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Multiplicativity Factors for Seminorms

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Let $\mathscr{A}$ be an algebra and let S be a seminorm on $\mathscr{A}$. In this paper we study multiplicativity factors for S, i.e., constants $\mu > 0$ for which $S(xy) \leqslant \mu S(x)\, S(y)$ for all $x, y \in \mathscr{A}$. We begin by investigating these factors in terms of the kernel of S. We then specialize our study to function algebras and to seminorms generated by the sup norm, where we provide a convenient characterization of multiplicativity factors.

1. INTRODUCTION

Let $\mathscr{A}$ be an algebra over a field $\mathbf{F}$ where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. Throughout the paper we exclude the trivial case where all products in $\mathscr{A}$ are zero. As usual, a function

$$S: \mathscr{A} \to \mathbf{R}$$

is called a *seminorm* if for all $x, y \in \mathscr{A}$ and $\lambda \in \mathbf{F}$:

$$S(x) \geqslant 0,$$

$$S(\lambda x) = |\lambda| \cdot S(x),$$

$$S(x + y) \leqslant S(x) + S(y).$$

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If in addition, S is positive definite, i.e.,

$$S(x) > 0, \qquad x \neq 0,$$

then S is a *norm*. We call a seminorm S *proper* if $S \neq 0$ and $S(x) = 0$ for some $x \neq 0$. Finally, we say that S is *multiplicative* if

$$S(xy) \leqslant S(x)\, S(y), \qquad \forall x,\ y \in \mathscr{A}.$$

Of special interest are, of course, operator algebras. A well-known example of a nonmultiplicative norm on such an algebra is the numerical radius [8, 9],

$$r(A) = \sup\{|(Ax, x)| : x \in \mathbf{H},\ (x, x) = 1\}, \tag{1.1}$$

defined on $\mathscr{B}(\mathbf{H})$, the algebra of bounded linear operators on a Hilbert space $\mathbf{H}$ over $\mathbf{C}$.

Another example of considerable interest is the l_p norm, $1 \leqslant p \leqslant \infty$, on the algebra $\mathbf{C}_{n \times n}$ of $n \times n$ complex matrices, defined by

$$\|A\|_p = \left(\sum_{i,j=1}^{n} |\alpha_{ij}|^p \right)^{1/p}, \qquad A = (\alpha_{ij}) \in \mathbf{C}_{n \times n}. \tag{1.2}$$

Ostrowski [12] has shown that this norm is multiplicative if and only if $1 \leqslant p \leqslant 2$.

Given a seminorm S on an arbitrary algebra and a fixed constant $\mu > 0$, then obviously $S_\mu \equiv \mu S$ is a seminorm too. Clearly, S_μ may or may not be multiplicative. If it is, we call μ a *multiplicativity factor* for S. That is, *μ is a multiplicativity factor for S if and only if*

$$S(xy) \leqslant \mu S(x)\, S(y), \qquad \forall x,\ y \in \mathscr{A}.$$

Evidently, if μ_0 is a multiplicativity factor for S, then so is any μ with $\mu \geqslant \mu_0$. Thus, having a seminorm S, the question is whether S has multiplicativity factors; and if so, is there a best (least) one?

If S is a norm, this question can be answered since Theorem 2.1 in [5] is valid for norms:

THEOREM 1.1. *Let $\mathscr{A}$ be an algebra, and let S be a norm on $\mathscr{A}$. Then*:

(a) *S has multiplicativity factors if and only if*

$$\mu_{\inf} \equiv \sup\{S(xy) : x,\ y \in \mathscr{A},\ S(x) = S(y) = 1\} < \infty. \tag{1.3}$$

(b) *If S has multiplicativity factors, then a constant $\mu > 0$ is a multiplicativity factor for S if and only if $\mu \geqslant \mu_{\inf}$.*

A standard compactness argument implies that *if $\mathscr{A}$ is finite dimensional and S is a norm, then μ_{inf} in* (1.3) *is finite, so S has multiplicativity factors.* As we shall see, this is not always the case if S is a proper seminorm.

It is shown, both in [5, Examples 2.1–2.4] and in Section 3 below, that in the infinite dimensional case, norms as well as proper seminorms may or may not have multiplicativity factors.

