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Analytical blowup solutions to the 2-dimensional isothermal Euler–Poisson equations of gaseous stars

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Abstract

We study the Euler–Poisson equations of describing the evolution of the gaseous star in astrophysics. Firstly, we construct a family of analytical blowup solutions for the isothermal case in R^2 . Furthermore the blowup rate of the above solutions is also studied and some remarks about the applicability of such solutions to the Navier–Stokes–Poisson equations and the drift-diffusion model in semiconductors are included. Finally, for the isothermal case ($\gamma = 1$), the result of Makino and Perthame for the tame solutions is extended to show that the life span of such solutions must be finite if the initial data is with compact support.

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1. Introduction

The evolution of a self-gravitating fluid such as gaseous stars can be formulated by the Euler–Poisson equation of the following form:

$$\begin{aligned}
 \rho_t + \nabla \cdot (\rho u) &= 0, \\
 (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P &= -\rho \nabla \Phi, \\
 S_t + u \cdot \nabla S &= 0, \\
 \Delta \Phi(t, x) &= \alpha(N) g \rho,
 \end{aligned} \tag{1}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N : $\alpha(1) = 2$; $\alpha(2) = 2\pi$. For simplicity, we take the constant term $g = 1$. For $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2) \frac{\pi^{N/2}}{\Gamma(N/2+1)},$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is a Gamma function. As usual, $\rho = \rho(t, x)$, $u = u(t, x) \in \mathbf{R}^N$ and $S(t, x)$ are the density, the velocity and the entropy respectively. $P = P(\rho)$ is the pressure. In the above system,

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the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density ρ through the Poisson equation. Equations (1)₁ and (1)₂ are the compressible Euler equation with forcing term. Equation (1)₃ is the Poisson equation through which the gravitational potential is determined by the density distribution of the gas itself. Thus, we called the system (1) the Euler–Poisson equations. Here, the viscosity term does not appear, that is, the viscous effect is neglected. In this case, the equations can be viewed as a perfect gas model. For $N = 3$, (1) is a classical (non-relativistic) description of a galaxy, in astrophysics. See [2,9] for a detail about the system.

If we take $S(t, x) = \ln K$, for some fixed $K > 0$, we have a γ -law on the pressure $P(\rho)$, i.e.

$$P(\rho) = K\rho^\gamma \doteq \frac{\rho^\gamma}{\gamma}, \tag{2}$$

which is a commonly the hypothesis. The constant $\gamma = c_p/c_v \geq 1$, where c_p, c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma = 1$. It can be used for constructing models with non-degenerate isothermal cores, which have a role in connection with the so-called Schonberg–Chandrasekhar limit [8].

For the physical dimension $N = 3$, we are interested in the hydrostatic equilibrium specified by $u = 0, S = \ln K$. According to [2], the ratio between the core density $\rho(0)$ and the mean density $\bar{\rho}$ for $6/5 < \gamma < 2$ is given by

$$\frac{\bar{\rho}}{\rho(0)} = \left(\frac{-3}{z} \dot{y}(z) \right)_{z=z_0}$$

where y is the solution of the Lane–Emden equation with $n = 1/(\gamma - 1)$,

$$\ddot{y} + \frac{2}{z}\dot{y} + y^n = 0, \quad y(0) = \alpha > 0, \quad \dot{y}(0) = 0, \quad n = \frac{1}{\gamma - 1},$$

and z_0 is the first zero of $y(z_0) = 0$. We can solve the Lane–Emden equation analytically for

$$y_{\text{anal}}(z) \doteq \begin{cases} 1 - \frac{1}{6}z^2, & n = 0, \\ \frac{\sin z}{z}, & n = 1, \\ \frac{1}{\sqrt{1+z^2/3}}, & n = 5, \end{cases}$$

and for the other values, only numerical values can be obtained. It can be shown that for $n < 5$, the radius of polytropic models is finite; for $n \geq 5$, the radius is infinite.

