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Estimates for eigenvalues of quasilinear elliptic systems

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Abstract

In this paper we introduce the generalized eigenvalues of a quasilinear elliptic system of resonant type. We prove the existence of infinitely many continuous eigencurves, which are obtained by variational methods. For the one-dimensional problem, we obtain an hyperbolic type function defining a region which contains all the generalized eigenvalues (variational or not), and the proof is based on a suitable generalization of Lyapunov's inequality for systems of ordinary differential equations. We also obtain a family of curves bounding by above the kth variational eigencurve.
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1. Introduction

In this work we are interested in the problem of find lower and upper bounds for the eigenvalues of nonlinear elliptic systems. There are various bounds for eigenvalues of a single elliptic equation, not necessarily linear, based on different techniques, see for example [2,15,17,20,21,23,24,30] and the references therein. However, the situation is different for elliptic systems, and there are few results in this case. We could cite the work of Protter [25],

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who introduced the notion of generalized spectrum, the $\lambda = (\lambda_1, \dots, \lambda_m) \in C^m$ such that the following system

$$\sum_{i=1}^{m} L_{ij} u + \lambda_j \sum_{i=1}^{m} r_{i,j} u_i = 0, \quad 1 \leqslant j \leqslant m, \tag{1.1}$$

has a nontrivial solution $u = (u_1, \dots, u_m)$ subject to a set of homogeneous boundary conditions, where L_{ij} are linear elliptic operators. Then, the generalized spectrum was extended in [6,7] to more general elliptic systems. With the same approach of Boggio and Barta, generalizing his own work [24], Protter obtained the following results:

(i) there exists a positive value $r_\Omega \in \mathbb{R}$ such that $r_\Omega < \sum_i \lambda_i^2$ for every λ in the generalized spectrum, and
(ii) for every $\lambda \in C^m$, if Ω is contained in a ball of sufficiently small radius, its size depending only on the coefficients, then there are no nontrivial solutions of the system.

Also, Cosner [8] considered the nonlinear eigenvalue problem,

$$\sum_{i=1}^{m} L_{ij} + \lambda f(x, u) = 0, \quad 1 \leqslant j \leqslant m, \tag{1.2}$$

where $f(x, u)$ is a nonlinearity satisfying certain growth conditions. He obtained relationships between the norm of a solution and λ, by using the Faber–Krahn and Sobolev inequalities (see [19]).

For a single p-Laplacian equation, some lower bounds were obtained by using an appropriate generalization of the Boggio inequality in [16]. Moreover, symmetrization techniques were applied to obtain bounds of eigenvalues in [3,9], see also the references therein. However, we are not aware of similar works for p-Laplacian systems.

We consider here a quasilinear elliptic system of resonant type:

$$\begin{cases} -\Delta_p u = \lambda \alpha r(x) |u|^{\alpha-2} u |v|^{\beta}, \\ -\Delta_q v = \mu \beta r(x) |u|^{\alpha} |v|^{\beta-2} v, \end{cases} \quad x \in \Omega, \tag{1.3}$$

where the functions u and v satisfy a Dirichlet boundary condition

$$u(x) = v(x) = 0, \quad x \in \partial\Omega.$$

Here, $\Omega \in \mathbb{R}^n$ is a domain with smooth boundary $\partial\Omega$, $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2} \nabla u)$, the exponents satisfy $1 < p, q < +\infty$, and the positive parameters α, β satisfy

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1. \tag{1.4}$$

For brevity, we restrict ourselves to only two equations, but the results follows with minor changes for m equations.

The eigenvalues of quasilinear elliptic system has deserved a great deal of attention in the last years, we may cite the works of Boccardo and de Figueiredo [4], Manasevich and Mawhin [18], Stavrakakis and Zographopoulos [28,31], and the references therein.

We define the generalized spectrum $\mathcal{S}$ of a nonlinear elliptic system as the set of pairs $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$ such that the eigenvalue problem (1.3) admits a nontrivial solution.

