Stability analysis of linear parabolic systems 
and removal of singularities in substructure:
Static feedback

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Abstract
We analyze stability property of a class of linear parabolic systems via static feedback. Stabilization via 
static feedback scheme is most difficult and challenging when both actuators and observation weights admit 
spillovers. This arises typically in the boundary observation–boundary feedback scheme. We propose a 
simple static feedback law containing a parameter $\gamma$, and enhance the stability property or achieve (slightly) 
stabilization. In some situations, the evolution of the substructure of finite dimension contains singularities 
regarding $\gamma$. We show that these singularities are removed as long as the dimension is not large.
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1. Introduction

In the last two decades the study of feedback stabilization for parabolic systems has gathered much attention both from mathematical and practical viewpoints. Let $H$ be a Hilbert space 
equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\| \cdot \|$. The control system with state $u$ is the 
differential equation in $H$ described by

$$\frac{du}{dt} + Lu = \sum_{k=1}^{N} f_k(t)h_k, \quad t > 0, \quad u(0) = u_0.$$  

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Here, $f_k(t)$ denote inputs; $h_k$ actuators; and $L$ a linear closed operator with dense domain $\mathcal{D}(L)$ such that the resolvent $(\lambda - L)^{-1}$ satisfies the decay estimate

$$
\|(\lambda - L)^{-1}\|_{\mathcal{L}(H)} \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \Sigma,
$$

where $\Sigma$ denotes some sector described by $\Sigma = \{\lambda - b; \theta_0 \leq |\arg \lambda| \leq \pi\}, 0 < \theta_0 < \pi/2, b \in \mathbb{R}$. The output of the system is a finite number of observations with weights $w_k \in H$:

$$
\langle u, w_k \rangle_H, \quad 1 \leq k \leq N. \quad (1.2)
$$

Setting $f_k(t) = \langle u, w_k \rangle_H, 1 \leq k \leq N$, we have the closed-loop feedback control system

$$
\frac{du}{dt} + Lu = \sum_{k=1}^{N} \langle u, w_k \rangle_H h_k, \quad t > 0, \quad u(0) = u_0. \quad (1.3)
$$

Let us briefly review the stabilization scheme in the literature. Given a prescribed $\mu > 0$, it is assumed that the set $\sigma(L) \cap \{\lambda \in \mathbb{C}; \Re \lambda < \mu\}$ consists only of the eigenvalues. The projection operator associated with these eigenvalues is denoted as $P$ with $\dim P < \infty$. The problem is to construct the $w_k$ and the $h_k$ in order that

$$
\|\exp(-t(L - \sum_{k=1}^{N} \langle \cdot, w_k \rangle_H h_k))\|_{\mathcal{L}(H)} \leq Me^{-\mu t}, \quad t \geq 0,
$$

where $M > 0$ denotes a constant depending on $\mu$. The problem is partially solved (see, for example, [7,12]), if

(i) $(L|_{PH}, \{Ph_1, \ldots, Ph_N\})$ is a controllable pair and the $w_k$ are freely constructed in the subspace $PH$; or as the dual assumption

(ii) $(L|_{PH}, \{Pw_1, \ldots, Pw_N\})$ is an observable pair and the $h_k$ are freely constructed in $PH$,

where $L|_{PH}$ denotes the restriction of $L$ onto the invariant subspace $PH$. Once the $w_k$ or the $h_k$ are constructed in $PH$, the classical perturbation arguments admit very small spillovers $(1 - P)w_k$ or $(1 - P)h_k$, the quantities of which are determined by the finite-dimensional structure. However, their construction in order that

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\|\exp(-t(L - \sum_{k=1}^{N} \langle \cdot, w_k \rangle_H h_k))\|_{\mathcal{L}(H)} \leq Me^{-\mu t}, \quad t \geq 0,
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(i) $(L|_{PH}, \{Ph_1, \ldots, Ph_N\})$ is a controllable pair and the $w_k$ are freely constructed in the subspace $PH$; or as the dual assumption

(ii) $(L|_{PH}, \{Pw_1, \ldots, Pw_N\})$ is an observable pair and the $h_k$ are freely constructed in $PH$,
such as in flexible structures. Although the static feedback scheme in (1.3) looks simple, the stabilization problem remains unsolved when the $w_k$ and the $h_k$ essentially contain spillovers.

Based on these observations, we study in this paper the stability improvement or the stabilization of (1.3) in the essential presence of the spillovers of the $w_k$ and the $h_k$, and generalize the result in [7, 12] to some extent. More precisely, when the $h_k$ satisfy the finite-dimensional controllability conditions, we construct the $w_k$ such that the stability property is improved and enhanced to some extent. In our study both finite- and infinite-dimensional structures are important factors. Especially the evolution of the semigroup in the finite-dimensional substructure plays the central role. The precise assumptions on the spectrum is that $\sigma(L)$ consists of two disjoint closed sets $\sigma_1$ and $\sigma_2$: $\sigma(L) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Here,

(i) $\sigma_1$ consists only of the eigenvalues $\lambda_i, 1 \leq i \leq n$, on the vertical line: $\text{Re}\, \lambda = \omega$.

(ii) For each $\lambda_i, 1 \leq i \leq n$, there is a set of the eigenvectors $\phi_{ij}, 1 \leq j \leq m_i (< \infty)$, such that the set $\{\phi_{ij}\}_{j=1}^{m_i}$ forms a basis for the subspace

$$\frac{-1}{2\pi \sqrt{-1}} \int_{C_i} (\lambda - L)^{-1} H d\lambda,$$

where $C_i$ denotes a small contour encircling $\lambda_i$.

(iii) $\min_{\lambda \in \sigma_2} \text{Re}\, \lambda > \omega$.

Let $\gamma > 0$ denote a gain parameter. By setting $f_k(t) = -\gamma \langle u, w_k \rangle_H$ in (1.1), our control system is, instead of (1.3), described as

$$\frac{du}{dt} + Lu = -\gamma \sum_{k=1}^{N} \langle u, w_k \rangle_H h_k, \quad t > 0, \quad u(0) = u_0. \quad (1.4)$$

When there is no control action, the semigroup of the unperturbed equation satisfies the estimate

$$\| e^{-tL} \|_{\mathcal{L}(H)} \leq c e^{-\omega t}, \quad t \geq 0. \quad (1.5)$$

Henceforth $c$ with or without subscript will denote a various positive constant. We show that the power $\omega$ is improved a little for the perturbed equation (1.4) in the essential presence of the spillovers of the $w_k$ and the $h_k$.

Some readers might be afraid that the control system (1.4) would not reflect the boundary observation–boundary feedback scheme. Let us show that a class of problems in this scheme is reduced to (1.4) with slight technical modifications. Let $\Omega$ be a bounded domain in $\mathbb{R}^m$ with the boundary $\Gamma$ which consists of a finite number of smooth components of $(m-1)$-dimension. The boundary control system with state $u(t, \cdot)$ is described by the differential equation

$$\begin{cases}
\frac{\partial u}{\partial t} + \mathcal{L} u = 0 & \text{in } \mathbb{R}_+^1 \times \Omega, \\
\tau u = -\gamma \sum_{k=1}^{N} \langle u, w_k \rangle_{\Gamma} h_k & \text{on } \mathbb{R}_+^1 \times \Gamma, \\
u(0, \cdot) = u_0(\cdot) & \text{in } \Omega.
\end{cases} \quad (1.6)$$
Here, $(\mathcal{L}, \tau)$ denotes the pair of differential operators defined by

$$\mathcal{L} u = - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u, \quad \tau u = \frac{\partial u}{\partial \nu} + \sigma(\xi) u, \quad (1.7)$$

and $a_{ij}(x) = a_{ji}(x)$ for $1 \leq i, j \leq m$, $x \in \overline{\Omega}$; for some positive $\delta$

$$\sum_{i,j=1}^{m} a_{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2, \quad \forall \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, \forall x \in \overline{\Omega};$$

and $\partial u/\partial \nu = \sum_{i,j=1}^{m} a_{ij}(\xi) v_i(\xi) \partial u/\partial x_j |_{\Gamma}$, where $(v_1(\xi), \ldots, v_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary conditions on the coefficients $a_{ij}$, $b_i$, $c$, and $\sigma$ are tacitly assumed. Set $H = L^2(\Omega)$. The inner products in $L^2(\Omega)$ and in $L^2(\Gamma)$ are denoted as $\langle \cdot, \cdot \rangle_{\Omega}$ and $\langle \cdot, \cdot \rangle_{\Gamma}$, respectively. As usual (see [1]), set $Lu = \mathcal{L} u$, $\mathcal{D}(L) = \{u \in H^2(\Omega); \tau u = 0 \text{ on } \Gamma\}$. Choose a large constant $c > 0$. It is well known that (see [3,4])

$$\mathcal{D}(L^\omega_c) = H^{2\omega}(\Omega), \quad 0 \leq \omega < \frac{3}{4}, \quad L_c = L + c.$$

Set $x(t) = L_c^{-1/4-\epsilon} u(t)$, where $0 < \epsilon < \frac{1}{4}$. Then $x(t)$, $t > 0$, belongs to $\mathcal{D}(L)$, and satisfies the differential equation in $H = L^2(\Omega)$:

$$\frac{dx}{dt} + Lx = -\gamma \sum_{k=1}^{N} \langle L_c^{1/4+\epsilon} x, w_k \rangle_{\Gamma} L_c^{3/4-\epsilon} \varphi_k, \quad x(0) = x_0 = L_c^{-1/4-\epsilon} u_0, \quad (1.8)$$

where $\varphi_k \in H^2(\Omega)$, $1 \leq k \leq N$, denote the unique solutions to the boundary value problems:

$$(c + \mathcal{L}) \varphi_k = 0 \quad \text{in } \Omega, \quad \tau \varphi_k = h_k \quad \text{on } \Gamma.$$

Introduce the operator $M$ by

$$M = L + \gamma \sum_{k=1}^{N} \langle L_c^{1/4+\epsilon} \cdot, w_k \rangle_{\Gamma} L_c^{3/4-\epsilon} \varphi_k, \quad \mathcal{D}(M) = \mathcal{D}(L).$$

By choosing a larger constant $c > 0$ if necessary, both $L_c$ and $M_c = M + c$ are $m$-accretive. Thus we see that (see [6])

$$\mathcal{D}(M_c^\omega) = \mathcal{D}(L_c^\omega), \quad 0 \leq \omega \leq 1.$$

Solutions $u(t, \cdot)$ to (1.6) are then expressed by

$$u(t, \cdot) = L_c^{\alpha/2} M_c^{-\alpha/2} e^{-tM} M_c^{\alpha/2} L_c^{-\alpha/2} u_0, \quad t \geq 0, \quad (1.9)$$
where $\alpha = \frac{1}{2} + 2\epsilon$, and both $L^{\alpha/2}_c M^{-\alpha/2}_c$ and $M^{\alpha/2}_c L^{-\alpha/2}_c$ are bounded. Thus the problem is reduced to the problem of the analytic semigroup $e^{-tM}$. In the operator $M$, the unboundedness arising from the boundary observation is merely of technical nature.

One of the main results is stated in Section 2, where the problem (1.8) is also discussed in parallel. We consider in Section 3 the case where the assumption of the $h_k$ is somewhat weakened. As long as the eigenvalues in $\sigma_1$ satisfy some restrictive algebraic conditions, we achieve the same result as in Section 2 (a part of these results have been reported in [9] without complete proofs). In Section 4, we consider the more general case where these algebraic restrictions are not fulfilled: We face, however, a difficulty: Singularities in $\gamma$ arise in some components of the evolution $e^{-t(\Lambda + H W)}$ in the substructure of finite dimension. It is shown that these singularities are cancelled and thus removed as long as the dimension is low.

