Criteria for the nonsingularity of partitioned matrices

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Abstract: From the applicational point of view, the most interesting criteria for the nonsingularity of a matrix are those which use the moduli of the elements and their simple combinations only. Some criteria of this type have been found by Pupkov, who generalized the Theorem of Hadamard and a result of Ostrowski. This paper generalizes the Pupkov criteria to the case of the partitioned matrices and presents some applications of the obtained results. Moreover, the paper generalizes some results of Pearce and Okuguchi concerning the matrices with dominating diagonal blocks.

Keywords: Partitioned complex matrix, nonsingular matrix, eigenvalues.

1. Preliminaries

Let \( N = \{1, \ldots, n\} \) be the set of indices and let \( \{ I_1, \ldots, I_p \} \), \( 1 \leq p \leq n \), be a partition of \( N \). Then, by \( A = (A_{i,j}) \), \( i, j = 1(1)p \), we denote a partitioned complex matrix, where \( A_{i,j} \) is a submatrix of \( A \) with typical entry \( a_{rs} \) such that \( r \in I_i \) and \( s \in I_j \). By \( \alpha_i \), \( i = 1(1)p \), we denote the smallest singular value of \( A_{i,i} \). In our considerations we will also use the following expressions:

\[
P_i(A) = \sum_{j \neq i} \| A_{i,j} \|, \quad i = 1(1)p,
\]

\[
Q_i = \sum_{j \neq i} \| A_{i,j} \|, \quad i = 1(1)p,
\]

where \( \| \cdot \| \) denotes the spectral matrix norm (the Euclidean vector norm depending on the argument of \( \| \cdot \| \)). We restrict ourselves to the spectral matrix norm, pointing out that it is possible to generalize the Pupkov criteria using Minkowski’s matrix norms induced by non-Euclidean vector norms [1,3]. Notice, that in such case it is not necessary to determine \( A_{i,i}^{-1} \), \( i = 1(1)p \), and therefore the obtained generalizations are useful in practice. Furthermore, by \( \Theta_i \), we denote the column null vector of dimension \( s \) and by \( (x_{I_1}, \ldots, x_{I_p})^T \) a vector partitioned with respect to \( \{ I_1, \ldots, I_p \} \).
2. Results

We start with the following result:

Lemma 2.1. If $A$ is singular, then for any partition $\{I_1, \ldots, I_p\}$, $1 \leq p \leq n$, of $N$ the following inequality holds:

$$\sum_{i=1}^{p} (\alpha_i - Q_{I_i}) \|x_{I_i}\| \leq 0,$$

where $x_{I_i}$ is the $i$th component of a nontrivial solution $x = (x_{I_1}, \ldots, x_{I_p})^T$ of the system $Ax = 0$.

Proof. Let $\{I_1, \ldots, I_p\}$ be a partition of $N$. Then, we can write (4) in the equivalent form

$$\sum_{j=1}^{p} A_{I_j} x_{I_j} = \Theta_I, \quad i = 1(1)p.$$

Taking into account the $i$th $(1 \leq i \leq p)$ equation of (5), we obtain

$$\|A_{I_i} x_{I_i}\| \leq \sum_{j \neq i} \|A_{I_j} x_{I_j}\|,$$

which, by applying an inequality of [9, p.111], takes the form

$$\alpha_i \|x_{I_i}\| \leq \sum_{j \neq i} \|A_{I_j} \| \cdot \|x_{I_j}\|.$$

Summing this over $i = 1, \ldots, p$, we get

$$\sum_{i=1}^{p} \alpha_i \|x_{I_i}\| \leq \sum_{i=1}^{p} \sum_{j \neq i} \|A_{I_j} \| \cdot \|x_{I_j}\|.$$

Changing the order of summation in the right-hand sum of (6) and using (2) we obtain (3).

Remark 2.1. If each $I_i$ consists of a single index the inequality (3) clearly reduces to the inequality

$$\sum_{i=1}^{n} \left( |a_{ii}| - \sum_{j \neq i} |a_{ji}| \right) \cdot |x_i| \leq 0,$$

given by Pupkov [6].