In Theorem 2.4 below we shall provide a characterization of multiplicativity factors for arbitrary seminorms in terms of the quantity μ_{inf} in (1.3). While this quantity is often difficult to compute, a more practical approach toward checking whether a given constant $\mu > 0$ is the best (least) multiplicativity factor for a given seminorm S is obviously by verifying that

$$S(xy) \leqslant \mu S(x)\, S(y) \qquad \forall x, y \in \mathscr{A},$$

with equality for some $x = x_0$, $y = y_0$ for which $S(x_0) \neq 0$, $S(y_0) \neq 0$.

Using this observation, it was shown in [10] and later in [4] that if $\mathbf{H}$ is a Hilbert space over $\mathbf{C}$ of dimension at least 2, and r is the numerical radius in (1.1), then the best multiplicativity factor for r is $\mu_{\text{inf}} = 4$.

Similarly, it was shown in [11] and later in [6] that the best multiplicativity factor for the l_p norm on $\mathbf{C}_{n \times n}$ defined in (1.2) is

$$\mu_{\text{inf}} = \begin{cases} 1, & 1 \leqslant p \leqslant 2 \\ n^{1-2/p}, & 2 \leqslant p \leqslant \infty. \end{cases}$$

As a final introductory remark let us point out that if μ_{inf} remains unknown, one may try to obtain multiplicativity factors through the following version of a theorem by Gastinel.

THEOREM 1.2 (Compare [3] and [5, Theorem 2.3]). *Let S, T be seminorms on an algebra $\mathscr{A}$. Let T be multiplicativity, and let $\tau \geqslant \sigma > 0$ be constants such that*

$$\sigma T(x) \leqslant S(x) \leqslant \tau T(x), \qquad \forall x \in \mathscr{A}.$$

Then any $\mu \geqslant \tau/\sigma^2$ in a multiplicative factor for S.

This result was utilized by Goldberg and Straus [5, 7] to obtain multiplicativity factors for certain generalizations of the numerical radius.

2. Multiplicativity Factors and Kernels

In this section we further discuss multiplicativity factors by studying kernels of seminorms. We start with the following observation.

THEOREM 2.1. *Let $\mathscr{A}$ be an algebra, and let S be a seminorm on $\mathscr{A}$. Then*:

(a) $\mathscr{K} = \operatorname{Ker} S$ *is a subspace of* $\mathscr{A}$.

(b) *If S has multiplicativity factors, then $\mathscr{K}$ is an ideal in $\mathscr{A}$.*

(c) *If $\mathscr{A}$ is a topological algebra, and S is continuous with multiplicativity factors, then $\mathscr{K}$ is a closed ideal of $\mathscr{A}$.*

Proof. $\mathscr{K}$ is obviously a subspace of $\mathscr{A}$. Now let $\mu > 0$ be a multiplicativity factor for S, and take any $x \in \mathscr{K}$, $y \in \mathscr{A}$. Then

$$S(xy) \leqslant \mu S(x)\, S(y) = 0.$$

That is, $S(xy) = 0$, i.e., $xy \in \mathscr{K}$. Similarly, $yx \in \mathscr{K}$, so $\mathscr{K}$ is an ideal. Finally, if $\mathscr{A}$ is a topological algebra and S is continuous, then this ideal, being where a continuous function vanishes, is closed. ∎

COROLLARY 2.1. (a) *If $\mathscr{A}$ is a simple algebra, then there are no multiplicative proper seminorms on $\mathscr{A}$.*

(b) *If $\mathscr{A}$ is a topological algebra that has no proper closed ideals, then there are no multiplicative, continuous proper seminorms on $\mathscr{A}$.*

Proof. Let S have multiplicativity factors. By part (b) of Theorem 2.1, $\mathscr{K} = \operatorname{Ker} S$ is an ideal. Since $\mathscr{A}$ is simple, then $\mathscr{K} = \{0\}$ or $\mathscr{K} = \mathscr{A}$. In the first case S is a norm, and in the second $S = 0$. Assertion (b) follows from part (c) of Theorem 2.1 in the same way. ∎

Since $\mathbf{F}_{n \times n}$, the algebra of $n \times n$ matrices over $\mathbf{F}$, is simple (e.g., [2, Theorem 10, p. 414]) we immediately obtain from Corollary 2.1:

THEOREM 2.2. *There are no multiplicative proper seminorms on* $\mathbf{F}_{n \times n}$.