2. Main result I

According to the assumptions on $\sigma(L)$, let $P$ denote the projection operator associated with the eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$P = \sum_{i=1}^{n} \frac{1}{2\pi \sqrt{-1}} \int_{C_i} (\lambda - L)^{-1} d\lambda.$$  

The subspace $PH$ is invariant relative to $L$. Set $L_1 = L|_{PH}$ and $L_2 = L|_{QH}$ with $D(L_2) = D(L) \cap QH$, where $Q = 1 - P$. We mainly consider the problem of (1.4), and then extend the result—via technical modifications—to the case of the boundary control system (1.6). By setting $u_1 = Pu$ and $u_2 = Qu$, (1.4) is decomposed into the system of differential equations:

$$\begin{cases} 
\frac{du_1}{dt} + L_1 u_1 = -\gamma \sum_{k=1}^{N} \langle u_1, Pw_k \rangle_H Ph_k - \gamma \sum_{k=1}^{N} \langle u_2, Qw_k \rangle_H Ph_k, \\
\frac{du_2}{dt} + L_2 u_2 = -\gamma \sum_{k=1}^{N} \langle u_1, Pw_k \rangle_H Qh_k - \gamma \sum_{k=1}^{N} \langle u_2, Qw_k \rangle_H Qh_k.
\end{cases} \tag{2.1}$$

We will rewrite (2.1) in appropriate form. According to the basis $\{\varphi_{ij}; \ 1 \leq i \leq n, \ 1 \leq j \leq m_i\}$ for the subspace $PH$, the quantities $u_1, h_k$, and $L_1$ in the first equation are equivalent to

$$u = \begin{pmatrix} u_{11} \\
\vdots \\
u_{ij} \\
\vdots \\
u_{nm_n}
\end{pmatrix}, \quad h_k = \begin{pmatrix} h_{11}^k \\
\vdots \\
h_{ij}^k \\
\vdots \\
h_{nm_n}^k
\end{pmatrix}, \quad 1 \leq k \leq N, \quad \text{and}$$

$$\Lambda = \text{diag}(\Lambda_1 \ \Lambda_2 \ \ldots \ \Lambda_n), \quad \Lambda_i = \text{diag}(\lambda_i \ \lambda_i \ \ldots \ \lambda_i),$$

where $\lambda_i = \frac{1}{2} \lambda_i + 2\epsilon_i$, and both $L^{\alpha/2}_c M^{-\alpha/2}_c$ and $M^{\alpha/2}_c L^{-\alpha/2}_c$ are bounded.

respectively. In the second equation let $F$ be the operator in $QH$ defined by

$$F = L_2 + \gamma \sum_{k=1}^{N} \langle \cdot, Qw_k \rangle Qh_k, \quad D(F) = D(L_2). \quad (2.2)$$

Then (2.1) is rewritten as the system of differential equations in $C^S \times QH$ ($S = m_1 + \cdots + m_n$):

$$
\begin{align*}
\frac{du}{dt} + (A + \gamma HW)u &= -\gamma \sum_{k=1}^{N} \langle u_2, Qw_k \rangle h_k, \quad u(0) = u_0, \\
\frac{du_2}{dt} + Fu_2 &= -\gamma (Qh_1 \cdots Qh_N)Wu, \quad u_2(0) = Qu_0,
\end{align*}
$$

(2.3)

where

$$H = (h_1 \ h_2 \ \cdots \ h_N) \quad \text{and} \quad W = \left( \langle \phi_{ij}, w_k \rangle_H; \quad k \downarrow 1, \ldots, N \quad (i, j) \to (1, 1), \ldots, (n, m_n) \right).$$

Setting

$$H_i = \left( h_{ij}^k; \quad k \to 1, \ldots, N \quad j \downarrow 1, \ldots, m_i \right) \quad \text{and} \quad W_i = \left( \langle \phi_{ij}, w_k \rangle_H; \quad k \downarrow 1, \ldots, N \quad j \to 1, \ldots, m_i \right),$$

we have the expression

$$H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix}, \quad W = (W_1 \ W_2 \ \cdots \ W_n).$$

The above matrices $H$ and $W$ are, respectively, the so-called controllability and the observability matrices. Changing the order of $\lambda_i$ if necessary, we may assume with no loss of generality that

$$m_1 \geq m_2 \geq \cdots \geq m_n. \quad (2.4)$$

Our first result is stated as follows:

**Theorem 2.1.** Set $N = m_1$. In (2.3), choose the $w_k$ and the $h_k$ such that

$$H_i = (H_{i1} \ 0), \quad H_{i1}; \quad m_i \times m_i, \quad \text{rank} \ H_i = m_i, \quad \text{and} \quad W_i = \begin{pmatrix} H_{i1}^{-1} \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n. \quad (2.5)$$

Then, as long as $\gamma > 0$ is small enough, there exist a constant $c > 0$ which is independent of $\gamma$ and an $O(\gamma^2)$, such that

$$\left\| \exp \left( -t \left( L + \gamma \sum_{k=1}^{N} \langle \cdot, Qw_k \rangle Qh_k \right) \right) \right\|_{\mathcal{L}(H)} \leq ce^{-(\omega + \gamma + O(\gamma^2))t}, \quad t \geq 0. \quad (2.6)$$
Remark. The essential difference between our result and those in the literature lies in the construction of the $w_k$ and the $h_k$ (see Section 1): The only requisite in our assertion is that the $w_k$ satisfy the finite-dimensional conditions: $W_i = (H_i^{-1})$. The resultant spillovers $Qw_k$ and $Qh_k$ are the quantities that we cannot manipulate in general: Thus they cannot remain in $PH$.

Corollary 2.2. Consider the simplest case where $m_1 = m_2 = \cdots = m_n$. Set $N = m_1$, and suppose that

$$\text{rank } H_i = m_i, \quad 1 \leq i \leq n.$$  \tag{2.7}

Choose the $w_k$ so that $W_i = H_i^{-1}$, $1 \leq i \leq n$. Then, as long as $\gamma > 0$ is small enough, there exists an $O(\gamma^2)$ such that the estimate (2.6) holds.

Proof of Theorem 2.1. Equation (2.3) is rewritten as the system of integral equations which is described by

$$\begin{cases}
  u(t) = e^{-t(\Lambda + \gamma HW)} u(0) - \gamma \int_0^t e^{-(t-s)(\Lambda + \gamma HW)} \sum_{k=1}^N \langle u_2(s), Qw_k \rangle_H h_k ds, \\
  u_2(t) = e^{-tF} u_2(0) - \gamma \int_0^t e^{-(t-s)F} (Qh_1 \cdots Qh_N)W u(s) ds.
\end{cases} $$  \tag{2.8}

Combining these equations, we will derive an integral inequality for $|u(t)|$.

Choose an arbitrary $\beta$ such that $\omega < \beta < \min_{\lambda \in \sigma_2} \text{Re } \lambda$. Then note that

$$\| e^{-tL_2} \|_{L(H)} \leq M_1 e^{-\beta t}, \quad t \geq 0.$$  \tag{2.9}

It is immediately seen via the standard perturbation argument that

$$\| e^{-tF} \|_{L(H)} \leq M_1 e^{-(\beta - M_1 c_1 \gamma)t}, \quad t \geq 0, \quad c_1 = \sum_{k=1}^N \| Qw_k \| \| Qh_k \|. $$  \tag{2.10}

The eigenvalues of $\Lambda + \gamma HW$ are nonlinear functions of $\gamma$. According to the choice of the $w_k$, we have the following proposition which forms the key to the theorem. The proof is to be given later.

Proposition 2.3. There exist a constant $M > 0$ and an $O(\gamma^2)$ such that

$$\| e^{-t(\Lambda + \gamma HW)} \|_{L(\mathcal{C}^S)} \leq Me^{-(\omega + \gamma + O(\gamma^2))t}, \quad t \geq 0,$$  \tag{2.11}

where $M$ is independent of $\gamma$.

Remark. If the number of the $w_k$ and the $h_k$ is increased, a much better estimate is obtainable. However, we are attempting the improvement of stability with the smallest number of them.
Set $\alpha(\gamma) = \gamma + O(\gamma^2)$. When $\gamma$ is small, we may assume that

$$\omega + \alpha(\gamma) < \beta - M_1 c_1 \gamma.$$ 

Based on the estimates (2.10), (2.11), and the integral equation (2.8), we can derive

$$|u(t)| \leq M e^{-(\omega+\alpha(\gamma)t)}|u(0)| + \frac{MM_1 c_3 \gamma}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} e^{-(\omega + \alpha(\gamma)t)}\|u_2(0)\|$$

$$+ MM_1 c_2 c_3 \gamma^2 \int_0^t K(t - \sigma)|u(\sigma)|d\sigma,$$

where

$$c_2 = \sum_{k=1}^N \|Pw_k\|\|Qh_k\|, \quad c_3 = \sum_{k=1}^N \|Qw_k\|\|Ph_k\|,$$

and

$$K(t) = \int_0^t e^{-(\omega+\alpha(\gamma))(t-\tau)} e^{-(\beta-M_1 c_1 \gamma)\tau} d\tau < \frac{e^{-(\omega+\alpha(\gamma)t)}}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)}, \quad t \geq 0.$$

Thus the estimate (2.12) is rewritten as

$$|u(t)| \leq M_2 e^{-(\omega+\alpha(\gamma)t)}\|u_0\| + \frac{MM_1 c_2 c_3}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} \gamma^2 \int_0^t e^{-(\omega+\alpha(\gamma)(t-\sigma))}|u(\sigma)|d\sigma.$$

Applying Gronwall’s inequality to the above, we see that

$$|u(t)| \leq M_2 \|u_0\| \exp\left((-\omega + \gamma + O(\gamma^2) - \frac{MM_1 c_2 c_3}{\beta - \omega - M_1 c_1 \gamma - \alpha(\gamma)} \gamma^2) t\right), \quad t \geq 0.$$

By going back to the equation for $u_2$ in (2.8), this leads to a similar estimate for $\|u_2(t)\|$. Thus we have proven the desired estimate (2.6).

**Proof of Proposition 2.3.** We calculate $e^{-t(\Lambda + \gamma HW)}$ according to the formula

$$e^{-t(\Lambda + \gamma HW)} = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} e^{-i\lambda(\Lambda - \gamma HW)^{-1}} d\lambda,$$

where $\Gamma$ denotes a counterclockwise contour encircling $\sigma(\Lambda + \gamma HW)$. We need to calculate and estimate the residue of the integrand at each singularity. Set
\[
\begin{pmatrix}
\lambda - \lambda_1 - \gamma & -\gamma & \cdots & -\gamma \\
-\gamma & \lambda - \lambda_2 - \gamma & \cdots & -\gamma \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma & -\gamma & \cdots & \lambda - \lambda_i - \gamma
\end{pmatrix}
\begin{pmatrix}
\lambda & a_{i1} & \cdots & a_{i1} \\
& a_{i2} & \cdots & a_{i2} \\
& \vdots & \ddots & \vdots \\
& a_{i1} & a_{i2} & \cdots & a_{ii}
\end{pmatrix}^{-1}
\]
\[
1 \leq i \leq n.
\] (2.15)

Extending the \( a_{ij}^k \) as
\[
a_{ij}^k = 0, \quad \text{if } i > k \text{ or } j > k,
\]
define the \( N \times N \) \((= m_1 \times m_1)\) diagonal matrices \( A_{ij} \) as
\[
A_{ij} =
\begin{pmatrix}
a_{ij}^n I_{m_n} & O & \cdots & O \\
O & a_{ij}^{n-1} I_{m_{n-1}-m_n} & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & a_{ij}^1 I_{m_1}-m_2
\end{pmatrix}.
\] (2.16)

Then we can show

**Lemma 2.4.** The resolvent of \( \Lambda + \gamma HW \) is expressed as
\[
(\lambda - \Lambda - \gamma HW)^{-1} =
\begin{pmatrix}
H_1 A_{11} W_1 & H_1 A_{12} W_2 & \cdots & H_1 A_{1n} W_n \\
H_2 A_{21} W_1 & H_2 A_{22} W_2 & \cdots & H_2 A_{2n} W_n \\
\vdots & \vdots & \ddots & \vdots \\
H_n A_{n1} W_1 & H_n A_{n2} W_2 & \cdots & H_n A_{nn} W_n
\end{pmatrix}.
\] (2.17)

**Proof.** We only have to show that, for each \( i \) and \( j \),
\[
K_{ij} = -\gamma H_i A_{ij} W_j - \gamma H_i W_2 H_2 A_{2j} W_j - \cdots + (\lambda - \lambda_i - \gamma) H_i A_{ij} W_j
\]
\[
- \cdots - \gamma H_i W_n H_n A_{nj} W_j
\]
\[
= \begin{cases}
I_{m_i}, & i = j, \\
O_{m_i}, & i \neq j.
\end{cases}
\] (2.18)

In view of the definition of the \( a_{ij}^k \), it is clear that \( W_i H_i A_{ij} = A_{ij} \). Then,
\[
K_{ij} = H_i \kappa_{ij} W_j, \quad \text{where}
\]
\[
\kappa_{ij} = -\gamma A_{1j} - \gamma A_{2j} - \cdots + (\lambda - \lambda_i - \gamma) A_{ij} - \cdots - \gamma A_{nj}.
\]

By recalling the definition of the \( a_{ij}^k \) again, each diagonal block of the \( N \times N \) matrix \( \kappa_{ij} \) is calculated, when \( i > j \) for example, as
the (1, 1)-block: \[ -\gamma a^n_{1j} - \gamma a^n_{2j} - \cdots + (\lambda - \lambda_i - \gamma) a^n_{ij} - \cdots - \gamma a^n_{nj} = 0, \]

the (2, 2)-block: \[ -\gamma a^{n-1}_{1j} - \gamma a^{n-1}_{2j} - \cdots + (\lambda - \lambda_i - \gamma) a^{n-1}_{ij} - \cdots - \gamma a^{n-1}_{(n-1)j} = 0, \]

\[ \ldots \]

the (n + 1 - i, n + 1 - i)-block: \[ -\gamma a^i_{1j} - \gamma a^i_{2j} - \cdots + (\lambda - \lambda_i - \gamma) a^i_{ij} = 0. \]

The other cases: \( i < j \) and \( i = j \) are similarly calculated. Consequently we see that

\[ \kappa_{ij} = \begin{cases} 
(O_{m_j \times N}), & i > j, \\
O_N, & i < j, \\
(I_{m_i} O_{N-m_i}), & i = j. 
\end{cases} \]

By our choice of the \( H_i \) and the \( W_j \), relation (2.18) is now clear. \( \square \)

Lemma 2.4 shows that each element of \((\lambda - \Lambda - \gamma HW)^{-1}\) is a rational function of \( \lambda \) with the denominator which is one of the following \( d_1, \ldots, d_n \):

\[ d_i = \begin{vmatrix} 
\lambda - \lambda_1 - \gamma & -\gamma & \cdots & -\gamma \\
-\gamma & \lambda - \lambda_2 - \gamma & \cdots & -\gamma \\
& \ddots & \ddots & \ddots \\
-\gamma & -\gamma & \cdots & \lambda - \lambda_i - \gamma 
\end{vmatrix}, \quad 1 \leq i \leq n. \quad (2.19) \]

