Available at www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 290 (2004) 481–496

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Nonoscillations for odd-dimensional systems of linear retarded functional differential equations ☆

Xueyan Liu and Binggen Zhang *

Department of Mathematics, Ocean University of China, Qingdao 266071, PR China

Received 15 June 2003

Submitted by S.R. Grace

Abstract

This paper is concerned with the nonoscillatory problems of odd-dimensional systems of linear retarded functional differential equations. Based upon the corresponding characteristic equations, we get some criteria for nonoscillations by utilizing the matrix measures.

Keywords: Nonoscillation; Functional differential equations; Systems; Matrix measure

1. Introduction

Recently there have been several papers concerning the study of the oscillations for linear functional differential systems; see, for example, [1–11] and references therein. Some explicit conditions for oscillation are investigated by exploiting the characteristic equations or by some other methods, such as the matrix measures (some authors also use the term logarithmic derivatives, or Lozinskii measures); see [1–4,6,9]. However, there are few results about the corresponding nonoscillation problems. Our purpose in this paper is to study some explicit nonoscillation criteria for certain linear functional differential systems.

Consider the linear system

$$x'(t) = Q_0 x(t) + \int_{-r}^{0} d\eta(\theta) x(t+\theta) \tag{1.1}$$

☆ This work is supported by NSF of Shandong Province of China.

* Corresponding author.
E-mail address: bgzhang@public.qd.sd.cn (B. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.10.007

and the system of neutral type

$$\big(x(t) - Ax(t-\tau)\big)' = Q_0 x(t) + \int_{-r}^{0} d\eta(\theta) x(t+\theta), \tag{1.2}$$

where $x(t) \in R^n$, $r > 0$, $\tau > 0$, $Q_0, A \in R^{n\times n}$, $\eta(\theta)$ is a left-continuous matrix-valued function of bounded variation on $[-r, 0]$ and vanishes at $\theta = 0$.

Obviously, if we assume $\eta(\theta) = \sum_{j=1}^{m} H(\theta + \tau_j) Q_j$, $\tau_j \in (0, r]$, where $H(\theta)$ is the Heaviside function and $Q_j \in R^{n\times n}$, $j = 1, 2, \ldots, m$, then Eq. (1.1) becomes

$$x'(t) = Q_0 x(t) + \sum_{j=1}^{m} Q_j x(t - \tau_j), \tag{1.3}$$

and Eq. (1.2) becomes

$$\big(x(t) - Ax(t-\tau)\big)' = Q_0 x(t) + \sum_{j=1}^{m} Q_j x(t - \tau_j). \tag{1.4}$$

We first give the definitions of oscillation and nonoscillation of Eq. (1.1). The corresponding definitions of the system (1.2) of neutral type are similar.

Definition 1.1. A nontrivial vector solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) : [-r, \infty) \to R^n$ of Eq. (1.1) is oscillatory if at least one of its nontrivial components $x_i(t)$, $1 \leqslant i \leqslant n$, has arbitrarily large zeros. We say Eq. (1.1) is oscillatory if all its nontrivial solutions are oscillatory. Otherwise, Eq. (1.1) is said to be nonoscillatory.

The following Lemmas 1.1 and 1.2 are due to Krisztin [12]. One also can see [4,5,8] for reference.

Lemma 1.1. *Equation* (1.1) *is oscillatory if and only if the characteristic equation*

$$\det\left(-\lambda I + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right) = 0 \tag{1.5}$$

has no real root.

Lemma 1.2. *Equation* (1.2) *is oscillatory if and only if the characteristic equation*

$$\det\left(-\lambda(I - Ae^{-\lambda\tau}) + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right) = 0 \tag{1.6}$$

has no real root.

For a matter of completeness, we recall the definitions of the matrix measures and their main properties. For any $A \in R^{n\times n}$, we denote by $\lambda_1(A)$, the eigenvalue with maximum real part.