A direct proof of this result was given by Goldberg and Straus in [4, Theorem 3].

In the finite dimensional case the converse of Theorem 2.1(b) holds, so we have:

THEOREM 2.3. *Let $\mathscr{A}$ be a finite dimensional algebra, and let S be a seminorm on $\mathscr{A}$. Then S has multiplicativity factors if and only if* Ker S *is an ideal.*

Proof. If S has multiplicativity factors, then $\mathscr{K} = \operatorname{Ker} S$ is an ideal by Theorem 2.1.

Conversely, let $\mathscr{K}$ be an ideal. Consider the quotient algebra $\mathscr{A}/\mathscr{K}$, and define

$$T(x + \mathscr{K}) = S(x), \qquad x \in \mathscr{A}.$$

Clearly, T is a seminorm on $\mathscr{A}/\mathscr{K}$. Further, T vanishes only for $x \in \mathscr{K}$, so in fact T is a norm on $\mathscr{A}/\mathscr{K}$. Since $\mathscr{A}/\mathscr{K}$ is finte dimensional, then by the remark following Theorem 1.1, T has multiplicativity factors. Thus, S has the same multiplicativity factors and the theorem follows. ∎

In Example 3.1 we shall show that in the infinite dimensional case, the converse of Theorem 2.1(b), (c) is false. That is, a norm or a proper seminorm S may fail to have multiplicativity factors even when Ker S is a (closed) ideal in $\mathscr{A}$.

We now give a characterization of multiplicativity factors for arbitrary seminorms.

THEOREM 2.4. *Let $\mathscr{A}$ be an algebra, and let $S \neq 0$ be a seminorm on $\mathscr{A}$. Then*:

(a) *S has multiplicativity factors if and only if $\mathscr{K} = \operatorname{Ker} S$ is an ideal in $\mathscr{A}$ and*

$$\mu_{\inf} \equiv \sup\{S(xy) : S(x) = S(y) = 1\} < \infty.$$

(b) *If S has multiplicativity factors and $\mu_{\inf} > 0$, then μ is a multiplicativity factor if and only if $\mu \geqslant \mu_{\inf}$.*

(c) *If S has multiplicativity factors and $\mu_{\inf} = 0$, then μ is a multiplicativity factor if and only if $\mu > 0$.*

Proof. By Theorem 2.1(b), if $\mathscr{K}$ is not an ideal, then S has no multiplicativity factors. Similarly, if $\mu_{\inf} = \infty$, then for each μ there are elements $x_0, y_0 \in \mathscr{A}$ such that

$$S(x_0 y_0) > \mu S(x_0)\, S(y_0), \qquad S(x_0) = S(y_0) = 1; \tag{2.1}$$

so again, S has no multiplicativity factors.

Conversely, suppose $\mathscr{K}$ is an ideal and $\mu_{\inf} < \infty$. If $\mu < \mu_{\inf}$ then as above, there exists $x_0, y_0 \in \mathscr{A}$ that satisfy (2.1), so μ is not a multiplicativity factor. However, if $\mu \geqslant \mu_{\inf}$, then we write

$$\mu_{\inf} = \sup\left\{\frac{S(xy)}{S(x)\, S(y)} : x, y \notin \mathscr{K}\right\}$$

and realize that

$$S(xy) \leqslant \mu_{\inf} S(x)\, S(y) \leqslant \mu S(x)\, S(y) \qquad \forall x, y \notin \mathscr{K}. \tag{2.2}$$

Since $\mathscr{K}$ is an ideal, we also have

$$S(xy) = 0 = \mu S(x)\, S(y) \qquad \text{for} \quad x \in \mathscr{K} \text{ or } y \in \mathscr{K},$$

which together with (2.2) implies

$$S(xy) \leqslant \mu S(x)\, S(y) \qquad \forall x, y \in \mathscr{A},$$

and the theorem follows without difficulty. ▮

If S is a norm then clearly, $\mu_{\text{inf}} > 0$ and Ker $S = \{0\}$ is an ideal in $\mathscr{A}$; hence in this case Theorem 2.4 says no more than Theorem 1.1.