The diagonal $\lambda = \mu$ of the generalized spectrum coincides with the eigenvalues of the elliptic system considered in [14]. For $\lambda = \mu$, the eigenvalue problem (1.3) has infinitely many eigenfunctions given by:

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{(u,v) \in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q}{\int_\Omega r(x)|u|^\alpha |v|^\beta}, \tag{1.5}$$

where $\mathcal{C}_k$ is the class of compact symmetric ($C = -C$) subsets of the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of genus greater or equal than k. We recall that for $C \in \mathcal{C}$ the Krasnoselskii genus $\mathrm{gen}(C)$ is defined as the minimum integer n such that there exists an odd continuous mapping $\varphi : C \to (\mathbb{R}^n - \{0\})$ (see for example [1]). However, it is not known if this set of eigenvalues exhaust the spectrum.

First, we will prove the existence of a sequence of continuous curves emanating from the eigenvalues $\{\lambda_k\}$:

Theorem 1.1. *There exist a sequence of continuous curves $(\lambda_k(t), \mu_k(t))$ emanating from (λ_k, λ_k), where λ_k is the kth variational eigenvalue given by* (1.5)*.*

The existence follows by adapting the arguments of Cantrell and Cosner in [7], which are similar to the techniques used for the Fucik spectrum by Cuesta, de Figueiredo and Gossez [10]. By fixing a line $\mu = t\lambda$, we find a set of variational eigenvalues $\{\lambda_k(t), \mu_k(t)\}_k$ in the generalized spectrum, and the continuity is proved varying the parameter t.

Also, we give an upper bound of the first variational eigenvalue $(\lambda_1(t), \mu_1(t))$:

Theorem 1.2. *Let $p > q$, $r(x) \geqslant m > 0$. Then, the first variational eigenvalue of problem* (1.3) *in the line $\mu = t\lambda$ satisfies*

$$\lambda_1 \leqslant \frac{\Lambda_1}{p} + \frac{m^{-1+q/p}}{qt}\left(\frac{p}{q}\right)^q \Lambda_1^{q/p},$$

where Λ_1 is the first variational eigenvalue of the Dirichlet problem

$$-\Delta_p u = \lambda r(x)|u|^{p-2}u$$

on the same domain Ω.

Moreover, we extend Theorem 1.2 obtaining upper bounds for all the variational eigenvalues in $\mathcal{S}$ in the one-dimensional case:

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \lambda\alpha r(x)|u|^{\alpha-2}u|v|^\beta, \\ -(|v'(x)|^{q-2}v'(x))' = \mu\beta r(x)|u|^\alpha|v|^{\beta-2}v \end{cases} \tag{1.6}$$

on the interval (a, b), with Dirichlet boundary conditions.

This generalization is possible due to the nodal domain structure of the kth eigenvalue of a single equation. We conjecture that the same result must valid in $\mathbb{R}^n$. We have the following theorem:

Theorem 1.3. *Let $p > q$, $r(x) \geqslant m > 0$. Then, the kth variational eigenvalue of problem* (1.6) *in the line $\mu = t\lambda$ satisfies*

$$\lambda_k(t) \leqslant \frac{\Lambda_k}{p} + \frac{m^{-1+q/p}}{qt}\left(\frac{p}{q}\right)^q \Lambda_k^{q/p},$$

where Λ_k is the kth variational eigenvalue of the Dirichlet problem

$$-\left(\left|u'(x)\right|^{p-2}u'(x)\right)' = \lambda r(x)|u|^{p-2}u \tag{1.7}$$

on the same interval (a, b).

Since the eigenvalues of (1.7) when $r(x) = 1$ have been computed explicitly by Drabek and Manasevich [13],

$$\Lambda_k = \left(\frac{k\pi_p}{b-a}\right)^p,$$

we obtain an explicit upper bound for the one-dimensional system whenever $r(x) \geqslant m > 0$.

