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Runge–Kutta methods with minimal dispersion and dissipation for problems arising from computational acoustics

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Abstract

In this paper a new Runge–Kutta method with minimal dispersion and dissipation error is developed. The Chebyshev pseudospectral method is utilized using spatial discretization and a new fourth-order six-stage Runge–Kutta scheme is used for time advancing. The proposed scheme is more efficient than the existing ones for acoustic computations.

Keywords: Dissipation; Dispersion; Computational acoustics

1. Introduction

There are many phenomena in nature that can be expressed by partial differential equations (PDEs). However, there is no general analytical solution for a well-defined PDE. As regards the wave propagation there are many recent works on numerical methods [2,7,4,8]. High-order methods are often used to reach accuracy requirements as well as low dissipation and dispersion errors [9].

Mead and Renaut [7] have constructed a six-stage fourth-order RK method with extended stability along the imaginary axes, which was of dissipative order five and of dispersive order four.

Hu et al. [4] propose a six-stage fourth-order RK method with minimal dissipation and dispersion.

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Bogey and Bailly [2] following the idea of Hu et al. [4], have obtained a six-stage second-order RK method with minimal dissipation and dispersion.

2. Basic theory

2.1. Modified Chebyshev pseudospectral method (MPS) [7]

Consider the one-dimensional wave equation $u_t = u_x$. If MPS is used in space, then

$$u_t = MSu \tag{1}$$

with

$$S_{i,j} = \frac{\mathrm{d}}{\mathrm{d}x} T_j(x)|_{x=x_i}, \tag{2}$$

where $T_j(x)$ and $x_i = \cos(i\pi/N)$ are the Chebyshev polynomials and the points respectively [7]. The entries of matrix M are dependent on the transformation proposed by Kosloff and Tal-Ezer [5], and are given by

$$M_{i,j} = \frac{\sin^{-1}(\alpha)\sqrt{1-(\alpha x_i)^2}}{\alpha}. \tag{3}$$

For our investigation the parameter α has been chosen to be equal to 0.99 (for details see [7]).

2.2. Dispersion and dissipation in Runge–Kutta methods

For the initial value problem

$$u_t = f(t,u) \tag{4}$$

the general s-stage Runge–Kutta method, is defined by

$$u_{n+1} = u_n + h\sum_{i=1}^{s} b_i k_i, \tag{5}$$

$$k_i = f\left(t_n + hc_i, u_n + h\sum_{j=1}^{s} a_{i,j}k_j\right), \tag{6}$$

where $c_i = \sum_{j=1}^{s} a_{i,j}$, $i = 1, \ldots, s$. The coefficients b_i, c_i, $a_{i,j}$ are dependent on the method used and can be presented by Butcher [3] table below

$$\begin{array}{c|c} c & A \\ \hline & b \end{array}$$

We use the linear test equation,

$$u_t = \lambda u, \quad \lambda = x + yi \tag{7}$$

which has the analytical solution

$$u(t+h) = \mathrm{e}^{h(x+yi)}u(t). \tag{8}$$

Following the procedure introduced by Albrecht [1], the RK solution can be written as

$$u_{n+1} = (1 + z\beta_1 + \cdots + z^s\beta_s)u_n = (P_s + \mathrm{i}F_s)u_n, \tag{9}$$

where $z = h\lambda$, $\beta_j = b^{\mathrm{T}}A^{j-1}e$, $e = (1, ..., 1) \in R^s$ (for more details see [7])

$$\begin{aligned} P_s =& 1 + hx\beta_1 + h^2\left(x^2 - y^2\right)\beta_2 + h^3\left(x^3 - 3xy^2\right)\beta_3, \\ &+ h^4\left(x^4 - 6x^2y^2 + y^4\right)\beta_4 + h^5\left(x^5 - 10x^3y^2 + 5xy^4\right)\beta_5 + \cdots, \\ F_s =& hy\beta_1 + 2h^2xy\beta_2 + h^3\left(3x^2y - y^3\right)\beta_3 + h^4\left(4x^3y - 4xy^3\right)\beta_4 \\ &+ h^5\left(5x^4y - 10x^2y^3 + y^5\right)\beta_5 + \cdots. \end{aligned} \tag{10}$$