Thus singularities of each element of \((\lambda - \Lambda - \gamma HW)^{-1}\) are simple poles as long as \( \gamma > 0 \) is small enough. Let \( \lambda_{i1}(\gamma), \ldots, \lambda_{ii}(\gamma) \) be the distinct solutions to the algebraic equation: \( d_i = 0 \), where each \( \lambda_{ij}(\gamma) \) is close to \( \lambda_j \) in a neighborhood of \( \gamma = 0 \). In order to know the behavior of the \( \lambda_{ij}(\gamma) \), differentiate the both sides of \( d_i = 0 \) with respect to \( \gamma \) and set \( \gamma = 0 \). Then we see that

\[ \frac{d}{d\gamma} \lambda_{ij}(\gamma) \bigg|_{\gamma=0} = 1, \quad 1 \leq j \leq i \leq n. \]

Thus,

\[ \lambda_{ij}(\gamma) = \lambda_j + \gamma + O(\gamma^2), \quad 1 \leq j \leq i \leq n. \]

Calculating the residue of \( e^{-t\lambda} a^k_{ij} \) at each possible pole in the integral (2.14), we obtain the estimate (2.11). This finishes the proof of Proposition 2.3. \( \square \)

The proof of Theorem 2.1 is thereby complete. \( \square \)

**Remark.** In Corollary 2.2, each \( A_{ij} \) is reduced to \( a^n_{ij} I_N \). Thus, \((\lambda - \Lambda - \gamma HW)^{-1}\) is expressed as
\[(\lambda - \Lambda - \gamma HW)^{-1} = \begin{pmatrix}
I_N & 0 & \cdots & 0 \\
0 & H_2H_1^{-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_nH_1^{-1}
\end{pmatrix} \begin{pmatrix}
a_{11}^I I_N & a_{12}^I I_N & \cdots & a_{1n}^I I_N \\
a_{21}^I I_N & a_{22}^I I_N & \cdots & a_{2n}^I I_N \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}^I I_N & a_{n2}^I I_N & \cdots & a_{nn}^I I_N
\end{pmatrix} \times 
\begin{pmatrix}
0 & \cdots & 0 \\
H_1^{-1} & \cdots & \vdots \\
0 & \cdots & \cdots \\
0 & \cdots & \cdots 
\end{pmatrix}.
\]

2.1. Application to a class of boundary control systems

Let us consider the boundary feedback control system (1.6) and obtain the same result as in Theorem 2.1. In (1.6), assume that \( \omega = \min_{\lambda \in \sigma(L)} \text{Re} \lambda > 0. \) Let \( \lambda_1, \ldots, \lambda_n \in \sigma(L) \) be on the vertical line: \( \text{Re} \lambda = \omega. \) As we have seen in (1.9), the problem is reduced to the estimate of the analytic semigroup \( e^{-tM}, t > 0 \) in \( H^\ast = L^2(\Omega), \) where

\[ M = L + \gamma \sum_{k=1}^{N} \langle L_{c}^{1/4+\epsilon} \cdot, w_k \rangle_f L_{c}^{3/4-\epsilon} \varphi_k. \] (2.20)

By our assumption in Section 1, the set of the eigenfunctions \( \{ \varphi_{ij} \}_{j=1}^{m_i} \) forms a basis for the eigenspace corresponding to \( \lambda_i. \) Since \( \lambda_i \) are the eigenvalues of \( L^\ast \) and the multiplicities are the same as those of \( \lambda_i \) for \( L, \) let \( \psi_{ij}, 1 \leq i \leq n, 1 \leq j \leq m_i, \) be the eigenfunctions of \( L^\ast, \) that is, \((\lambda_i - L^\ast)\psi_{ij} = 0.\) Set \( P\varphi_k = \sum_{i=1}^{n} \sum_{j=1}^{m_i} h_{ij}^k \varphi_{ij}. \) It is easily seen via Green’s formula that

\[ \begin{pmatrix}
h_{i1}^k \\
\vdots \\
h_{imb}^k
\end{pmatrix} = \Theta_i \begin{pmatrix}
\langle h_k, \psi_{i1} \rangle_f \\
\vdots \\
\langle h_k, \psi_{imb} \rangle_f
\end{pmatrix}, \quad 1 \leq i \leq n, \] (2.21)

where \( \Theta_i \) denote the \( m_i \times m_i \) nonsingular matrices consisting of the elements \( \langle \varphi_{ij}, \psi_{il} \rangle_{\Omega} = \langle \varphi_{ij}, \psi_{il} \rangle_H, 1 \leq j, l \leq m_i. \)

Setting \( \alpha = \frac{1}{2} + 2\epsilon, \) \( x_1 = Px, \) and \( x_2 = Qx \) in (1.8), we obtain the system of differential equations in \( \mathbb{C}^S \times QH \) (see (2.3) for comparison)

\[ \begin{cases}
\frac{dx}{dt} + (\Lambda + \gamma \hat{H}\hat{W})x = -\gamma \hat{H}(L_{c}^{1/2} x_2, w)_{f,}, \quad x(0) = x_0, \\
\frac{dx_2}{dt} + \hat{F}x_2 = -\gamma (QL_{c}^{1/2} \varphi_1 \cdots QL_{c}^{1/2} \varphi_n)\hat{W}x, \quad x_2(0) = Qx_0,
\end{cases} \] (2.22)

where

\[ \hat{H} = \begin{pmatrix}
(\lambda_1 + c)^{1-\alpha/2} H_1 \\
\vdots \\
(\lambda_n + c)^{1-\alpha/2} H_n
\end{pmatrix}, \quad \hat{F}_i = \Theta_i \begin{pmatrix}
\langle h_k, \psi_{ij} \rangle_f; \quad k \to 1, \ldots, N \\
\downarrow \\
j \to 1, \ldots, m_i
\end{pmatrix}, \]
\[
\hat{W} = (\lambda_1 + c)^{\alpha/2}W_1 \ldots (\lambda_n + c)^{\alpha/2}W_n, \quad W_i = \left(\varphi_{ij}, w_k\right)_\Gamma; \quad k \downarrow 1, \ldots, N_j \rightarrow 1, \ldots, m_i,
\]

\[
\langle L^{\alpha/2}x_2, w \rangle_\Gamma = \begin{pmatrix} \langle L^{\alpha/2}x_2, w_1 \rangle_\Gamma \\ \vdots \\ \langle L^{\alpha/2}x_2, w_N \rangle_\Gamma \end{pmatrix}, \quad \text{and} \quad \hat{F} = L_2 + \gamma \sum_{k=1}^{N} L^{\alpha/2}_c w_k Q L^{-1-\alpha/2}_c \varphi_k.
\] (2.23)

In Theorem 2.1, we set \( N = m_1 \) and choose the \( w_k \) and the \( h_k \) such that

\[
H_i = (H_{i1} \ 0), \quad H_{i1}; \ m_i \times m_i,
\]

\[
\text{rank } H_i = m_i, \quad \text{and} \quad W_i = \begin{pmatrix} (\lambda_i + c)^{-1} H_{i1}^{-1} \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n. \quad (2.24)
\]

Then we obtain the estimate (see (2.11))

\[
\left\| e^{-t(A+\gamma \hat{W})} \right\|_{L(C^5)} \leq M e^{-(\omega+\gamma + O(\gamma^2))t}, \quad t \geq 0.
\] (2.25)

Let us turn to the estimate of \( e^{-t\hat{F}}. \) An estimate of the resolvent \((\lambda - L_2)^{-1}\)—via the well-known moment inequality—shows a rough estimate:

\[
\left\| e^{-t\hat{F}} \right\| \leq M_0 e^{-\beta t}, \quad t \geq 0,
\] (2.26)

as long as \( \gamma \) is small enough. Here, \( M_0 \) denotes a constant independent of \( \gamma. \) We also need to obtain a more accurate \( L^1(0, \infty) \)-estimate of \( \|e^{-t\hat{F}}\|. \) As a technical issue, \( \hat{F} \) contains an unbounded perturbation. We note that

\[
\left\| e^{-tL_2} \right\| \leq M_1 e^{-\beta t}, \quad t \geq 0, \quad \text{and} \quad \left\| L^{\alpha} e^{-tL_2} \right\| \leq \frac{M_1 e^{-\beta t}}{t^\alpha}, \quad t > 0.
\]

We recall that (the trace theorem)

\[
\|u\|_{L^2(\Gamma)} \leq c_0 \|u\|_{H^\alpha(\Omega)} \leq c_1 \left\| L^{\alpha/2}_c u \right\|.
\]

Set \( y(t) = L^\alpha_c e^{-t\hat{F}} y_0, \) \( y_0 \in Q L^2(\Omega). \) Then, \( y(t) \) satisfies the integral inequality:

\[
\left\| y(t) \right\| \leq M_1 e^{-\beta t} \|y_0\| + \gamma c_2 \int_0^t M_1 e^{-\beta(t-s)} \left\| y(s) \right\| ds, \quad c_2 = c_1 \sum_{k=1}^{N} \|w_k\|_{L^2(\Gamma)} \left\| Q L^{-1-\alpha}_c \varphi_k \right\|.
\]

The following estimates will be immediate (see [5] for details on estimates of similar kinds):

\[
ee^{\beta t} \|y(t)\| \leq M_1 \|y_0\| \sum_{k=0}^{\infty} \frac{(\gamma c_2 M_1)^k \Gamma(1-\alpha)^{k+1}}{\Gamma((k+1)(1-\alpha))}, \quad (k+1)(1-\alpha) - 1,
\]

\[
\int_0^\infty e^{\delta t} \left\| L^\alpha_c e^{-t\hat{F}} \right\| dt \leq \frac{M_1 \Gamma(1-\alpha)}{(\beta - \delta)^{1-\alpha} - \gamma c_2 M_1 \Gamma(1-\alpha)}, \quad (2.27)
\]
where \( \gamma > 0 \) is assumed to be small: \( \beta^{1-\alpha} > \gamma c_2 M_1 \Gamma(1-\alpha) \), and \( \beta > 0 \) is chosen so that \( (\beta - \delta)^{1-\alpha} > \gamma c_2 M_1 \Gamma(1-\alpha) \). Thus we see that

\[
\int_0^\infty e^{\delta t} \left\| e^{-t(\Lambda + \gamma \hat{H})} \right\| dt \leq \int_0^\infty e^{\delta t} \left\| e^{-L_2} \right\| dt + \gamma c_2 \int_0^\infty e^{\delta t} dt \int_0^t \left\| e^{-(t-s)L_2} \right\| L_c^\alpha e^{-s \hat{F}} ds \leq \int_0^\infty M_1 e^{-(\beta - \delta)t} dt + \gamma c_2 M_1 \int_0^\infty e^{\delta s} \left\| L_c^\alpha e^{-s \hat{F}} \right\| ds \int_0^\infty e^{-(\beta - \delta)(t-s)} dt \leq \frac{1}{(\beta - \delta)^\alpha} \cdot \frac{M_1}{(\beta - \delta)^{1-\alpha} - \gamma c_2 M_1 \Gamma(1-\alpha)}.
\]

(2.28)

Based on the estimates of \( e^{-t(\Lambda + \gamma \hat{H} \hat{W})} \) and \( e^{-t \hat{F}} \), we evaluate \((x(t), x_2(t))\) in (2.22). It is immediate to see that

\[
\left\| x(t) \right\| \leq \left\| e^{-t(\Lambda + \gamma \hat{H} \hat{W})} \right\|_{\mathcal{L}(C^S)} |x_0| + \gamma c_3 \int_0^t \left\| e^{-(t-s)(\Lambda + \gamma \hat{H} \hat{W})} \right\|_{\mathcal{L}(C^S)} ds \times \left\{ \left\| L_c^\alpha e^{-s \hat{F}} \right\| \left\| Qx_0 \right\| + \gamma c_3 \int_0^s \left\| L_c^\alpha e^{-(s-\sigma) \hat{F}} \right\| \left\| x(\sigma) \right\| d\sigma \right\}
\]

\[
\leq \left\| e^{-t(\Lambda + \gamma \hat{H} \hat{W})} \right\|_{\mathcal{L}(C^S)} |x_0| + \gamma c_3 \left\| Qx_0 \right\| P(t) + \gamma^2 c_3^2 \int_0^t P(t - \sigma) \left\| x(\sigma) \right\| d\sigma,
\]

(2.29)

where

\[
P(t) = \int_0^t \left\| e^{-(t-s)(\Lambda + \gamma \hat{H} \hat{W})} \right\|_{\mathcal{L}(C^S)} \left\| L_c^\alpha e^{-s \hat{F}} \right\| ds,
\]

and the constant \( c_3 \), like \( c_2 \), is determined by the quantities: \( \|w_k\|_{L^2(\Gamma)}\), \( \|PL_1^{1-\alpha} \varphi_k\| \), and \( \|QL_1^{1-\alpha} \varphi_k\| \). For a fixed \( \eta \), \( 0 < \eta < 1 \), set \( \delta = \omega + (1 - \eta) \gamma < \beta \). In view of the estimates (2.25) and (2.27), we see that

\[
\int_0^\infty e^{\delta t} P(t) dt \leq \frac{M_1 \Gamma(1-\alpha)}{(\beta - \omega - (1 - \eta) \gamma)^{1-\alpha} - \gamma c_2 M_1 \Gamma(1-\alpha)} \cdot \frac{M}{\eta \gamma + O(\gamma^2)}.
\]

