On the Meromorphic and Schechter–Shapiro Operational Calculi

MANUEL GONZALEZ AND VICTOR M. ONIEVA

Facultad de Ciencias, Santander, Spain

Submitted by Ky Fan

The well-known Dunford–Taylor operational calculus for closed linear operators in Banach spaces has been generalized in two different ways: First, by H. A. Gindler (Nagoya Math. J. 26 (1966), 31–38) and B. Nagy (Acta Math. Acad. Sci. Hungar. 33 (1979), 379–390), to include meromorphic functions with poles in the extended spectrum which are not eigenvalues. Second, by M. Schechter and J. Shapiro (Trans. Amer. Math. Soc. 175 (1973), 439–467), to include functions analytic on a neighborhood of the Fredholm spectrum. In this paper we give several results about the meromorphic calculus, in particular, spectral mapping theorems for essential spectra; and we apply our results to the study of the solutions of some integro-differential equations. Moreover, for functions admissible in both calculi, we verify that with suitable choice of quasi-resolvent and contour of integration, the Schechter–Shapiro calculus yields an extension of the operator provided by the meromorphic calculus. © 1986 Academic Press, Inc.

0. INTRODUCTION

Let $X$ be a complex Banach space, $C(X)$ the set of all closed linear operators with domain and range in $X$, and $T \in C(X)$ such that the resolvent set $\rho(T)$ is non-empty.

The well-known Dunford–Taylor operational calculus for $T$ has been generalized in two different ways: First, H. A. Gindler [2] considered meromorphic functions with poles in the extended spectrum $\sigma_e(T)$ which does not belong to the point spectrum $\sigma_p(T)$, and he associated each of them with an operator $f(T) \in C(X)$; B. Nagy [7] has developed this calculus. Second, M. Schechter and J. Shapiro [9] associate an operator in $L(X)$ with a function analytic on a neighborhood of the Fredholm spectrum and analytic in $\infty$ for each choice of quasi-resolvent and of contour of integration; different choices of these concepts lead to operators in $L(X)$ which differ in a compact operator. We note that Müller-Hörrig [6] has also studied this calculus for $T \in L(X)$ and functions not necessarily defined in $\infty$.

In this paper, by means of a factorization of the operator provided by
the meromorphic calculus and some additional results relating domains, kernels and ranges of the operators involved, we complete the study realized in [1, 2, 7].

Particularly, by defining extended essential spectra for any operator in $C(X)$, we are able to give spectral mapping theorems. We also present an application of the calculus to the study of the solutions of a kind of integro-differential equations.

On the other hand, by considering admissible functions for both calculi, we show that the Schechter-Shapiro calculus, with a suitable choice of quasi-resolvent and contour of integration, gives an extension of the operator provided by the meromorphic calculus.

Now, we collect some notations used in the paper: other notations will be used without explanation because their meaning is obvious.

$\text{Op}(X)$ will denote the set of all linear operators $T$ with domain $D(T)$ and range $R(T)$ in $X$; $N(T)$ will be the kernel of $T$. Also, $L(X) := \{ T \in C(X) \mid D(T) = X \}$ and $\text{Co}(X) := \{ T \in L(X) \mid T \text{ compact} \}$.

$B, RS, F, A_+, A_-, SF_+, SF_-$ denote the subclasses of $C(X)$ of all bijective, Riesz-Schauder, Fredholm, Atkinson with finite nullity, Atkinson with finite defect, semi-Fredholm with finite nullity and semi-Fredholm with finite defect, respectively. Remember that $T$ is Riesz-Schauder if it is Fredholm with finite ascent and descent, and $T$ is Atkinson if it is semi-Fredholm with kernel and range complemented.

If $M$ is a linear subspace of $X$, $T \mid M$ will be the restriction of $T$ to $M$. $\mathcal{P}$ will denote the class of all polynomials, and the canonical factorization of $P \in \mathcal{P}$ will be written $P(z) = \prod_{i=1}^{k} (\lambda_i - z)^{m_i}$.

Finally, for properties of the Dunford-Taylor calculus we refer to [11].

1. Polynomials in a Linear Operator

In this section $X$ will be a complex linear space and $A \in \text{Op}(X)$. For each polynomial $P \in \mathcal{P}$ we consider the operator $P(A)$ defined as usual. If $P$ has degree $d$, it is evident that

\[ N[(\lambda - A)^m] \subset D[P(A)] = D(A^d), \quad \text{where } \lambda \in C, \ m \in N, \]

\[ P(A)[D(A^m)] = R(P(A)) \cap D(A^{m-d}), \quad \text{where } m \geq d. \tag{1.0} \]

The following result is proved in [12] when $D(A) = X$.

\begin{equation}
(1.1) \text{ Theorem. } \text{ "Let } P(A) = \prod_{i=1}^{k} (\lambda_i - A)^{n_i}, \text{ where } \lambda_i \neq \lambda_j \text{ for } i \neq j. \text{ Then:} \]

(i) \ $N[P(A)] = \sum_{i=1}^{k} N[(\lambda_i - A)^{n_i}]$ (algebraic direct sum).

(ii) \ $R[P(A)] = \bigcap_{i=1}^{k} R[(\lambda_i - A)^{n_i}].$"
Proof. (i) See [12].

(ii) First we shall prove

\[(\lambda - A)^m x \in R[(\mu - A)^m] \iff x \in R[(\mu - A)^m],\]

where \(m, n \in \mathbb{N}, \lambda, \mu \in C, \lambda \neq \mu\) and \(x \in D(A^n)\). (1.2)

If \(x \in R[(\mu - A)^m]\), then \((\lambda - A)^m x \in R[(\lambda - A)^n(\mu - A)^m] \subset R[(\mu - A)^m]\).

Conversely, if \(m = n = 1\) and \(x' \in D(A)\) with \((\lambda - A) x = (\mu - A) x'\), it is clear that \(x = (\lambda - \mu)^{-1}(\mu - A)(x' - x) \in R(\mu - A)\). Suppose the converse to be true for \(1 = n \leq m - 1\) and let \((\lambda - A) x \in R[(\mu - A)^m] \subset R[(\mu - A)^{m-1}]\); then \(x \in R[(\mu - A)^{m-1}]\), and choosing \(x' \in R[(\mu - A)^{m-1}] \cap D(A)\) such that \((\lambda - A) x = (\mu - A) x'\) we obtain \(x = (\lambda - \mu)^{-1}(\mu - A)(x' - x) \in R[(\mu - A)^m]\). Now, if the converse is true for \(m\) and \(n - 1 \geq 1\), and \((\lambda - A)^n x \in R[(\mu - A)^m]\), then \((\lambda - A) x \in R[(\mu - A)^m]\) and consequently \(x \in R[(\mu - A)^m]\). Thus, (1.2) is shown.

