Stronger Form of an $M$-Part Sperner Theorem

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1. Introduction

Let $X$ be an $n$-element set and let $\mathcal{F}$ be a family of its different subsets. The well-known Sperner theorem [13] states that if the members $F_1, F_2$ satisfy $F_1 \not\subset F_2$ then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}. \quad (1)$$

Katona [7] and Kleitman [11] independently discovered a sharpening of this theorem: Take a partition $X_1 \cup X_2$ of $X$ and suppose that there is no pair $F_1, F_2 \in \mathcal{F}, F_1 \subset F_2$ such that $F_2 - F_1 \subset X_i$ for some $i = 1, 2$. Under this weaker condition the same inequality (1) can be proved. This statement is called the two-part Sperner theorem. Analogously, if the partition $X_1 \cup \cdots \cup X_M = X$ is considered one may exclude the pairs $F_1, F_2 \in \mathcal{F}, F_1 \subset F_2$ such that $F_2 - F_1 \subset X_i$ for some $i \ (1 \leq i \leq M)$. Easy counterexamples show that this condition does not imply (1) even in the case $M = 3$. [10] and [6] give some additional conditions (for $M = 3$) ensuring this implication. The exact maximum of $|\mathcal{F}|$ under this general condition is unknown.

On the other hand Erdős [3] proved that if $\mathcal{F}$ does not contain $l + 1$ different members satisfying $F_1 \subset F_2 \subset \cdots \subset F_{l+1}$ then $|\mathcal{F}|$ does not exceed the sum of the $l$ largest binomial coefficients of order $n$. A natural combination of the above two conditions is the following one:

$\mathcal{F}$ does not contain $l + 1$ different members

$F_1, F_2, \ldots, F_{l+1}$ satisfying $F_1 \subset \cdots \subset F_{l+1}$ and

$F_{l+1} - F_1 \subset X_i$ for some $i \ (1 \leq i \leq M). \quad (2)$

Griggs [5] proved that condition (2) implies the inequality

$$|\mathcal{F}| \leq 2^{M-2l} \binom{n}{\lfloor n/2 \rfloor}. \quad (3)$$

The aim of this paper is to improve this estimate.

**Theorem 1.** If $X_1 \cup \cdots \cup X_M$ is a partition of the $n$-element set $X$ and $\mathcal{F}$ is a family of subsets of $X$ satisfying condition (2) then

$$|\mathcal{F}| \leq Ml \binom{n}{\lfloor n/2 \rfloor}.$$
a symmetric chain order (see [8] and [4]) if $\mathcal{P}$ has a partition $\mathcal{P} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_T$ where each $\mathcal{C}_i = \{a_1, \ldots, a_{n_i}\}$ is a symmetric chain, that is,
\[
\begin{align*}
r(a_i) &= r(a_{i-1}) + 1 = \cdots = r(a_1) + l_i - 1; \\
r(a_1) + r(a_{n_i}) &= n = \max_{a \in \mathcal{P}} r(a)
\end{align*}
\]
This terminology can be used to formulate the following generalization of Theorem 1:

**Theorem 2.** Let $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_M$ be the direct product of the symmetric chain orders $\mathcal{P}_1, \ldots, \mathcal{P}_M$. Suppose that $\mathcal{F} \subseteq \mathcal{P}$ contains no $l + 1$ elements $f_1 < \cdots < f_{l+1}$ such that $f_i$ and $f_{i+1}$ are equal in $M - 1$ components. Then
\[
|\mathcal{F}| \leq M!w(\mathcal{P})
\]
where $w(\mathcal{P})$ is the number of elements of rank $\lfloor n/2 \rfloor$.

