langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + C \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p) \\
 &\quad + C \sum_{0 < \gamma \leq \alpha} \|\partial^\gamma u\|_{L^p(\mathbb{R}^2)}^2 \|\langle v \rangle^k \partial_\beta^{\alpha-\gamma} f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2(p-2)}{p}} \|\langle v \rangle^k \partial_\beta^{\alpha-\gamma} \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{4}{p}} \\
 &\quad \times \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{p-2}, \tag{4.6}
 \end{aligned}$$

where we applied (4.5) to  $|\partial_\beta^{\alpha-\gamma} f|$  and Hölder’s inequality to  $2/p + (p - 2)/p = 1$ . By Young’s inequality to  $2(p - 2)/p^2 + 4/p^2 + (p - 2)/p = 1$  and an induction hypothesis, we have

$$\mathcal{I}_{42} \leq C \epsilon (\|u\|_{L^\infty(\mathbb{R}^2)} + 1) \|\langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + C_{N-(m'+1)} \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p.$$

Collecting the above estimates, we can see that there exist  $C_1, C_2$  depending on  $C_N$  such that

$$\frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2)}^p + C_1 \|\langle v \rangle^{\frac{kp}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \partial_\beta^\alpha \nabla_v f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \leq C_2 \|\langle v \rangle^k \partial_\beta^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p$$

for  $|\beta| \leq m' + 1$  and  $|\alpha| + |\beta| = N$ . From Gronwall’s inequality, we conclude Theorem 3.  $\square$

**Proof of Lemma 12.** In Section 2, we have already (4.3) for  $m = 1$ . Assume (4.3) holds for  $m \geq 1$ . For  $|\alpha| \leq m + 1$  we take  $\partial^\beta$  to the vorticity equation with  $|\beta| = m$  to have

$$\begin{aligned}
 &\partial_t \partial^\beta \omega - \Delta \partial^\beta \omega + (u \cdot \nabla) \partial^\beta \omega + \int_{\mathbb{R}^2} \partial^\beta \omega f dv \\
 &= - \sum_{\gamma \leq \beta, |\gamma| \geq 1} C_\gamma^\beta (\partial^\gamma u \cdot \nabla) \partial^{\beta-\gamma} \omega - \sum_{\gamma \leq \beta, |\gamma| \geq 1} \int_{\mathbb{R}^2} \partial^{\beta-\gamma} \omega \partial^\gamma f dv \\
 &\quad - \sum_{\gamma \leq \beta} \int_{\mathbb{R}^2} \partial^{\beta-\gamma} u \times \nabla_x \partial^\gamma f dv + \int_{\mathbb{R}^2} (v \times \nabla_x) \partial^\beta f dv. \tag{4.7}
 \end{aligned}$$

Taking inner product with  $|\partial^\beta \omega|^{p-2} \partial^\beta \omega$  on (4.7), we have



$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^p + C_p \|\partial^\beta \omega\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2}} \|\partial^\beta \nabla \omega\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\partial^\beta \omega|^p f \, dv \, dx \\
 & \leq C \sum_{0 < \gamma \leq \beta} \left| \int_{\mathbb{R}^2} (\partial^\gamma u \cdot \nabla) \partial^{\beta-\gamma} \omega |\partial^\beta \omega|^{p-2} \partial^\beta \omega \, dx \right| \\
 & \quad + C \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\partial^\beta(\omega f) - f \partial^\beta \omega) |\partial^\beta \omega|^{p-2} \partial^\beta \omega \, dv \, dx \right| \\
 & \quad + C \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial^\beta(u \times \nabla_x f) |\partial^\beta \omega|^{p-2} \partial^\beta \omega \, dv \, dx \right| \\
 & \quad + C \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v \times \nabla_x) \partial^\beta f |\partial^\beta \omega|^{p-2} \partial^\beta \omega \, dv \, dx \right| \\
 & := J_{41} + J_{42} + J_{43} + J_{44}.
 \end{aligned} \tag{4.8}$$

We estimate  $J_{41}$  by

$$\begin{aligned}
 J_{41} & \leq C \sum_{0 < \gamma \leq \beta} \|\partial^\gamma u\|_{L^\infty(\mathbb{R}^2)} \|\nabla \partial^{\beta-\gamma} \omega\|_{L^p(\mathbb{R}^2)} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C \sum_{|\beta|=m} \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^p + C \sum_{\gamma \leq \beta, |\gamma| > 1} \|\partial^\gamma u\|_{L^\infty(\mathbb{R}^2)} \|\partial^{\beta-\gamma} \nabla \omega\|_{L^p(\mathbb{R}^2)} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \left( \sum_{|\beta|=m} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^p \right) + C_m^{\frac{1}{p}} \sum_{\gamma \leq \beta, |\gamma| > 1} \|\partial^\gamma u\|_{L^p(\mathbb{R}^2)}^{\frac{p-2}{p}} \|\nabla \partial^\gamma u\|_{L^p(\mathbb{R}^2)}^{\frac{2}{p}} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C_m^{\frac{1}{p}} (1 + C_m) \|\omega\|_{W^{m,p}(\mathbb{R}^2)}^p.
 \end{aligned}$$