In order to prove (ii), we observe that \(\bigcap_{i=1}^k R[(\lambda_i - A)^m] \subset R[Q(A)]\) is evident for \(k = 1\); suppose it to be true for \(k - 1 \geq 1\) and let \(y \in \bigcap_{i=1}^k R[(\lambda_i - A)^m]\); then \(y = (\lambda_1 - A)^m y_1 \in \bigcap_{i=2}^k R[(\lambda_i - A)^m]\) and so \(y \in R[P(A)]\) since \(y_1 \in \bigcap_{i=2}^k R[(\lambda_i - A)^m] = R[\prod_{i=2}^k (\lambda_i - A)^m]\) by virtue of (1.2). As \(R[P(A)] \subset \bigcap_{i=1}^k R[(\lambda_i - A)^m]\) is obvious, the proof is complete.

(1.3) Corollary. "If \(P, Q\) are polynomials with no common zeros, then:

(i) \(P(A) x \in R[Q(A)] \iff x \in R[Q(A)]\), where \(x \in D[Q(A)]\).

(ii) \(P(A)(N[Q(A)]) = N[Q(A)];\) hence \(N[Q(A)] \subset R[P(A)]\)."

Proof: (i) It is an obvious consequence of (1.1) and (1.2).

(ii) By virtue of (1.1) we have

\[P(A) x \in N[Q(A)] \iff x \in N[P(A) Q(A)] \iff x \in N[P(A)] \cap N[Q(A)].\]

Thus, it suffices to show that \(N[Q(A)] \subset R[P(A)]\), or that \(N[(\lambda - A)^n] \subset R[(\mu - A)^m]\) with \(m, n \in \mathbb{N}\) and \(\lambda \neq \mu\). For this, if \(x \in N[(\lambda - A)^n]\), then \((\lambda - A)^n x = 0 \in R[(\mu - A)^m]\), and so \(x \in R[(\mu - A)^m]\) by (1.2).

2. Application to the Dunford–Taylor Calculus

Throughout this section \(X\) will denote a complex Banach space, \(\mathcal{C}\) the extended complex plane, \(T \in C(X)\) with \(\rho(T) \neq \emptyset\) and \(A(T)\) the set of all functions \(f: \mathcal{C} \to \mathcal{C}\) with domain an open set \(\Delta(f)\) such that \(\sigma_s(T) \subset \Delta(f)\).
and \( f \) holomorphic on \( A(f) \); let \( f(T) \in L(X) \) the operator defined by the Dunford–Taylor calculus.

(2.1) DEFINITION. When \( f \in A(T) \) we consider two open sets \( A, A' \) such that \( A(f) = A \cup A', A \cap A' = \emptyset \), \( f \) is identically 0 on \( A' \) and is not identically 0 on each connected component of \( A \); then \( \sigma_e(T) = \sigma \cup \sigma' \), where \( \sigma' := \sigma_e(T) \cap A' \) and \( \sigma := \sigma_e(T) \backslash \sigma' \). Let \( E_\sigma \) be the projector corresponding to the function \( e \in A(T) \) such that \( e(A) = \{1\} \) and \( e(A') = \{0\} \).

Note that the zeros of \( f \) in \( A \) are isolated with finite multiplicity, and there is only a finite number of them in \( \sigma \) because \( \sigma \) is compact, say \( c_0 = \infty, c_1, \ldots, c_k \) with finite orders \( m_0 \geq 0, m_i > 0, i = 1, \ldots, k. \) Let \( m := m_0 + m_1 + \cdots + m_k \) and \( P(z) := \prod_{i=1}^k (c_i - z)^{m_i} \).

Choose \( \alpha \in \rho(T) \) and consider the function \( F_\alpha \in A(T) \) defined by

\[
F_\alpha(z) := f(z) \frac{1}{P(z)(\alpha - z)^m} \quad \text{if} \quad z \in A
\]

\[
:= 1 \quad \text{if} \quad z \in A'.
\]

Clearly \( F_\alpha \) has no zeros in \( \sigma_e(T) \), hence \( F_\alpha(T) \in L(X) \) is invertible in \( L(X) \). Since \( P(z)(\alpha - z)^{-m} \in A(T) \) and \( f(z) = F_\alpha(z) \cdot P(z)(\alpha - z)^{-m} \cdot e(z) \), we obtain

\[
f(T) = F_\alpha(T) \frac{P(T)(\alpha - T)^{-m}}{E_\sigma},
\]

which will be termed associated factorization with \( f \in A(T) \), or factorization of \( f(T) \). We observe that \( F_\alpha(T) \), \( P(T)(\alpha - T)^{-m} \) and \( E_\sigma \) are commuting operators, and that \( f(T)x = F_\alpha(T)(\alpha - T)^{-m} E_\sigma P(T)x \) for every \( x \in D(T^{m-m_0}) = D(P(T)] \), where \( F_\alpha(T) \), \( (\alpha - T)^{-m} \) and \( E_\sigma \) commute.

The following statement expresses \( N[f(T)] \) and \( R[f(T)] \) in terms of the factorization of \( f(T) \).

(2.2) THEOREM. "Let \( f \in A(T) \) and \( f(T) = F_\alpha(T) P(T)(\alpha - T)^{-m} E_\sigma \) the factorization of \( f(T) \). Then

\[
N[f(T)] = N[P(T)] \oplus N(E_\sigma)
\]

and

\[
R[f(T)] = R[P(T)] \cap R(E_\sigma) \cap D(T^{m_0})."
\]

**Proof:** Let \( T_\sigma := T \mid R(E_\sigma) \) and \( T_\sigma' := T \mid N(E_\sigma) \). As \( (R(E_\sigma), N(E_\sigma)) \) completely reduces \( T, P(T) \) and \( h(T) \) for every \( h \in A(T) \), we have \( P(T) \mid R(E_\sigma) = P(T_\sigma), f(T) \mid R(E_\sigma) = f(T_\sigma), h(T) \mid R(E_\sigma) = h(T_\sigma) \), and analogously with \( T_\sigma' \). We derive from the factorization of \( f(T) \) that \( f(T_\sigma) = F_\alpha(T_\sigma) P(T_\sigma)(\alpha - T_\sigma)^{-m} \), and, according to (1.0),

\[
R[f(T_\sigma)] = R[P(T_\sigma)] \cap D(T_\sigma^{m_0}) = R[P(T)] \cap R(E_\sigma) \cap D(T^{m_0});
\]
moreover, by (1.3),
\[ x \in N[P(T_\sigma)(x - T_\sigma)^{-m}] \iff (x - T_\sigma)^{-m} x \in N[P(T_\sigma)] \iff x \]
\[ = (x - T_\sigma)^m(x - T_\sigma)^{-m} x \in N[P(T_\sigma)], \]
and so \( N[f(T_\sigma)] = N[P(T_\sigma)]. \)