In concluding this section we prove:

COROLLARY 2.2. *Let $\mathscr{A}$ and $S \neq 0$ be as in Theorem* 2.4. *Then S has multiplicativity factors and*

$$\mu_{\text{inf}} \equiv \sup\{S(xy) : S(x) = S(y) = 1\} = 0 \tag{2.3}$$

if and only if

$$xy \in \text{Ker } S \qquad \forall x, y \in \mathscr{A}. \tag{2.4}$$

Proof. If S has multiplicativity factors then by Theorem 2.4, $\mathscr{K} = \text{Ker } S$ is an ideal, so

$$xy \in \mathscr{K} \qquad \text{if} \quad x \in \mathscr{K} \text{ or } y \in \mathscr{K}. \tag{2.5}$$

Further, if $\mu_{\text{inf}} = 0$ then $S(xy) = 0$ for $x, y \notin \mathscr{K}$, i.e.,

$$xy \in \mathscr{K} \qquad \forall x, y \notin \mathscr{K}$$

which together with (2.5) yields (2.4).

Conversely, if (2.4) holds, then clearly, any $\mu > 0$ will serve as a multiplicativity factor, and by (2.3), $\mu_{\text{inf}} = 0$. ▮

In view of Theorem 2.4 and Corollary 2.2, we realize that the only nontrivial seminorms on $\mathscr{A}$ that have multiplicativity factors, but not a least one, are those satisfying (2.4). In this case, the multiplicativity factors are just the positive reals.

For example, take $\mathscr{A} = \mathbf{C}^2$ with multiplication defined by

$$xy = (\xi_1 \eta_1, 0), \qquad x = (\xi_1, \xi_2),\ y = (\eta_1, \eta_2) \in \mathscr{A},$$

and consider the proper seminorm

$$S(x) = |\xi_2|.$$

Evidently, (2.4) holds; hence the multiplicativity factors of S form the interval $(0, \infty)$.

3. A Special Class of Seminorms

Let $\mathscr{A}$ be an algebra with a multiplicative seminorm S. Fix an element $c \in \mathscr{A}$ and define

$$S_c(x) = S(cx), \qquad x \in \mathscr{A}. \tag{3.1}$$

Obviously, S_c is a seminorm on $\mathscr{A}$. In fact, it is quite evident that S_c *in* (3.1) *is a norm* on $\mathscr{A}$ *if and only if* S *is a norm and* c *is not a zero divisor.*

We prove:

THEOREM 3.1. *Let $\mathscr{A}$ be an algebra with a multiplicative norm S, and let S_c, $0 \neq c \in \mathscr{A}$ fixed, be the seminorm in* (3.1). *Then S_c has multiplicativity factors if either*

(a) *$\mathscr{A}$ has a unit and c is invertible, or*

(b) (*compare* [1, p. 462]) *c is in the center of $\mathscr{A}$ and $c = c^2 d$ for some d in $\mathscr{A}$.*

Proof. (a) We have

$$S_c(x) = S(cx) \leqslant S(c)\, S(x),$$
$$S(x) = S(c^{-1}cx) \leqslant S(c^{-1})\, S(cx) = S(c^{-1})\, S_c(x);$$

hence S_c is equivalent to S, and by Theorem 1.2, (a) follows.

(b) For all $x, y \in \mathscr{A}$,

$$cxy = c^2dxy = dcxcy.$$

Thus,

$$S(cxy) \leqslant S(d)\, S(cx)\, S(cy),$$

i.e.,

$$S_c(xy) \leqslant \mu S_c(x)\, S_c(y)$$

with $\mu = S(d)$. ∎

In what follows we specialize our discussion to certain function algebras where S in (3.1) is the sup norm. In this case we are able to give simple characterizations of multiplicativity factors for S_c.