Finally, we extend the results of Protter to the one-dimensional system (1.6), and we improve the lower bounds on the eigenvalues. Instead of a ball, we find an hyperbola type curve enclosing the region which contains the eigenvalues. Our main result is the following theorem:

Theorem 1.4. *There exist a function $h(\lambda)$ such that $\mu \geqslant h(\lambda)$ for every generalized eigenvalue (λ, μ) of problem* (1.6)*, where $h(\lambda)$ is given by*:

$$h(\lambda) = \frac{1}{\beta}\left(\frac{C}{\lambda^{\alpha/p}\int_a^b r(x)\,dx}\right)^{q/\beta},$$

and the constant C is given by

$$C = \frac{2^{\alpha+\beta}}{\alpha^{\alpha/p}(b-a)^{\alpha/p'+\beta/q'}}.$$

The proof is based on the following extension of the Lyapunov inequality for systems:

Theorem 1.5. *Let us assume that there exists a positive solution of the system*

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x)|u|^{\alpha-2}u|v|^{\beta}, \\ -(|v'(x)|^{q-2}v'(x))' = g(x)|u|^{\alpha}|v|^{\beta-2}v \end{cases}$$

on the interval (a,b)*, with Dirichlet boundary conditions. Then, we have that*:

$$2^{\alpha+\beta} \leqslant (b-a)^{\alpha/p'+\beta/q'} \left(\int_a^b f(x)\,dx \right)^{\alpha/p} \left(\int_a^b g(x)\,dx \right)^{\beta/q}. \tag{1.8}$$

To our knowledge, there are no previous work on Lyapunov inequalities for elliptic systems. Lyapunov inequalities for linear problems can be found in [26], see the references therein for different proofs.

By using this inequality, we prove a lower bound for the first eigenvalue in each line $\lambda = \mu t$. This enable us to define the curve bounding the region which contains the generalized spectrum. Hence, combining this result with Theorem 1.2, we have obtained a region containing the first curve in the generalized spectrum, which cannot be too close to the origin, nor too far.

The paper is organized as follows: In Section 2 we review some well-known results that we shall need in the sequel, we introduce the generalized spectrum, and we prove Theorem 1.1. In Section 3 we prove Theorems 1.2 and 1.3. Section 4 is devoted to the Lyapunov inequality, and we prove Theorems 1.4 and 1.5.

2. Existence of generalized eigenvalues

In this section we prove Theorem 1.1. In order to obtain the existence of eigenvalues in each line $\mu = t\lambda$, we shall apply the following abstract theorem due to H. Amann [1], and the continuity of the eigenvalue curve follows from Lemma 2.3.

Theorem 2.1. *Suppose that the following hypotheses are satisfied*:

- *X is a real Banach space of infinite dimension, that is uniformly convex.*
- *$A: X \to X^*$ is an odd potential operator (i.e. A is the Gateaux derivative of $\mathcal{A}: X \to \mathbb{R}$) which is uniformly continuous on bounded sets, and satisfies the condition $(S)_1$: If $u_j \rightharpoonup u$ (weakly in X) and $A(u_j) \to v$, then $u_j \to u$ (strongly in X).*
- *For a given constant $\alpha > 0$ the level set*

$$M_m = \{u \in X\colon\ \mathcal{A}(u) = m\}$$

is bounded and each ray through the origin intersects M_m. Moreover, for every $u \neq 0$, $\langle A(u), u\rangle > 0$ and there exists a constant $\rho_m > 0$ such that $\langle A(u), u\rangle \geqslant \rho_m$ on M_m.

- *The mapping $B: X \to X^*$ is a strongly sequentially continuous odd potential operator (with potential $\mathcal{B}$), such that $\mathcal{B}(u) \neq 0$ implies that $B(u) \neq 0$.*

Let

$$\beta_k = \sup_{C \in \mathcal{C}_k,\, C \subset M_m} \inf_{u \in C} \mathcal{B}(u),$$

where $\mathcal{C}_k$ is the class of compact symmetric ($C = -C$) subsets of the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of genus greater or equal than k.

Then if $\beta_k > 0$, there exists an eigenfunction $u_k \in M_m$ with $\mathcal{B}(u) = \beta_k$.