On the other hand \( N[f(T_\sigma)] = N(E_\sigma) \) and \( R[f(T_\sigma)] = \{0\} \) because \( f \) is identically 0 on the neighborhood \( A' \) of \( \sigma' = \sigma_\epsilon(T_\sigma) \); and \( N[P(T_\sigma)] = \{0\} \) since \( P \) has no zeros in \( \sigma' \), hence \( N[P(T)] = N[P(T_\sigma)]. \) Therefore

\[ N[f(T)] = N[f(T_\sigma)] \oplus N[f(T_\sigma)] = N[P(T)] \oplus N(E_\sigma), \]
\[ R[f(T)] = R[f(T_\sigma)] \oplus R[f(T_\sigma)] = R[P(T)] \cap R(E_\sigma) \cap D(T^m). \]

(2.3) Corollary. "Let \( f, g \in A(T) \cup \mathcal{P} \) with no common zeros in \( \sigma_\epsilon(T). \) Then:

(i) \( N[f(T)] \subset R[g(T)]. \)

(ii) \( f(T) x \in R[g(T)] \iff x \in R[g(T)], \) where \( x \in D[f(T)]. \)"

Proof. If \( H \) is a polynomial of degree \( n \) and \( z \in \rho(T) \) is not zero of \( H, \) it is clear that \( H(z)(\alpha - z)^{-n} \in A(T), \) \( R[H(T)(\alpha - T)^{-n}] = R[H(T)] \) and \( N[H(T)(\alpha - T)^{-n}] = (\alpha - T)^n N[H(T)] = N[H(T)] \) by (1.3). Consequently, it suffices to prove the statement for \( f, g \in A(T). \n
(i) Consider the factorizations \( f(T) = F_\epsilon(T) P(T)(x - T)^{-m} E_\sigma, \)
\( g(T) = G_\epsilon(T) Q(T)(x - T)^{-m} E_\epsilon, \) and the decomposition \( \sigma_\epsilon(T) = \zeta \cup \sigma' \cup \tau' \)
determined by the spectral sets \( \sigma', \tau' \). If \( X = X_\zeta \oplus X_\sigma \oplus X_\tau \) is the associated direct sum, we note that \( X_\zeta = X_\zeta \oplus X_{\epsilon'} \) and \( X_{\epsilon'} = X_\zeta \oplus X_\sigma '; \) and the isolated zeros of \( f \) and \( g \) belongs to \( \zeta. \)

We have
\[ N[f(T_\zeta)] = N[P(T_\zeta)] \subset R[Q(T_\zeta)] \cap D(T_\zeta^m) = R[g(T_\zeta)], \]
\[ N[f(T_\sigma')] = X_\sigma' = R[g(T_\sigma')] \quad \text{and} \quad N[f(T_{\epsilon'})] = \{0\}; \]
hence we obtain
\[ N[f(T)] = N[f(T_\zeta)] \oplus N[f(T_\sigma')] \oplus N[f(T_{\epsilon'})] \]
\[ \subset R[g(T_\sigma)] \oplus R[g(T_{\epsilon'})] \oplus R[g(T_{\epsilon'})] = R[g(T)]. \]

(ii) If \( x = g(T) x' \in R[g(T)], \) then \( f(T) x = g(T) f(T) x' \in R[g(T)]. \)

Conversely, if \( f(T) x \in R[g(T)], \) by virtue of (2.2) it is
\[ R[f(T)] = R[P(T)] \cap D(T^m) \cap (X_\zeta \oplus X_{\epsilon'}). \]
and
\[ R[g(T)] = R[Q(T)] \cap D(T^{m_0}) \cap (X_\varepsilon \oplus X_{\sigma}), \]
where \( \min(m_0, n_0) = 0 \) since \( \infty \) is not a common zero of \( f \) and \( g \).
Define \( p := \max(m_0, n_0) \); then
\[ R[f(T)] \cap R[g(T)] = R[P(T)Q(T)] \cap D(T^p) \cap X_\varepsilon = R[f(T)g(T)] \]
and \( f(T)x \in R[f(T)g(T)] \). It follows that
\[ x \in R[g(T)] + N[f(T)] \subset R[g(T)]. \]

(2.4) Remark. Let \( \alpha \in \rho(T) \), \( k \in \mathbb{N} \) and \( f \in A(T) \) such that \( f(\infty) \neq 0 \).
From (2.3) we derive
\[ x \in D(T^k) = R[(\alpha - T)^{-k}] \Leftrightarrow f(T)x \in D(T^k). \]

3. THE MEROMORPHIC OPERATIONAL CALCULUS

Let \( X \) be a complex Banach space, \( T \in \mathcal{C}(X) \) with \( \rho(T) \neq \emptyset \), \( M(T) \)
the set of all functions \( f: \Delta(f) \subset \bar{C} \to \bar{C} \) such that \( \Delta(f) \) is an open
set, \( \sigma_e(T) \subset \Delta(f) \) and \( f \) is meromorphic on \( \Delta(f) \) having no poles in \( \sigma_e(T) \).
Since \( \sigma_e(T) \) is compact, then \( f \) has (at most) a finite number of poles in \( \sigma_e(T) \), say \( p_0 = \infty \), \( p_1, \ldots, p_h \) with orders \( n_0 \geq 0 \), \( n_j > 0 \), \( j = 1, \ldots, h \); let
\( n := n_0 + n_1 + \cdots + n_h \).
If \( f \in M(T) \), \( \alpha \in \rho(T) \) and \( Q(z) := \prod_{j=1}^{h} (p_j - z)^{n_j} \), we define \( G_\alpha(z) := f(z)Q(z)(\alpha - z)^{-n} \) with the usual conventions; clearly \( G_\alpha \in A(T) \). Then,
Gindler [2] defined the operator \( f(T) \) by
\[ f(T) := G_\alpha(T)(\alpha - T)^{-n} Q(T)^{-1}. \]

It is easy to verify that \( f(T) \) is independent of \( \alpha \in \rho(T) \). In fact, if \( \alpha, \beta \in \rho(T) \) we have \( G_\alpha(z) = G_\beta(z)(\beta - z)^n(\alpha - z)^{-n} \); since \( (\beta - z)^n(\alpha - z)^{-n} \in A(T) \), \( (\beta - z)^n \) has degree \( n \) and \( \infty \) is zero of \( (\alpha - z)^{-n} \) with order \( n \), the Dunford–Taylor calculus gives \( (\beta - T)^n(\alpha - T)^{-n} \), hence \( G_\alpha(T) = G_\beta(T)(\beta - T)^n(\alpha - T)^{-n} \), and in consequence \( G_\alpha(T)(\alpha - T)^n = G_\beta(T)(\beta - T)^n \).