THEOREM 3.2. *Let* **T** *be a set and let $\mathscr{A}$ be the algebra of bounded functions*

$$x\colon \mathbf{T} \to \mathbf{F},$$

with the usual multiplication

$$xy(t) = x(t)\, y(t); \qquad x, y \in \mathscr{A};\ t \in \mathbf{T}.$$

For $0 \neq c \in \mathscr{A}$ *define the seminorm*

$$S_c(x) = \sup_{t \in \mathbf{T}} |c(t)\, x(t)|. \tag{3.2}$$

Then:

(a) S_c *has multiplicativity factors if and only if*

$$\varepsilon \equiv \inf\{|c(t)| : t \in \mathbf{T},\ c(t) \neq 0\} > 0. \tag{3.3}$$

(b) *If* $\varepsilon > 0$, *then* $\mu > 0$ *is a multiplicativity factor for* S_c *if and only if* $\mu \geqslant \varepsilon^{-1}$.

Proof. Evidently, Ker S_c is an ideal in $\mathscr{A}$; and $c \neq 0$ implies that

$$\mu_{\text{inf}} \equiv \sup\{S_c(xy) : x, y \in \mathscr{A};\ S_c(x) = S_c(y) = 1\} \geqslant \frac{S_c(c^2)}{S_c(c)^2} > 0.$$

Thus, by Theorem 2.4, S_c has multiplicativity factors if and only if $\mu_{\text{inf}} < \infty$; and if $\mu_{\text{inf}} < \infty$ then μ is a multiplicativity factor if and only if $\mu \geqslant \mu_{\text{inf}}$.

It therefore suffices to prove that if $\varepsilon > 0$ then $\mu_{\text{inf}} = \varepsilon^{-1}$; and if S_c has multiplicativity factors then $\varepsilon > 0$.

Suppose $\varepsilon > 0$. Put

$$\mathbf{E} = \{t \in \mathbf{T} : c(t) \neq 0\}. \tag{3.4}$$

Then

$$\varepsilon = \inf_{t \in \mathbf{E}} |c(t)|,$$

and

$$\begin{aligned}
\mu_{\text{inf}} &= \sup\{\sup_{t \in \mathbf{E}} |c(t)\, x(t)\, y(t)| : x, y \in \mathscr{A};\ \sup_{t \in \mathbf{E}} |c(t)\, x(t)| \\
&= \sup_{t \in \mathbf{E}} |c(t)\, y(t)| = 1\} \\
&\leqslant \sup\{\sup_{t \in \mathbf{E}} |c(t)\, x(t)| \cdot \sup_{t \in \mathbf{E}} |y(t)| : \sup_{t \in \mathbf{E}} |c(t)\, x(t)| \\
&= \sup_{t \in \mathbf{E}} |c(t)\, y(t)| = 1\} \\
&= \sup\{\sup_{t \in \mathbf{E}} |y(t)| : \sup_{t \in \mathbf{E}} |c(t)\, y(t)| = 1\} = \varepsilon^{-1}.
\end{aligned} \tag{3.5}$$

Now define

$$d(t)=\begin{cases}1/c(t), & t\in\mathbf{E}\\ 0, & t\in\mathbf{G}\equiv\mathbf{T}\backslash\mathbf{E}.\end{cases} \tag{3.6}$$

Surely $d\in\mathscr{A}$, since $\varepsilon>0$. Moreover, $S_c(d)=1$. Hence

$$\mu_{\text{inf}}\geqslant S_c(d^2)=\sup_{t\in\mathbf{T}}|c(t)\,d(t)^2|=\sup_{t\in\mathbf{E}}|d(t)|=\varepsilon^{-1},$$

and so, using (3.5),

$$\mu_{\text{inf}}=\varepsilon^{-1}.$$

Next, assume that $\mu>0$ is a multiplicativity factor for S_c, and let us show that $\varepsilon>0$. Choose a sequence $t_n\in\mathbf{E}$ such that

$$|c(t_n)|\equiv\varepsilon_n\xrightarrow[n\to\infty]{}\varepsilon.$$

Let $\mathbf{V}_n$ be the subset of $\mathbf{T}$ where

$$|c(t)|\leqslant 2\varepsilon_n,$$

and choose an element $u_n\in\mathscr{A}$ with

$$|u_n(t)|\leqslant 1,\qquad t\in\mathbf{T}, \tag{3.7a}$$

$$u_n(t_n)=1, \tag{3.7b}$$

$$u_n(t)=0,\qquad t\notin\mathbf{V}_n. \tag{3.7c}$$

Then

$$|c(t_n)\,u_n(t_n)|=\varepsilon_n,$$

and

$$|c(t)\,u_n(t)|\leqslant|c(t)|\leqslant 2\varepsilon_n,\qquad t\in\mathbf{V}_n.$$