We used an induction hypothesis, Young’s inequality, and the Sobolev inequality (4.5). We estimate  $J_{42}$  by

$$\begin{aligned}
 J_{42} & \leq C \int_{\mathbb{R}^2} \sum_{\gamma < \beta} \|\partial^\gamma \omega \partial^{\beta-\gamma} f\|_{L^p(\mathbb{R}^2)} \, dv \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C \left( \sum_{\gamma < \beta} \|\partial^\gamma \omega\|_{L^p(\mathbb{R}^2)} \int_{\mathbb{R}^2} \|\partial^{\beta-\gamma} f\|_{L_x^p(\mathbb{R}^2)}^{\frac{p-2}{p}} \|\partial^{\beta-\gamma} \nabla_x f\|_{L_x^p(\mathbb{R}^2)}^{\frac{2}{p}} \, dv \right) \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C \|\omega\|_{W^{m-1,p}(\mathbb{R}^2)} \left( \sum_{|\beta| \leq m} \|\langle v \rangle^k \partial^\beta \nabla_x f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)} \right) \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 & \leq C_m^{\frac{1}{p}} (\|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + \|\omega\|_{W^{m,p}(\mathbb{R}^2)}^p).
 \end{aligned}$$

We estimate  $J_{43}$  by

$$\begin{aligned}
 J_{43} &\leq C \left( \int_{\mathbb{R}^2} \|\partial^\beta (u \times \nabla_x f)\|_{L_x^p(\mathbb{R}^2)} dv \right) \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 &\leq C \left( \int_{\mathbb{R}^2} \|u\|_{L^\infty(\mathbb{R}^2)} \|\nabla_x f\|_{W_x^{m,p}(\mathbb{R}^2)} + \|u\|_{W^{m,p}(\mathbb{R}^2)} \|\nabla_x f\|_{L_x^\infty(\mathbb{R}^2)} dv \right) \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 &\leq C (\|u\|_{L^\infty(\mathbb{R}^2)} \|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)} + C_m^{\frac{1}{p}} \|\langle v \rangle^k f\|_{W_x^{2,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}) \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^{p-1} \\
 &\leq C_m (\|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + \|\omega\|_{W^{m,p}(\mathbb{R}^2)}^p).
 \end{aligned}$$

Summing up the estimates for  $J_{41}$ ,  $J_{42}$ ,  $J_{43}$  and

$$J_{44} \leq C \|\langle v \rangle^k \nabla_x \partial^\beta f\|_{L^p(\mathbb{R}^2)} \|\omega\|_{W^{m,p}(\mathbb{R}^2)}^{p-1},$$

we find (4.8) to be

$$\begin{aligned}
 &\frac{1}{p} \frac{d}{dt} \|\partial^\beta \omega\|_{L^p(\mathbb{R}^2)}^p + C_p \|\partial^\beta \omega\|^{\frac{p-2}{2}} \|\partial^\beta \nabla \omega\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\partial^\beta f|^p f dv dx \\
 &\leq C_m^{\frac{1}{p}} (1 + C_m) (\|\omega\|_{W^{m,p}(\mathbb{R}^2)}^p + \|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p).
 \end{aligned} \tag{4.9}$$

Next we consider the equation for  $f$  and take  $\partial^\alpha$  derivatives with  $|\alpha| = m + 1$ ,

$$\partial_t \partial_x^\alpha f + (v \cdot \nabla_x \partial_x^\alpha f) - \nabla_v \cdot (v \partial_x^\alpha f) + \sum_{\gamma \leq \alpha} \partial^\gamma u \cdot \nabla_v \partial_x^{\alpha-\gamma} f - \Delta_v (\partial_x^\alpha f) = 0.$$

Multiplying  $\langle v \rangle^{pk} |\partial^\alpha f|^{p-2} \partial^\alpha f$  on the both sides of the above equation, integrating over  $\mathbb{R}^2 \times \mathbb{R}^2$ , we have

$$\begin{aligned}
 &\frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \partial^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_p \|\langle v \rangle^{\frac{kp}{2}} |\partial^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\
 &= \sum_{\gamma \leq \alpha} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} \partial^\gamma u \cdot \nabla_v \partial_x^{\alpha-\gamma} f |\partial_x^\alpha f|^{p-2} \partial_x^\alpha f dx dv \\
 &\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^k \nabla_v \cdot (v \partial^\alpha f) |\partial^\alpha f|^{p-2} \partial^\alpha f dv dx.
 \end{aligned}$$