Theorem 2.1. Assume that for a complex matrix $A$ there exists a partition $\{I_1, \ldots, I_p\}$ of $N$ and positive numbers $d_{I_i}$, $i = 1(1)p$, such that for any $i = 1, \ldots, p$ the following inequality holds:

$$d_i \alpha_i > \min \left\{ \sum_{j \neq i} d_{I_j} \|A_{I_j}\|, d_i Q_{I_i} + \sum_{j \in N^-} d_{I_j} (Q_{I_j} - \alpha_j) \right\},$$

where $N^- = \{ k \in \{1, \ldots, p\} : \alpha_k \leq Q_{I_k} \}$.

Then $A$ is nonsingular.
Proof. Suppose the contrary. Then there exists a nontrivial solution \( x = (x_1, \ldots, x_p)^T \) of (4). Let \( D = \text{diag}(D_1, \ldots, D_p) \) be the diagonal partitioned matrix, where the submatrix \( D_i, i = 1(1)p \), is square diagonal matrix of order \( I_i \) with identical positive diagonal entries \( d_i \). Then we can write (4) as

\[
A' y = \Theta_n,
\]

where \( A' = (A_i'_{ij}) = AD = (A_i, D_i) \) is singular and \( y = (y_1, \ldots, y_p)^T = (d_i^{-1}x_i, \ldots, d_i^{-1}x_p)^T \) is nontrivial. Using the partition \( \{ I_1, \ldots, I_p \} \), we can write (8) as

\[
\sum_{j=1}^{p} A_i'_{ij} y_j = \Theta_{I_i}, \quad i = 1(1)p.
\]

By the nontriviality of \( y \) there exists a component \( y_{I_i} \) (1 \( \leq \) s \( \leq \) p) the Euclidean norm of which is positive and such that

\[
\| y_{I_s} \| \geq \| y_{I_j} \|, \quad j = 1(1)p.
\]

Assume that

\[
\sum_{j \neq s} d_{i,j} A_i'_{ij} \| y_{I_s} \| \leq \sum_{j \neq s} \| A_i'_{ij} \| \cdot \| y_{I_s} \|,
\]

where \( \alpha'_j \) is connected with \( A_i'_{ij} \). Dividing both sides of (11) by \( \| y_{I_s} \| \) and observing that

\[
\alpha'_j = d_{i,j} \alpha_j, \quad \| A_i'_{ij} \| = d_{i,j} \| A_i_{ij} \|, \quad j = 1(1)p,
\]

we arrive at a contradiction, provided (10) holds.

On the other hand, assuming that

\[
Q_{i'} d_{i,j} + \sum_{j \in N^-} d_{i,j} (Q_{i'} - \alpha_j) = \min \left\{ \sum_{j \neq s} d_{i,j} \| A_i_{ij} \|, d_{i,j} Q_{i'} + \sum_{j \in N^-} d_{i,j} (Q_{i'} - \alpha_j) \right\}
\]

and applying Lemma 2.1 to \( A' \), we get

\[
0 \geq \frac{1}{\| y_{I_s} \|} \sum_{i=1}^{p} (\alpha'_i - Q_{i'}) \| y_{I_s} \| \geq \alpha'_i - Q_{i'} + \sum_{j \in N^-} (\alpha'_j - Q_{i'_j}),
\]

which, by (12), contradicts (7). \( \square \)

Remark 2.2. When \( A \) is unpartitioned then, by Pupkov's Theorem 2, for the nonsingularity of \( A \) it suffices that for any \( i = 1, \ldots, n \) at least one of the following inequalities holds:

\[
|a_{ii}| > P_i(A),
\]

\[
|a_{ii}| + \sum_{j \in N^-} |a_{ij}| > Q_i + \sum_{j \in N^-} Q_j.
\]
So, Theorem 2.1 generalizes the Pupkov criterion. Moreover, Theorem 2.1 generalizes the
Theorem of Okuguchi [3] when A is partitioned.