Thus, as long as \( \gamma > 0 \) is small, we have

\[
\gamma^2 c_3^2 \int_0^\infty e^{\delta t} P(t) dt < 1.
\]
It follows from (2.29) that
\[
\int_0^\infty e^{\delta t} |x(t)| \, dt \leq \int_0^\infty \| e^{-(t-s)(A+\gamma H\hat{\omega})} \|_{L(C^S)} \, dt \| x_0 \| + \gamma c_3 \| Qx_0 \| \int_0^\infty e^{\delta t} P(t) \, dt \\
+ \gamma^2 c_3^2 \int_0^\infty dt \int_0^t e^{\delta(t-\sigma)} P(t-\sigma) e^{\delta \sigma} \| x(\sigma) \| \, d\sigma,
\]
from which we have the estimate
\[
\int_0^\infty e^{\delta t} |x(t)| \, dt \leq O\left( \frac{1}{\gamma} \right) \| x_0 \|, \quad \delta = \omega + (1 - \eta)\gamma < \beta.
\]
Via the integral inequality for \( \| L_\alpha^c x^2(t) \| \) and the estimate just above, we similarly obtain
\[
\int_0^\infty e^{\delta t} \| L_\alpha^c x^2(t) \| \, dt \leq M_1 \Gamma(1 - \alpha) \left( \| Qx_0 \| + \gamma c_3 \int_0^\infty e^{\delta t} |x(t)| \, dt \right) \\
\leq c_4 \| x_0 \|.
\]
Via the integral inequality for \( |x(t)| \) and the estimate just above, we obtain
\[
e^{\delta t} |x(t)| \leq e^{\delta t} \| e^{-(t-s)(A+\gamma H\hat{\omega})} \|_{L(C^S)} |x_0| \\
+ \gamma c_3 \int_0^t e^{\delta(t-s)} \| e^{-(t-s)(A+\gamma H\hat{\omega})} \|_{L(C^S)} e^{\delta s} \| L_\alpha^c x^2(s) \| \, ds \\
\leq M e^{-(\eta\gamma + O(\gamma^2))t} \| x_0 \| + \gamma c_3 M c_4 \| x_0 \|,
\]
or
\[
|x(t)| \leq c_5 e^{-(\omega + (1 - \eta)\gamma)t} \| x_0 \|, \quad t \geq 0. \tag{2.30}
\]
As for the estimate of \( x_2(t) \), we calculate as (see (2.28))
\[
e^{\delta t} \| x_2(t) \| \leq e^{\delta t} M_0 e^{-\beta t} \| Qx_0 \| + \gamma c_3 \int_0^t e^{\delta(t-s)} \| e^{-(t-s)\hat{F}} \| e^{\delta s} |x(s)| \, ds \\
\leq c_6 \| x_0 \|, \quad t \geq 0. \tag{2.31}
\]
By the estimates (2.30) and (2.31) we finally obtain the desired estimate:
\[
\| e^{-tM} \| \leq \text{const} e^{-(\omega + (1 - \eta)\gamma)t}, \quad t \geq 0. \tag{2.32}
\]
3. Main result II (generalization)

We will extend in this section Theorem 2.1 to some extent. The assumption on the $h_k$ will be somewhat weakened (see (3.2) below). For positive integers $i$ and $j$ with $2 \leq i < j \leq n$ and $\lambda \in \mathbb{C}$, set

$$
\Xi(i,j)(\lambda) = (\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_j) + \cdots + (\lambda - \lambda_i) \cdots (\lambda - \lambda_{j-1})
= \prod_{k=i}^{j}(\lambda - \lambda_k) \cdot \left(\frac{1}{\lambda - \lambda_i} + \cdots + \frac{1}{\lambda - \lambda_j}\right).
$$

(3.1)

Here it is assumed that $n \geq 3$. Then we have

**Theorem 3.1.** Take $N = m_1$, and assume that

$$
H_i = (H_{i1} \ H_{i2}), \quad H_{i1}; \ m_i \times m_i,
$$

$$
det H_{i1} \neq 0 \ (\text{and thus rank } H_i = m_i), \quad \text{and} \quad W_i = \left(\begin{array}{c}
H_{i1}^{-1}
\end{array}\right), \quad 1 \leq i \leq n.
$$

(3.2)

Assume finally that

$$
\Xi(i,j)(\lambda_h) \neq 0, \quad 1 \leq h < i < j \leq n, \quad 1 \leq i \leq n.
$$

(3.3)

Then the assertion of Theorem 2.1 holds.

**Remark 1.** When $n = 3$, for example, (3.3) means that $\lambda_1 \neq (\lambda_2 + \lambda_3)/2$, and when $n = 4$, that

$$
\lambda_1 \neq \frac{\lambda_2 + \lambda_3}{2}, \quad \lambda_1 \neq \frac{\lambda_3 + \lambda_4}{2}, \quad \lambda_2 \neq \frac{\lambda_3 + \lambda_4}{2}, \quad \text{and}
$$

$$
\sum_{2 \leq i < j \leq 4} A_i(\lambda_1) A_j(\lambda_1) = \sum_{2 \leq i < j \leq 4} (\lambda_1 - \lambda_i)(\lambda_1 - \lambda_j) \neq 0, \quad \text{where } A_i(\lambda) = \lambda - \lambda_i.
$$

**Remark 2.** Assumption (3.3) is posed for a technical reason, and seems not essential for our theorem. In fact, when (3.3) is not satisfied, it is shown that an estimate a little weaker than (2.11)

$$
\| e^{-t(\Lambda + \gamma HW)} \|_{L^2(\mathbb{C^d})} \leq Me^{-(\omega + \gamma/2 + O(\gamma^2))t}, \quad t \geq 0,
$$

(3.4)

holds for $n = 3, 4$. We discuss on the removement of (3.3) later in Section 4.

**Proof of Theorem 3.1.** Since the key idea is to obtain the estimate (2.11) for the semigroup $e^{-t(\Lambda + \gamma HW)}$, we concentrate hereafter on the behavior of the resolvent $(\lambda - \Lambda - \gamma HW)^{-1}$ in the neighborhood of each singularity. The rest of the proof is the same as in Theorem 2.1, and thus omitted.
The presence of the terms $H_{12}$ rather complicates the structure of the inverse $(\lambda - \Lambda - \gamma HW)^{-1}$: It seems difficult in this case to obtain the expression similar to (2.17). An alternative means is to reduce $\lambda - \Lambda - \gamma HW$ to an upper-triangular matrix in the identity relation:

$$(\lambda - \Lambda - \gamma HW)(\lambda - \Lambda - \gamma HW)^{-1} = 1.$$  \hspace{1cm} (3.5)

Define the $N \times N$ matrices $\langle k \rangle$ as

$$\langle k \rangle = \gamma A_1 \cdots A_{k-1} W_k H_k + d_k I_N, \quad k = 2, \ldots, n,$$

where $A_i = \lambda - \lambda_i$, and let $\Psi(\lambda, \gamma)$ be the $S \times S$ lower-triangular matrix ($S = m_1 + \cdots + m_n$):

$$\Psi(\lambda, \gamma) = \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\
\gamma H_2 H_1 & d_1 I & 0 & \cdots & 0 \\
\gamma H_3 & \gamma A_1 H_3 W_2 & d_2 I & \cdots & 0 \\
\gamma H_4 & \gamma A_1 H_4 W_3 & \gamma A_2 H_4 W_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma H_n & \prod_{k=1}^{n-1} \frac{\langle k \rangle}{d_{k-1}} - H_1 & \gamma A_1 H_n & \gamma A_2 H_n & \cdots & n-1 I \end{pmatrix}.$$  \hspace{1cm} (3.6)

Then we have

**Lemma 3.2.**

$$\Psi(\lambda, \gamma)(\lambda - \Lambda - \gamma HW) = \begin{pmatrix} d_1 I & -\gamma H_1 W_2 & -\gamma H_1 W_3 & -\gamma H_1 W_4 & \cdots & -\gamma H_1 W_n \\
0 & d_2 I & -\gamma A_1 H_2 W_3 & -\gamma A_1 H_2 W_4 & \cdots & -\gamma A_1 H_2 W_n \\
0 & 0 & d_3 I & -\gamma A_1 A_2 H_3 W_4 & \cdots & -\gamma A_1 A_2 W_n \end{pmatrix}. \hspace{1cm} (3.7)$$

The inverse of the last matrix is denoted by $\Phi(\lambda, \gamma)$. Then,

$$\Phi(\lambda, \gamma) = \begin{pmatrix} \frac{1}{d_1} I & \frac{\gamma}{d_1 d_2} H_1 W_2 & \frac{\gamma A_2}{d_1 d_3} H_1 W_3 & \frac{\gamma A_2 A_3}{d_1 d_4} H_1 W_4 & \cdots & \frac{\gamma}{d_1 \prod_{k=1}^{n-1} A_k} H_1 W_n \\
0 & \frac{1}{d_2} I & \frac{\gamma A_1}{d_2 d_3} H_2 W_3 & \frac{\gamma A_1 A_2}{d_2 d_4} H_2 W_4 & \cdots & \frac{\gamma}{d_2 \prod_{k=1}^{n-1} A_k} H_2 W_n \\
0 & 0 & \frac{1}{d_3} I & \frac{\gamma A_1 A_2}{d_3 d_4} H_3 W_4 & \cdots & \frac{\gamma}{d_3 \prod_{k=1}^{n-1} A_k} H_3 W_n \\
0 & 0 & 0 & \frac{1}{d_4} I & \cdots & \frac{\gamma}{d_4 \prod_{k=1}^{n-1} A_k} H_4 W_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{d_n} I \end{pmatrix}. \hspace{1cm} (3.8)$$
Proof. The relation (3.7) means that

\[
\text{the } i\text{th row of } \Psi(\lambda, \gamma) \times \text{the } j\text{th column of } (\lambda - \Lambda - \gamma H W) = \begin{cases} 0, & i > j, \\ d_i, & i = j, \\ -\gamma(A_1 \cdots A_{i-1})H_i W_j, & i < j. \end{cases}
\]

It is elementary but tedious to show the above. Let us begin with the case of \(i > j\), where we have to show that

\[
-\gamma^2 H_i \frac{(i - 1) \cdots (2)}{d_{i-2} \cdots d_1} W_j - \gamma^2 A_1 H_i \frac{(i - 1) \cdots (3)}{d_{i-2} \cdots d_2} W_2 H_2 W_j - \cdots - \gamma^2 (A_1 \cdots A_{j-2}) H_i \frac{(i - 1) \cdots (j)}{d_{i-2} \cdots d_{j-1}} W_{j-1} H_{j-1} W_j + \gamma (A_1 \cdots A_{j-1}) H_i \frac{(i - 1) \cdots (j + 1)}{d_{i-2} \cdots d_j} (A_j - \gamma) W_j H_j W_j - \gamma^2 (A_1 \cdots A_j) H_i \frac{(i - 1) \cdots (j + 2)}{d_{i-2} \cdots d_{j+1}} W_{j+1} H_{j+1} W_j - \cdots - \gamma d_{i-1} H_i W_j = 0.
\]

The key through the proof is the relation

\[W_l H_l W_j = W_j, \quad 1 \leqslant l \leqslant j.\]

Deleting the common factor \(\gamma H_i\) for simplicity, we calculate as

the first term + the second term = \(-\gamma \frac{(i - 1) \cdots (3)}{d_{i-2} \cdots d_2} \left( \frac{1}{d_1} (d_2 + \gamma A_1 W_2 H_2) + A_1 \right) W_j = -\gamma \frac{(i - 1) \cdots (3)}{d_{i-2} \cdots d_2} \left( \frac{1}{d_1} (d_2 + \gamma A_1) + A_1 \right) W_j = -\gamma \frac{(i - 1) \cdots (3)}{d_{i-2} \cdots d_2} (d_1 A_2 + A_1) W_j = -\gamma \frac{(i - 1) \cdots (3)}{d_{i-2} \cdots d_2} \Xi(1, 2) W_j.
\]

Inductively we find that

\[
\sum_{k=1}^{j-1} \text{ (the } k\text{th term)} = -\gamma \frac{(i - 1) \cdots (j)}{d_{i-2} \cdots d_{j-1}} \Xi(1, j-1) W_j.
\]

Thus, \(\sum_{k=1}^{j} \text{ (the } k\text{th term)}\) becomes
\[
\frac{(i - 1) \cdots (j + 1)}{d_{i-2} \cdots d_j} \left( -\gamma \mathcal{S}_{(1, j-1)} \frac{\langle j \rangle}{d_{j-1}} + (A_1 \cdots A_{j-1})(A_j - \gamma) \right) W_j \\
= \frac{(i - 1) \cdots (j + 1)}{d_{i-2} \cdots d_j} \\
\times \left( -\gamma \mathcal{S}_{(1, j-1)} \left( d_j + \gamma (A_1 \cdots A_{j-1}) W_j H_j \right) + (A_1 \cdots A_{j-1})(A_j - \gamma) \right) W_j \\
= \frac{(i - 1) \cdots (j + 1)}{d_{i-2} \cdots d_j} \left( -\gamma \mathcal{S}_{(1, j-1)} \left( d_j + \gamma (A_1 \cdots A_{j-1}) \right) \frac{\langle j \rangle}{d_{j-1}} + (A_1 \cdots A_{j-1})(A_j - \gamma) \right) W_j \\
= \frac{(i - 1) \cdots (j + 1)}{d_{i-2} \cdots d_j} W_j.
\]