Gamblin [6] and Bezarad [1] obtained the existence results about the explicitly stationary solution ($u = 0$) for $\gamma = 6/5$:

$$\rho = \left(\frac{3KA^2}{2\pi} \right)^{5/4} (1 + A^2r^2)^{-5/2}. \tag{3}$$

The Poisson equation (1)₃ can be solved as

$$\Phi(t, x) = \int_{\mathbb{R}^N} G(x - y)\rho(t, y) dy,$$

where G is the Green’s function for the Poisson equation in the N -dimensional spaces defined by

$$G(x) \doteq \begin{cases} |x|, & N = 1, \\ \log|x|, & N = 2, \\ \frac{-1}{|x|^{N-2}}, & N \geq 3. \end{cases}$$

In the following, we always seek solutions in spherical symmetry. Thus, the Poisson equation (1)₃ is transformed to

$$r^{N-1}\Phi_{rr}(t, x) + (N - 1)r^{N-2}\Phi_r = \alpha(N)\rho r^{N-1},$$

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r \rho(t, s)s^{N-1} ds.$$

Definition 1 (Blowup). We say a solution blows up if one of the following conditions is satisfied:

- (1) The solution becomes infinitely large at some point x and some finite time T ;
- (2) The derivative of the solution becomes infinitely large at some point x and some finite time T .

In this paper, we concern about blowup solutions for the N -dimensional isothermal Euler–Poisson equations, which may describe the phenomenon called the core collapsing in physics. And our aim is to construct a family of such blowup solutions to it.

Historically in astrophysics, Goldreich and Weber [7] constructed the analytical blowup solution (collapsing) solution of the 3-dimensional Euler–Poisson equation for $\gamma = 4/3$ for the non-rotating gas spheres. After that, Makino [9] obtained the rigorously mathematical proof of the existence of such kind of blowup solutions. And in [5], we find the extension of the above blowup solutions to the case $N \geq 3$ and $\gamma = (2N - 2)/N$. In [11], the solutions with a from is written as

$$\left\{ \begin{aligned} \rho(t, r) &= \begin{cases} \frac{1}{a(t)^N} y\left(\frac{r}{a(t)}\right)^{N/(N-2)}, & \text{for } r < a(t)Z_\mu, \\ 0, & \text{for } a(t)Z_\mu \leq r, \end{cases} & u(t, r) = \frac{\dot{a}(t)}{a(t)}r, & S(t, r) = \text{Ln } K, \\ \ddot{a}(t) &= -\frac{\lambda}{a(t)^{N-1}}, & a(0) = a_0 > 0, & \dot{a}(0) = a_1, \\ \ddot{y}(z) + \frac{N-1}{z}\dot{y}(z) + \frac{N(N-2)^2V(N)}{(2N-2)K}y(z)^{N/(N-2)} &= \mu, & y(0) = \alpha > 0, & \dot{y}(0) = 0, \end{aligned} \right. \tag{4}$$

where $\mu = [N(N - 2)\lambda]/(2N - 2)K$ and the finite Z_μ is the first zero of $y(z)$. And no other analytical blowup solution for the Euler–Poisson equation has been obtained. Through there are a lot of numerical simulation results on the Euler–Poisson equation.

In Section 2, we obtain the blowup solutions for the Euler–Poisson equation in spherical symmetry in the 2-dimensional case,

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u &= 0, \\ \rho(u_t + uu_r) + K\rho_r &= -\frac{2\pi\rho}{r} \int_0^r \rho(t, s)s \, ds, \end{aligned} \tag{5}$$

in the form of the following theorem.

Theorem 2. For the 2-dimensional Euler–Poisson equations in radial symmetry (5), there exists a family of solutions,

$$\left\{ \begin{aligned} \rho(t, r) &= \frac{1}{a(t)^2} e^{y(r/a(t))}, & u(t, r) &= \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) &= -\frac{\lambda}{a(t)}, & a(0) = a_0 > 0, & \dot{a}(0) = a_1, \\ \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{\alpha(N)}{K}e^{y(x)} &= \mu, & y(0) = \alpha, & \dot{y}(0) = 0, \end{aligned} \right. \tag{6}$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small λ and α are constants.