Definition 1.2 [13,14]. For $A \in R^{n\times n}$, we define the induced norms

$$\|A\|_i = \sup_{x\in R^n,\, x\neq 0} \frac{\|Ax\|_i}{\|x\|_i} \quad \text{for each } i = 1, 2, \ldots, \infty,$$

where

$$x = (x_1, x_2, \ldots, x_n)^T, \qquad \|x\|_i = \left(\sum_{j=1}^{n} |x_j|^i\right)^{1/i}, \quad i < \infty,$$

and

$$\|x\|_\infty = \max_{1\leqslant j\leqslant n} \{|x_j|\}.$$

The corresponding matrix measures $\mu_i : R^{n\times n} \to R$, for $i = 1, 2, \ldots, \infty$, are defined by

$$\mu_i(A) = \lim_{\varepsilon\to 0^+} \frac{\|I + \varepsilon A\|_i - 1}{\varepsilon}.$$

It has been proved that $\mu_i(A)$, $i = 1, 2, \ldots, \infty$, exist for any $A \in R^{n\times n}$ and can be explicitly evaluated for $i = 1, 2, \infty$ as follows:

$$\mu_1(A) = \sup_j \left\{a_{jj} + \sum_{i, i\neq j} |a_{ij}|\right\}, \qquad \mu_2(A) = \lambda_1\left(\frac{1}{2}(A + A^T)\right),$$

$$\mu_\infty(A) = \sup_i \left\{a_{ii} + \sum_{i, j\neq i} |a_{ij}|\right\}.$$

In general, without specification, we denote by $\mu(\cdot)$ any one of $\mu_i(\cdot)$, $i = 1, 2, \ldots, \infty$. Independently of the considered norm, a matrix measure $\mu(\cdot)$ has the following basic properties:

(i) $-\|A\| \leqslant -\mu(-A) \leqslant \mu(A) \leqslant \|A\|$, $\forall A \in R^{n\times n}$;
(ii) $\mu(\alpha A) = \alpha\mu(A)$, $\forall \alpha > 0$, $\forall A \in R^{n\times n}$;
(iii) $\max\{\mu(A) - \mu(-B), -\mu(-A) + \mu(B)\} \leqslant \mu(A + B) \leqslant \mu(A) + \mu(B)$, $\forall A, B \in R^{n\times n}$;
(iv) $-\mu(-A) \leqslant \mathrm{Re}\,\lambda \leqslant \mu(A)$, where λ is an eigenvalue of A, $\forall A \in R^{n\times n}$.

2. Preliminaries

Throughout this paper, we denote by $BV[a, b]$ the set of $n \times n$ matrix-valued functions of bounded variation on $[a, b]$.

Definition 2.1 [1]. Let $\eta \in BV[a, b]$. We say that $\mu(d\eta(\theta)) \leqslant 0$ on $[a, b]$, if $\mu(\eta(d) - \eta(c)) \leqslant 0$, $\forall c, d \in [a, b]$ such that $c \leqslant d$.

Lemma 2.1 [1]. *Let $\eta \in BV[a, b]$ and $\mu(d\eta(\theta)) \leqslant 0$ on $[a, b]$. Then $\mu(\int_a^b d\eta(\theta)) \leqslant 0$.*

Lemma 2.2 [1]. *Let $f, g \in C([a,b], R)$ and $\eta \in BV[a,b]$ such that $f(\theta) \geqslant g(\theta)$ and $\mu(d\eta(\theta)) \leqslant 0$ on $[a,b]$. Then*

$$\mu\left(\int_a^b f(\theta)\,d\eta(\theta)\right) \leqslant \mu\left(\int_a^b g(\theta)\,d\eta(\theta)\right).$$

Definition 2.2. Let $\eta \in BV[a,b]$. We say that $\mu(d\eta(\theta)) \geqslant 0$ on $[a,b]$, if $\mu(\eta(c) - \eta(d)) \leqslant 0$, $\forall c, d \in [a,b]$ such that $c \leqslant d$.