If

$$\gamma\big(\{u \in M_m\colon \mathcal{B}(u) \neq 0\}\big) = \infty, \tag{2.1}$$

where the genus over compact sets $\gamma(S)$ *is defined by*:

$$\gamma(S) = \sup\big\{\mathrm{gen}(C)\colon C \subset S,\ C \in \mathcal{C},\ C \textit{ compact}\big\},$$

then there exist infinitely many eigenfunctions.

In our problem, we shall work in the Banach space:

$$W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$$

equipped with the norm

$$\big\|(u,v)\big\|_W = \sqrt{\|u\|_p^2 + \|v\|_q^2}.$$

As each factor is uniformly convex, we can conclude that W is uniformly convex (see [11]). Given $(u^*, v^*) \in W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega)$ we may think it as an element of W^*:

$$\big\langle (u^*, v^*), (u, v)\big\rangle = \langle u^*, u\rangle + \langle v^*, v\rangle.$$

Then we have $W^* \cong W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega)$ (isometric isomorphism), where the norm in W^* is given by:

$$\big\|(u^*, v^*)\big\|_{W^*} = \sqrt{\|u^*\|^2 + \|v^*\|^2}.$$

With the notations of Theorem 2.1, we define:

$$\mathcal{A}_t(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{tq}\int_\Omega |\nabla v|^q,$$

$$A_t(u,v) = \left(-\Delta_p u, \frac{-1}{t}\Delta_q v\right),$$

$$\mathcal{B}(u,v) = \int_\Omega r(x)|y|^\alpha |v|^\beta,$$

$$B(u,v) = \big(r(x)\alpha|u|^{\alpha-2}u|v|^\beta, |u|^\alpha\beta|v|^{\beta-2}v\big). \tag{2.2}$$

It is easy to check that these functionals satisfy all the conditions of Theorem 2.1 by adapting the arguments in [13]. Hence, for each t, we obtain a sequence $\beta_k(t)$ given by:

$$\beta_k(t) = \sup_{C\in\mathcal{C}_k,\, C\in M_m(t)}\ \inf_{(u,v)\in C} \mathcal{B}(u,v),$$

where

$$M_m(t) = \{(u, v) \in W\colon \mathcal{A}_t(u, v) = \alpha\},$$

and the kth eigenvalue is given by

$$\lambda_k(t) = 1/\beta_k(t).$$

However, we introduce a different characterization of the eigenvalues, by fibering the space W with the level sets of the functional $\mathcal{B}$ instead of the functional $\mathcal{A}_t$. Hence, we may define a sequence of eigenvalues $\hat{\lambda}_k(t)$ by the Rayleigh quotient,

$$\hat{\lambda}_k(t) = \inf_{C \in \mathcal{C}_k} \sup_{(u,v) \in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{qt}\int_\Omega |\nabla v|^q}{\int_\Omega r(x)|u|^\alpha |v|^\beta}.$$

This approach is due to Browder [5], and following Riddell [27], it is easy to see that

$$\lambda_k(t) = \hat{\lambda}_k(t).$$

In fact, it is clear that

$$\beta_k(t) \leqslant \sup_{C \in \mathcal{C}_k,\, C \subset W - \{0\}} \inf_{(u,v) \in C} \frac{\mathcal{B}(u, v)}{\mathcal{A}_t(u, v)}.$$

On the other hand, if $C \subset W - \{0\}$, we construct a set $\tilde{C}$ in M_m by taking the image of C by the retraction

$$(u, v) \mapsto \left(\frac{u}{\mathcal{A}_t^{1/p}(u, v)}, \frac{v}{\mathcal{A}_t^{1/q}(u, v)}\right),$$

where the different powers is due to the (p, q) homogeneity of $\mathcal{B}(u, v)$. Then $\operatorname{gen}(C) = \operatorname{gen}(\tilde{C})$ and

$$\inf_{u \in C} \frac{\mathcal{B}(u, v)}{\mathcal{A}_t(u, v)} = \inf_{(u,v) \in \tilde{C}} \mathcal{B}(u, v).$$