- (1) When $\lambda > 0$, the solutions blow up in a finite time T ;
- (2) When $\lambda = 0$, if $a_1 < 0$, the solutions blow up at $t = -a_0/a_1$.

The blowup rate of the solution (6) is discussed in Section 3. Actually, we have

Theorem 3. The blowup rate of the solution (6) is

$$\lim_{t \rightarrow T} \rho(t, 0)(T - t)^\eta \geq O(1),$$

with $\eta < 2$.

In the last section, we can rewrite (1) in scalar form,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \sum_{k=1}^N u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^N \frac{\partial u_k}{\partial x_k} &= 0, \\ \rho \left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial P}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_i} &= 0, \quad \text{for } i = 1, 2, \dots, N. \end{aligned} \tag{7}$$

The tame solution is introduced in here.

Definition 4. A solution (ρ, u) for (7) is called “tame” solution if

- (1) $(\rho, u) \in C^1([0, T] \times \mathbb{R}^N)$, $\rho \geq 0$, $\rho(t)$ has compact support, and
- (2) $\rho^{(\gamma-1)/2} \in C^1([0, T] \times \mathbb{R}^N)$, $u \in C([0, T]; B^0)$, and

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial \Phi}{\partial x_i} = 0, \quad \text{for } i = 1, 2, \dots, N, \tag{8}$$

holds on the exterior of the support of ρ .

If only the condition (1) is assumed, the solution is called “classical.” For $\gamma > 1$, under the transformation

$$w = \frac{2\sqrt{K\gamma}}{\gamma - 1} \rho^{(\gamma-1)/2},$$

Makino and Perthame [10] showed that the life span of tame solution of the Euler–Poisson equation in spherical symmetry is finite. As we are interested in non-global existence for the isothermal case ($\gamma = 1$) in 2 dimension, i.e.

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u &= 0, \\ \rho(u_t + uu_r) + K\rho_r &= -\frac{2\pi\rho}{r} \int_0^r \rho(t, s)s \, ds, \end{aligned} \tag{9}$$

the result of Makino and Perthame [10] for the tame solutions is extended as the following theorem.

Theorem 5. Let $(\rho(t), u(t))$ be a radially symmetric tame solution of (9) on $0 \leq t < T$. If the support of $(\rho(0), u(0))$ is compact and $\rho(0)$ is not identically equal to zero, then T must be finite.

2. Separable blowup solutions

In this section, before presenting the proof of Theorem 2, we prepare some lemmas. First, we obtain a general class of solutions for the continuity equation of mass in radial symmetry (5)₁.

Lemma 6. For the equation of conservation of mass in radial symmetry

$$\rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0, \tag{10}$$

there exist solutions,

$$\rho(t, r) = \frac{f(r/a(t))}{a(t)^2}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)}r, \tag{11}$$

with the form with $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

Proof. We just plug (11) into (10). Then

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u &= \frac{-2\dot{a}(t)f(r/a(t))}{a(t)^3} - \frac{\dot{a}(t)r\dot{f}(r/a(t))}{a(t)^4} + \frac{\dot{a}(t)r}{a(t)}\frac{\dot{f}(r/a(t))}{a(t)^3} + \frac{f(r/a(t))\dot{a}(t)}{a(t)^2} \frac{\dot{a}(t)}{a(t)} \\ &\quad + \frac{1}{r}\frac{f(r/a(t))}{a(t)^2}\frac{\dot{a}(t)}{a(t)}r \\ &= 0. \end{aligned}$$

The proof is completed. \square

Secondly, we obtain an estimate to the equation,

$$\ddot{a}(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1. \tag{12}$$

Lemma 7. For the Emden equation (12), we have:

- (1) If $\lambda > 0$, there exists a finite time $T_- < +\infty$ such that $a(T_-) = 0$;
- (2) If $\lambda = 0$, it holds that for any $t \geq 0$,

$$a(t) = a_0 + a_1t.$$

Proof. (1) By integrating (12), we have

$$0 \leq \frac{1}{2}\dot{a}(t)^2 = -\lambda \ln a(t) + \theta \tag{13}$$

where $\theta = \lambda \ln a_0 + \frac{1}{2}a_1^2$.