Continuing further the calculation, we see that
\[
\sum_{k=1}^{i-3} \text{(the kth term)} = \frac{(i - 1)(i - 2)}{d_{i-2}} W_j;
\]
\[
\sum_{k=1}^{i-2} \text{(the kth term)} = \frac{(i - 1)}{d_{i-2}} \left( (i - 2) - \gamma (A_1 \cdots A_{i-3}) W_{i-2} H_{i-2} \right) W_j \\
= \frac{(i - 1)}{d_{i-2}} W_j = (i - 1) W_j;
\]
and finally
\[
\sum_{k=1}^{i} \text{(the kth term)} = (i - 1) W_j - \gamma (A_1 \cdots A_{i-2}) W_{i-1} H_{i-1} W_j - d_{i-1} W_j \\
= (i - 1 - \gamma (A_1 \cdots A_{i-2}) W_{i-1} H_{i-1} - d_{i-1}) W_j \\
= (d_{i-1} - d_{i-1}) W_j = 0.
\]

Let us consider the case of \( i = j \). Similarly we calculate inductively as (counting the factor \( \gamma H_i \) at this time)
\[
\sum_{k=1}^{i-2} \text{(the kth term)} = -\gamma^2 H_i \frac{(i - 1)}{d_{i-2}} \mathcal{S}_{(1, i-2)} W_i.
\]

Then, recalling that \( W_{i-1} H_{i-1} W_i = W_i \),
\[
\sum_{k=1}^{i-1} \text{(the kth term)} = -\gamma^2 H_i \frac{\mathcal{S}_{(1, i-2)}}{d_{i-2}} (i - 1) W_i - \gamma^2 (A_1 \cdots A_{i-2}) H_i W_{i-1} H_{i-1} W_i \\
= -\gamma^2 H_i \left( \frac{\mathcal{S}_{(1, i-2)}}{d_{i-2}} \left( d_{i-2} + \gamma (A_1 \cdots A_{i-2}) \right) \right) W_i
\]
\[= -\gamma^2 H_i \left( \frac{\mathcal{S}_{(1,i-2)}}{d_{i-2}} d_{i-2} A_{i-1} + (A_1 \cdots A_{i-2}) \right) W_i \]

\[= -\gamma^2 \mathcal{S}_{(1,i-1)} H_i W_i = -\gamma^2 \mathcal{S}_{(1,i-1)}, \]

and finally we have

\[\sum_{k=1}^{i} \text{ (the kth term)} = -\gamma^2 \mathcal{S}_{(1,i-1)} + d_{i-1} (A_i - \gamma) \]

\[= -\gamma^2 \mathcal{S}_{(1,i-1)} + (A_1 \cdots A_{i-1}) - \gamma \mathcal{S}_{(1,i-1)} (A_i - \gamma) \]

\[= (A_1 \cdots A_i) - \gamma (A_1 \cdots A_{i-1}) - \gamma A_i \mathcal{S}_{(1,i-1)} \]

\[= d_i. \]

Let us consider the final case of \( i < j \). We inductively calculate as

\[\sum_{k=1}^{i-2} \text{ (the kth term)} = -\gamma^2 H_i \frac{\mathcal{S}_{(1,i-2)}}{d_{i-2}} (i-1) W_j. \]

Then,

\[\sum_{k=1}^{i-1} \text{ (the kth term)} = -\gamma^2 H_i \frac{\mathcal{S}_{(1,i-2)}}{d_{i-2}} (i-1) W_j - \gamma^2 (A_1 \cdots A_{i-2}) H_i W_{i-1} H_{i-1} W_j \]

\[= -\gamma^2 H_i \left( \frac{\mathcal{S}_{(1,i-2)}}{d_{i-2}} (d_{i-1} + \gamma (A_1 \cdots A_{i-2})) + (A_1 \cdots A_{i-2}) \right) W_j \]

\[= -\gamma^2 H_i \left( \frac{\mathcal{S}_{(1,i-2)}}{d_{i-2}} d_{i-2} A_{i-1} + (A_1 \cdots A_{i-2}) \right) W_j \]

\[= -\gamma^2 \mathcal{S}_{(1,i-1)} H_i W_j, \]

and finally we have

\[\sum_{k=1}^{i} \text{ (the kth term)} = -\gamma^2 \mathcal{S}_{(1,i-1)} H_i W_j - \gamma d_{i-1} H_i W_j \]

\[= -\gamma (A_1 \cdots A_{i-1}) H_i W_j. \]

The calculation of \( \Phi(\lambda, \gamma) \) is similarly carried out. Thus it is omitted. \qed

In view of the identity relation (3.5) and Lemma 3.2, we obtain the following decomposition expression:

\[
(\lambda - \Lambda - \gamma HW)^{-1} = \Phi(\lambda, \gamma) \Psi(\lambda, \gamma). \tag{3.9}
\]

In calculating the contour integral in (2.14), we estimate each element of \( (\lambda - \Lambda - \gamma HW)^{-1} \) at every singularity. The following lemma holds regardless of the assumption (3.3).
Lemma 3.3. When $\gamma \neq 0$ is small enough, we have the implication

$$ d_i = 0 \Rightarrow d_j \neq 0, \quad i \neq j. \quad (3.10) $$

Thus, the singularities of each element of the matrices $\Phi(\lambda, \gamma)$ and $\Psi(\lambda, \gamma)$ consist of simple poles.

Proof. By definition,

$$ d_i = A_1 \cdots A_i - \gamma \mathcal{S}(1,i). $$

As long as $\gamma \neq 0$ is small enough, the solutions $\lambda$ to $d_i = 0$ are close to, but not equal to any of $\lambda_1, \ldots, \lambda_i$. This follows from the fact $\lambda'(0) = 1$. Thus when $d_i = 0$, we see that

$$ A_j = \lambda(\gamma) + \lambda_j \neq 0, \quad 1 \leq j \leq n. $$

Note that

$$ d_i = d_1 A_2 A_3 \cdots A_i - \gamma A_1 \mathcal{S}(2,i) = d_2 A_3 \cdots A_i - \gamma A_1 A_2 \mathcal{S}(3,i) $$

$$ = \cdots = d_{i-1} A_i - \gamma A_1 A_2 \cdots A_{i-1}, $$

a part of which has been used in Lemma 3.2. Let $\lambda(\gamma)$ be one of the solutions to the equation $d_i = 0$. For any pair of integers $1 \leq p < q \leq n$, consider the function $\mathcal{S}_{(p,q)}(\lambda(\gamma))$. As an analytic function of $\gamma$, we show, as a general result, that

$$ \exists \ell \geq 0; \quad \frac{d^\ell}{d\gamma^\ell} \mathcal{S}_{(p,q)}(\lambda(0)) \neq 0. \quad (3.12) $$

In fact, if this were not true, we would obtain $\mathcal{S}_{(p,q)}(\lambda(\gamma)) \equiv 0$. Set $m = q - p$, and let

$$ \mathcal{S}_{(p,q)}(\lambda) = (m+1)\lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-2}\lambda^2 + a_{m-1}\lambda + a_m, $$

where the coefficients $a_i$ are the polynomials of $\lambda_p, \ldots, \lambda_q$, especially $a_1 = -m(\lambda_p + \cdots + \lambda_q)$. Differentiating the both sides of $\mathcal{S}_{(p,q)}(\lambda(\gamma)) \equiv 0$ with respect to $\gamma$, we have

$$ (m+1)m\lambda^{m-1}\lambda' + (m-1)a_1\lambda^{m-2}\lambda' + \cdots + 2a_{m-2}\lambda\lambda' + a_{m-1}\lambda' \equiv 0. $$

Noting that $\lambda'(0) = 1$, we have, through analytic continuation,

$$ (m+1)m\lambda^{m-1}(\gamma) + (m-1)a_1\lambda^{m-2}(\gamma) + \cdots + 2a_{m-2}\lambda(\gamma) + a_{m-1} \equiv 0. $$

Continuing the same procedure repeatedly in the above relation, we finally obtain

$$ (m+1)!\lambda + (m-1)!a_1 \equiv 0 \quad \text{or} \quad \lambda(\gamma) = \frac{-a_1}{(m+1)m} = \frac{1}{q-p+1} (\lambda_p + \cdots + \lambda_q), $$

which contradicts the property $\lambda'(0) = 1$. Thus there is an integer $\ell$ satisfying (3.12). As a result, there is a function $f_{(p,q)}(\gamma)$ which is analytic at $\gamma = 0$ such that

$$ \mathcal{S}_{(p,q)}(\lambda(\gamma)) = \gamma^\ell f_{(p,q)}(\gamma), \quad f_{(p,q)}(0) \neq 0. $$
We go back to (3.11). When \( d_i = 0 \), then (3.11) implies that
\[
d_j = \frac{\gamma A_1 \cdots A_j}{A_{j+1} \cdots A_i} \mathcal{S}_{(j+1,i)} = \frac{A_1 \cdots A_j}{A_{j+1} \cdots A_i} \gamma^{\ell+1} f_{(j+1,i)}(\gamma).
\]
As long as \( \gamma \neq 0 \) is small enough, the above expression implies that \( d_j \neq 0 \), \( 1 \leq j \leq i - 1 \). Similarly we see that \( d_j \neq 0 \), \( i + 1 \leq j \leq n \), when \( d_i = 0 \). \( \square \)

Based on Lemma 3.3, it is convenient to write down Table 2 (the \( d_i-d_j \) table) which describes the behavior of the \( d_j \), \( j \neq i \), when \( d_i = 0 \). The table is written down in the end of the paper. In view of Lemma 3.3, the singularities of \( \Phi(\lambda, \gamma) \) arise at points different from those of \( \Psi(\lambda, \gamma) \). We first consider the singularities of \( \Phi(\lambda, \gamma) \). The \( m \)th column has the singularities at the points where \( d_{m-1} = 0 \) and \( d_m = 0 \). We need to calculate the residues of the matrix \( e^{-\iota(\lambda - \Lambda - \gamma HW)^{-1}} \) when \( d_{m-1} = 0 \) and \( d_m = 0 \). When \( d_{m-1} = 0 \), we see that\(^1\)
\[
\left\| \left[ d_{m-1} \Phi(\lambda, \gamma)(i,m) \right] \right\| \leq \begin{cases} O(1) \left( \frac{\gamma A_1 \cdots A_{m-1}}{A_i d_m} \right) \leq O\left( \frac{1}{\gamma} \right), & 1 \leq i \leq m - 1, \\ 0, & m \leq i \leq n. \end{cases}
\]
Similarly, when \( d_m = 0 \), the element \( \Phi(\lambda, \gamma)(i,m) \) times \( d_m \) is estimated as follows:
\[
\left\| d_m \Phi(\lambda, \gamma)(i,m) \right\| \leq \begin{cases} O(1) \left( \frac{\gamma A_1 \cdots A_{m-1}}{A_i d_{m-1}} \right) \leq O\left( \frac{1}{\gamma} \right) |A_m|, & 1 \leq i \leq m - 1, \\ 0, & i = m, \\ 1, & m + 1 \leq i \leq n. \end{cases}
\]
In the calculation of the residues, the corresponding terms are the \( m \)th row of \( \Psi(\lambda, \gamma) \). By (3.6), they are written down as
\[
\Psi(\lambda, \gamma)(m,j) = \begin{cases} H_m \gamma \prod_{k=m-1}^{2} \frac{(k)}{d_{k-1}} H_{1-j}, & j = 1, \\ H_m \gamma A_1 \cdots A_{j-1} \prod_{k=m-1}^{j+1} \frac{(k)}{d_{k-1}} W_j, & 2 \leq j \leq m - 2, \\ H_m \gamma A_1 \cdots A_{m-2} W_{m-1}, & j = m - 1, \\ d_{m-1}, & j = m, \\ O, & m + 1 \leq j \leq n. \end{cases}
\]
As we have seen in the estimate of \( d_{m-1} \Phi(\lambda, \gamma)(i,m) \) and \( d_m \Phi(\lambda, \gamma)(i,m) \), \( 1 \leq i \leq n \), the above elements \( \Psi(\lambda, \gamma)(m,j) \) are desired to be at least of order \( O(\gamma) \) when \( d_{m-1} = 0 \) and \( d_m = 0 \). To examine this we need to know the behavior of the matrices \( \frac{(k)}{d_{k-1}} = \gamma A_1 \cdots A_{k-1} W_k H_k + d_k I_N \), as the functions of \( \gamma \sim 0 \), when \( d_i = 0 \). According to Table 2 (the \( d_i-d_j \) table), we complete Table 3 (the \( d_i-(j) \) table) which is also written down in the end of the paper. In view of this table, it is immediately seen that, when \( d_{m-1} = 0 \) and \( d_m = 0 \),
\[
\left\| \frac{(k)}{d_{k-1}} \right\| = O(1) |A_k|, \quad 2 \leq k \leq m - 1.
\]
\(^1\) We are calculating the residue: \( \lim_{\lambda \to \lambda_{(m-1)}} \gamma^{(\lambda - \Lambda_{(m-1)})(\gamma)} \Phi(\lambda, \gamma)(i,m) \), \( 1 \leq j \leq m - 1 \). The same convention appears just below when \( d_m = 0 \).
Thus we see that
\[ \| \Psi(\lambda, \gamma)(m,j) \| = O(\gamma), \quad 1 \leq j \leq n, \; j \neq m, \quad \text{when } d_{m-1} = 0 \text{ and } d_m = 0. \]