Hence,

$$\beta_k(t) = \sup_{C \in \mathcal{C}_k,\, C \subset W - \{0\}} \inf_{(u,v) \in C} \frac{\mathcal{B}(u, v)}{\mathcal{A}_t(u, v)}.$$

It follows that

$$\frac{1}{\beta_k(t)} = \inf_{C \in \mathcal{C}_k,\, C \subset W - \{0\}} \sup_{(u,v) \in C} \frac{\mathcal{A}_t(u, v)}{\mathcal{B}(u, v)}.$$

We conclude that $\lambda_k(t) = \hat{\lambda}_k(t)$.

Remark 2.2. By the regularity theory of Tolksdorf and Dibenedetto [12,29], the generalized eigenfunctions are of class $C^{1,a}(\Omega)$ for some $0 < a < 1$. This fact will be used in Section 3 in the proof of Theorem 1.5.

In order to conclude the proof of Theorem 1.1, let us prove now the continuity of the eigenvalue curve with respect to t.

Lemma 2.3. *The curve* $(\lambda_k(t), \mu_k(t))$ *is continuous. Moreover,* $\lambda_k(t)$ (*respectively* $\mu_k(t)$) *is nonincreasing* (*respectively nondecreasing*) *in* t.

Proof. Clearly, since

$$\lambda_k(t) = \inf_{C \in \mathcal{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{qt}\int_\Omega |\nabla v|^q}{\int_\Omega r(x)|u|^\alpha |v|^\beta},$$

and $1/qt$ is decreasing, $\lambda_k(t)$ is nonincreasing.

Let us prove now the continuity of $\lambda_k(t)$. We fix t_0, and for every $\varepsilon > 0$, there exist a compact symmetric set C_ε such that:

$$\sup_{(u,v)\in C_\varepsilon} \frac{\mathcal{A}_{t_0}(u,v)}{\mathcal{B}(u,v)} \leqslant \lambda_k(t_0) + \varepsilon. \tag{2.3}$$

We consider now t_1, and we have:

$$\lambda_k(t_1) \leqslant \sup_{(u,v)\in C_\varepsilon} \frac{\mathcal{A}_{t_1}(u,v)}{\mathcal{B}(u,v)} \leqslant \sup_{(u,v)\in C_\varepsilon} \left| \frac{\mathcal{A}_{t_1}(u,v)}{\mathcal{B}(u,v)} - \frac{\mathcal{A}_{t_0}(u,v)}{\mathcal{B}(u,v)} \right| + \lambda_k(t_0) + \varepsilon.$$

Since C_ε is compact, there exist $(u_\varepsilon, v_\varepsilon)$ where the supremum

$$\sup_{(u,v)\in C_\varepsilon} \left| \frac{\mathcal{A}_{t_1}(u,v)}{\mathcal{B}(u,v)} - \frac{\mathcal{A}_{t_0}(u,v)}{\mathcal{B}(u,v)} \right| = \sup_{(u,v)\in C_\varepsilon} \frac{|t_1 - t_0|}{t_0 t_1} \frac{\int_\Omega |\nabla v|^q}{\mathcal{B}(u,v)}$$

is attained.

Therefore,

$$\lambda_k(t_1) \leqslant \frac{|t_1 - t_0|}{t_0 t_1 q} \frac{\int_\Omega |\nabla v_\varepsilon|^q}{\mathcal{B}(u_\varepsilon, v_\varepsilon)} + \lambda_k(t_0) + \varepsilon.$$

Let us note that, from Eq. (2.3), the term

$$\frac{1}{t_0 q} \frac{\int_\Omega |\nabla v_\varepsilon|^q}{\mathcal{B}(u_\varepsilon, v_\varepsilon)}$$

is bounded by $\lambda_k(t_0) + \varepsilon$, and

$$\lambda_k(t_1) \leqslant \lambda_k(t_0) + \varepsilon + \big(\lambda_k(t_0) + \varepsilon\big) \frac{|t_1 - t_0|}{t_1}. \tag{2.4}$$