Note that \( Q(T)(\alpha - T)^{-n} \in \mathcal{L}(X) \) and then \( (\alpha - T)^n Q(T)^{-1} \in \mathcal{C}(X) \). Moreover
\[ D[f(T)] = D[(\alpha - T)^n Q(T)^{-1}] = R[Q(T)(\alpha - T)^{-n}] = Q(T)(D(T^n)) \]
\[ = R[Q(T)] \cap D(T^{m_0}), \]
\[ R[f(T)] = R[G_\alpha(T)]. \]

The following result has been proved in [7]; our proof is simpler.
(3.1) **Theorem.** "If \( f \in M(T) \), then \( f(T) = (\alpha - T)^n Q(T)^{-1} G_{\alpha}(T) \). Consequently \( f(T) \in C(X) \)."

**Proof.** Firstly we shall see that \( D[f(T)] = D[(\alpha - T)^n Q(T)^{-1} G_{\alpha}(T)] \). If \( n_0 > 0 \), we have \( G_{\alpha}(\infty) \neq 0 \), and by using (2, 3), (2, 4), we obtain

\[
x \in D[f(T)] = R[Q(T)] \cap D(T^{n_0}) \iff G_{\alpha}(T) x \in R[Q(T)] \cap D(T^{n_0})
\]

\[
= D[(\alpha - T)^n Q(T)^{-1}] \iff x \in D[(\alpha - T)^n Q(T)^{-1} G_{\alpha}(T)].
\]

If \( n_0 = 0 \), then

\[
x \in D[f(T)] = R[Q(T)] \iff G_{\alpha}(T) x \in R[Q(T)] = D[(\alpha - T)^n Q(T)^{-1}]
\]

\[
\iff x \in D[(\alpha - T)^n Q(T)^{-1} G_{\alpha}(T)].
\]

Now, if \( y \in D[f(T)] \) and \( x \in D(T^n) \) are such that \( y = Q(T) x \), by \([11; V.8.6]\) we have

\[
(\alpha - T)^n Q(T)^{-1} G_{\alpha}(T) y = (\alpha - T)^n Q(T)^{-1} Q(T) G_{\alpha}(T) x
\]

\[
= G_{\alpha}(T)(\alpha - T)^n x = f(T) y. \quad \square
\]

(3.2) **Corollary.** "Given \( f \in M(T) \), we have \( N[f(T)] = N[G_{\alpha}(T)]."\]

**Proof.** Obvious.

(3.3) **Remark.** Let \( P, Q \) be polynomials of degrees \( m, n \), respectively, with no common zeros and such that each zero \( \lambda \) of \( Q \) satisfies \( \lambda \notin \sigma_{\alpha}(T) \). Clearly \( P/Q \in M(T) \) and the associated function is \( G_{\alpha}(z) = P(z)/(\alpha - z)^{-p} \) with \( p := \max(m, n) \). Then, the meromorphic calculus yields the closed operator \( P(T) Q(T)^{-1} \), which is the usually defined.

The following is a technical lemma required for the spectral mapping theorem.

(3.4) **Lemma.** "Let \( f \in M(T), k \in N \) and \( g_k(z) := f(z)^k \). Then \( g_k \in M(T) \) and \( g_k(T) = f(T)^k \)."

**Proof.** Obviously \( g_k \in M(T) \) and \( D[g_k(T)] = R[Q(T)^k] \cap D(T^{n_0 k}) \) for every \( k \in N \). Then, the equality \( g_k(T) = f(T)^k \) is trivial for \( k = 1 \); suppose it to be true for \( k \) and show it for \( k + 1 \).

In fact, we have

\[
x \in D[g_{k+1}(T)] = R[Q(T)^{k+1}] \cap D(T^{n_0(k+1)})
\]

\[
\iff (\alpha - T)^n Q(T)^{-1} x \in R[Q(T)^k] \cap D(T^{n_0 k})
\]

\[
\iff f(T) x \in R[Q(T)^k] \cap D(T^{n_0 k});
\]
hence \( D[g_{k+1}(T)] = D[g_k(T)f(T)] \). Now, the equality \( g_{k+1}(T) = g_k(T)f(T) \) follows from the commutativity of the operators in the definition of \( g_k(T) \) and \( f(T) \).

Next, we define extended essential spectra. Our definition covers those considered in [4] for \( T \in L(X) \) or \( T \) unbounded.

(3.5) Definition. Suppose \( T \in C(X) \) and let \( \sigma_i(T) := \{ \lambda \in \mathbb{C} \mid \lambda - T \notin i \} \), where \( i \) runs over the set \( \{ Bi, RS, F, A_+, A_-, SF_+, SF_- \} \). We define the extended essential spectrum \( \sigma_{ie}(T) \) by

\[
\sigma_{ie}(T) := \begin{cases} 
\sigma_i(T) & \text{if each injective operator in } L(X) \text{ with} \\
\text{range } D(T) \text{ belongs to } i, \\
\sigma_i(T) \cup \{ \infty \} & \text{otherwise.}
\end{cases}
\]

If \( \rho(T) \neq \emptyset \) and \( \alpha \in \rho(T) \), we have \( \infty \in \sigma_{ie}(T) \) iff \( (\alpha - T)^{-1} \notin i \).

Concerning to the spectra it is shown in [3] the following spectral mapping theorem for functions \( f \in A(T) \) by using the factorization of \( f(T) \) and well-known properties of products of operators in the corresponding classes. We observe that this result is proved in [4] for the particular cases \( T \in L(X) \) or \( T \) unbounded; our approach covers the general case and it is more unified.

(3.6) Theorem [3]. "Let \( f \in A(T) \) and \( i \in \{ Bi, RS, F, A_+, A_-, SF_+, SF_- \} \). Then \( f(\sigma_{ie}(T)) = \sigma_{ie}(f(T)) \)."