The second term is bounded by  $C \|\langle v \rangle^k \partial^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p$ . By the integration by parts, we estimate the first term by

$$\begin{aligned}
 &C \sum_{\gamma \leq \alpha} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk-1} |\partial^\gamma u| |\partial^{\alpha-\gamma} f| |\partial^\alpha f|^{p-1} dx dv \\
 &\quad + C \sum_{\gamma \leq \alpha} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle v \rangle^{pk} |\partial^\gamma u| |\partial^{\alpha-\gamma} f| |\partial^\alpha f|^{p-2} |\nabla_v \partial^\alpha f| dx dv := I_{41} + I_{42}.
 \end{aligned}$$

In the same way for estimating  $\mathcal{I}_{41}, \mathcal{I}_{42}$  in (4.4), (4.6) replacing  $\partial_\beta^{\alpha-\gamma} f, \partial_\beta^\alpha f$  with  $\partial^{\alpha-\gamma} f, \partial^\alpha f$  instead, we obtain

$$I_{41} \leq C \|u\|_{L^\infty(\mathbb{R}^2)} \|\langle v \rangle^k \partial^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p + C_m \|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p,$$

$$I_{42} \leq C \epsilon (\|u\|_{L^\infty(\mathbb{R}^2)} + 1) \|\langle v \rangle^{\frac{pk}{2}} |\partial^\alpha f|^{\frac{p-2}{2}} |\nabla_v \partial^\alpha f|\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + C_m \|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p.$$

By (4.9) and the above estimates, we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \left( \sum_{|\alpha| \leq m} \|\partial^\alpha \omega\|_{L^p(\mathbb{R}^2)}^p + \sum_{|\alpha| \leq m+1} \|\langle v \rangle^k \partial^\alpha f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p \right) \\ & + C_p \left( \sum_{|\alpha|=m+1} \|\langle v \rangle^{\frac{kp}{2}} |\partial^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial^\alpha f\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + \sum_{|\alpha|=m} \|\partial^\alpha \omega\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2}} \nabla \partial^\alpha \omega\|_{L^2(\mathbb{R}^2)}^2 \right) \\ & \leq C_m (\|\omega\|_{W^{m,p}(\mathbb{R}^2)}^p + \|\langle v \rangle^k f\|_{W_x^{m+1,p} L_v^p(\mathbb{R}^2 \times \mathbb{R}^2)}^p). \end{aligned}$$

Using Gronwall’s inequality and the induction hypothesis, we conclude the lemma.  $\square$

Next we present the proof of Theorem 4, which is the uniqueness theorem for two dimensional Navier–Stokes–Vlasov–Fokker–Planck system.

**Proof of Theorem 4.** We set  $\tilde{u} = u_1 - u_2, \tilde{p} = p_1 - p_2,$  and  $\tilde{f} = f_1 - f_2.$  Then  $\tilde{u}, \tilde{p},$  and  $\tilde{f}$  solve the following equations:

$$\partial_t \tilde{u} + (u_1 \cdot \nabla) \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \int_{\mathbb{R}^2} v \tilde{f} dv - \int_{\mathbb{R}^2} \tilde{u} f_1 dv - \int_{\mathbb{R}^2} u_2 \tilde{f} dv - (\tilde{u} \cdot \nabla) u_2, \quad \text{div } \tilde{u} = 0, \tag{4.10}$$

$$\partial_t \tilde{f} + (v \cdot \nabla_x) \tilde{f} - \Delta_v \tilde{f} + \nabla_v \cdot (\tilde{u} \tilde{f}) + \nabla_v \cdot (u_2 \tilde{f}) + \nabla_v \cdot (\tilde{u} f_2) - \nabla_v \cdot (v \tilde{f}) = 0. \tag{4.11}$$

Let  $k > 2.$  Multiplying (4.10) by  $\tilde{u}$  and then integrating in spatial variables, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \int_{\mathbb{R}^2} |v| \|\tilde{f}\|_{L^2(\mathbb{R}^2)} \|\tilde{u}\|_{L^2(\mathbb{R}^2)} dv + \int_{\mathbb{R}^2} \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\tilde{f}\|_{L^2(\mathbb{R}^2)} \|\tilde{u}\|_{L^p(\mathbb{R}^2)} dv \\ & \quad + C \|\tilde{u}\|_{L^p(\mathbb{R}^2)} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \\ & \leq C \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \left( \int_{\mathbb{R}^2} \frac{\|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2}{\langle v \rangle^{2(k-1)}} dv \right)^{\frac{1}{2}} \\ & \quad + C \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p}} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{p}} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \\ & \quad + C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^p + \epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \end{aligned}$$

$$\begin{aligned} &\leq C \|\tilde{u}\|_{L^2(\mathbb{R}^2)} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} + C \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^{\frac{2p}{p+2}} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{4}{p+2}} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2p}{p+2}} \\ &\quad + C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \|u_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^p + 2\epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq C (\|u_2\|_{L^q(\mathbb{R}^2)}^p + 1) \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + C \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + 2\epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