From (13), we get

$$a(t) \leq e^{\theta/\lambda}.$$

If the statement (1) is not true, we have

$$0 < a(t) \leq e^{\theta/\lambda}, \quad \text{for all } t \geq 0.$$

But since

$$\ddot{a}(t) = -\frac{\lambda}{a(t)} \leq \frac{-\lambda}{e^{\theta/\lambda}},$$

we integrate this twice to deduce

$$a(t) \leq \int_0^t \int_0^\tau \frac{-\lambda}{e^{\theta/\lambda}} ds d\tau + C_1t + C_0 = \frac{-\lambda t^2}{2e^{\theta/\lambda}} + C_1t + C_0.$$

By taking t large enough, we get

$$a(t) < 0.$$

As a contradiction is met, the statement (1) is true. It is trivial to verify (2). \square

Remark 8. The modified systems of the above Emden equation may be referred more detailedly in [2,5,11].

Lemma 9. There exists a sufficiently small $x_0 > 0$, such that the equation

$$\begin{cases} \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \sigma e^{y(x)} = \mu, \\ y(0) = \alpha, \quad \dot{y}(0) = 0, \end{cases} \tag{14}$$

where $\sigma > 0$, μ , and α are constants, has a solution $y = y(x, \mu) \in C^2[0, x_0]$.

Proof. The lemma can be proved by the fixed point theorem. Multiply (14) by x , to give

$$\frac{d}{dx}(x\dot{y}(x)) = x(\mu - \sigma e^{y(x)}).$$

Notice $\dot{y}(0) = 0$, we have

$$\dot{y}(x) = \frac{1}{x} \int_0^x s(\mu - \sigma e^{y(s)}) ds.$$

By using $y(0) = \alpha$, (14) is reduced to

$$\dot{y}(x) = \frac{1}{x} \int_0^x s(\mu - \sigma e^{y(s)}) ds, \quad y(0) = \alpha.$$

Set

$$f(x, y(x)) = \frac{1}{x} \int_0^x s(\mu - \sigma e^{y(s)}) ds.$$

Then for any $x_0 > 0$, we get $f \in C^1[0, x_0]$, and for any $y_1, y_2 \in C^2[0, x_0]$, we have

$$|f(x, y_1(x)) - f(x, y_2(x))| = \frac{\sigma |\int_0^x s(e^{y_2(s)} - e^{y_1(s)}) ds|}{x}.$$

As e^y is a C^1 function of y , we can show that the function e^y , is Lipschitz-continuous. And we get

$$|f(x, y_1(x)) - f(x, y_2(x))| = \frac{O(1) \int_0^x s |y_2(s) - y_1(s)| ds}{x} \leq O(1)x_0 \sup_{0 \leq s \leq x_0} |y_1(s) - y_2(s)|,$$

where $\tau \in [0, x] \subseteq [0, x_0]$. Let

$$Ty(x) = \alpha + \int_0^x f(s, y(s)) ds.$$

We have $Ty \in C[0, x_0]$ and

$$|Ty_1(x) - Ty_2(x)| = \left| \int_0^x f(s, y_1(s)) ds - \int_0^x f(s, y_2(s)) ds \right| \leq O(1)x_0 \sup_{0 \leq s \leq x_0} |y_1(s) - y_2(s)|.$$

By choosing $x_0 > 0$ to be a sufficiently small number, such that $O(1)x_0 < 1$, this shows that the mapping $T : C[0, X_0] \rightarrow C[0, x_0]$, is a contraction with the sup-norm. By the fixed point theorem, there exists a unique $y(x) \in C[0, x_0]$, such that $Ty(x) = y(x)$. The proof is completed. \square

Lemma 10. *The equation,*

$$\begin{cases} \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \sigma e^{y(x)} = 0, \\ y(0) = \alpha, \quad \dot{y}(0) = 0, \end{cases} \quad (15)$$

where $\sigma > 0$ and α are constants, has a solution in $[0, +\infty)$ and $\lim_{x \rightarrow +\infty} y(x) = -\infty$.