Hence, interchanging the roles of t_0 and t_1, we get

$$\left|\lambda_k(t_0)-\lambda_k(t_1)\right| \leqslant \varepsilon + \begin{cases} (\lambda_k(t_0)+\varepsilon)\frac{|t_1-t_0|}{t_1}, \\ (\lambda_k(t_1)+\varepsilon)\frac{|t_1-t_0|}{t_0}. \end{cases}$$

By using inequality (2.4), we have:

$$\left|\lambda_k(t_0)-\lambda_k(t_1)\right| \leqslant \varepsilon + \begin{cases} (\lambda_k(t_0)+\varepsilon)\frac{|t_1-t_0|}{t_1}, \\ \left([\lambda_k(t_0)+\varepsilon]\left[1+\frac{|t_1-t_0|}{t_1}\right]+\varepsilon\right)\frac{|t_1-t_0|}{t_0}. \end{cases}$$

Clearly, when $t_1 \to t_0$, both terms goes to zero, and since the ε is arbitrary, the lemma follows. $\square$

Remark 2.4. Let us observe that the beautiful proof of [7, Theorem 3.3] cannot be adapted to our problem since it is no clear that there exists a compact symmetric set of genus greater or equal than k where the kth variational eigenvalue is attained.

3. Upper bounds for generalized eigenvalues

As usual, it is easier to find upper bounds for eigenvalues than lower bounds. In fact, the results in this section follows by using elementary inequalities.

Proof of Theorem 1.2. We will find an upper bound for the first eigenvalue of the system (1.3) in a fixed line $\mu = t\lambda$. We call it $\lambda_1(t)$, and the generalized eigenvalue is $(\lambda_1(t), t\lambda_1(t))$.

We note that

$$\lambda_1(t) \leqslant \frac{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{qt}\int_\Omega |\nabla v|^q}{\int_\Omega r(x)|u|^\alpha |v|^\beta}.$$

We choose $u=\varphi_1$ and $v=\varphi_1^{p/q}$, where φ_1 denotes the first Dirichlet eigenfunction of the single equation:

$$-\Delta_p w = \Lambda r(x)|w|^{p-2}w$$

in Ω with the normalization

$$\int_\Omega r(x)|\varphi_1|^p = 1.$$

Hence, Λ_1 is the first eigenvalue of the single equation, which is given by

$$\Lambda_1 = \int_\Omega |\nabla\varphi_1|^p.$$

Now,

$$\nabla v = \frac{p}{q}\varphi_1^{p/q-1}\nabla\varphi_1$$

and we get:

$$\int_\Omega r(x)|u|^\alpha|v|^\beta = \int_\Omega r(x)|\varphi_1|^p = 1.$$

On the other hand, by Hölder inequality:

$$\begin{aligned}\int_\Omega |\nabla v|^q &= \int_\Omega \left|\frac{p}{q}\varphi_1^{p/q-1}\nabla\varphi_1\right|^q = \int_\Omega \left(\frac{p}{q}\right)^q |\varphi_1|^{p-q}|\nabla\varphi_1|^q \\ &\leqslant \left(\frac{p}{q}\right)^q \left(\int_\Omega |\nabla\varphi_1|^p\right)^{1/s}\left(\int_\Omega |\varphi_1|^{s'(p-q)}\right)^{1/s'}\end{aligned}$$

with $s = p/q$. Hence,

$$s'(p-q) = p$$

and we get

$$\lambda_1 \leqslant \frac{\Lambda_1}{p} + \frac{m^{-1+q/p}}{qt}\left(\frac{p}{q}\right)^q \Lambda_1^{q/p}. \qquad \square$$

We close the section with the proof of Theorem 1.3. Let us note that the proof could be easily adapted to the problem in $\mathbb{R}^n$ whenever the kth eigenvalue of a single equation has exactly k nodal domains. This fact is well known for the second eigenvalue (see [8]); it is also possible to find a domain where the first k eigenvalues satisfy this property, but is not valid for every eigenvalue even in the linear case $p = 2$, see [22].