Lemma 2.3. *Let $\eta \in BV[a,b]$ and $\mu(d\eta(\theta)) \geqslant 0$ on $[a,b]$. Then $\mu(\int_a^b d\eta(\theta)) \geqslant 0$.*

Proof. Since $\mu(d\eta(\theta)) \geqslant 0$ on $[a,b]$ and $\mu((-\eta(d)) - (-\eta(c))) = \mu(\eta(c) - \eta(d))$, $\forall c, d \in [a,b]$ such that $c \leqslant d$, one can easily observe that $\mu(d(-\eta(\theta))) \leqslant 0$ on $[a,b]$. Therefore, from Lemma 2.1, we know $\mu(\int_a^b d(-\eta(\theta))) \leqslant 0$, which, together with the property (i) of matrix measures, shows

$$\mu\left(\int_a^b d\eta(\theta)\right) \geqslant -\mu\left(\int_a^b d\big(-\eta(\theta)\big)\right) \geqslant 0. \qquad \square$$

Lemma 2.4. *Let $f, g \in C([a,b], R)$ and $\eta \in BV[a,b]$ such that $f(\theta) \geqslant g(\theta)$ and $\mu(d\eta(\theta)) \geqslant 0$ on $[a,b]$. Then*

$$\mu\left(\int_a^b f(\theta)\,d\eta(\theta)\right) \geqslant \mu\left(\int_a^b g(\theta)\,d\eta(\theta)\right).$$

Proof. It follows from $\mu(d\eta(\theta)) \geqslant 0$ and $f(\theta) \geqslant g(\theta)$ on $[a,b]$ that $\mu(d(-\eta(\theta))) \leqslant 0$ and $-g(\theta) \geqslant -f(\theta)$ on $[a,b]$. Therefore, by applying Lemma 2.2, we have that

$$\mu\left(\int_a^b \big(-g(\theta)\big)d\big(-\eta(\theta)\big)\right) \leqslant \mu\left(\int_a^b \big(-f(\theta)\big)d\big(-\eta(\theta)\big)\right),$$

that is,

$$\mu\left(\int_a^b f(\theta)\,d\eta(\theta)\right) \geqslant \mu\left(\int_a^b g(\theta)\,d\eta(\theta)\right). \qquad \square$$

3. The nonoscillation of Eq. (1.1)

Ferreira and Györi [3] first use the general matrix measures to investigate the oscillation criteria for Eq. (1.1), but the explicit conditions were only applicable with the particular matrix measure μ_2. Kong [1] and Tian et al. [2] use some new techniques to extend their results by the general matrix measures. The following propositions are adopted from [2], which improve the results in [1].

Proposition 3.1. *Assume* $\mu(d\eta(\theta)) \leqslant 0$ *on* $[-r, 0]$. *If*

$$\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta} \, d\eta(\theta)\right) < -\frac{1}{e}, \tag{3.1}$$

then Eq. (1.1) *is oscillatory.*

Proposition 3.2. *Assume* $\mu(Q_j) \leqslant 0$, $j = 1, 2, \ldots, m$. *If*

$$\mu\left(\sum_{j=1}^{m} \tau_j e^{-\mu(Q_0)\tau_j} Q_j\right) < -\frac{1}{e}, \tag{3.2}$$

then Eq. (1.3) *is oscillatory.*

With respect to the nonoscillation of Eq. (1.1), we will investigate it in two different cases: $\mu(d\eta(\theta)) \leqslant 0$, and $\mu(d\eta(\theta)) \geqslant 0$ on $[-r, 0]$.

Throughout this section, we denote $F(\lambda) = -\lambda I + Q_0 + \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$ and the eigenvalues of $F(\lambda)$ and $-F(\lambda)$ by $\bar{\lambda}_{F(\lambda)}$ and $\bar{\lambda}_{-F(\lambda)}$, respectively.

Theorem 3.1. *Let n be odd, and assume* $\mu(d\eta(\theta)) \leqslant 0$ *on* $[-r, 0]$. *If*

$$r \cdot \mu\left(-\int_{-r}^{0} e^{-\mu(-Q_0)\theta} \, d\eta(\theta)\right) \leqslant \frac{1}{e}, \tag{3.3}$$

then Eq. (1.1) *is nonoscillatory. Furthermore, if* $\mu(Q_0) \leqslant 0$, *then Eq.* (1.1) *has at least one bounded nonoscillatory solution.*

Proof. According to Lemma 1.1, it suffices to prove that Eq. (1.5) has at least one real root, that is, there exists $\lambda_0 \in R$ such that $\det F(\lambda_0) = 0$. Assume for the sake of contradiction that $\det F(\lambda) \neq 0$, for all $\lambda \in R$; then we have either $\det F(\lambda) < 0$ for all $\lambda \in R$, or $\det F(\lambda) > 0$ for all $\lambda \in R$, since $\det F(\lambda)$ is continuous about λ.