where Hölder's inequality, Sobolev's inequality, and Young's inequality are used. Here we also used that  $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tilde{u}|^2 f_1 \, dv \, dx$  is nonnegative and  $\|\tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} < C$ . On the other hand, multiplying (4.11) by  $\langle v \rangle^{2k} \tilde{f}$  and integrating in  $v$  and  $x$  variables, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\ &\leq C \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (\tilde{u} \tilde{f}) \langle v \rangle^{2k} \tilde{f} \, dx \, dv \right| \\ &\quad + \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (u_2 \tilde{f}) \langle v \rangle^{2k} \tilde{f} \, dx \, dv \right| + \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (\tilde{u} f_2) \langle v \rangle^{2k} \tilde{f} \, dx \, dv \right|. \end{aligned}$$

The second and third terms in the right side can be estimated as in  $J_{11}$  in Proposition 3. Indeed, for the second one, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (\tilde{u} \tilde{f}) \langle v \rangle^{2k} \tilde{f} \, dx \, dv \right| &\leq \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{u} \langle v \rangle^{2k-1} |\tilde{f}|^2 \, dx \, dv \right| \\ &\leq C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{k}} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{k}} \|\tilde{f}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{1}{k}} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2k-1}{k}} \\ &\leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{k}} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2k-1}{k}}, \end{aligned}$$

where we used that  $\|\tilde{u}\|_{L^2(\mathbb{R}^2)}$  and  $\|\tilde{f}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}$  are uniformly bounded. Similarly, the third term is estimated as follows:

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (u_2 \tilde{f}) \langle v \rangle^{2k} \tilde{f} \, dx \, dv \right| \leq C \|\nabla u_2\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{k}} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2k-1}{k}}.$$

It remains to estimate the last term. Due to integration by parts, we have

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \cdot (\tilde{u} f_2) \langle v \rangle^{2k} \tilde{f} \right| \leq \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{u} f_2 \nabla_v \langle v \rangle^{2k} \tilde{f} \right| + \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{u} f_2 \langle v \rangle^{2k} \nabla_v \tilde{f} \right| := I + II.$$

Consider the first term  $I$  and, due to the Hölder inequality, the Young inequality, we estimate  $I$  as follows:

$$I \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tilde{u} f_2 \langle v \rangle^{2k-1} \tilde{f}| \leq C \int_{\mathbb{R}^2} \|\tilde{u}\|_{L^p(\mathbb{R}^2)} \|\langle v \rangle^{k-1} f_2\|_{L^q(\mathbb{R}^2)} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2)} \, dv$$

$$\begin{aligned} &\leq C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p}} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \int_{\mathbb{R}^2} \|\langle v \rangle^{k-1} f_2\|_{L^q(\mathbb{R}^2)} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2)} dv \\ &\leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \|\langle v \rangle^{k+\alpha} f_2\|_{L^q(\mathbb{R}^2 \times \mathbb{R}^2)} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \\ &\leq \epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + C \|\langle v \rangle^{k+\alpha} f_2\|_{L^q(\mathbb{R}^2 \times \mathbb{R}^2)}^{q'} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{q'}, \end{aligned}$$

where  $q'$  is Hölder conjugate of  $q$ , i.e.  $q' = \frac{q}{q-1}$  and uniform bound of  $\|\tilde{u}\|_{L^2}$  is used. Using again the Hölder inequality, the Young inequality, and the interpolation inequality, we estimate  $II$  as follows:

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{u} f_2 \langle v \rangle^{2k} \nabla_v \tilde{f} dx dv \right| \\ &\leq C \|\tilde{u}\|_{L^p(\mathbb{R}^2)} \int_{\mathbb{R}^2} \|\langle v \rangle^k f_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2)} dv \\ &\leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{p}} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p}} \int_{\mathbb{R}^2} \|\langle v \rangle^k f_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2)} dv \\ &\leq \epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{4}{p-2}} \left( \int_{\mathbb{R}^2} \|\langle v \rangle^k f_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2 dv \right)^{\frac{p}{p-2}} \left( \int_{\mathbb{R}^2} \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2)}^2 dv \right)^{\frac{p}{p-2}} \\ &\leq \epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + \epsilon \left( \int_{\mathbb{R}^2} \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2)}^2 dv \right) + C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \left( \int_{\mathbb{R}^2} \|\langle v \rangle^k f_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2 dv \right)^{\frac{p}{2}} \\ &\leq \epsilon \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + \epsilon \left( \int_{\mathbb{R}^2} \|\langle v \rangle^k \nabla_v \tilde{f}\|_{L^2(\mathbb{R}^2)}^2 dv \right) + C \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \left( \int_{\mathbb{R}^2} \|\langle v \rangle^{k+\alpha} f_2\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2 dv \right)^{\frac{p-2}{2}}. \end{aligned}$$