Proof of Theorem 1.3. We will find an upper bound for the kth variational eigenvalue $(\lambda_k(t), t\lambda_k(t))$ in the fixed line $\mu = t\lambda$ as in Theorem 1.2.

Let Λ_k be the kth variational eigenvalue of

$$-\left(\left|u'(x)\right|^{p-2}u'(x)\right)' = \lambda r(x)|u|^{p-2}u,$$

and let φ_k be the corresponding eigenfunction. Then, by the Sturm–Liouville theory for the p-Laplacian (see [30]), the eigenfunction φ_k has $k+1$ zeros at $\{x_j\}_{j=0}^k$, $a = x_0 < x_1 < \cdots < x_k = b$.

Let $w_i(x) = \varphi_k(x)$ if $x \in (x_{i-1}, x_i)$, and $w_i(x) = 0$ elsewhere. Let S_m be the sphere in $W^{1,p}(\Omega)$ of radius m. Then, the set

$$C^k = \left(\operatorname{span}\{w_1, \dots, w_k\} \cap S_m\right) \times \left\{\left|\varphi_k(x)\right|^{p/q-1}\varphi_k(x)\right\} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$$

has genus k, and is admissible in the variational characterization of $(\lambda_k(t), t\lambda_k(t))$.

Now,

$$\lambda_k(t) = \inf_{C\in\mathcal{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q^t}\int_a^b |v'|^q}{\int_a^b r(x)|u|^\alpha |v|^\beta} \leqslant \sup_{(u,v)\in C^k} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q^t}\int_a^b |v'|^q}{\int_a^b r(x)|u|^\alpha |v|^\beta}, \qquad (3.1)$$

and replacing u, v we obtain

$$\lambda_k(t) \leqslant \frac{\Lambda_k}{p} + \frac{m^{-1+q/p}}{qt}\left(\frac{p}{q}\right)^q \Lambda_k^{q/p},$$

and the proof is finished. □

4. The Lyapunov inequality and lower bounds

In this section we obtain lower bounds for the first eigencurve in the generalized spectra, using similar techniques to the ones in [25]. We prove first a version of the classical Lyapunov inequality for elliptic systems of resonant type.

Proof of Theorem 1.5. Let us consider the system

$$\begin{cases} -(|u'|^{p-2}u')' = f(x)|u|^{\alpha-2}|v|^\beta u, \\ -(|v'|^{q-2}v')' = g(x)|u|^\alpha |v|^{\beta-2} v, \end{cases}$$

with Dirichlet boundary conditions:

$$u(a) = u(b) = v(a) = v(b) = 0,$$

and

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1. \qquad (4.1)$$

For any $c \in [a,b]$, we have:

$$2|u(c)| = \left|\int_a^c u(x)\,dx\right| + \left|\int_c^b u(x)\,dx\right| \leqslant \int_a^b |u'(x)|\,dx.$$

By using the Hölder inequality,

$$2|u(c)| \leqslant (b-a)^{1/p'}\left(\int_a^b |u'(x)|^p\,dx\right)^{1/p} = (b-a)^{1/p'}\left(\int_a^b f(x)|u(x)|^\alpha |v(x)|^\beta\,dx\right)^{1/p},$$

where $1/p + 1/p' = 1$. Now choosing c as the point where $|u(x)|$ is maximum, and d as the point where $|v(x)|$ is maximum, we have that:

$$2\left|u(c)\right| \leqslant (b-a)^{1/p'}\left|u(c)\right|^{\alpha/p}\left|v(d)\right|^{\beta/p}\left(\int_a^b f(x)\,dx\right)^{1/p}, \tag{4.2}$$

$$2\left|v(d)\right| \leqslant (b-a)^{1/q'}\left|u(c)\right|^{\alpha/q}\left|v(d)\right|^{\beta/q}\left(\int_a^b g(x)\,dx\right)^{1/q}. \tag{4.3}$$