The only exception is the case where \( j = m \). When \( d_m = 0 \), we remark however that, in the neighborhood of \( \gamma = 0 \),
\[ \| d_m \Phi(\lambda, \gamma)(i,m) \cdot \Psi(\lambda, \gamma)(m,m) \| = O\left( \frac{1}{\gamma} \right) |A_m| \cdot |d_{m-1}| = O(1). \]

These are the desired estimates, and
\[ |\text{the residues of each element of } e^{-t\lambda}(\lambda - \Lambda - \gamma HW)^{-1}| = O(1)e^{-(\omega + \gamma + O(\gamma^2))t}, \quad \text{when } d_{m-1} = 0 \text{ and } d_m = 0. \] (3.14)

Let us turn to the singularities of \( \Psi(\lambda, \gamma)(m,j) \) in (3.13). The corresponding terms in \( \Phi(\lambda, \gamma) \) are the \( m \)th column. We have to evaluate
\[ H_i \gamma A_1 \cdots A_{m-1} W_m \Psi(\lambda, \gamma)(m,j), \quad 1 \leq i \leq m-1, \quad \text{and } \frac{1}{d_m} \Psi(\lambda, \gamma)(m,j), \]
which amounts to the estimate of the residues of
\[
\begin{align*}
\gamma & \prod_{k=m}^{2} \frac{\langle k \rangle}{d_{k-1}} \cdot \frac{\gamma A_1 \cdots A_{m-1}}{d_{m-1}d_m} \frac{1}{A_i}, \\
& \quad 1 \leq i \leq m-1, \quad j = 1,
\end{align*}
\]
and
\[
\begin{align*}
\gamma A_1 \cdots A_{j-1} & \prod_{k=m}^{j+1} \frac{\langle k \rangle}{d_{k-1}} \cdot \frac{\gamma A_1 \cdots A_{m-1}}{d_{m-1}d_m} \frac{1}{A_i}, \\
& \quad 1 \leq i \leq m-1, \quad 2 \leq j \leq m-2,
\end{align*}
\] (3.15)

when \( dl = 0, \; 1 \leq l \leq m-2 \). Let us consider first (3.15). It is plain that, when \( dl = 0, \; 1 \leq l \leq m-2 \),
\[
\begin{align*}
\| \frac{\langle k \rangle}{d_{k-1}} \| &= O(1)|A_k|, \quad 2 \leq k \leq l, \\
\| (l+1) \| &= \gamma |A_1 \cdots A_l|, \\
\| \frac{\langle k \rangle}{d_{k-1}} \| &= O(1), \quad l+2 \leq k \leq m-1.
\end{align*}
\]

Thus, we see that
\[
\left\| d_1 \left( \gamma \prod_{k=m-1}^{2} \frac{\langle k \rangle}{d_{k-1}} \frac{\gamma^i A_1 \cdots A_{m-1}}{d_{m-1}d_m} \right) \right\|_1 = O(1) \left( \frac{\gamma A_2 \cdots A_{m-1}}{|A_i|} \right) = O(1), \quad 1 \leq i \leq m-1,
\]

when \( d_1 = 0 \). Each term of (3.16) satisfies a similar estimate. Thus we see that

\[
\left\| d_1 \left( \gamma \prod_{k=m-1}^{2} \frac{\langle k \rangle}{d_{k-1}} \right) \right\|_1 = O(1) \gamma^{A_2 \cdots A_{m-1}} |A_i|, \quad 1 \leq i \leq m-1,
\]

when \( d_1 = 0 \).

Combining this with (3.14), we finally obtain the desired estimate (2.11). The proof of Theorem 3.1 is thereby complete.

\[\square\]

4. On removal of the assumption (3.3): \( \Xi(i,j)(\lambda_h) \neq 0, 1 \leq h < i < j \leq n, 1 \leq i \leq n \)

In this section, we consider the case where \( n \leq 4 \), and show that Theorem 3.1 holds without the additional assumption (3.3). When (3.3) is lost, however, some singularities regarding \( \gamma \sim 0 \) generally arise in calculation of the residues of \( e^{-t\lambda} (\lambda - \Lambda - \gamma HW)^{-1} \). We need to guarantee the estimate (2.11) in Proposition 2.3.

**Theorem 4.1.** Suppose that \( n \leq 4 \). Then Theorem 3.1 holds without the additional assumption (3.3).

**Proof.** Only the case \( n = 4 \) is considered. The other case is easy. We only have to concentrate on the estimate (2.11) in Proposition 2.3. We have four eigenvalues \( \lambda_1, \ldots, \lambda_4 \) in question. When (3.3) is lost, there are four possibilities: \( \Xi(2,3)(\lambda_1) = 0, \Xi(2,4)(\lambda_1) = 0, \Xi(3,4)(\lambda_1) = 0, \) and \( \Xi(3,4)(\lambda_2) = 0 \). The matrix \( (\lambda - \Lambda - \gamma HW)^{-1} \) has sixteen blocks according to the decomposition (3.5). We have to examine these blocks in each case. However, singularities of \( (\lambda - \Lambda - \gamma HW)^{-1} \) arise only in the \( (1,1) \)- and \( (2,2) \)-blocks. The other blocks have no problem, the proof of which is omitted to save spaces. Otherwise we have to write down \( 16 \times 4 = 64 \) cases altogether. The calculations in the following are lengthy and tedious, but seem not avoidable.

**On the residues of the block \( e^{-t\lambda} (\lambda - \Lambda - \gamma HW)^{-1} \) \((1,1)\):**

According to the expression (3.5)–(3.7), we see that

\[
(\lambda - \Lambda - \gamma HW)^{-1} \mid_{(1,1)} = \frac{1}{d_1} I + \frac{\gamma^2}{d_1d_2} H_1 W_2 H_2 H_4^{-1} + \frac{\gamma^2 A_2}{d_1d_2d_3} H_1 W_3 H_3 H_4(2) H_4^{-1} + \frac{\gamma^2 A_2 A_3}{d_1d_2d_3d_4} H_1 W_4 H_4(3) (2) H_4^{-1}.
\]

Thus we have to evaluate the residues of the functions:
Thus we only prove typical cases. The other calculations will be left to the readers.

(i) The case where $\mathcal{E}_{2,3}(\lambda_1) = 2\lambda_1 - \lambda_2 - \lambda_3 = 0$.

We note that $\mathcal{E}_{3,4}(\lambda_1) \neq 0$ and $\mathcal{E}_{2,4}(\lambda_1) \neq 0$ in this case. However, there is a possibility that $\mathcal{E}_{3,4}(\lambda_2) = 0$. Based on Table 2 (the $d_i - d_j$ table), we need to analyze precisely the properties of the $d_j$, $j \neq i$, when $d_i = 0$.

\begin{align}
\frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4}, \quad \frac{\gamma^3 A_1 A_2^2 A_3}{d_1 d_3 d_4}, \quad \frac{\gamma^3 A_1 A_2 A_3}{d_1 d_2 d_4}, \quad \frac{\gamma^2 A_2 A_3}{d_1 d_4}, \quad \frac{\gamma^2 A_2}{d_1 d_3}, \quad \text{and} \quad \frac{\gamma^2}{d_1 d_2}
\end{align}

Many times $e^{-t\lambda}$ at each singularity. In the following, we encounter similar calculations repeatedly. Thus we only prove typical cases. The other calculations will be left to the readers.

The residues of $\frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t\lambda}$.

As we see below, singularities regarding $\gamma$ arise when $d_1 = 0$ and $d_3 = 0$. In view of the behaviors of the $d_i$ in the above table, no singularity arises in the residues at $\lambda = \lambda_{21}(\gamma), \lambda_{22}(\gamma)$ (or $d_2 = 0$), $\lambda_{32}(\gamma), \lambda_{23}(\gamma)$, and $\lambda_{41}(\gamma), \ldots, \lambda_{44}(\gamma)$ (or $d_4 = 0$). At $\lambda = \lambda_{11}(\gamma) = \lambda_1 + \gamma$, we calculate the residue as

$$\text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t\lambda}; \lambda_1 + \gamma \right) = \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_2 d_3 d_4} e^{-t\lambda} \bigg|_{\lambda = \lambda_1 + \gamma} = -\frac{1}{2\gamma} (\lambda_1 - \lambda_2 + O(\gamma)) e^{-(\lambda_1 + \gamma)t}. $$

At $\lambda = \lambda_{31}(\gamma) \sim \lambda_1$, the residue is

$$\text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t\lambda}; \lambda_{31}(\gamma) \right) = \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 (\lambda - \lambda_{32})(\lambda - \lambda_{33})d_4} e^{-t\lambda} \bigg|_{\lambda = \lambda_{31}} = -\frac{1}{2\gamma} (\lambda_1 - \lambda_2 + O(\gamma)) e^{-\lambda_{31}t}. $$
Recalling that $\lambda_1(\gamma) = \lambda_1 + \gamma + O((\gamma^2)^2)$ and $2\lambda_1 = \lambda_2 + \lambda_3$, we obtain the estimate:

\[
\left| \text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t\lambda}; \lambda_1 + \gamma \right) \right| + \left| \text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t\lambda}; \lambda_3(\gamma) \right) \right| \\
\leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2))t}, \quad t \geq 0.
\]  

Thus the sum of these residues consequently reveal no singularity regarding $\gamma$.

The residues of $\frac{\gamma^3 A_1^2 A_2^2 A_3}{d_1 d_2 d_3} e^{-t\lambda}$, $\frac{\gamma^2 A_1 A_2}{d_1 d_2} e^{-t\lambda}$, $\frac{\gamma A_1}{d_1} e^{-t\lambda}$, $\frac{\gamma^2 A_2}{d_1 d_2} e^{-t\lambda}$, and $\frac{\gamma^2 A_3}{d_1 d_4} e^{-t\lambda}$, and $\frac{\gamma^2}{d_1 d_2} e^{-t\lambda}$.

In the first three functions, cancellation of the singularities can be proven in exactly the same manner as above. In the last three, no singularity arises in the calculation of the residues.

Combining these estimates together, we obtain in (2.14)

\[
\left\| e^{-t(\Lambda + \gamma HW)} \right\|_{(1,1)} = \left\| \frac{1}{2\pi e^{-t(\lambda - \Lambda - \gamma HW)^{-1}}(1,1) d\lambda} \right\| \\
\leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2))t}, \quad t \geq 0.
\]  

(ii) The case where $\mathcal{E}(2,4)(\lambda_1) = \sum_{2 \leq i < j \leq 4} (\lambda_1 - \lambda_i)(\lambda_1 - \lambda_j) = 0$.

We note that $\mathcal{E}(2,3)(\lambda_1) \neq 0$ and $\mathcal{E}(3,4)(\lambda_1) \neq 0$ in this case. However, there is a possibility such that $\mathcal{E}(3,4)(\lambda_2) = 0$. As in the previous case (i), we need to know precisely the properties of $d_j$, $j \neq i$, when $d_i = 0$. The table is similar to, but slightly different from the previous one:

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1(\gamma) = \lambda_1 + \gamma$</td>
<td>0</td>
<td>$O(\gamma^2)$</td>
<td>$O(\gamma^2)$</td>
</tr>
<tr>
<td>$\lambda_2(\gamma) = \lambda_2(\gamma) - \lambda_2$</td>
<td>0</td>
<td>$O(\gamma^2)$</td>
<td>$O(\gamma^2)$</td>
</tr>
<tr>
<td>$\lambda_3(\gamma) = \lambda_3(\gamma) - \lambda_3$</td>
<td>0</td>
<td>$O(\gamma^2)$</td>
<td>$O(\gamma^2)$</td>
</tr>
<tr>
<td>$\lambda_4(\gamma) = \lambda_4(\gamma) - \lambda_4$</td>
<td>0</td>
<td>$O(\gamma^2)$</td>
<td>$O(\gamma^2)$</td>
</tr>
</tbody>
</table>

\[ a \] This arises only when $\mathcal{E}(3,4)(\lambda_2) = 0$.

\[ b \] This expression can be sharpened: $\lambda_3(\gamma) = \lambda_1 + \gamma + O(\gamma^3)$. See the consideration in the case: $n = 5$. 

The residues of $\frac{γ^4A_1^2A_2^2A_3}{d_1d_2d_3d_4}e^{-t\lambda}$.