The explicit form of the coefficients β_j for methods having up to six stages are given by (see [8])

$$\begin{aligned} &\beta_1 = \sum b_i, \quad \beta_4 = \sum b_i a_{ij} a_{jk} c_k, \\ &\beta_2 = \sum b_i c_i, \quad \beta_5 = \sum b_i a_{ij} a_{jk} a_{kl} c_l, \\ &\beta_3 = \sum b_i a_{ij} c_j, \quad \beta_6 = \sum b_i a_{ij} a_{jk} a_{kl} a_{lm} c_m. \end{aligned} \tag{11}$$

Definition 1. (Van Der Howen and Sommeijer [10]). The RK method defined by (9) is dissipative of order p if

$$\mathrm{e}^{xh} - |P_s + \mathrm{i}F_s| = \mathrm{O}(h^{p+1}) \tag{12}$$

and dispersive of order q if

$$hy - \tan^{-1}(F_s/P_s) = \mathrm{O}(h^{q+1}). \tag{13}$$

The proposed coefficients by Mead and Renaut [7] method are $\beta_5 = 0.00556$ and $\beta_6 = 0.00093$.

Hu et al. [4], minimizing the $|r - r_e|^2$, where $r = (u_{n+1})/u_n = 1 + z\beta_1 + \cdots + z^s\beta_s$ (9) and $r_e = u(t + h)/u(t) = \mathrm{e}^{h(x+yi)}$ (9), propose the coefficients: $\beta_5 = 0.00781005$ and $\beta_6 = 0.00132141$.

The proposed coefficients by Bogey and Bailly [2] method are $\beta_3 = 0.165919771368$, $\beta_4 = 0.040919732041$, $\beta_5 = 0.007555704391$ and $\beta_6 = 0.000891421261$.

3. New method

For a four order-six stage method and for the case of the test equation (8) with $x = 0$ [10], P_6 and F_6 can be written as

$$\begin{aligned} &P_6 = 1 - \beta_2(hy)^2 + \beta_4(hy)^4 - \beta_6(hy)^6, \\ &F_6 = hy - \beta_3(hy)^3 + \beta_5(hy)^5, \end{aligned} \tag{14}$$

where $\beta_2 = \frac{1}{2}$, $\beta_3 = \frac{1}{6}$ and $\beta_4 = \frac{1}{24}$, while the RK method is of order four.

Table 1

Order of dissipation	Order of dispersion	β_5	β_6
9	4	1/128	1/1152
5	8	1/120	1/840

Based on the definitions given above, the dissipation error can be written as

$$\mathrm{EDS}(hy) = 1 - |P_6 + iF_6| \tag{15}$$

and the dispersion error is given by

$$\mathrm{EDP}(hy) = hy - \tan^{-1}(F_6/P_6). \tag{16}$$

Expanding (15) and (16) via Taylor Series, we have

$$\begin{aligned}\mathrm{EDS}(hy) = {} & h^6y^6\left(\frac{1}{144} - \beta_5 + \beta_6\right) \\ & + h^8y^8\left(-\frac{1}{1152} + \frac{\beta_5}{6} - \frac{\beta_6}{2}\right) \\ & + h^{10}y^{10}\left(-\frac{\beta_5^2}{2} + \frac{\beta_6}{24}\right) \\ & + h^{12}y^{12}\left(\frac{1}{41472} + \frac{\beta_5^2}{2} + \beta_5\left(-\frac{1}{144} - \frac{\beta_6}{6}\right) + \frac{\beta_6}{144}\right) + \cdots,\end{aligned} \tag{17}$$