Proof. By integrating (15), we have

$$\dot{y}(x) = -\frac{\sigma}{x} \int_0^x s e^{y(s)} ds \leq 0. \quad (16)$$

Thus, for $0 < x < x_0$, $y(x)$ has a uniform lower upper bound

$$y(x) \leq y(0) = \alpha.$$

As we obtained the local existence in Lemma 9, there are two possibilities:

- (1) $y(x)$ only exists in some finite interval $[0, x_0]$:
 - (1a) $\lim_{x \rightarrow x_0^-} y(x) = -\infty$;
 - (1b) $y(x)$ has a uniformly lower bound, i.e. $y(x) \geq \alpha_0$ for some constant α_0 .
- (2) $y(x)$ exists in $[0, +\infty)$:
 - (2a) $\lim_{x \rightarrow +\infty} y(x) = -\infty$;
 - (2b) $y(x)$ has a uniformly lower bound, i.e. $y(x) \geq \alpha$ for some constant α_0 .

We claim that possibility (1) does not exist. We need to reject (1b) first: If the statement (1b) is true, (16) becomes

$$-\sigma x e^\alpha = -\frac{\sigma}{x} \int_0^x x e^\alpha ds \leq \dot{y}(x). \tag{17}$$

Thus, $\dot{y}(x)$ is bounded in $[0, x_0]$. Therefore, we can use the fixed point theorem again to obtain a large domain of existence, such that $[0, x_0 + \delta]$ for some positive number δ . There is a contradiction. Therefore, (1b) is rejected.

Next, we do not accept (1a) because of the following reason: It is impossible that $\lim_{x \rightarrow x_0^-} y(x) = -\infty$, as from (17), $\dot{y}(x)$ has a lower bound in $[0, x_0]$:

$$-\sigma x_0 e^\alpha \leq \dot{y}(x). \tag{18}$$

Thus, (18) becomes

$$y(x_0) = y(0) + \int_0^{x_0} \dot{y}(x) dx \geq \alpha - \int_0^{x_0} \sigma x_0 e^\alpha dx = \alpha - \sigma x_0^2 e^\alpha.$$

Since $y(x)$ is bounded below in $[0, x_0]$, it contradicts the statement (1a), such that $\lim_{x \rightarrow x_0^-} y(x) = -\infty$. So, we can exclude the possibility (1).

We claim that the possibility (2b) does not exist. It is because

$$\dot{y}(x) = -\frac{\sigma}{x} \int_0^x s e^{y(s)} ds \leq -\frac{\sigma}{x} \int_0^x e^{\alpha_0} s ds = -\frac{\sigma e^{\alpha_0} x}{2}.$$

Then, we have

$$y(x) \leq \alpha - \frac{\sigma e^{\alpha_0}}{4} x^2. \tag{19}$$

By letting $x \rightarrow \infty$, (19) turns out to be

$$y(x) = -\infty.$$

Since a contradiction is established, we exclude the possibility (2b). Thus, Eq. (15) exists in $[0, +\infty)$ and $\lim_{x \rightarrow +\infty} y(x) = -\infty$. This completes the proof. \square

Now transferring (14) to the first-order system,

$$\begin{cases} \frac{dy}{dx} = \dot{y}, \\ \frac{d\dot{y}}{dx} = -\frac{1}{x} \dot{y} - \sigma e^y + \mu, \end{cases}$$

we consider the system along the solution curve $y = y(x, 0)$, $\dot{y} = \dot{y}(x, 0)$, $0 < x_0 \leq x < +\infty$. Since the right-hand side of this system is continuously differentiable in x , y , \dot{y} and μ . We apply the comparison theorem (Theorem 7.4) in [3], to obtain the following lemma.

Lemma 11. For a sufficiently small ϵ such that $|\mu| < \epsilon$, the equation,

$$\begin{cases} \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \sigma e^{y(x)} = \mu, \\ y(0) = \alpha, \quad \dot{y}(0) = 0, \end{cases}$$

exists in $[0, \infty)$, and $\lim_{x \rightarrow +\infty} y(x) = -\infty$.

Now, we give the proof of Theorem 2.