At this time, cancellation of the singularities is proven for the residues different from those of the case (i): In fact, no singularity arises in the residues at $\lambda = \lambda_{21}(γ)$, $\lambda_{22}(γ)$, $\lambda_{31}(γ)$, ..., $\lambda_{33}(γ)$, and $\lambda_{42}(γ)$, ..., $\lambda_{44}(γ)$. At $\lambda = \lambda_{11}(γ)$ and $\lambda_{41}(γ) \sim \lambda_1$, the residues are calculated, respectively, as

$$\text{Res}\left(\frac{γ^4A_1^2A_2^2A_3}{d_1d_2d_3d_4}e^{-t\lambda};\lambda_1 + γ\right) = \left(\frac{-(λ_1 - λ_2)^2(λ_1 - λ_3)}{2α(2λ_1 - λ_2 - λ_3)γ} + O(1)\right)e^{-(λ_1 + γ)t} \quad \text{and}$$

$$\text{Res}\left(\frac{γ^4A_1^2A_2^2A_3}{d_1d_2d_3d_4}e^{-t\lambda};\lambda_{41}(γ)\right) = \left(\frac{(λ_1 - λ_3)(λ_1 - λ_4)^2}{2α(2λ_1 - λ_3 - λ_4)γ} + O(1)\right)e^{-(λ_{41})t}.$$

By the condition $Ξ(2,4)(λ_1) = 0$, we note that

$$\frac{(λ_1 - λ_2)^2(λ_1 - λ_3)}{2α(2λ_1 - λ_2 - λ_3)γ} + \frac{(λ_1 - λ_3)(λ_1 - λ_4)^2}{2α(2λ_1 - λ_3 - λ_4)γ} = 0.$$

Thus, recalling that $λ_{41}(γ) = λ_1 + γ + O(γ^2)$, we obtain the estimate

$$\left|\text{Res}\left(\frac{γ^4A_1^2A_2^2A_3}{d_1d_2d_3d_4}e^{-t\lambda};\lambda_1 + γ\right) + \text{Res}\left(\frac{γ^4A_1^2A_2^2A_3}{d_1d_2d_3d_4}e^{-t\lambda};\lambda_{41}(γ)\right)\right| \leq \text{const} \, e^{-(ω + γ/2 + O(γ^2)t)}, \quad t \geq 0.$$

The residues of $\frac{γ^3A_1A_2^2A_3}{d_1d_3d_4}e^{-t\lambda}$, $\frac{γ^3A_1A_2A_3}{d_1d_2d_4}e^{-t\lambda}$, $\frac{γ^2A_2A_3}{d_1d_4}e^{-t\lambda}$, $\frac{γ^3A_1A_2}{d_1d_2d_3}e^{-t\lambda}$, $\frac{γ^2A_2}{d_1d_3}e^{-t\lambda}$, and $\frac{γ^2}{d_1d_2}e^{-t\lambda}$.

In the first three functions, cancellation of the singularities can be proven in exactly the same manner as above. In the last three, on the other hand, no singularity arises in the calculation of the residues.

Combining these estimates together, we obtain the estimate (4.4) in this case, too.

(iii) The case where $Ξ(3,4)(λ_1) = 2λ_1 - λ_3 - λ_4 = 0$.

We note that $Ξ(2,3)(λ_1) \neq 0$, $Ξ(2,4)(λ_1) \neq 0$, and $Ξ(3,4)(λ_2) \neq 0$ in this case. As before, we write down the properties of the $d_j$, $j \neq i$, when $d_i = 0$:
The case where $d_1 = 0$ ($\lambda_{11}(\gamma) = \lambda_1 + \gamma$)

\[
\begin{array}{cccc}
\lambda_{11}(\gamma) & 0 & O(\gamma^2) & O(\gamma^2) \\
\end{array}
\]

The case where $d_2 = 0$ ($\lambda = \lambda_{21}(\gamma) \sim \lambda_1, \lambda_{22}(\gamma) \sim \lambda_2$)

\[
\begin{array}{cccc}
\lambda_{21}(\gamma) & O(\gamma^2) & 0 & O(\gamma^2) \\
\lambda_{22}(\gamma) & O(1) & 0 & O(\gamma^2) \\
\end{array}
\]

The case where $d_3 = 0$ ($\lambda = \lambda_{31}(\gamma) \sim \lambda_1, \lambda_{32}(\gamma) \sim \lambda_2, \lambda_{33}(\gamma) \sim \lambda_3$)

\[
\begin{array}{cccc}
\lambda_{31}(\gamma) & O(\gamma^2) & O(\gamma^2) & 0 \\
\lambda_{32}(\gamma) & O(1) & O(\gamma^2) & 0 \\
\lambda_{33}(\gamma) & O(1) & O(1) & O(\gamma^2) \\
\end{array}
\]

The case where $d_4 = 0$ ($\lambda = \lambda_{41}(\gamma) \sim \lambda_1, \lambda_{42}(\gamma) \sim \lambda_2, \lambda_{43}(\gamma) \sim \lambda_3, \lambda_{44}(\gamma) \sim \lambda_4$)

\[
\begin{array}{cccc}
\lambda_{41}(\gamma) & O(\gamma^2) & \frac{2(\lambda_1 - \lambda_2)\gamma^3}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} (1 + O(\gamma)) & O(\gamma^2) \\
\lambda_{42}(\gamma) & O(1) & O(\gamma^2) & 0 \\
\lambda_{43}(\gamma) & O(1) & O(1) & O(\gamma^2) \\
\lambda_{44}(\gamma) & O(1) & O(1) & O(1) \\
\end{array}
\]

The residues of $\frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t \lambda}$.

No singularity arises in the residues at $\lambda = \lambda_{11}(\gamma), \lambda_{22}(\gamma), \lambda_{31}(\gamma), \ldots, \lambda_{33}(\gamma)$, and $\lambda_{42}(\gamma), \ldots, \lambda_{44}(\gamma)$. At $\lambda = \lambda_{21}(\gamma)$ and $\lambda_{41}(\gamma)$, the residues are calculated, respectively, as

\[
\begin{align*}
\text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t \lambda}; \lambda_{21}(\gamma) \right) &= \left( \frac{\lambda_1 - \lambda_3}{2\gamma} + O(1) \right) e^{-\lambda_{21} t} \quad \text{and} \\
\text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t \lambda}; \lambda_{41}(\gamma) \right) &= \left( \frac{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)^2}{2 \Xi(2,4)(\lambda_1) \gamma} + O(1) \right) e^{-\lambda_{41} t}.
\end{align*}
\]

By the condition: $\Xi(3,4)(\lambda_1) = 0$, note that

\[
\frac{\lambda_1 - \lambda_3}{2\gamma} + \frac{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)^2}{2 \Xi(2,4)(\lambda_1) \gamma} = 0.
\]

Then we have the estimate

\[
\left| \text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t \lambda}; \lambda_{21}(\gamma) \right) + \text{Res} \left( \frac{\gamma^4 A_1^2 A_2^2 A_3}{d_1 d_2 d_3 d_4} e^{-t \lambda}; \lambda_{41}(\gamma) \right) \right| \leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2)) t}, \quad t \geq 0.
\]

The residues of $\frac{\gamma^3 A_1 A_2 A_3}{d_1 d_2 d_4} e^{-t \lambda}, \frac{\gamma^3 A_1 A_2 A_3}{d_1 d_2 d_4} e^{-t \lambda}, \frac{\gamma^3 A_1 A_2}{d_1 d_3 d_4} e^{-t \lambda}, \frac{\gamma^2 A_2 A_3}{d_1 d_4} e^{-t \lambda}, \frac{\gamma^2 A_2}{d_1 d_3} e^{-t \lambda}$, and $\frac{\gamma^2}{d_1 d_2} e^{-t \lambda}$.

In the first function, cancellation of the singularities can be shown in exactly the same manner as above. In the last five, no singularity arises in the calculation of the residues.

Combining these estimates together, we obtain the estimate (4.4) in this case, too.

(iv) The case where $\Xi(3,4)(\lambda_2) = 2\lambda_2 - \lambda_3 - \lambda_4 = 0$. 


We note that \( \mathcal{E}_{(3,4)}(\lambda_1) \neq 0 \). However, there is a possibility that \( \mathcal{E}_{(2,3)}(\lambda_1) = 0 \) or \( \mathcal{E}_{(2,4)}(\lambda_1) = 0 \). As we have already seen in (i) and (ii), no singularity arises in the sum of the residues (see the footnotes in the tables).

On the residues of the block \( e^{-t\lambda}(\lambda - \Lambda - \gamma HW)^{-1}|_{(2,2)} \).

According to the expression (3.5)–(3.7), we see that

\[
(\lambda - \Lambda - \gamma HW)^{-1}|_{(2,2)} = \frac{d_1}{d_2} I + \frac{\gamma^2 A_1^2}{d_2 d_3} H_2 W_3 H_3 W_2 + \frac{\gamma^2 A_1^3 A_2^2}{d_2 d_3 d_4} H_2 W_4 H_4(3) W_2. \tag{4.5}
\]

Thus we have to evaluate the residues of the functions:

\[
\frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4}, \quad \frac{\gamma^2 A_1^2 A_3}{d_2 d_4}, \quad \frac{\gamma^2 A_1^2}{d_2 d_3}, \quad \text{and} \quad \frac{d_1}{d_2} \tag{4.6}
\]
times \( e^{-t\lambda} \) at each singularity. Singularities arise only in the case where \( \mathcal{E}_{(3,4)}(\lambda_2) = 0 \).

The case where \( \mathcal{E}_{(3,4)}(\lambda_2) = 2\lambda_2 - \lambda_3 - \lambda_4 = 0 \).

The residues of \( \frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4} e^{-t\lambda} \).

The residues except at \( \lambda = \lambda_{22}(\gamma) \) and \( \lambda_{42}(\gamma) \) contain no singularity. At \( \lambda = \lambda_{22}(\gamma) \) and \( \lambda_{42}(\gamma) \), the residues are calculated as

\[
\text{Res}\left( \frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4} e^{-t\lambda}; \lambda_{22}(\gamma) \right) = \left( \frac{\lambda_2 - \lambda_3}{2\gamma} + O(1) \right) e^{-\lambda_{22} t} \quad \text{and}
\]

\[
\text{Res}\left( \frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4} e^{-t\lambda}; \lambda_{42}(\gamma) \right) = \left( \frac{\lambda_2 - \lambda_4}{2\gamma} + O(1) \right) e^{-\lambda_{42} t},
\]

respectively. By the condition: \( \mathcal{E}_{(3,4)}(\lambda_2) = 0 \) and \( \lambda_{22}(\gamma), \lambda_{42}(\gamma) = \lambda_2 + \gamma + O(\gamma^2) \), we obtain

\[
\left| \text{Res}\left( \frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4} e^{-t\lambda}; \lambda_{22}(\gamma) \right) + \text{Res}\left( \frac{\gamma^3 A_1^3 A_2 A_3}{d_2 d_3 d_4} e^{-t\lambda}; \lambda_{42}(\gamma) \right) \right| \leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2)) t}, \quad t \geq 0.
\]

The residues of \( \frac{\gamma^2 A_1^2 A_3}{d_2 d_4} e^{-t\lambda}, \frac{\gamma^2 A_1^2}{d_2 d_3} e^{-t\lambda}, \text{and} \frac{d_1}{d_2} e^{-t\lambda} \).

In the first function, cancellation of the singularities can be shown in exactly the same manner as above. In the last two, no singularity arises in the calculation of the residues.

Combining these estimates together, we obtain the estimate

\[
\| e^{-t(\Lambda + \gamma HW)} \|_{(2,2)} \leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2)) t}, \quad t \geq 0, \tag{4.7}
\]

in this case, too. We have shown that, when \( n = 4 \), Theorem 3.1 holds without the additional condition (3.3).

We would try to extend Theorem 4.1 to general cases: \( n \geq 5 \), where more serious singularities however arise. When \( n = 5 \) and (3.3) is lost, we can examine and guarantee the estimate (2.11)
for some elements through more complicated calculations. We will show this just in the limited situation later. Based on this observation, we come to the following statement as a conjecture.

**Conjecture.** When \( n \geq 3 \), Theorem 3.1 generally holds without the additional assumption (3.3).

Let \( n = 5 \) and consider the residues of the \((1, 1)\)-block of \( e^{-t\lambda}(\lambda - \Lambda - \gamma HW)^{-1} \). According to the expression (3.5)–(3.7), we see that

\[
e^{-t\lambda}(\lambda - \Lambda - \gamma HW)^{-1}\bigg|_{n=5}^{(1,1)} = e^{-t\lambda}(\lambda - \Lambda - \gamma HW)^{-1}\bigg|_{n=4}^{(1,1)} + e^{-t\lambda} \frac{\gamma^2 A_2 A_3 A_4}{d_1d_2d_3d_4d_5} H_1 W_5 H_5 (3)(2) H_1^{-1}.
\]

(4.8)

The first term of the right-hand side of (4.8) has been already examined. The second term contains eight functions. Among others it contains the functions

\[
f_1(\lambda) = e^{-t\lambda} \frac{\gamma^5 A_1^3 A_2^3 A_3 A_4}{d_1d_2d_3d_4d_5}, \quad f_2(\lambda) = e^{-t\lambda} \frac{\gamma^4 A_1^2 A_2^2 A_3 A_4}{d_1d_2d_3d_5}, \quad \text{and}
\]

\[
f_3(\lambda) = e^{-t\lambda} \frac{\gamma^4 A_1^2 A_2^2 A_3 A_4}{d_1d_3d_4d_5}.
\]

We will see that singularities of order \( \gamma^{-2} \) and \( \gamma^{-1} \) appear in the residues of these functions. In the residues of the other five functions, the situation is simple and similar to the case \( n = 4 \). Only the singularities of order \( \gamma^{-1} \) appear and they cancel each other.