Raising Eq. (4.2) to a power e_1, Eq. (4.3) to a power e_2 and multiplying the resulting equations, we obtain:

$$2^{e_1+e_2} \leqslant (b-a)^{e_1/p'+e_2/q'}\left|u(c)\right|^{(\alpha/p-1)e_1+(\alpha/q)e_2}\left|v(d)\right|^{(\beta/p)e_1+(\beta/q-1)e_2}$$
$$\times\left(\int_a^b f(x)\,dx\right)^{e_1/p}\left(\int_a^b g(x)\,dx\right)^{e_2/q}.$$

Now we choose $e_1$and e_2 such that $|u(c)|$ and $|u(d)|$ cancels out, i.e. e_1, e_2 solves the homogeneous linear system:

$$\begin{cases} \left(\frac{\alpha}{p}-1\right)e_1+\frac{\alpha}{q}e_2=0, \\ \frac{\beta}{p}e_1+\left(\frac{\beta}{q}-1\right)e_2=0. \end{cases}$$

We observe that by condition (4.1) this system admits a nontrivial solution, indeed both equations are equivalent to

$$e_1\beta = e_2\alpha.$$

Hence, we may take $e_1=\alpha$, $e_2=\beta$ and we get:

$$2^{\alpha+\beta} \leqslant (b-a)^{\alpha/p'+\beta/q'}\left(\int_a^b f(x)\,dx\right)^{\alpha/p}\left(\int_a^b g(x)\,dx\right)^{\beta/q},$$

which proves Theorem 1.5. □

Remark 4.1. Let us observe that for $\beta=0$ and $\alpha=p$ (or $\alpha=0$, $\beta=q$) we get the result for the case of a single equation.

Let us prove now Theorem 1.4:

Proof of Theorem 1.4. Let (λ,μ) be a generalized eigenpair, and u, v the corresponding nontrivial solutions. We have:

$$\begin{cases} -(|u'(r)|^{p-2}u'(r))' = \lambda\alpha r(x)|u|^{\alpha-2}|v|^{\beta}u, \\ -(|v'(r)|^{q-2}v'(r))' = \mu\beta s(x)|u|^{\alpha}|v|^{\beta-2}v. \end{cases}$$

Now, let us call

$$M = \frac{2^{\alpha+\beta}}{(b-a)^{\alpha/p'+\beta/q'}}.$$

Hence, by replacing in the Lyapunov inequality (1.8) the functions

$$f(x) = \lambda\alpha r(x), \qquad g(x) = \mu\beta r(x),$$

we have:

$$M \leqslant \left(\int_a^b \lambda\alpha r(x)\,dx\right)^{\alpha/p} \left(\int_a^b \mu\beta r(x)\,dx\right)^{\beta/q}.$$

Rearranging the terms, and by using condition (1.4), we obtain

$$M \leqslant (\lambda\alpha)^{\alpha/p}(\mu\beta)^{\beta/q} \int_a^b r(x)\,dx$$

which gives

$$\left(\frac{M}{(\lambda\alpha)^{\alpha/p}\int_a^b r(x)\,dx}\right)^{q/\beta} \leqslant \mu\beta.$$

Hence, we have

$$\mu \geqslant \frac{1}{\beta}\left(\frac{C}{\lambda^{\alpha/p}\int_a^b r(x)\,dx}\right)^{q/\beta},$$

where $C = M/\alpha^{\alpha/p}$, and the proof is finished. □

Remark 4.2. Since h is a continuous function, and $h(\lambda) \to +\infty$ as $\lambda \to 0^+$, it is clear that there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior.

Also, by using that

$$\mu^{\beta/q}\lambda^{\alpha/p} \geqslant \frac{1}{\beta}\left(\frac{C}{\int_a^b r(x)\,dx}\right),$$

and the right-hand side goes to infinity when the interval collapses, we obtain the desired generalizations of Protter results for one-dimensional nonlinear elliptic systems.

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