Proof of Theorem 2. From Lemma 11, we easily get that (6) satisfy (5)₁. For the momentum equation (5)₂, we get

$$\begin{aligned} \rho(u_t + uu_r) + K\rho_r + \frac{2\pi\rho}{r} \int_0^r \rho(t, s)s ds &= \frac{\rho}{a(t)} \left[\ddot{a}(t)r + K\dot{y}\left(\frac{r}{a(t)}\right) + \frac{2\pi}{ra(t)} \int_0^r e^{y(s/a(t))} s ds \right] \\ &= \frac{\rho}{a(t)} \left[-\frac{\lambda r}{a(t)} + K\dot{y}\left(\frac{r}{a(t)}\right) + \frac{2\pi}{\left(\frac{r}{a(t)}\right)} \int_0^{r/a(t)} e^{y(s)} s ds \right] \\ &= \frac{\rho}{a(t)} Q\left(\frac{r}{a(t)}\right). \end{aligned}$$

Here, we use the property of $a(t)$:

$$\ddot{a}(t) = -\frac{\lambda}{a(t)},$$

and denote

$$Q\left(\frac{r}{a(t)}\right) = Q(x) = -\lambda x + K\dot{y}(x) + \frac{2\pi}{x} \int_0^x e^{y(s)} s ds.$$

Differentiate $Q(x)$ with respect to x ,

$$\begin{aligned} \dot{Q}(x) &= -\lambda + K\ddot{y}(x) + 2\pi e^{y(x)} - \frac{2\pi}{x} \int_0^x e^{y(s)} s ds \\ &= -\lambda + K\left(-\frac{1}{x}\dot{y}(x) + \mu\right) - \frac{2\pi}{x^2} \int_0^x e^{y(s)} s ds \\ &= -\frac{1}{x} \left(\lambda x + K\dot{y}(x) - K\mu x + \frac{2\pi}{x} \int_0^x e^{y(s)} s ds \right) \\ &= -\frac{1}{x} Q(x). \end{aligned}$$

The above result is due to the fact that we choose the following ordinary differential equation:

$$\begin{cases} \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{2\pi}{K}e^{y(x)} = \mu, \quad \mu = \frac{2\lambda}{K}, \\ y(0) = \alpha, \quad \dot{y}(0) = 0. \end{cases}$$

With $Q(0) = 0$, this implies that $Q(x) = 0$. By using Lemma 7 about $a(t)$ and Lemma 10 about $y(x)$, we have shown that the family of the solutions blows up in finite time T under the prescribed conditions of Theorem 2. This completes the proof. \square

Remark 12. Besides, the above solution requiring $y(0) > 0$ is not necessary for (6) in contrast to (4) in [5,9]. And the mass of the above solution is

$$M = \int_{\mathbb{R}^2} \rho(t, s) ds = \int_0^{+\infty} \frac{2\pi}{a(0)^2} e^{y(s/a(0))} s ds = 2\pi \int_0^{+\infty} e^{y(s)} s ds,$$

where $\eta(N)$ is the measure of a unit ball. Thus the mass of the solution (6) depends on the initial data $y(0)$. And it is not easy to determine if the mass is finite or infinite with different datum. It is different for the finite mass of the family of solutions (4) for $\gamma = 4/3$ and $N = 3$ which is independent of $y(0)$ in [4].

Remark 13. Now, we consider the stationary solutions of (1), in spherical symmetry, i.e.

$$P_r = -\rho\Phi_r. \tag{20}$$

For $\gamma = 1$, (20) becomes

$$\rho = e^{-\Phi/K+C},$$

where C is a constant.

For our convenience, $C = 0$ is chosen

$$\rho = e^{-\Phi/K}.$$

From the Poisson equation (1)₃, we obtain

$$\Delta\Phi = \alpha(N)e^{-\Phi/K}. \tag{21}$$

As the solution is in radially symmetric, (21) becomes

$$\Phi_{rr} + \frac{1}{r}\Phi_r - 2\pi e^{-\Phi/K} = 0.$$

Under the transformation

$$y(x) = -\frac{\Phi(r)}{K},$$

we have

$$\ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{2\pi}{K}e^{y(x)} = 0. \tag{22}$$

Furthermore by setting $\lambda = 0$ and $\dot{a}(0) = 0$ in Theorem 2, we easily have the following corollary for the non-trivial stationary solutions.