Here  $\alpha$  is any number with  $\alpha p > 2$ . Combining estimates for  $\tilde{u}$  and  $\tilde{f}$ , we obtain

$$\begin{aligned} &\frac{d}{dt} (\|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2) \\ &\leq C(1 + \|u_2\|_{L^q(\mathbb{R}^2)}^p + \|\langle v \rangle^{k+\alpha} f_2\|_{L^q(\mathbb{R}^2 \times \mathbb{R}^2)}^q) \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + C \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 + C(\|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{k}} + \|\nabla u_2\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{k}}) \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{2k-1}{k}} \\ &\quad + C \|\langle v \rangle^{k+\alpha} f_2\|_{L^q(\mathbb{R}^2 \times \mathbb{R}^2)}^{q'} \|\langle v \rangle^k \tilde{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^{q'}. \end{aligned}$$

The Gronwall type inequality implies that  $\tilde{u} = 0$  and  $\tilde{f} = 0$ . This completes the proof.  $\square$

**Remark 5.** As mentioned in Remark 1, if the initial data  $(f_0, u_0)$  satisfy (1.2) and furthermore,  $\langle v \rangle^{k+\alpha} f_0 \in L^q(\mathbb{R}^2 \times \mathbb{R}^2)$  with  $\alpha p > 2$ ,  $k > 2$ , and  $\frac{2}{p} + \frac{2}{q} = 1$ , then there exists a weak solution  $(f, u)$  satisfying the condition of Theorem 1 and  $\langle v \rangle^{k+\alpha} f \in L^\infty(0, T; L^q(\mathbb{R}^2 \times \mathbb{R}^2))$  from Proposition 3. We then observe that the condition of Theorem 4 is also satisfied, and therefore weak solution  $(f, u)$  becomes unique.

As an application of the uniqueness result of Theorem 4, we shall provide a strong solution for Navier–Stokes–Vlasov–Fokker–Planck system defined on a half-space  $\mathbb{R}_+^2 \times \mathbb{R}^2$ , where  $\mathbb{R}_+^2 = \{(x_1, x_2) \mid x_1 > 0\}$  in case of the slip boundary condition for  $u$  and the specular reflection for  $f$  on the boundary  $\{x_1 = 0\}$ . Namely, we consider the following system: For  $(t, x, v) \in (0, T) \times \mathbb{R}_+^2 \times \mathbb{R}^2$

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p - \int_{\mathbb{R}^2} (v - u) f \, dv = 0, & \operatorname{div} u = 0, \\ \partial_t f + (v \cdot \nabla_x) f + \nabla_v \cdot ((u - v) f - \nabla_v f) = 0, \end{cases} \tag{4.12}$$

where boundary conditions are

$$f(t, 0, x_2, v_1, v_2) = f(t, 0, x_2, -v_1, v_2), \quad u^1(t, 0, x_2) = \partial_{x_1} u^2(t, 0, x_2) = 0. \tag{4.13}$$

The compatibility conditions for the initial data  $f_0, u_0$  are

$$f_0(0, x_2, v_1, v_2) = f_0(0, x_2, -v_1, v_2), \quad u_0^1(0, x_2) = \partial_{x_1} u_0^2(0, x_2) = 0. \tag{4.14}$$

The following theorem is the uniqueness result of weak solutions for the system (4.12)–(4.14).

**Proposition 4.** *Let  $(f, u)$  be a weak solution of the Navier–Stokes–Vlasov–Fokker–Planck equations (4.12) with boundary conditions (4.13). Assume that initial data  $(f_0, u_0)$  satisfy the hypothesis of Theorem 2 and the compatibility condition (4.14) in the half-space. Then  $(f, u)$  becomes unique and strong to the system (4.12)–(4.14) and furthermore satisfies (1.4) in the half-space.*

**Proof.** For a given initial data  $(f_0, u_0)$  for (4.12), let us define  $(\bar{f}_0, \bar{u}_0)$  the extension of  $(f_0, u_0)$  across  $\{x_1 = 0\}$  as

$$\begin{aligned} \bar{f}_0(x, v) &= f_0(-x_1, x_2, -v_1, v_2) \quad \text{if } x_1 < 0, \\ \bar{u}_0^1(x) &= -u_0^1(-x_1, x_2), \quad \bar{u}_0^2(x) = u_0^2(-x_1, x_2) \quad \text{if } x_1 < 0. \end{aligned}$$

We note that  $(\bar{f}_0, \bar{u}_0)$  satisfies the initial hypothesis of Theorem 2, hence there exists a strong solution  $(\bar{f}, \bar{u}, \bar{p})$  satisfying (1.4) in the whole space.