Griggs [5] also proved this general theorem with $2^{M-2}$ in place of $M$. The crucial point of Griggs' proof was a lemma. We are able to improve the statement of this lemma:

**Lemma.** Let $Q$ be the set of vectors $(x_1, \ldots, x_M)$ with integer components satisfying $0 \leq x_i \leq k_i$, where $k_i$ are given positive integers. Suppose that $F \subseteq Q$ contains no $l + 1$ different elements equal in $M - 1$ components. Then:
\[
|F| \leq M!w(Q)
\]
where $w(Q)$ is the number of elements with component-sum equal to $\lfloor \frac{1}{2} \sum_{i=1}^{M} k_i \rfloor$.

Theorem 2 (and therefore Theorem 1) follows from this lemma exactly as in Griggs' paper [5]. Therefore we will prove only the lemma (Section 2). In Section 3 we add some remarks concerning the Littlewood-Offord problem.

**2. Proof of the Lemma**

The set $Q$ can be endowed with an ordering in a natural way: $(x_1, \ldots, x_M) \leq (y_1, \ldots, y_M)$ iff $x_i \leq y_i$ for all $i$. It is known [1] that the poset obtained in such a way is a symmetric chain order. However, the decomposition given by Greene and Kleitman [4] will be analyzed to prove the lemma. First, a repeat is given of their construction.

To each element $x = (x_1, \ldots, x_M)$ of $Q$ is associated a sequence $p(x)$ of $\sum_{i=1}^{M} k_i$ parentheses, as follows:
\[
\begin{align*}
x_1 \text{ right parentheses}, & \quad \text{then } k_1 - x_1 \text{ left parentheses}, \quad \text{then} \\
x_2 \text{ right parentheses}, & \quad \text{then } k_2 - x_2 \text{ left parentheses}, \quad \text{then} \\
\vdots & \\
x_M \text{ right parentheses}, & \quad \text{then } k_M - x_M \text{ left parentheses.}
\end{align*}
\]
The set of parentheses listed in the $j$-th row will be called the $j$-th block. That is, the $j$-th block consists of the parentheses standing on the $\sum_{i=1}^{j-1} (k_i + 1)$st, $\ldots$, $\sum_{i=1}^{j} k_i$th spaces. A sub-sequence of parentheses is said to be monotonic if begins with some right parentheses followed by some left ones. Any block of $p(x)$ is monotonic.
$p(x)$ has a unique 'parenthesization' which is made in the following way: first the adjacent left–right pairs are closed, then those ones which are separated only by other closed pairs, and repeat this process until no further pairing is possible. Two sequences are said to have the same parenthesization if any parenthesis is paired with the similar one in both sequences, they can differ in the remaining unpaired ones. Notice that these unpaired parentheses will always form a monotonic sequence. On the other hand, if the system of unpaired parentheses can be changed for any monotonic sequence in $p(x)$, then the modified sequence $p^*(x)$ is equal to $p(y)$ for some $y \in Q$. This can be verified by showing that $p^*(x)$ is monotonic in each block. Suppose that, in contrast, there is a left parenthesis followed by a right one in a block of $p^*(x)$. They cannot both be unpaired, because the subsequence of the unpaired parentheses is monotonic by supposition. They cannot be both paired, because $p^*(x)$ and $p(x)$ are identical in the sequence of closed parentheses. If the left parenthesis is paired, the right one is not, then the right pair of the left one must precede the unpaired right one; that is, a pair of parentheses appears in the same block contradicting the construction of $p(x)$. If the right one is paired and the left one is not a contradiction is obtained in the same way.

Now one class $C_{fi}$ of the chain-decomposition is defined, as the set of elements of $Q$ having the same parenthesization. As outlined above, the subsequences of impaired parentheses in such a class look like

\[
(((
\cdots
((, \\
)\cdots
((, \\
))\cdots
))(, \\
))))\cdots
),
\]

in this order. The rank $r(x)$ of $x = (x_1, \ldots, x_M)$ in $Q$ is $\sum_{i=1}^{M} x_i$, that is, the number of right parentheses in $p(x)$. It may be concluded that the rank increases one by one in a chain. Moreover, let $a$ denote the number of closed pairs, and $b$ the number of unpaired parentheses. Then the minimum rank in the chain is $a$, the maximum rank is $a + b$. Their sum $2a + b = \sum_{i=1}^{M} k_i$ is maximum rank in $Q$. The chains are symmetric, indeed.