We now present the result for functions \( f \) in \( M(T) \); the particular case \( i = Bi \) has been proved in [7].

(3.7) Spectral Mapping Theorem. "Let \( f \in M(T) \) and \( i \in \{ Bi, RS, F, A_+, A_-, SF_+, SF_- \} \). Then \( f(\sigma_{ie}(T)) = \sigma_{ie}(f(T)) \)."

Proof. First, we observe that if \( G_\alpha(T) \) is the operator associated with \( f(T) \) in (3.0), then \( G_\alpha(T)^k \) is the operator associated with \( f(T)^k \) for each \( k \in \mathbb{N} \). Therefore \( N[f(T)^k] = N[G_\alpha(T)^k] \) and \( R[f(T)^k] = R[G_\alpha(T)^k] \); hence, \( f(T) \) and \( G_\alpha(T) \) have the same nullity, defect, ascent and descent.

Let \( \mu \in \mathbb{C} \) and \( g_\mu(z) := \mu - f(z) \in M(T) \). Then

\[
g_\mu(T) := G_{\mu e}(T)(\alpha - T)^n Q(T)^{-1} = \mu I - f(T) = [\mu Q(T)(\alpha - T)^{-n} - G_\alpha(T)](\alpha - T)^n Q(T)^{-1}.
\]

Hence, by virtue of (3.6), \( \mu \notin \sigma_i(f(T)) \Leftrightarrow \mu I - f(T) \in i \Leftrightarrow G_{\mu e}(T) \in i \Leftrightarrow G_\mu(z) \neq 0 \), \( \forall z \in \sigma_{ie}(T) \Leftrightarrow f(z) \neq \mu \), \( \forall z \in \sigma_{ie}(T) \Leftrightarrow \mu \notin f(\sigma_{ie}(T)) \). Moreover, as
\[ D[f(T)] = R[Q(T)(\alpha - T)^{-n}], \] we have \( \infty \notin \sigma_{ie}(f(T)) \Leftrightarrow Q(T)(\alpha - T)^{-n} \in i \Leftrightarrow (p_j \notin \sigma_{ie}(T), j = 1, \ldots, h) \) and \( \infty \notin \sigma_{ie}(T) \) if \( n_0 > 0 \) \( \Leftrightarrow \infty \notin f[\sigma_{ie}(T)]. \]

(3.8) COROLLARY. "If \( f \in M(T) \) and \( \tau \) is a spectral set of \( f(T) \), then \( \sigma := \sigma_\tau(T) \cap f^{-1}(\tau) \) is a spectral set of \( T \)."

Proof. Let \( \sigma' := \sigma_\tau(T) \setminus \sigma \) and \( \tau' := \sigma_{f(T)}(\tau) \setminus \tau \). By (3.7) we have \( f(\sigma \cup \sigma') = \tau \cup \tau' \), hence \( \sigma' = \sigma_\tau(T) \cap f^{-1}(\tau') \). Since \( f \) is continuous, \( \sigma \) and \( \sigma' \) are clopen in \( \sigma_\tau(T) \), and so \( \sigma, \sigma' \) are complementary spectral sets of \( T \). |

We are going to examine the conditions under which it may happen that \( f(T) = 0 \) where \( f \in M(T) \). The following minimal equation theorem is consequence of the corresponding to the Dunford–Taylor calculus.

(3.9) THEOREM. "Suppose that \( f \in M(T) \). Then \( f(T) = 0 \) if and only if \( f \) vanishes identically in a neighborhood of each point of \( \sigma_\tau(T) \) except perhaps for a finite number of isolated points which are zeros of \( f \) and poles of the resolvent of \( T \) with order as poles not exceeding the order as zeros."

Proof. Let \( G_\tau \in A(T) \) the function associated with \( f \) and \( f(T) := G_\tau(T)(\alpha - T)^{-n}Q(T)^{-1} \). Since \( R[f(T)] = R[G_\tau(T)] \) it follows \( f(T) = 0 \) iff \( G_\tau(T) = 0 \). Now the statement is derived from [11; V.10.7] by observing that \( f(z) \) and \( G_\tau(z) \) have the same zeros with equal orders. |

Next we give some conditions under which \( f(T) \in L(X) \) or it is bounded or \( f(T) \)-compact operator.

(3.10) PROPOSITION. "Let \( f \in M(T) \) and \( f(T) := G_\tau(T)(\alpha - T)^{-n}Q(T)^{-1} \). The following properties are equivalent:

(i) \( f(T) \in L(X) \).
(ii) \( R[Q(T)] \cap D(T^{n_0}) = X \).
(iii) \( R[G_\tau(T)] \subseteq R[Q(T)] \cap D(T^{n_0}) \).
(iv) \( f \in A(T) \)."

Proof. (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii). Note that \( D[f(T)] = R[Q(T)] \cap (T^{n_0}) \) and \( f(T) = (\alpha - T)^{-n}Q(T)^{-1}G_\tau(T) \).

(i) \( \Rightarrow \) (iv). If \( f(T) \in L(X) \) then \( D(T^{n_0}) = X = R[Q(T)] \); hence \( n_0 = 0 \) or \( D(T) = X \), and \( R(p_j - T) = X \) for each pole \( p_j \) in \( \sigma(T) \); since \( N(p_j - T) = \{0\} \), there are no poles of \( f \) in \( \sigma(T) \). Thus \( f \in A(T) \).

(iv) \( \Rightarrow \) (i). If \( f \in A(T) \) we have \( D(T) = X \) or \( n_0 = 0 \); moreover \( Q(T) = I \). Therefore \( D[f(T)] = X \). |
(3.11) **PROPOSITION.** "Let \( f \in M(T) \). The following statements are equivalent:

(i) \( f(T) \) is continuous, or equivalently,
\[ D[f(T)] = D(T^n) \cap R[Q(T)] \]
is closed.

(ii) \( R(p_i - T) \) is closed for \( i = 1, \ldots, h \), and \( D(T) \) is closed when \( n_0 > 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). We know that \( Q(T)(\alpha - T)^{-n} \in L(X) \) with
\[ R[Q(T)(\alpha - T)^{-n}] = D[f(T)] \]
and \( N[Q(T)(\alpha - T)^{-n}] = \{0\} \). If \( f(T) \) is continuous, then \( Q(T)(\alpha - T)^{-n} \in SF_+ \) and so \( 0 \notin \sigma_{ie}[Q(T)(\alpha - T)^{-n}] = \{Q(z)(\alpha - z)^{-n} \mid z \in \sigma_{ie}(T)\} \) for \( i = SF_+ \); therefore \( p \notin \sigma_{ie}(T) \) for \( i = SF_+ \) and every zero \( p \) of \( Q(z)(\alpha - z)^{-n} \). But the zeros of this function are \( p_1, \ldots, p_h \), and also \( p_0 = \infty \) if \( n_0 > 0 \); hence (ii) follows.