On the other hand, we extend a weak solution  $(f, u)$  to a whole space in the following manner: We define  $(\bar{f}, \bar{u})$  in  $(0, T) \times \mathbb{R}^2 \times \mathbb{R}^2$  by

$$\begin{aligned} (\bar{f}, \bar{u}) &= (f, u) \quad \text{if } x_1 \geq 0, \\ \bar{f}(t, x, v) &= f(t, -x_1, x_2, -v_1, v_2) \quad \text{if } x_1 < 0, \\ \bar{u}^1(t, x) &= -u^1(t, -x_1, x_2), \quad \bar{u}^2(t, x) = u^2(t, -x_1, x_2) \quad \text{if } x_1 < 0. \end{aligned} \tag{4.15}$$

To extend the pressure function, we set  $\bar{g}(t, x) = -\operatorname{div}(\bar{u} \otimes \bar{u})(t, x)$ . Let  $\bar{j} = \int_{\mathbb{R}^2} v \bar{f} \, dv$  and  $\bar{n} = \int_{\mathbb{R}^2} \bar{f} \, dv$ . We then consider the scalar function  $q(t, x)$  in  $(0, T) \times \mathbb{R}^2$  satisfying the equation  $\Delta q = \operatorname{div}(\bar{j} - \bar{n}\bar{u}) + \operatorname{div} \bar{g}$ . Especially we fix  $q$  as an integral representation

$$q(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| (\operatorname{div}(\bar{j} - \bar{n}\bar{u})(t, y) + \operatorname{div} \bar{g}(t, y)) \, dy.$$

Then one can see that  $(\bar{f}, \bar{u}, q)$  satisfies the Navier–Stokes–Vlasov–Fokker–Planck equations in distribution sense in  $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$ , namely  $(\bar{f}, \bar{u}, q)$  is a weak solution of the following system in  $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$ :

$$\begin{cases} \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} - \Delta \bar{u} + \nabla q - \int_{\mathbb{R}^2} (v - \bar{u}) \bar{f} \, dv = 0, & \operatorname{div} u = 0, \\ \partial_t \bar{f} + (v \cdot \nabla_x) \bar{f} + \nabla_v \cdot ((\bar{u} - v) \bar{f} - \nabla_v \bar{f}) = 0. \end{cases} \tag{4.16}$$

Then, with the aid of the uniqueness result of Theorem 4, it is straightforward that  $(\bar{f}, \bar{u}) = (\tilde{f}, \tilde{u})$ . Therefore,  $(f, u)$  is a strong solution and satisfying the boundary condition (4.13) on  $\{x_1 = 0\}$ . Uniqueness of solutions in the half-space can be obtained in a similar manner as above and thus its details are omitted. This completes the proof.  $\square$

### 5. Strong solution for 3D Vlasov–Stokes system

In this section, we consider the three dimensional Vlasov–Stokes system and Vlasov–Fokker–Planck–Stokes system (1.7). As in the previous sections, we provide the a priori estimates using the known estimates of weak solutions. As mentioned in Remark 4, weak solutions  $(f, u)$  satisfy

$$u \in L^2(0, T; W^{2,3/2}(\mathbb{R}^3)), \quad f \in L^\infty(0, T; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)), \quad \text{and} \\ |v|^3 f \in L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$$

under (1.2) and  $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^3 f_0 \, dx \, dv < C$ . At first, we provide high moments estimate for three dimensional Vlasov–Stokes system and Vlasov–Fokker–Planck–Stokes system (1.7).

**Proposition 5.** *Suppose a pair  $(f, u)$  is a weak solution of Eq. (1.7). Furthermore, if the initial datum  $f_0$  satisfies*

$$\| \langle v \rangle^k f_0 \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} < \infty, \quad \text{for any } k > 0, \, p \geq 2,$$

where  $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$ , then  $f$  satisfies

$$\langle v \rangle^k f \in L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)).$$

**Proof.** We follow the proof of Proposition 3 line by line. In fact, we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \| \langle v \rangle^k f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + C_p \sigma \| \langle v \rangle^{\frac{pk}{2}} \nabla_v |f|^{\frac{p}{2}} \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\ &= -\frac{1}{p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u \cdot \nabla_v) f^p \langle v \rangle^{kp} \, dv \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_v \cdot (vf) \langle v \rangle^{kp} f^{p-1} \, dx \, dv \\ &+ \frac{1}{p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^p \nabla_v^2 \langle v \rangle^{kp} \, dv \, dx := \tilde{J}_{11} + \tilde{J}_{12} + \tilde{J}_{13}. \end{aligned} \tag{5.1}$$

Similarly with the proof of Proposition 3 by choosing  $\epsilon \in (0, \frac{kp}{kp+3})$ , we have

$$\tilde{J}_{12}, \tilde{J}_{13} \leq C \| \langle v \rangle^k f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p,$$

and

$$\begin{aligned} \tilde{J}_{11} &\leq C \int_{\mathbb{R}^3} |u| \left( \int_{\mathbb{R}^3} \langle v \rangle^{kp} f^p \frac{kp}{kp-\epsilon} dv \right)^{\frac{kp-\epsilon}{kp}} \left( \int_{\mathbb{R}^3} \frac{1}{\langle v \rangle^{kp \cdot \frac{(1-\epsilon)}{\epsilon}}} dv \right)^{\frac{\epsilon}{kp}} dx \\ &\leq C \|u\|_{L^{\frac{kp}{\epsilon}}(\mathbb{R}^3)} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{kp-\epsilon}{k}} \|f\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{\epsilon}{k}} \\ &\leq C \|u\|_{L^2(\mathbb{R}^3)}^{\frac{2\epsilon}{kp}} \|\Delta u\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{1-2\epsilon}{kp}} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{kp-\epsilon}{k}}. \end{aligned}$$