Now the chain-decomposition into symmetric chains is constructed. One only has to notice a simple property of it: any chain can be decomposed into $M$ parts (corresponding to the $M$ blocks) in such a way that the $f$-th part contains elements of the form $(x_1, \ldots, x'_f, \ldots, x_M)$, where only $x'_f$ is changing, the other ones are fixed. Therefore such a part contains at most $l$ elements of $F$, that is, no chain can contain more than $Ml$ elements of $F$. If we see that the number of chains in the decomposition is $w(Q)$, this completes the proof of the lemma. However, the above statement follows by the facts that (1) any element with component-sum $\frac{1}{2} \sum_{i=1}^{M} k_i$ is contained by exactly one chain and (2) any chain contains such an element by the symmetry of the chain. The proof is complete.

3. REMARKS

Theorem 1 can be applied for the well known Littlewood–Offord problem [12]:

Suppose $a_1, \ldots, a_n$ are vectors of length at least one in $m$-dimensional real space, not necessarily distinct. Suppose $S$ is an $m$-dimensional open sphere of diameter $d$. Let
$f_m(n, d)$ denote the maximum, over all choices of the $a_i - S$ and $S$, of the number of sums $\sum_{i=1}^{\ell} \epsilon_i a_i$ which lie in $S$, where each $\epsilon_i$ is 0 or 1.

With this denotation the first result in this problem was discovered by Erdös [3] applying the Sperner theorem:

$$f_1(n, 1) = \binom{n}{\lfloor n/2 \rfloor}.$$  

Then Katona and Kleitman independently proved the two-part Sperner theorem, which implied that $f_2(n, 1) = \binom{n}{\lfloor n/2 \rfloor}$. After this, Kleitman devised an elegant construction to establish that the same bound holds, regardless of dimension.

On the other hand it is easily seen that $f_1(n, d) =$ the sum of $\lfloor d \rfloor$ largest binomial coefficients in $n$, for $d > 1$. In more than one dimension Katona and Kleitman have obtained several results for $1 < d \leq 5^{1/2}$. For the complete references see [5].

In general, Griggs [5] proved that

$$f_m(n, d) \leq 2^{m-1-2} \binom{dm^{1/2}}{\lfloor n/2 \rfloor}.$$  

The stronger $M$-part theorem implies by Griggs' method:

$$f_m(n, d) \leq 2^{m-1} \binom{dm^{1/2}}{\lfloor n/2 \rfloor}.$$  

However, a general result of Enger ([2], Corollary 3.1.1) implies the stronger estimate

$$f_m(n, d) \leq O \left( d \binom{n}{\lfloor n/2 \rfloor} \right).$$  

I am indebted to G. Halász for calling my attention to this result. Unfortunately, it shows that analytic methods have worked better until now than combinatorial ones.

On the other hand, for $l = 1$ there is a lower bound in Theorem 2:

$$\max |\mathcal{F}| \geq O(M^{1/2}) w(\mathcal{P}).$$

When $P, s$ are two element chains and $l = 1$ then the maximal size of $\mathcal{F}$ satisfying the conditions of Theorem 2 is $O(M^{1/2}) w(\mathcal{P})$. This can be verified by showing this is the same problem which is considered by Katona [9]: Let $\mathcal{G}$ be a family of subsets of an $M$-element set and there is no pair $G_1, G_2 \in \mathcal{G}, G_1 \subseteq G_2$ such that $|G_2 \setminus G_1| = 1$. Then maximal size of $\mathcal{G}$ is $2^{M-1}$, equal to maximal size of $\mathcal{F}$ and $w(\mathcal{P}) = \binom{M}{\lfloor M/2 \rfloor}$.

REFERENCES


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