$$\begin{aligned}\mathrm{EDP}(hy) = {} & h^5y^5\left(\frac{1}{120} - \beta_5\right) \\ & + h^7y^7\left(-\frac{1}{336} + \frac{\beta_5}{2} - \beta_6\right) \\ & + h^9y^9\left(-\frac{1}{5184} - \frac{\beta_5}{24} + \frac{\beta_6}{6}\right) \\ & + h^{11}y^{11}\left(\frac{1}{19008} + \beta_5^2 + \beta_5\left(-\frac{1}{72} - \beta_6\right)\right) + \cdots.\end{aligned} \tag{18}$$

For a four order-six stage method, the maximum order of dissipation and dispersion and the resulting coefficients are provided in Table 1.

Definition 2. We define the estimation of the error for the dissipation error (SDS) and the estimation of the error for the dispersion error (SDP), using the coefficients of the powers of hy in expressions (17)

Table 2

Method	Error of dissipation	Error of dispersion	Total error
Mead and Renaut [7]	2.35×10^{-3}	2.99×10^{-3}	5.345×10^{-3}
F.Q. Hu et al. [4]	5.09×10^{-4}	6.60×10^{-4}	1.169×10^{-3}
Bogey and Bailly [2]	4.33×10^{-5}	8.495×10^{-4}	8.298×10^{-4}
New	1.48×10^{-4}	9.91×10^{-5}	2.470×10^{-4}

and (18), by the formulae:

$$\begin{aligned}\mathrm{SDS}(\beta_5,\beta_6)=&\left(\frac{1}{144}-\beta_5+\beta_6\right)^2+\left(-\frac{1}{1152}+\frac{\beta_5}{6}-\frac{\beta_6}{2}\right)^2\\&+\left(-\frac{\beta_5^2}{2}+\frac{\beta_6}{24}\right)^2+\left(\frac{1}{41472}+\frac{\beta_5^2}{2}+\beta_5\left(-\frac{1}{144}-\frac{\beta_6}{6}\right)+\frac{\beta_6}{144}\right)^2+\cdots,\end{aligned} \tag{19}$$

$$\begin{aligned}\mathrm{SDP}(\beta_5,\beta_6)=&\left(\frac{1}{120}-\beta_5\right)^2+\left(-\frac{1}{336}+\frac{\beta_5}{2}-\beta_6\right)^2\\&+\left(-\frac{1}{5184}-\frac{\beta_5}{24}+\frac{\beta_6}{6}\right)^2+\left(\frac{1}{19008}+\beta_5^2+\beta_5\left(-\frac{1}{72}-\beta_6\right)\right)^2+\cdots\end{aligned} \tag{20}$$

and the estimation of the total error (SDSDP) by the formula

$$\mathrm{SDSDP}(\beta_5,\beta_6)=\mathrm{SDS}(\beta_5,\beta_6)+\mathrm{SDP}(\beta_5,\beta_6). \tag{21}$$

Minimizing the SDSDP(β_5, β_6) using the Levenberg Marquardt method [6], the resulting coefficients are

$$\beta_5 = 0.0082673837508637 93 \quad \text{and} \quad \beta_6 = 0.00121166825454822479.$$

In Table 2 we present the estimation of the errors SDS, SDP and SDSDP of the new method, the Bogey and Bailly method [2], the Hu et al. method [4] and Mead and Renaut method [7]. In Fig. 1 we present the formulae EDS and EDP for the same methods.