Hence,

$$\varepsilon_n\leqslant\sup_{t\in\mathbf{T}}|c(t)\,u_n(t)|=\sup_{t\in\mathbf{V}_n}|c(t)\,u_n(t)|\leqslant 2\varepsilon_n.$$

Thus,

$$\varepsilon_n\leqslant S_c(u_n)\leqslant 2\varepsilon_n.$$

Now u_n^2 satisfies the same conditions (3.7) as u_n, so as before

$$\varepsilon_n\leqslant S_c(u_n^2)\leqslant 2\varepsilon_n.$$

By hypothesis,

$$S_c(u_n^2) \leqslant \mu S_c(u_n)^2;$$

hence

$$\varepsilon_n \leqslant S_c(u_n^2) \leqslant \mu S_c(u_n)^2 \leqslant \mu 4\varepsilon_n^2.$$

Therefore, $4\mu\varepsilon_n \geqslant 1$, and consequently, $\varepsilon \geqslant (4\mu)^{-1} > 0$. ∎

A similar result is the following:

THEOREM 3.3. *Let* **T** *be a topological space and let* $\mathscr{A}$ *be the algebra of bounded, continuous function*

$$x : \mathbf{T} \to \mathbf{F}.$$

For $0 \neq c \in \mathscr{A}$ *let* S_c *be the seminorm in* (3.2). *Then conclusions* (a) *and* (b) *of Theorem* 3.2 *hold.*

Proof. The proof goes precisely along the lines of the previous proof, subject to the following two clarifications:

First, we must show that if (3.3) holds then the function d in (3.6) is continuous on **T**, hence belonging to $\mathscr{A}$. Indeed, since $|c|$ is continuous, the set **E** in (3.4) is open, being where $|c(t)| > 0$. Since $\varepsilon > 0$, the complement **G** of **E** is actually where $|c(t)| < \varepsilon$, and is thus open too. Since **E** and **G** complement each other and are open, they are also closed. Now d is continuous on **E** and on **G**. Since **E** and **G** have no common limit points, d is continuous on $\mathbf{E} \cup \mathbf{G} = \mathbf{T}$, so $d \in \mathscr{A}$.

We must also show that the functions u_n satisfying (3.7) can be chosen to be continuous on **T**. This can be done, for example, by taking the composite functions

$$u_n(t) = \frac{1}{\varepsilon_n} \varphi_n(|c(t)|),$$

where φ_n is the real-valued continuous function on $[0, \infty)$:

$$\varphi_n(\lambda) = \begin{cases} \lambda, & 0 \leqslant \lambda \leqslant \varepsilon_n, \\ 2\varepsilon_n - \lambda, & \varepsilon_n \leqslant \lambda \leqslant 2\varepsilon_n, \\ 0, & \lambda \geqslant 2\varepsilon_n. \end{cases}$$

This completes the proof. ∎

If **T** is connected, Theorem 3.3 takes the following form:

THEOREM 3.4. *Let* $\mathbf{T}$, $\mathscr{A}$, c, *and* S_c *be as in Theorem* 3.3. *Suppose* $\mathbf{T}$ *is connected. Then*

(a) *The following are equivalent:*

(i) S_c *has multiplicativity factors.*

(ii) $\delta \equiv \inf\{|c(t)| : t \in \mathbf{T}\} > 0$.

(iii) c *is invertible.*

(b) *If* $\delta > 0$ *then*

(i) S_c *is a norm on* $\mathscr{A}$.

(ii) *A constant* $\mu > 0$ *is a multiplicativity factor for* S_c *if and only if* $\mu \geq \delta^{-1}$.