Corollary 14. For the 2-dimensional isothermal Euler–Poisson equation, there exists a family of stationary solution, i.e.

$$\begin{cases} \rho(t, r) = \frac{1}{a^2}e^{y(r/a)}; \\ \ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{2\pi}{K}e^{y(x)} = \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0, \end{cases} \tag{23}$$

where α is an arbitrary positive constant.

Remark 15. Our blowup solutions only work for the 2-dimensional case. But we do not know what will happen after the critical time that the solutions blow up.

If we consider the system of conservation laws with viscosity, i.e. with $\nu > 0$,

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P &= -\rho \nabla \Phi + \nu \Delta u, \\ \Delta \Phi(t, x) &= 2\pi \rho. \end{aligned}$$

We get the corresponding Navier–Stokes–Poisson equation. The equation describes the situation closer to the real model compared with the Euler–Poisson equations in gaseous stars. And our family of solutions is also suitable for it. This is due to the fact in spherical symmetry for the vector Laplacian in $u(t, r)$,

$$\Delta u = u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u.$$

Remark 16. In [11], we have the lemma to control the modified Emden equation with $\beta > 0$,

$$\ddot{a}(t) + \beta a(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1.$$

The similar blowup results may be obtained for the 2-dimensional Euler–Poisson equation with frictional damping or Navier–Stokes–Poisson equation with frictional damping or not, i.e.

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P &= -\rho \nabla \Phi - \beta \rho u + \nu \Delta u, \\ \Delta \Phi(t, x) &= 2\pi \rho, \end{aligned}$$

where $\beta > 0$ and $\nu \geq 0$. The corresponding family of blowup solution is

$$\begin{cases} \rho(t, r) = \frac{1}{a(t)^2} e^{y(r/a(t))}, & u(t, r) = \frac{\dot{a}(t)}{a(t)} r; \\ \ddot{a}(t) + \beta a(t) = -\frac{\lambda}{a(t)}, & a(0) = a_0 > 0, \quad \dot{a}(0) = a_1; \\ \ddot{y}(x) + \frac{1}{x} \dot{y}(x) + \frac{2\pi}{K} e^{y(x)} = \mu, & y(0) = \alpha, \quad \dot{y}(0) = 0, \end{cases} \tag{24}$$

where $\mu = 2\lambda/K$ and α are constants.

Remark 17. Besides, if we consider the drift-diffusion model in semiconductors,

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P &= +\rho \nabla \Phi - \beta \rho u + \nu \Delta u, \\ \Delta \Phi(t, x) &= 2\pi \rho, \end{aligned}$$

the special solutions with infinite mass may be obtained as follows

$$\begin{cases} \rho(t, r) = \frac{1}{a(t)^2} e^{y(r/a(t))}, & u(t, r) = \frac{\dot{a}(t)}{a(t)} r; \\ \ddot{a}(t) + \beta a(t) = -\frac{\lambda}{a(t)}, & a(0) = a_0 > 0, \quad \dot{a}(0) = a_1; \\ \ddot{y}(x) + \frac{1}{x} \dot{y}(x) - \frac{2\pi}{K} e^{y(x)} = \mu, & y(0) = \alpha, \quad \dot{y}(0) = 0. \end{cases} \tag{25}$$

3. Blowup rate

In this section, we present the confirmation of Theorem 3. The blowup rate of the constructed solutions (6), for the Euler–Poisson equation is studied. It is interesting to investigate how fast the blowup solution tends to infinity as the time tends to the critical value T .