To avoid similar calculations, we limit ourselves to the case of \( f_1(\lambda) \). To calculate the residues of \( f_1(\lambda) \), we need more information on the behavior of the solutions \( \lambda_{ij}^{(\gamma)} \) to the equation \( d_i = 0 \). Differentiating in \( \gamma \) twice the both sides of the equations \( d_3 = 0 \) and \( d_5 = 0 \) and then setting \( \gamma = 0 \), we obtain \( \lambda_{31}''(0) = \lambda_{51}''(0) = 0 \). Thus the behaviors of \( \lambda_{31}(\gamma) \) and \( \lambda_{51}(\gamma) \) in the neighborhood of \( \gamma = 0 \) are characterized by the following expansions

\[
\lambda_{31}(\gamma) = \lambda_1 + \gamma + O(\gamma^2), \quad \lambda_{51}(\gamma) = \lambda_1 + \gamma + O(\gamma^3).
\]

(4.9)

Let us consider first the case where \( \mathcal{E}_{(2,3)}(\lambda_1) = \mathcal{E}_{(4,5)}(\lambda_1) = 0 \). If \( \mathcal{E}_{(2,3)}(\lambda_1) \neq 0 \), in addition, then note that \( \mathcal{E}_{(2,5)}(\lambda_1) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \mathcal{E}_{(4,5)}(\lambda_1) \neq 0 \). The calculation for \( f_1(\lambda) \) in this case is the same as in the residues of \( e^{-t\lambda} \frac{\gamma^4 A_1^2 A_2 A_3}{d_4d_2d_3d_5} \). Only the singularities of order \( \gamma^{-1} \) appear and they cancel each other (see (4.3)). Thus we begin with the following case:

(i) *The case where* \( \mathcal{E}_{(2,3)}(\lambda_1) = \mathcal{E}_{(4,5)}(\lambda_1) = 0 \).

We first consider the residues at the points which are close to \( \lambda_1 \). Note that

\[
\mathcal{E}_{(2,3)}(\lambda_1 + \gamma) = 2\gamma,
\]

\[
\mathcal{E}_{(2,4)}(\lambda_1 + \gamma) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) + 2(\lambda_1 - \lambda_4)\gamma + 3\gamma^2, \quad \text{and}
\]

\[
\mathcal{E}_{(2,5)}(\lambda_1 + \gamma) = 2\gamma((\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) + (\lambda_1 - \lambda_4)(\lambda_1 - \lambda_5))(1 + O(\gamma^2)).
\]

At \( \lambda = \lambda_{21}(\gamma) \) and \( \lambda_{41}(\gamma) \), the residues reveal no singularity. Recalling (4.9), on the other hand, we calculate the residues at \( \lambda = \lambda_{11}(\gamma) = \lambda_1 + \gamma, \lambda_{31}(\gamma), \) and \( \lambda_{51}(\gamma) \) as
respectively. Note that \( \gamma \) arises only when \( \lambda - \lambda_2 + \lambda_1 - \lambda_3 = 0 \) and \( \lambda - \lambda_4 + \lambda_1 - \lambda_5 = 0 \). Then the sum of the coefficients (except for the exponentials) of \( \gamma^{-2} \) in (4.10) is equal to 0. Similarly the sum of the coefficients of \( \gamma^{-1} \) is equal to 0. Thus we see that

\[
|\text{Res}(f_1(\lambda); \lambda_1 + \gamma) + \text{Res}(f_1(\lambda); \lambda_3(\gamma)) + \text{Res}(f_1(\lambda); \lambda_5(\gamma))| \\
\leq \text{const} e^{-(\omega + \gamma^2/2 + O(\gamma^2))}, \quad t \geq 0. \tag{4.11}
\]

Similar calculations show the estimate (4.11) for \( f_2(\lambda) \) and \( f_3(\lambda) \).

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>The ( \lambda_2(\gamma)-d_j ) table</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda_2 (\gamma) )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( d_3 )</th>
<th>( d_4 )</th>
<th>( d_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_2 (\gamma) )</td>
<td>( O(1) )</td>
<td>0</td>
<td>( O(\gamma^2) )</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{a}} ))</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{b}} ))</td>
</tr>
<tr>
<td>( \lambda_3 (\gamma) )</td>
<td>( O(1) )</td>
<td>( O(\gamma^2) )</td>
<td>0</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{a}} ))</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{b}} ))</td>
</tr>
<tr>
<td>( \lambda_4 (\gamma) )</td>
<td>( O(1) )</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{a}} ))</td>
<td>( O(\gamma^2) )</td>
<td>0</td>
<td>( O(\gamma^2) )</td>
</tr>
<tr>
<td>( \lambda_5 (\gamma) )</td>
<td>( O(1) )</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{b}} ))</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{b}} ))</td>
<td>( O(\gamma^2) ) (or ( O(\gamma^3)^{\text{b}} )</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^{\text{a}}\) This arises only when \( \mathcal{E}_{(3,4)}(\lambda_2) = 0 \).

\(^{\text{b}}\) This arises only when \( \mathcal{E}_{(3,5)}(\lambda_2) = 0 \).

\(^{\text{c}}\) This arises only when \( \mathcal{E}_{(4,5)}(\lambda_2) = 0 \).
Table 2
The $d_i - d_j$ table

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>...</th>
<th>$d_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = 0$</td>
<td>0</td>
<td>$-\gamma A_1$</td>
<td>$-\gamma A_1 \xi(2,3)$</td>
<td>$-\gamma A_1 \xi(2,4)$</td>
<td>$-\gamma A_1 \xi(2,5)$</td>
<td>...</td>
<td>$-\gamma A_1 \xi(2,n)$</td>
</tr>
<tr>
<td>$d_2 = 0$</td>
<td>$\frac{\gamma A_1}{A_2}$</td>
<td>0</td>
<td>$-\gamma A_1 A_2$</td>
<td>$-\gamma A_1 A_2 \xi(3,4)$</td>
<td>$-\gamma A_1 A_2 \xi(3,5)$</td>
<td>...</td>
<td>$-\gamma A_1 A_2 \xi(3,n)$</td>
</tr>
<tr>
<td>$d_3 = 0$</td>
<td>$\frac{\gamma A_1}{A_2 A_3}$</td>
<td>$\frac{\gamma A_1 A_2}{A_3}$</td>
<td>0</td>
<td>$-\gamma A_1 A_2 A_3$</td>
<td>$-\gamma A_1 A_2 A_3 \xi(4,5)$</td>
<td>...</td>
<td>$-\gamma A_1 A_2 A_3 \xi(4,n)$</td>
</tr>
<tr>
<td>$d_4 = 0$</td>
<td>$\frac{\gamma A_1}{A_2 A_3 A_4}$</td>
<td>$\frac{\gamma A_1 A_2}{A_3 A_4}$</td>
<td>$\frac{\gamma A_1 A_2 A_3}{A_4}$</td>
<td>0</td>
<td>$-\gamma A_1 A_2 A_3 A_4$</td>
<td>...</td>
<td>$-\gamma A_1 A_2 A_3 A_4 \xi(5,n)$</td>
</tr>
<tr>
<td>$d_5 = 0$</td>
<td>$\frac{\gamma A_1}{A_2 A_3 A_4 A_5}$</td>
<td>$\frac{\gamma A_1 A_2}{A_3 A_4 A_5}$</td>
<td>$\frac{\gamma A_1 A_2 A_3}{A_4 A_5}$</td>
<td>$\frac{\gamma A_1 A_2 A_3 A_4}{A_5}$</td>
<td>0</td>
<td>...</td>
<td>$-\gamma A_1 A_2 A_3 A_4 A_5 \xi(6,n)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_n = 0$</td>
<td>$\frac{\gamma A_1}{A_2 A_3 \cdots A_n}$</td>
<td>$\frac{\gamma A_1 A_2}{A_3 \cdots A_n}$</td>
<td>$\frac{\gamma A_1 A_2 A_3}{A_4 \cdots A_n}$</td>
<td>$\frac{\gamma A_1 A_2 A_3 A_4}{A_5 \cdots A_n}$</td>
<td>$\frac{\gamma A_1 A_2 A_3 A_4 A_5}{A_6 \cdots A_n}$</td>
<td>$\xi(6,n)$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3
The $d_i - (j)$ table

<table>
<thead>
<tr>
<th></th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>...</th>
<th>$(n - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = 0$</td>
<td>$\gamma A_1 (W_2 H_2 - 1)$</td>
<td>$\gamma A_1 A_2 \left( W_3 H_3 - \frac{\xi(2,3)}{A_2} \right)$</td>
<td>$\gamma A_1 A_2 A_3 \left( W_4 H_4 - \frac{\xi(2,4)}{A_2 A_3} \right)$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} - \frac{\xi(2,n-1)}{A_2 \cdots A_{n-2}} \right)$</td>
</tr>
<tr>
<td>$d_2 = 0$</td>
<td>$\gamma A_1 W_2 H_2$</td>
<td>$\gamma A_1 A_2 (W_3 H_3 - 1)$</td>
<td>$\gamma A_1 A_2 A_3 \left( W_4 H_4 - \frac{\xi(3,4)}{A_2 A_3} \right)$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} - \frac{\xi(3,n-1)}{A_3 \cdots A_{n-2}} \right)$</td>
</tr>
<tr>
<td>$d_3 = 0$</td>
<td>$\gamma A_1 \left( A_3 W_2 H_2 + \frac{A_2}{A_3} \right)$</td>
<td>$\gamma A_1 A_2 W_3 H_3$</td>
<td>$\gamma A_1 A_2 A_3 (W_4 H_4 - 1)$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} - \frac{\xi(4,n-1)}{A_4 \cdots A_{n-2}} \right)$</td>
</tr>
<tr>
<td>$d_4 = 0$</td>
<td>$\gamma A_1 \left( W_2 H_2 + \frac{A_2 \xi(3,4)}{A_3 A_4} \right)$</td>
<td>$\gamma A_1 A_2 \left( W_3 H_3 + \frac{A_3}{A_4} \right)$</td>
<td>$\gamma A_1 A_2 A_3 W_4 H_4$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} - \frac{\xi(5,n-1)}{A_5 \cdots A_{n-2}} \right)$</td>
</tr>
<tr>
<td>$d_5 = 0$</td>
<td>$\gamma A_1 \left( W_2 H_2 + \frac{A_2 \xi(3,5)}{A_3 A_4 A_5} \right)$</td>
<td>$\gamma A_1 A_2 \left( W_3 H_3 + \frac{A_3 \xi(4,5)}{A_4 A_5} \right)$</td>
<td>$\gamma A_1 A_2 A_3 \left( W_4 H_4 + \frac{A_4}{A_5} \right)$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} - \frac{\xi(6,n-1)}{A_6 \cdots A_{n-2}} \right)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_n = 0$</td>
<td>$\gamma A_1 \left( W_2 H_2 + \frac{A_2 \xi(3,n)}{A_3 A_4 \cdots A_n} \right)$</td>
<td>$\gamma A_1 A_2 \left( W_3 H_3 + \frac{A_3 \xi(4,n)}{A_4 \cdots A_n} \right)$</td>
<td>$\gamma A_1 A_2 A_3 \left( W_4 H_4 + \frac{A_4 \xi(5,n)}{A_5 \cdots A_n} \right)$</td>
<td>...</td>
<td>$\gamma A_1 \cdots A_{n-2} \left( W_{n-1} H_{n-1} + \frac{A_{n-1}}{A_n} \right)$</td>
</tr>
</tbody>
</table>
Let us turn to the residues at the points which are close to $\lambda_2$. We calculate the residues $\text{Res}(f_i(\lambda); \lambda_i(\gamma))$, $2 \leq i \leq 5$, where all of $\lambda_i(\gamma)$ are close to $\lambda_2$. Table 1 ($\lambda_i(\gamma)-d_j$ table), just a version of Table 2 (the $d_i-d_j$ table), describes the behaviors of the $d_j$ when $\lambda = \lambda_i(\gamma)$.

Via calculations by this table we find that the residues at $\lambda_i(\gamma)$ reveal no singularity. Completing the $\lambda_i(\gamma)-d_j$ and the $\lambda_i(\gamma)-d_j$ tables similarly, we also find that no singularity appears in the residues $\text{Res}(f_i(\lambda); \lambda_i(\gamma))$, $3 \leq i \leq 5$; $\text{Res}(f_i(\lambda); \lambda_i(\gamma))$, $i = 4, 5$; and $\text{Res}(f_1(\lambda); \lambda_{55}(\gamma))$.

(ii) The case where $\mathcal{E}(2,4)(\lambda_1) = 0$.

As already seen, we have $\mathcal{E}(2,3)(\lambda_1) \neq 0$; $\mathcal{E}(3,4)(\lambda_1) \neq 0$; $\mathcal{E}(3,5)(\lambda_1) \neq 0$; and $\mathcal{E}(2,5)(\lambda_1) = \prod_{i=2}^{4}(\lambda_1 - \lambda_i) \neq 0$. Then, the residues have singularities only of order $\gamma^{-1}$. The residues at $\lambda = \lambda_1 + \gamma$ and $\lambda_4(\gamma)$ contain the singularities, but

$$|\text{Res}(f_1(\lambda); \lambda_1 + \gamma) + \text{Res}(f_1(\lambda); \lambda_4(\gamma))| \leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2)) t}, \quad t \geq 0.$$  
Furthermore, if $\mathcal{E}(4,5)(\lambda_1) = 0$, we have to add the following:

$$|\text{Res}(f_1(\lambda); \lambda_3(\gamma)) + \text{Res}(f_1(\lambda); \lambda_5(\gamma))| \leq \text{const} e^{-(\omega + \gamma/2 + O(\gamma^2)) t}, \quad t \geq 0.$$  

(iii) The other cases.

The residues of $f_1(\lambda)$ in the other cases such as $\mathcal{E}(2,5)(\lambda_1) = 0$, $\mathcal{E}(3,4)(\lambda_1) = 0$, $\mathcal{E}(3,4)(\lambda_2) = 0$, etc., can be calculated in a similar manner. In all of these cases, the singularities, if they arise, are canceled just like the above.

References