**Theorem 2.2.** Assume that for a complex matrix A there exists a partition \( \{ I_1, \ldots, I_p \} \) of \( N \) and positive numbers \( d_{i,k} \), \( i = 1(1)p \), such that for any couple \( I_i, I_k \) \((i \neq k, i, k = 1(1)p)\) at least one of the following conditions holds:

\[
\begin{align*}
&d_i d_k \alpha_i \alpha_k > \left( \sum_{j \neq i}^p d_j \| A_{I_iI_j} \| \right) \left( \sum_{j \neq k}^p d_j \| A_{I_iI_j} \| \right), \\
&i, k \in N^+ \quad \text{and} \quad d_i \alpha_i + d_k \alpha_k + \sum_{j \in N^-} d_j \alpha_j > d_i Q_i + d_i Q_i + \sum_{j \in N^-} d_j Q_j,
\end{align*}
\]

\( i \in N^+ \), \( k \in N^- \), \( d_i \alpha_i + \sum_{j \in N^-} d_j \alpha_j > d_i Q_i + \sum_{j \in N^-} d_j Q_j \), and

\[
\alpha_k > (Q_k - \alpha_k) \left[ d_i \alpha_i - d_k Q_k + \sum_{j \in N^- \setminus \{k\}} d_j (\alpha_j - Q_j) \right]^{-1} \left( \sum_{j \neq k}^p d_j \| A_{I_iI_j} \| \right),
\]

where \( N^+ = \{ j \in \{1, \ldots, p\} : \alpha_j > Q_j \} \).

Then A is nonsingular.

**Proof.** Suppose the contrary. Then, by a reasoning similar to that of Theorem 2.1, there exists a nontrivial solution \( y = (y_{i_1}, \ldots, y_{i_p})^T \) of (8). Furthermore, a pair of components of \( y \), indexed by \( I_i \) and \( I_k \), satisfies

\[
\| y_{I_i} \| \geq \| y_{I_i} \| \geq \| y_{I_k} \| , \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, p,
\]

with \( \| y_{I_i} \| > 0 \). Indeed, if \( \| y_{I_i} \| = 0 \) for any \( i \neq s \), then, by the \( s \)th equation of (9), we get

\[
d_i \alpha_s \| y_{I_i} \| = 0.
\]

It implies that either \( d_i = 0 \) or \( \alpha_s = 0 \), which is a contradiction (by the assumption all \( \alpha_i \) are positive). So, \( \| y_{I_i} \| > 0 \).

Assume that for the couple \( I_i, I_s \), (15) holds. Then, taking into account the \( s \)th and \( t \)th equation of (9), by (18), we obtain

\[
\alpha_s \| y_{I_i} \| \leq P_{I_i}(A') \| y_{I_i} \| , \quad \alpha_t \| y_{I_i} \| \leq P_{I_i}(A') \| y_{I_i} \| ,
\]

which implies

\[\alpha_s \alpha_t \leq P_{I_i}(A') P_{I_i}(A').\]

This, together with (1) and (12), leads us to a contradiction.

Assume that for the couple \( I_i, I_s \), (16) holds. Then, using (18) and applying Lemma 2.1 to \( A' \), we obtain

\[
0 > \frac{1}{\| y_{I_i} \|} \sum_{i=1}^p \left( \alpha_s - Q_s i \right) \| y_{I_i} \| \]

\[
\geq \left( \alpha_s - Q_s i + \frac{\| y_{I_i} \|}{\| y_{I_i} \|} (\alpha_s - Q_s i) \right) \sum_{j \in N^-} (\alpha_j - Q_j) \| y_{I_i} \| \]

\[
\geq \frac{\| y_{I_i} \|}{\| y_{I_i} \|} \left( \alpha_s - Q_s i + \alpha_t - Q_t i + \sum_{j \in N^-} (\alpha_j - Q_j) \right),
\]