In the last inequality, we used  $\int_{\mathbb{R}^3} \frac{1}{\langle v \rangle^{kp \cdot \frac{(1-\epsilon)}{\epsilon}}} dv < \infty$  due to the assumption  $\frac{kp(1-\epsilon)}{\epsilon} > 3$ . We also used the estimates  $f \in L^\infty(0, T; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$  and  $u \in L^2(0, T; W^{2,3/2}(\mathbb{R}^3))$ . By using the Gronwall inequality, we have the conclusion.  $\square$

Next, using Proposition 5 and Giga–Sohr’s estimate for Stokes system [10], we provide the proof of Theorem 5.

**Proof of Theorem 5.** Since we have

$$\int_{\mathbb{R}^3} v f dv \leq \left( \int_{\mathbb{R}^3} \langle v \rangle^{kp} f^p dv \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^3} \frac{1}{\langle v \rangle^{\frac{p(k-1)}{p-1}}} dv \right)^{\frac{p-1}{p}} \leq C \left( \int_{\mathbb{R}^3} \langle v \rangle^{kp} f^p dv \right)^{\frac{1}{p}}, \tag{5.2}$$

and

$$\left\| u \int_{\mathbb{R}^3} f dv \right\|_{L^q(\mathbb{R}^3)} \leq \|u\|_{L^{\frac{pq}{p-q}}(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3} f dv \right\|_{L^p(\mathbb{R}^3)} \leq \|u\|_{L^{\frac{pq}{p-q}}(\mathbb{R}^3)} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}, \tag{5.3}$$

for  $p > q$  and  $k > 4 - \frac{3}{p}$ , we conclude from Proposition 5 ( $\epsilon < 3/4$ ) that

$$\int_{\mathbb{R}^3} v f dv \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad u \int_{\mathbb{R}^3} f dv \in L^\infty(0, T; L^q(\mathbb{R}^3))$$

for any  $p > q$ . From Giga and Sohr’s classical results [10] on the Stokes system, we find

$$\begin{aligned} &\|u\|_{L^r(0, T; W^{2,q}(\mathbb{R}^3))} + \|\partial_t u\|_{L^r(0, T; L^q(\mathbb{R}^3))} + \|\nabla p\|_{L^r(0, T; L^q(\mathbb{R}^3))} \\ &\leq C (\|nu\|_{L^\infty(0, T; L^q(\mathbb{R}^3))}, \|j\|_{L^\infty(0, T; L^p(\mathbb{R}^3))}, \|u_0\|_{W^{1,q}(\mathbb{R}^3)}), \end{aligned} \tag{5.4}$$

where  $r \in (1, \infty)$ ,  $j = \int_{\mathbb{R}^3} v f dv$  and  $n = \int_{\mathbb{R}^3} f dv$ . Hence we have

$$\begin{aligned} \|u\|_{L^r(0, T; L^\infty(\mathbb{R}^3))} &\leq C \|u\|_{L^r(0, T; W^{2,q}(\mathbb{R}^3))} \leq C \quad \text{for } p > q > 2, r \in (1, \infty), \\ \|\nabla u\|_{L^r(0, T; L^\infty(\mathbb{R}^3))} &\leq C \|u\|_{L^r(0, T; W^{2,q}(\mathbb{R}^3))} \leq C \quad \text{for } p > q > 3, r \in (1, \infty). \end{aligned} \tag{5.5}$$

Next, we provide a priori estimates for  $\nabla_x f$ . Similarly with (3.5), we have



$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \sigma C_p \|\langle v \rangle^{\frac{kp}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\
 &= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_v \cdot (\nabla_x(uf)) \langle v \rangle^{kp} |\nabla_x f|^{p-2} \nabla_x f \, dv \, dx \\
 & \quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_v \cdot (v \nabla_x f) \langle v \rangle^{kp} |\nabla_x f|^{p-2} \nabla_x f \, dv \, dx - \frac{1}{p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_v (|\nabla_x f|^p) \nabla_v \langle v \rangle^{kp} \, dv \, dx \\
 & := I_{31} + I_{32} + I_{33}.
 \end{aligned} \tag{5.6}$$