(ii) \( \Rightarrow \) (i). It is obvious from (1.0) and (1.1).

(3.12) **PROPOSITION.** "Let \( f \in M(T) \). The following statements are equivalent:

(i) \( f(T) \) is \((\alpha - T)^n Q(T)^{-1}\)-compact.

(ii) \( f(T) \) is \( f(T) \)-compact.

(iii) \( G_\alpha(T) \) is compact."

**Proof.** (i) \( \Leftrightarrow \) (ii). Note that \( f(T) \) and \((\alpha - T)^n Q(T)^{-1}\) are closed operators with the same domain; hence their graph topologies coincide.

(ii) \( \Leftrightarrow \) (iii). Consider the \( f(T) \)-norm in \( D[f(T)] \). Then the equivalence is an easy consequence of the following result [11; V.7.5]: Let \( E, F \) be Banach spaces, \( G \) a normed space, \( A \in L(F, G) \) and \( K \in Co(E, G) \) such that \( R(A) = R(K) \); then \( A \in Co(F, G) \).

(3.13) **Remark.** Since \( G_\alpha(z) \) and \( f(z) \) have the same polynomial of zeros, [6; Th. 5.1] provides another characterization of \( f(T) f(T) \)-compact. Moreover, it is simple to derive characterizations of \( T \) compact or \( T \in Co(X) \) from (3.10), (3.11) and (3.12).

We shall end this section with a result about the conjugate operator of \( f(T) \).

(3.14) **THEOREM.** "Let \( T \in C_p(X) := \{A \in C(X) \mid D(A) = X\} \) and \( f \in M(T) \) such that \( f(T) \in C_p(X) \). Then \( f \in M(T') \) and \( f(T') = f(T)' \)."

**Proof.** For each pole \( p_i \) of \( f \) we have \( R(p_i - T) = X \) because \( f(T) \in C_p(X) \) and \( D[f(T)] = R[Q(T)] \cap D(T^n) \); hence \( N(p_i - T') = \{0\} \), \( i = 1, \ldots, h \), and consequently \( f \in M(T') \).
Since \((\alpha - T)^{-n}Q(T) \in \mathcal{Q}(T)(\alpha - T)^{-n} \in \mathcal{L}(X)\) and \(D[(\alpha - T)^{-n} Q(T)]\) is dense, we have \(\mathcal{Q}(T)(\alpha - T)^{-n} = [(\alpha - T)^{-n} Q(T)]' = \mathcal{Q}(T')(\alpha - T')^{-n}\) (for the last equality see [5; III, pr. 5.26]); moreover, the Dunford–Taylor calculus yields \(G_a(T)' = G_a(T')\).

Now, as \(\mathcal{Q}(T)(\alpha - T)^{-n} \in \mathcal{L}(X)\), by using [5; III, pr. 5.26], we have
\[
f(T)' = [((\alpha - T)^{-n} Q(T)^{-1})'] G_a(T)' = [((\alpha - T)^{-n} Q(T)^{-1})']^{-1} G_a(T')
\]
\[
= [Q(T')(\alpha - T')^{-n}]^{-1} G_a(T') = (\alpha - T')^{-n} Q(T')^{-1} G_a(T') = f(T').
\]

4. COMPARISON OF THE MEROMORPHIC AND SCHECHTER–SHAPIRO CALCULI

The Schechter–Shapiro operational calculus [9] can be applied to the functions \(f \in M(T)\) such that \(p_1, \ldots, p_h \in \sigma_r(T) \setminus \sigma_f(T)\) and \(n_0 = 0\); note that in such a case
\[
p_j - T \in \mathcal{F}(X), \quad N(p_j - T) = \{0\}, \quad i(p_j - T) > 0, \quad j = 1, \ldots, h,
\]
where \(i(p_j - T)\) is the index of \(p_j - T\). Since \(n_0 = 0\) and \(R(p_j - T)\) is closed for each \(j\), the operator \(f(T)\) provided by the meromorphic calculus is continuous. Moreover, since
\[
f(z) = F(z) + \sum_{j=1}^{h} \sum_{i=1}^{n_j} a_{ji}(p_j - z)^{-i}, \quad \text{where } F(z) \text{ is analytic on } \sigma_e(T),
\]
we have
\[
f(T) = F(T) + \sum_{j=1}^{h} \sum_{i=1}^{n_j} a_{ji}(p_j - T)^{-i}.
\]

We shall prove that with suitable choices of quasi-resolvent and curve of integration, the Schechter–Shapiro calculus gives us an extension of \(f(T)\).

First, we shall construct a suitable quasi-resolvent \(R'_i\), [9]. Remember that in each connected component \(\Phi_i\) of the Fredholm resolvent \(\rho_f(T)\) there exists a set \(S_i\) of isolated points \(\mu_i\) with no accumulation point in \(\Phi_i\) such that the nullity of \(\mu_i - T\) is greater than the constant value in \(\Phi_i\) of the nullity of \(\lambda - T\) when \(\lambda \in \Phi_i - S_i\). Each choice of a point \(\lambda_i \in \Phi_i - S_i\) and a quasi-inverse \(T_i \in \mathcal{L}(X)\) of \(\lambda_i - T\) determines a set \(\Phi_i^0\) of isolated points \(v \in \Phi_i\) such that \(-(v - \lambda_i)^{-1} \in \sigma(T_i)\) having no accumulation point in \(\Phi_i\) and satisfying \(S_i \subseteq \Phi_i^0\). Then, the quasi-resolvent \(R'_i(T)\) is defined in \(\Phi_i - \Phi_i^0\) by \(R'_i(T) := T_i[(\lambda - \lambda_i) T_i + I]^{-1}\), and it is analytic on \(\Phi_i - \Phi_i^0\).