which, together with (12), contradicts the inequality in (16).
Finally, assume that for the couple \( I_i, I_s \), (17) holds. If \( t \in N^+, s \in N^- \),
\[
d_i a_t + \sum_{j \in N^-} d_j a_j > d_i Q_{i_t} + \sum_{j \in N^-} d_j Q_{i_j},
\]
than, similarly as above, we obtain
\[
0 \geq \alpha'_t - Q'_i + \frac{\|y_{i_t}\|}{\|y_{i_t}\|} \left( \alpha'_t - Q'_i + \sum_{j \in N^- \setminus \{s\}} (\alpha'_j - Q'_j) \right),
\]
which, after slight manipulations, yields
\[
\frac{\|y_{i_t}\|}{\|y_{i_t}\|} \leq \left( Q'_i - \alpha'_t \right) \left[ \alpha'_t - Q'_i + \sum_{j \in N^- \setminus \{s\}} (\alpha'_j - Q'_j) \right]^{-1}.
\]
Observing that, by (18) and the \( s \)th equation of (9), we get
\[
\alpha'_s \| y_{i_t} \| \leq P_i(A') \| y_{i_t} \|,
\]
and multiplying both sides of (19) by \( P_i(A') \), (19) becomes
\[
\alpha'_t \leq \left( Q'_i - \alpha'_t \right) \left[ \alpha'_t - Q'_i + \sum_{j \in N^- \setminus \{s\}} (\alpha'_j - Q'_j) \right]^{-1} P_i(A').
\]
The last inequality, together with (17) and (1), contradicts the inequality in (17). If \( t \in N^-, s \in N^+ \),
\[
d_i a_t + \sum_{j \in N^-} d_j a_j > d_i Q_{i_t} + \sum_{j \in N^-} d_j Q_{i_j},
\]
then, using (18) and applying Lemma 2.1 to \( A' \), we obtain
\[
\alpha'_t \leq \left( Q'_i - \alpha'_t \right) \left[ \alpha'_t - Q'_i + \sum_{j \in N^- \setminus \{s\}} (\alpha'_j - Q'_j) \right]^{-1} P_i(A').
\]
The last inequality, together with (17) and (1), contradicts the inequality in (17). Remarks 2.3. If each \( I_i \) consists of a single index, the conditions (15)-(17) clearly reduce to the following ones:
\[
d_i d_k | a_{i_t} | | a_{k_k} | > P_i(A) P_k(A),
\]
\[
\text{if } i, k \in N^+, d_i | a_{i_t} | + d_k | a_{k_k} | + \sum_{j \in N^-} d_j | a_{j_j} | > d_i Q_i + d_k Q_k + \sum_{j \in N^-} d_j Q_j.
\]
\[
\text{if } i \in N^+, k \in N^-, d_i | a_{i_t} | + \sum_{j \in N^-} d_j | a_{j_j} | > d_i Q_i + \sum_{j \in N^-} d_j Q_j,
\]
and
\[
| a_{k_k} | > (Q_k - | a_{k_k} |) \left[ d_i | a_{i_t} | - Q_i d_i + \sum_{j \in N^- \setminus \{k\}} d_j (| a_{j_j} | - Q_j) \right]^{-1},
\]
and the theorem does not coincide with Theorem 5 of Pupkov [6] (differs from the Pupkov result in the third condition).
3. Applications

Now similarly as in [5], we apply our results to localize the eigenvalues of $A$.

**Theorem 3.1.** Assume that for a complex matrix $A$ there exists a partition $\{ I_1, \ldots, I_p \}$ of $N$ such that the diagonal submatrices $A_{I_i I_i}$ are Hermitian positive (negative) definite and that $A$ satisfies the assumptions of Theorem 2.1 or 2.2 with $d_i = 1$, $i = 1(1)p$. Then, all eigenvalues of $A$ have positive (negative) real parts.

**Proof.** Assume that $A$ satisfies the assumptions of Theorem 2.1 and apply a reasoning similar to that of [5, Theorem 3]. So, suppose the contrary. Let $\lambda = \rho + i\mu$, $i = \sqrt{-1}$, be a complex eigenvalue of $A$ with nonpositive (nonnegative) real part. Consider the smallest singular value of $A_{I_i I_i} - \lambda J_{I_i}$, $i = 1(1)p$, where $J_{I_i}$ is the identity matrix of order $I_i$. So, taking into account the positive (negative) definiteness of $A_{I_i I_i}$, a direct calculation yields

$$
(A_{I_i I_i} - \lambda J_{I_i})^H(A_{I_i I_i} - \lambda J_{I_i}) = A_{I_i I_i}^2 - 2\rho A_{I_i I_i} + |\lambda|^2 J_{I_i},
$$

where $A_{I_i I_i}^H$ is the conjugate transpose of $A_{I_i I_i}$.