For a six-stage fourth-order RK method we have a nonlinear system of 10 equations and 10 unknowns and can be reduced to 10 equations and 10 unknowns if we assume the method is of the form

$$\begin{array}{c|cccccc} 0 & 0 \\ c_2 & c_2 \\ c_3 & 0 & c_3 \\ c_4 & 0 & 0 & c_4 \\ c_5 & 0 & 0 & 0 & c_5 \\ c_6 & 0 & 0 & 0 & 0 & c_6 \\ \hline & b_1 & b_2 & b_3 & b_4 & 0 & b_6 \end{array}$$

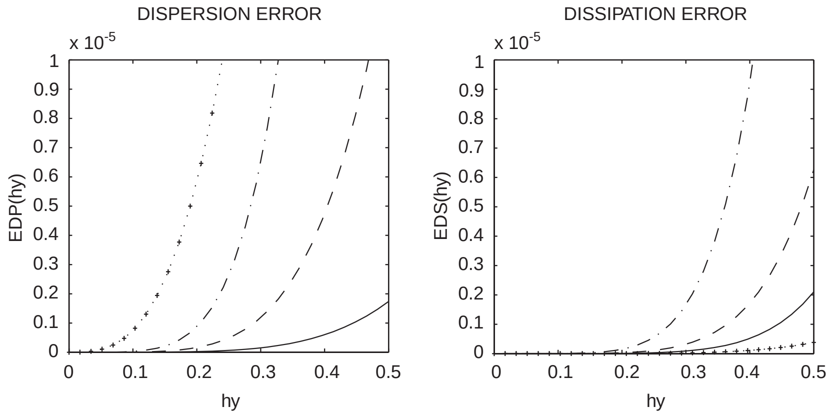

Fig. 1. (left)-Dispersion error, (right)-Dissipation error. Methods used: (i) · + ·, Bogey and Bailly [2] RK method of six stage-second order; (ii) - · -, Mead and Renaut [7] RK method of six stage-fourth order; (iii) - - - Hu et al. [4] RK method of six stage-fourth order; (iv) — New RK method of six stage-fourth order.

The values of the RK coefficients are given by

$$
\begin{aligned}
b_1 &= -3.948108158716446278687309660 01274,\\
b_2 &= 6.159333607199251372096155952597 97,\\
b_3 &= -8.744661007032283695137195023554 56,\\
b_4 &= 4.073877573976834298637571349895 27,\\
b_6 &= 3.459557984572644303090777381074 06,\\
c_2 &= 0.146560059513582781412187360597 05,\\
c_3 &= 0.271910317083483602336154516281 33,\\
c_4 &= 0.069368193985232337413393532103 66,\\
c_5 &= 0.258979400866361391119483868317 59,\\
c_6 &= 0.489210969984636592435769953273 96.
\end{aligned}
$$

In a similar way the values of the RK coefficients for the Hu et al. method [4] and Bogey and Bailly method [2] was obtained.

4. Numerical examples

The new method is compared with the Hu et al. method [4], Mead and Renaut method [7] and Bogey and Bailly method [2]. The modified Chebyshev pseudospectral method for the spatial discretization for $N = 270$ is used [7].

Table 3
Convective wave problem

Temporal approx.	Step size	Error	Time[a]
Bogey and Bailly [2]	0.2	4.84×10^{-4}	2 min 9 s
Mead and Renaut [7]	0.25	1.31×10^{-4}	1 min 46 s
F.Q. Hu et al. [4]	0.4	1.57×10^{-4}	1 min 7 s
New	0.5	6.68×10^{-5}	50 s

[a]Execution times, are for Fortran code running on a IBM 400 MHz system.