In our case, since \(N(p_j - T) = \{0\}\) for each pole \(p_j\) of \(f\), we can choose \(\lambda_i\) to be such a pole in each component \(\Phi_i\) with poles of \(f\). The choice of a
suitable quasi-inverse for each pole \( p_j \) is basic in the comparison: If \( N_j \) is a complement of \( R(p_j - T) \), we define \( T_j \in L(X) \) by means of
\[
T_j x := (p_j - T)^{-1} x \quad \text{if} \quad x \in R(p_j - T),
\]
\[
:= 0 \quad \text{if} \quad x \in N_j;
\]
(4.2)

\( T_j \) is, in fact, a quasi-inverse of \( p_j - T \) since \( T_j \in L(X) \), \( R(T_j) = D(T) \), \( T_j(p_j - T) = I - P_j \), where \( P_j \) is the projector on \( N_j \) with kernel \( R(p_j - T) \) [9].

If there is more than one pole in a component \( \Phi_i \), we have at least two choices of \( \lambda_i \) in \( \Phi_i \), and of course more than one quasi-resolvent. Nevertheless, chosen \( p_i \) in \( \Phi_i \), we are going to verify that, with an appropriate quasi-inverse, every pole in \( \Phi_i \) belongs to \( \Phi_i - \Phi_i^0 \), and that the choice of a different pole \( p \neq p_i \) in \( \Phi_i \) yields a quasi-resolvent which coincides on a neighborhood of \( p \) with that constructed by means of \( p_i \).

Let \( \Phi_k, \; k = 1, \ldots, m \), be the components of \( \rho_f(T) \) with poles of \( f \), and, rearranging with double index, let \( p_{ki}, \; i = 1, \ldots, r_k \), the poles in \( \Phi_k \); that is,
\[
p_{ki} \in \Phi_k, \quad i = 1, \ldots, r_k, \quad k = 1, \ldots, m \quad \text{and} \quad r_1 + \cdots + r_m = h.
\]

First, we shall give two preliminary lemmas.

**Lemma (4.3)**. "Given a component \( \Phi_k \) of \( \rho_f(T) \) with poles of \( f \), there exists a finite dimensional subspace \( N_k \) which is a complement of \( R[p_{ki} - T] \) for each \( i = 1, \ldots, r_k \)."

**Proof.** Since \( p_{ki} \in \Phi_k \) and \( N(p_{ki} - T) = \{0\} \), \( i = 1, \ldots, r_k \), it is clear that the subspaces \( R[p_{ki} - T] \) have the same finite codimension \( d_k \geq 1 \).

Clearly there is a subspace \( N_1 \) of dimension 1 such that \( N_1 \setminus R[p_{ki} - T] = \{0\} \) for \( i = 1, \ldots, r_k \).

Let \( d_k \geq s > 1 \) and suppose that there exists a subspace \( N_{s-1} \) of dimension \( s - 1 \) such that \( N_{s-1} \setminus R[p_{ki} - T] = \{0\} \), \( i = 1, \ldots, r_k \). Choosing \( x_0 \notin R[p_{ki} - T] \oplus N_{s-1} \), \( i = 1, \ldots, r_k \), it is clear that \( N_s := N_{s-1} \oplus [x_0] \) verifies \( N_s \setminus R[p_{ki} - T] = \{0\} \), \( i = 1, \ldots, r_k \). Therefore \( N_k := N_{d_k} \) satisfies the statement.

**Lemma (4.4)**. "Let \( T_k := T_{ki} \) the quasi-inverse of \( p_{ki} - T \) defined in (4.2). Then \( -(p_{ki} - p_{ki})^{-1} \in \rho(T_k) \) for \( i = 2, \ldots, r_k \)."

**Proof.** Note that
\[
T_k(p_{ki} - T) = T_k(p_{ki} - T) + (p_{ki} - p_{ki}) T_k = I + (p_{ki} - p_{ki}) T_k
\]
\[
= (p_{ki} - p_{ki})[ - (p_{ki} - p_{ki})^{-1} I - T_k];
\]
by virtue of the choice of $N_k$, we have $N[T_k(p_{k i} - T)] = \{0\}$ and $R[T_k(p_{k i} - T)] = D(T)$. Then $I + (p_{k i} - p_{k i}) T_k$ is a Fredholm operator with index 0 because $p_{k i} - T, p_{k i} - T \in F(X)$ and $[I + (p_{k i} - p_{k i}) T_k](p_{k i} - T) = p_{k i} - T$; moreover, it is injective on $R[p_{k i} - T]$ and applies identically $N_k$ onto $N_k$. Therefore $I + (p_{k i} - p_{k i}) T_k$ is bijective and, in consequence, $-(p_{k i} - p_{k i})^{-1} \in \rho(T_k)$.

Now, consider the operators $T_k := T_{k i}$ and $T_k^i, i = 2, \ldots, r_k$, associated with the poles $p_{k i}$ and $p_{k i}$ by means of (4.2), and the operators $R_k(\lambda) := T_k[(\lambda - p_{k i}) T_k + I]^{-1}$ and $T_k[(\lambda - p_{k i}) T_k + I]^{-1}$.

(4.5) Proposition. "There exists $\sigma_k > 0$ such that $R_k(\lambda)$ and $T_k[(\lambda - p_{k i}) T_k + I]^{-1}$ coincide on the open ball $B(p_{k i}, \delta_k$) for $i = 2, \ldots, r_k."

Proof. Since $\{\lambda \in \Phi_k \mid -(\lambda - p_{k i})^{-1} \in \rho(T_k) \cup \{\infty\}\} = \Phi_k - \Phi_k^0$ is open and $p_{k i} \in \Phi_k - \Phi_k^0, i = 1, \ldots, r_k$, there exists $\delta_k > 0$ such that $B(p_{k i}, \delta_k) \subset \Phi_k - \Phi_k^0$ for $i = 1, \ldots, r_k$.

First, we shall show that $R_k(p_{k i}) = T_k^i$.

In fact, since $R_k(p_{k i})[(p_{k i} - p_{k i}) T_k + I] = T_k$, we have

$$R_k(p_{k i})(p_{k i} - T) = R_k(p_{k i})[(p_{k i} - p_{k i}) + (p_{k i} - T)]$$

$$= R_k(p_{k i})[(p_{k i} - p_{k i}) T_k + I](p_{k i} - T) = I_{D(T)};$$

hence $R_k(p_{k i})$ coincides with $T_k^i$ on $R(p_{k i} - T)$. On the other hand $I + (p_{k i} - p_{k i}) T_k$ applies identically $N_k$ onto $N_k$, and so $T_k[(p_{k i} - p_{k i}) T_k + I]^{-1} = R_k(p_{k i})$ vanishes on $N_k$. Thus $R_k(p_{k i}) = T_k^i$.