Therefore, be the nonpositivity (nonnegativity) of $\rho$, the smallest singular value of $A_{I_i I_i} - \lambda J_{I_i}$ is not less than the one of $A_{I_i I_i}$. Notice that a similar reasoning can be repeated for $A$ satisfying the assumptions of Theorem 2.2. $\Box$

**Corollary 3.1.** Theorem 3.1 yields a criterion for $A$ to be a stable matrix and, in the case when off-diagonal entries of $A$ are nonpositive, to be an $M$-matrix.

**Proof.** The proof follows by the definition of a stable matrix and by a result of Varga [9], respectively. $\Box$

**Remark 3.1.** Theorem 3.1, with the assumptions of Theorem 2.1, generalizes (coincides with) Theorem 3 of Pearce [5] when $A$ is partitioned and non-Hermitian (Hermitian).

Before stating our next result we recall, for the convenience of the reader, the definition of positive (negative) quasi-definiteness.

**Definition** (Murata [2, p.61]). An $n \times n$ real matrix $A$ is said to be positive (or negative) quasi-definite if $x^T Ax$ is positive (or negative) for any real nonzero column $n$-vector $x$, or equivalently, if $A + A^T$ is positive (or negative) definite since

$$
x^T Ax = \frac{1}{2}(x^T (A + A^T) x).
$$

**Theorem 3.2.** Let $A$ be a real square matrix. Assume that there exists a partition $\{ I_1, \ldots, I_p \}$ of $N$ such that $A + A^T$ satisfies, with $d_i = 1$, the assumptions of Theorem 2.1 or 2.2 and its diagonal submatrices are positive (negative) definite. Then

(i) $A$ is positive (negative) quasi-definite,

(ii) all eigenvalues of $A$ have positive (negative) real parts,

(iii) all rth-order principal minors of $A$ are positive (of sign $(-1)^r$).
Proof. At first we notice that if \( A + A^T \) satisfies the assumptions of Theorem 2.1 with \( d_i = 1 \) and its diagonal submatrices are negative definite, the assertion coincides with [5, Theorem 5]. So, assume that \( A + A^T \) satisfies the assumptions of Theorem 2.2. Then, by a reasoning similar to that of Theorem 3.1, it follows that all eigenvalues of \( A + A^T \) are positive (negative) which, together with the definition, proves (i).

It is known [2, p.61] that all eigenvalues of a negative quasi-definite matrix have negative real parts. This, together with the observation that if \(-A\) is negative quasi-definite, then \(A\) is positive quasi-definite, proves (ii).

Finally, observe that the positive (negative) quasi-definiteness of \(A\) implies the same property of any principal submatrix of \(A\). So, by the reasoning employed above, all eigenvalues of any principal submatrix of \(A\) have positive (negative) real parts, which proves (iii) \(\square\)

Corollary 3.2. Theorem 3.2 yields a criterion for \(A\) to be a P-matrix (an NP-matrix).

Proof. The proof follows by (iii) and the definition of a P-matrix (an NP-matrix) (see e.g. [1, p.88]). \(\square\)

We close this section by the following result:

Theorem 3.3. Let \(A\) be a square complex matrix; assume that there exists a partition \(\{I_1, \ldots, I_p\}\) of \(N\) such that the diagonal square submatrices of order \(I_i\) in \(A^HA - I\) satisfy, with \(d_i = 1\), the assumptions of Theorem 2.1 or 2.2 and are negative (positive) definite. Then all eigenvalues of \(A\) lie within (outside) the unit circle of the complex plane.

Proof. At first we notice that if \(A^HA - I\) satisfies, with \(d_i = 1\), the assumptions of Theorem 2.1, then the assertion coincides with [5, Theorem 6]. So, assume that \(A^HA - I\) satisfies the assumptions of Theorem 2.2. Then, applying a reasoning similar to that of [5, Theorem 6], by Theorem 3.1, all eigenvalues of \(A^HA - I\) are negative (positive) and equal to the eigenvalues of \(A^HA\) minus 1. But the eigenvalues of \(A^HA\) are equal to the squares of the singular values of \(A\). Observing that the greatest of them is equal to the spectral norm of \(A\), the assertion follows by the Hirsch Theorem [7, p.395] (it is well known that the smallest singular value of \(A\) is not greater than the smallest, with respect to the modulus, eigenvalue of it). \(\square\)

References