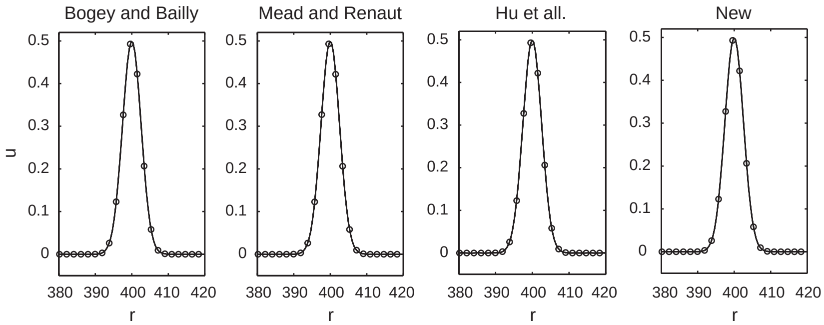

Fig. 2. Solution of convective wave problem (see [4,8]) with modified Chebyshev pseudospectral at $t = 400$ and $N = 270$ (see [7]). The true solution(-) and the computed solution (o). Methods used: (i) Bogey and Bailly [2] RK method of six stage-second order with $h = 0.2$; (ii) Mead and Renaut [7] RK method of six stage-fourth order with $h = 0.25$; (iii) Hu et al. [4] RK method of six stage-fourth order with $h = 0.4$ and; (iv) New RK method of six stage-fourth order with $h = 0.5$.

4.1. *Convective wave equation*

Consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{22}$$

The initial value when $t = 0$ is a Gaussian profile $u_0 = 0.5\mathrm{e}^{-\ln 2(x/3)^2}$ and the domain extends from $x = -50$ to $x = 450$. The maximum norm of the error $L_\infty = \max |u_{\text{calculated}} - u_{\text{exact}}|$ at the time $t = 400$, for several different values of step size h, is given in Table 3. Fig. 2 illustrates the solutions of the four compared methods.

4.2. *Spherical wave problem*

Consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0, \quad 5 \leqslant r \leqslant 315, \quad t > 0,$$

Table 4
Spherical wave problem

Temporal approx.	Step size	Error	Time[a]
Bogey and Bailly [2]	0.2	2.06×10^{-3}	1 min 25 s
Mead and Renaut [7]	0.2	2.02×10^{-3}	1 min 25 s
F.Q. Hu et al. [4]	0.2	1.99×10^{-3}	1 min 25 s
New	0.3	1.97×10^{-3}	54 s

[a]Execution times, are for Fortran code running on a IBM 400 MHz system.

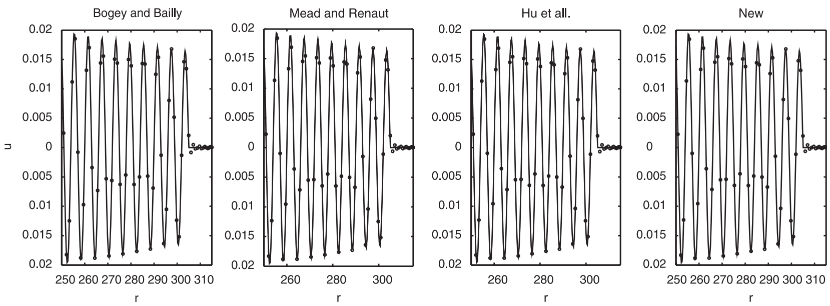

Fig. 3. Solution of spherical wave problem with modified Chebyshev pseudospectral at $t=300$ and $N=270$ (see [7]). The true solution(-) and the computed solution (o). Methods used: (i) Bogey and Bailly [2] RK method of six stage-second order with $h=0.2$; (ii) Mead and Renaut [7] RK method of six stage-fourth order with $h=0.2$; (iii) Hu et al. [4] RK method of six stage-fourth order with $h=0.2$ and; (iv) New RK method of six stage-fourth order with $h=0.3$.

$$u(r,0)=0, \quad 5\leqslant r\leqslant 315,$$

$$u(5,t)=\sin(\pi t/3), \quad 0<t<300.$$

The analytic solution is given by

$$u(r,t)=\begin{cases}0, & r>t+5,\\ 5[\sin(\pi(t-r+5)/3)]/r, & r\leqslant t+5.\end{cases}$$

The maximum norm of the error $L_\infty=\max|u_{\text{calculated}}-u_{\text{exact}}|$ at time $t=300$, for several different values of step size h, is given in Table 4. Fig. 3 illustrates the solutions of the four compared methods.

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