Now, if $\lambda \in B(p_{k i}, \delta_k)$ it is easy to verify

$$T_k[(\lambda - p_{k i}) T_k + I]^{-1} = T_k[(\lambda - p_{k i}) T_k + I]^{-1} := R_k(\lambda).$$

We shall choose the quasi-resolvent $R'$ to be $R_k(\lambda)$ in $\Phi_k - \Phi_k^0, k = 1, \ldots, m$; in a component $\Phi$, with no poles of $f$, we choose $\lambda_i$ without restrictions other than those of the calculus.

We now show how to choose the curve of integration. If

$$\varepsilon_k := \min \{l p_{k i} - p_{k j} \mid i \neq j, i, j = 1, \ldots, r_k\},$$

$$\varepsilon_k := \min \{\delta_k \varepsilon_k \frac{1}{2}, \frac{1}{2} l T_{k i} l \mid i = 1, \ldots, r_k\}$$

and $D_0$ is a Cauchy domain such that $\sigma(\infty) \subset D_0 \subset D_0 \subset A(f)$, then we consider the domain $D := D_0 \setminus \bigcup_{k = 1}^{m} \bigcup_{i = 1}^{r_k} \{z \in C \mid l z - p_{k i} l \leq \varepsilon_k\}$; let $\partial D_0, \partial D$ be the boundaries of $D_0$ and $D$, and $\Sigma_k := \{z \in C \mid l z - p_{k i} l = \varepsilon_k\}$. Obviously $D$ is an admissible domain for the Schecther-Shapiro calculus and $\partial D = \partial D_0 \cup (+ I_k).$
Once the curve of integration is chosen and the choice of quasi-resolvent is made, the calculus yields the operator

\[ \tilde{f}(T) := f(\infty) I + \frac{1}{2\pi i} \int_{\partial D} f(\lambda) R'_\lambda \, d\lambda \] 

if \( \infty \in D \),

\[ := \frac{1}{2\pi i} \int_{\partial D} f(\lambda) R'_\lambda \, d\lambda \]

otherwise.

\[ \text{(4.6) THEOREM.} \quad \tilde{f}(T) \text{ is an extension of } f(T). \]

\[ \text{Proof.} \] By virtue of (4.5), for every \( z \in B(p_{ki}, 2\varepsilon_k) \) we have

\[ R'_z = T_{ki}[(z - p_{ki}) T_{ki} + I]^{-1} = T_{ki} \sum_{s=0}^{\infty} (p_{ki} - z)^s T_{ki}, \]

and so \( (1/2\pi i) \int_{I_k} R'_z(p_{ki} - z)^{-s} \, dz = T_{ki} \) for \( i = 1, \ldots, r_k \) and \( k = 1, \ldots, m \). Moreover, it is clear that \( \int_{I_k} R'_z(p_{ki} - z)^{-s} \, dz = 0 \) for \( i \neq j, i, j = 1, \ldots, r_k \) and \( k = 1, \ldots, m \).

On the other hand, when \( \infty \in D \) the points of \( \sigma_e(T) \) have index 0 with respect to \( \partial D_0 \), and consequently \( \int_{\partial D_0} (p_{ki} - z)^{-s} R'_z \, dz = 0 \) for \( i = 1, \ldots, r_k \) and \( k = 1, \ldots, m \); if \( \infty \notin D \), taking \( r \) greater than the spectral radius of \( T \) we have \( \int_{\partial D_0} R'_z(p_{ki} - z)^{-s} \, dz = \int_{|z| = 2r} (z - T)^{-1}(p_{ki} - z)^{-s} \, dz = 0 \) because \( \sigma(T) \subset D_0 \), each point in \( \sigma(T) \) has index 1 with respect to \( \partial D_0 \) (in particular \( p_{ki} \)) and \( \lim_{z \to \infty} |z| \cdot |(z - T)^{-1}| \cdot |p_{ki} - z|^{-s} = 0 \).

Furthermore, for the analytic function \( F(z) \) on \( \sigma_e(T) \), we have \( \int_{I_k} R'_z F(z) \, dz = 0, \ j = 1, \ldots, r_k, \ k = 1, \ldots, m \). Consequently, recuperating the initial notations for the poles and taking account of (4.0), we obtain

\[ \tilde{f}(T) = F(T) + \sum_{j=1}^{h} \sum_{l=1}^{n_j} a_{jl}(p_j - T)^{-l}, \]

and, by (4.2), \( f(T) \subset \tilde{f}(T) \); the proof is complete.

5. Application

We use the meromorphic operational calculus to obtain some results about existence and uniqueness of the solutions of certain integro-differential equations.

Let \( X \) be a Banach space of functions, and \( A \) the differentiation operator defined by

\[ D(A) := \{ x \in X \mid x' \in X \}; \quad Ax = x'; \quad x \in D(A). \]

We assume that \( A \in C_p(X) \) and \( \rho(A) \neq \emptyset \).
For \( f \in M(A) \) we consider the equation

\[ f(A)x = y, \quad y \in X \text{ fixed.} \quad (5.1) \]

We shall see that (5.1) is a kind of integro-differential equation.

As \( R[f(A)] = R[G_{\sigma}(A)] \) and \( N[f(A)] = N[G_{\sigma}(A)] \), we can establish the following results:

(i) There exists a solution \( x \) for each \( y \in X \) if and only if \( R(c - A) = X \) for each zero \( c \) of \( f \); note that if \( \infty \) is zero of \( f \), then \( D(A) = X \).

(ii) If there exists solution, this is unique if and only if \( N(c - A) = \{0\} \) for each zero \( c \) of \( f \).

(iii) There exists a unique solution \( x \) for each \( y \in X \) if and only if \( c \in \rho(A) \) for each zero of \( f \); note that \( D(A) = X \) if \( \infty \) is zero of \( f \).

As an example, let \( X = \{ f \in C[0,1] \mid f(0) = 0 \} \) and \( D(A) = \{ f \in X \mid f' \in X \} \), \( Af = f' \); in this case we have \( A \in C_D(X) \) and \( \rho(A) = C \) [11; pr. V.3.7]. If \( P \) is a polynomial and \( F \) is analytic on a neighborhood of \( \infty \), then \( f := P + F \in M(A) \) and \( [f(A)x](t) = [P(A)x](t) + F(\infty)x(t) + \int_0^t x(s)g(t-s)\,ds \), where \( g(t-s) = (1/2\pi i) \int_{|z|=r} F(z) e^{z(t-s)}\,dz \) with \( r > 0 \) sufficiently large.

REFERENCES