We decompose  $I_{31}$  into two terms as follows

$$\begin{aligned}
 I_{31} &= -\frac{1}{p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} (u \cdot \nabla_v) (|\nabla_x f|^p) \langle v \rangle^{kp} \, dv \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla_x u \cdot \nabla_v) f |\nabla_x f|^{p-2} \nabla_x f \langle v \rangle^{kp} \, dv \, dx \\
 & := \tilde{I}_{31} + \tilde{I}_{32}.
 \end{aligned}$$

Here  $\tilde{I}_{31}$  and  $\tilde{I}_{32}$  can be estimated as follows:

$$\tilde{I}_{31} \leq C \|u\|_{L^\infty(\mathbb{R}^3)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p,$$

and

$$\tilde{I}_{32} \leq C \|\nabla u\|_{L^\infty(\mathbb{R}^3)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^{p-1} \|\langle v \rangle^k \nabla_v f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}.$$

We also have

$$I_{32}, I_{33} \leq C \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p.$$

From (5.6), we have the inequality

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \sigma C_p \|\langle v \rangle^{\frac{kp}{2}} \nabla_v |\nabla_x f|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\
 & \leq C (\|u\|_{L^\infty(\mathbb{R}^3)} + 1) \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p \\
 & \quad + C \|\nabla u\|_{L^\infty(\mathbb{R}^3)} \|\langle v \rangle^k \nabla_x f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^{p-1} \|\langle v \rangle^k \nabla_v f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}.
 \end{aligned} \tag{5.7}$$

We also consider the estimates for  $\nabla_v f$ . In the same way to obtain (4.4), (4.6) for  $\alpha = 0$ , we have

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \|\langle v \rangle^k \nabla_v f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \sigma C_p \|\langle v \rangle^{\frac{kp}{2}} |\nabla_v f|^{\frac{p-1}{2}} \nabla_v^2 f\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\
 & \leq C \|u\|_{L^\infty(\mathbb{R}^3)} \|\langle v \rangle^k \nabla_v f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + C \|\langle v \rangle^k \nabla_v f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p.
 \end{aligned} \tag{5.8}$$

Adding (5.7) and (5.8), we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} (\| \langle v \rangle^k \nabla_x f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \| \langle v \rangle^k \nabla_v f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p) \\ & + \sigma C_p (\| \langle v \rangle^{\frac{kp}{2}} \nabla_v | \nabla_x f |^{\frac{p}{2}} \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 + \| \langle v \rangle^{\frac{kp}{2}} | \nabla_v f |^{\frac{p-1}{2}} \nabla_v^2 f \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2) \\ & \leq C (\| u \|_{L^\infty(\mathbb{R}^3)} + \| \nabla u \|_{L^\infty(\mathbb{R}^3)} + 1) (\| \langle v \rangle^k \nabla_x f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \| \langle v \rangle^k \nabla_v f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p). \end{aligned}$$

From (5.5) for  $q > 3$  and Gronwall’s inequality, we have

$$\begin{aligned} & \| \langle v \rangle^k \nabla_x f \|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))}^p + \| \langle v \rangle^k \nabla_v f \|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))}^p \\ & + \sigma C \| \langle v \rangle^{\frac{kp}{2}} \nabla_v | \nabla_x f |^{\frac{p}{2}} \|_{L^2(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3))}^2 + \| \langle v \rangle^{\frac{kp}{2}} | \nabla_v f |^{\frac{p-1}{2}} \nabla_v^2 f \|_{L^2(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3))}^2 \\ & \leq C (\| \langle v \rangle^k \nabla_x f_0 \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p + \| \langle v \rangle^k \nabla_v f_0 \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}^p) \\ & \times \exp(T + \| u \|_{L^1(0, T; L^\infty(\mathbb{R}^3))} + \| \nabla u \|_{L^1(0, T; L^\infty(\mathbb{R}^3))}) < \infty. \end{aligned} \tag{5.9}$$

Notice that the estimate (5.9) is uniform with respect to  $\sigma \geq 0$ . This completes the proof.  $\square$

**Remark 6.** From the regularity theory of the Stokes system and linear property of the Vlasov system, we find that if the initial data  $(u_0, f_0)$  to (1.7) satisfy

$$\langle v \rangle^k \partial_x^\alpha f_0, \langle v \rangle^k \nabla_v \partial_x^\alpha f_0 \in L^p(\mathbb{R}^3 \times \mathbb{R}^3), \quad \partial_x^\alpha u_0 \in L^p(\mathbb{R}^3),$$

for any  $\alpha$  satisfying  $|\alpha| \leq m$ ,  $p \in (3, \infty)$  and  $k > 4 - \frac{3}{p}$ , then there exist a solution pair  $(u, f)$  to (1.7) satisfying

$$u \in L^\infty(0, T; W^{m+2, q}(\mathbb{R}^3)) \cap H^1(0, T; W^m(\mathbb{R}^3)), \quad \langle v \rangle^k \partial_x^\alpha f \in L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)),$$

and  $\langle v \rangle^k \nabla_v \partial_x^\alpha f \in L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))$  for any  $|\alpha| \leq m$  and  $q < p$ .

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