Proof of Theorem 3. We choose a finite time t to make $a(t)$ sufficiently small enough, such that $-\lambda \ln a + \theta > 0$. From the Lemma 7, it is clear for that there exists a finite time $t_0 > 0$ such that $\dot{a}(t) < 0$ for $t > t_0$. Next, since

$$\ddot{a}(t) = -\frac{\lambda}{a(t)} < 0,$$

when $t > t_0 + \epsilon = t_1$, this means there exists a constant $\epsilon > 0$, such that

$$a(t) \leq e^{\theta/\lambda} - \delta,$$

where δ is a sufficiently small positive constant, we have a smaller upper bound of $a(t)$. On the other hand, as

$$\dot{a} = -\sqrt{-2\lambda \ln a + \theta},$$

it is easy to see, that

$$t = t_1 + \int_{t_1}^t ds = t_1 - \int_{a(t_1)}^{a(t)} \frac{da}{\sqrt{-2\lambda \ln a + \theta}} \geq t_1 + \int_0^{a(t)} \frac{O(1) da}{\sqrt{-\ln a}}. \tag{26}$$

We denote δ as a sufficiently small positive number, such that $x \in (0, \delta]$,

$$-\ln x \leq \frac{1}{x^S},$$

where the constant $S > 0$. Then, (26) becomes

$$T - t_1 \geq \int_0^{a(t)} \frac{O(1) da}{\sqrt{1/a^S}} = O(1)a(t)^{S/2+1}.$$

By letting $t_1 \rightarrow T_-$, we get

$$\lim_{t_1 \rightarrow T_-} a(t)(T - t_1)^{-2/(S+2)} \leq O(1).$$

As $y(0) = \alpha$, we can estimate the blowup rate at the origin by

$$\lim_{t \rightarrow T_-} \rho(t, 0)(T - t)^{4/(S+2)} \geq O(1).$$

Thus, we obtain

$$\lim_{t \rightarrow T_-} \rho(t, 0)(T - t)^\eta \geq O(1),$$

with $\eta < 2$, and complete the proof. \square

4. Non-global existence for tame solutions

In this section, we present the proof for Theorem 5. The technique to Makino and Perthame’s [10] is similar.

Proof of Theorem 5. We denote the radius of the compact support of the tame solution by $R(t)$. Under the assumption that the initial data has a compact support, i.e. $\rho(0, r) = u(0, r) = 0$ for $r \geq R(0)$, from (9)₂, we must have

$$K\rho_r = 0.$$

And Eq. (9)₂ becomes

$$u_t + uu_r = -\frac{2\pi}{r} \int_0^r \rho(t, s) s ds,$$

along the curve $r = r(t; 0, y)$ where $r = r(t; t_0, r_0)$ is the solution of the characteristic equation outside the support of the tame solution, i.e.

$$\frac{dr}{dt} = u(t, r), \quad r|_{t=t_0} = r_0.$$

Clearly we have for $y \geq R(t)$,

$$\frac{d}{dt}u(t, r(t; 0, y)) = u_t + uu_r = -\Phi_r = G(r) \doteq -\frac{2\pi}{r} \int_0^r \rho(t, s)s \, ds \leq 0.$$

Therefore, we get

$$\frac{d}{dt}r(t; 0, y) = u(t, r(t; 0, y)) = \int_0^t G(\tau, r(\tau; 0, y)) \, d\tau \leq 0,$$

as $u(0, y) = 0$. Hence $r(t; 0, y) \leq y$ for $0 \leq t < T$, $R(t) \leq y$. For $\rho(t) \equiv 0$ outside the compact support of the tame solution, this implies

$$G(t, r(t, 0, R(0))) = -\frac{M}{r(t, 0, R(0))},$$

where

$$M = \int_{\mathbb{R}^2} \rho(t, x) \, dx = 2\pi \int_0^\infty \rho(t, s)s \, ds,$$

is the total mass and independent of t . And If $\rho(0)$ is not identically equal to zero, as the solution belongs C^1 , we have $M > 0$. Therefore $r = r(t; 0, R(0))$ satisfies the equation of free fall

$$\frac{d^2r}{dt^2} = -\frac{M}{r}, \quad r(0) = R(0), \quad \dot{r}(0) = 0. \quad (27)$$

From this it follows

$$0 \leq r \leq R(0) - \frac{M}{2R(0)}t^2.$$

This implies

$$T \leq \sqrt{2R(0)^2/M}.$$

The proof is completed. \square

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