Proof. Evidently, c is invertible if and only if $\delta > 0$. Moreover, since $\mathbf{T}$ is connected and c is continuous on $\mathbf{T}$ then $\delta = \varepsilon$, where ε is given in (3.3). By Theorem 3.3, this completes the proof of (a) and of (b)(ii). For (b)(i) we note that if $\delta > 0$ then c is not a zero divisor; hence S_c is a norm on $\mathscr{A}$. ∎

Our last result states exactly when S_c has multiplicativity factors in terms of the generalized invertibility condition in Theorem 3.1(b).

THEOREM 3.5. *Let* $\mathscr{A}$ *be the function algebra in Theorem* 3.2 *or* 3.3. *Then* S_c *in* (3.2) *has multiplicativity factors if and only if* c *satisfies the condition* (b) *of Theorem* 3.1.

Proof. One-half of this is Theorem 3.1(b) with S being the sup norm. Conversely, if S_c has multiplicativity factors then, by Theorems 3.2 and 3.3, (3.3) holds. So by the proof of these theorems, the function d in (3.6) belongs to $\mathscr{A}$; where obviously $c = c^2 d$. ∎

EXAMPLE 3.1. Consider l^∞, the algebra of bounded sequences $x = \{\xi_j\}_{j=1}^{\infty}$ over $\mathbf{F}$, with the usual Hadamard multiplication

$$xy = \{\xi_j \eta_j\}, \qquad x = \{\xi_j\},\ y = \{\eta_j\} \in l^\infty.$$

Fix an element $c = \{\gamma_j\} \in l^\infty$, $c \neq 0$, and define the seminorm,

$$S_c(x) = \sup_j |\gamma_j \xi_j|, \qquad x \in l^\infty.$$

Obviously, S_c is a norm on l^∞ if and only if

$$\gamma_j \neq 0, \qquad j = 1, 2, 3, \ldots.$$

Otherwise S_c is a proper seminorm.

By Theorem 3.2 (here $\mathbf{T} = \mathbf{Z}^+ = \{1, 2, 3, ...\}$), S_c has multiplicativity factors if and only if

$$\varepsilon \equiv \inf_{\gamma_j \neq 0} |\gamma_j| > 0,$$

and if $\varepsilon > 0$ then the best (least) multiplicativity factor for S_c is $\mu_{\min} = \varepsilon^{-1}$.

The four simple selections

$$\gamma_j = 1, \qquad j = 1, 2, 3, ... \tag{3.8a}$$

$$\gamma_j = j^{-1}, \qquad j = 1, 2, 3, ... \tag{3.8b}$$

$$\gamma_1 = 0; \qquad \gamma_j = 1, j = 2, 3, 4, ... \tag{3.8c}$$

$$\gamma_1 = 0; \qquad \gamma_j = j^{-1}, j = 2, 3, 4, ... \tag{3.8d}$$

show, as indicated in the Introduction, that in the infinite dimensional case, norms and proper seminorms may or may not have multiplicativity factors.

Further, in cases (3.8b) and (3.8d) we have

$$\text{Ker } S_c = \{0\}$$

and

$$\text{Ker } S_c = \{x = \{\xi_j\} : \xi_j = 0, j = 2, 3, 4, ...\},$$

respectively. Hence, as mentioned in Section 2, in the infinite dimensional case, both norms and seminorms may fail to have multiplicativity factors even when the kernel is a (closed) ideal in $\mathscr{A}$.

EXAMPLE 3.2. Let $\mathscr{C}[0, 1]$ be the algebra of continuous functions on $[0, 1]$, and consider the seminorm

$$S_c(x) = \sup_{0 \leq t \leq 1} |c(t)\, x(t)|, \qquad x \in \mathscr{C}[0, 1],$$

where $0 \neq c \in \mathscr{C}[0, 1]$ is fixed.

By Theorem 3.4 ($\mathbf{T} = [0, 1]$), S_c has multiplicativity factors if and only if

$$\delta \equiv \min_{0 \leq t \leq 1} |c(t)| > 0,$$

and if $\delta > 0$ then S_c is a norm whose best multiplicativity factor is $\mu_{\min} = \delta^{-1}$.

We remark that the converse of Theorem 3.4(b)(i) is false. For instance, if $c(t) = t$ then c is not a zero devisor in $\mathscr{C}[0, 1]$, so S_c in (3.2) is a norm. On the other hand, $\delta = 0$ for this c.

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