(i) $\det F(\lambda) < 0$ for all $\lambda \in R$.

It is well known that the determinant of a matrix equals the product of all of its eigenvalues, so we can obtain that for each $\lambda \in R$, $F(\lambda)$ has at least one negative real eigenvalue. Then from the property (iv) of matrix measures, we know

$$-\mu\big(-F(\lambda)\big) \leqslant \operatorname{Re} \bar{\lambda}_{F(\lambda)} \leqslant \mu\big(F(\lambda)\big),$$

and so

$$\mu\big(-F(\lambda)\big) > 0,$$

for each $\lambda \in R$.

On the other hand, when $r(\lambda + \mu(-Q_0)) = -1$, from the property (iii) of matrix measures and Lemma 2.2, we have

$$\begin{aligned}
\mu\big(-F(\lambda)\big) &= \mu\left(\lambda I - Q_0 - \int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \leqslant \lambda + \mu(-Q_0) + \mu\left(-\int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \\
&= \lambda + \mu(-Q_0) + \mu\left(-\int_{-r}^{0} e^{(\lambda+\mu(-Q_0))\theta} e^{-\mu(-Q_0)\theta}\,d\eta(\theta)\right) \\
&\leqslant \lambda + \mu(-Q_0) + e^{-r(\lambda+\mu(-Q_0))}\mu\left(-\int_{-r}^{0} e^{-\mu(-Q_0)\theta}\,d\eta(\theta)\right) \\
&\leqslant \lambda + \mu(-Q_0) + \frac{1}{re} e^{-r(\lambda+\mu(-Q_0))} \\
&= \frac{1}{re}\big(re\big(\lambda + \mu(-Q_0)\big) + e^{-r(\lambda+\mu(-Q_0))}\big) = 0.
\end{aligned}$$

This is a contradiction.

(ii) $\det F(\lambda) > 0$ for all $\lambda \in R$.

Since n is odd, we have $\det(-F(\lambda)) = (-1)^n \det F(\lambda) < 0$ for all $\lambda \in R$. So $-F(\lambda)$ has at least one negative real eigenvalue for each $\lambda \in R$. It follows from $-\mu(F(\lambda)) \leqslant \operatorname{Re}\bar{\lambda}_{-F(\lambda)} \leqslant \mu(-F(\lambda))$ that $\mu(F(\lambda)) > 0$ for each $\lambda \in R$.

On the other hand, when $\lambda = \mu(Q_0)$,

$$\begin{aligned}
\mu\big(F(\lambda)\big) &= \mu\left(-\lambda I + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \leqslant -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \\
&\leqslant -\lambda + \mu(Q_0) + \min\{1, e^{-\lambda r}\}\mu\left(\int_{-r}^{0} d\eta(\theta)\right) \\
&= \min\{1, e^{-\lambda r}\}\mu\left(\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0,
\end{aligned}$$

which leads to a contradiction.

Therefore, Eq. (1.1) is nonoscillatory.

One can see from [12] that a necessary and sufficient condition for all bounded solutions of Eq. (1.1) to be oscillatory is that the corresponding characteristic equation (1.5) has no real nonpositive root. Hence in order to prove Eq. (1.1) has at least one bounded nonoscillatory solution, it suffices to show that there exists $\lambda_0 \leqslant 0$, such that $\det F(\lambda_0) = 0$. To achieve a contradiction we assume $\det F(\lambda) \neq 0$ for all $\lambda \leqslant 0$; then we have either $\det F(\lambda) < 0$ for all $\lambda \leqslant 0$, or $\det F(\lambda) > 0$ for all $\lambda \leqslant 0$.

From the former proof (i), we know that when $\lambda = -1/r - \mu(-Q_0) \leqslant -1/r + \mu(Q_0) \leqslant -1/r < 0$, a contradiction leads to the invality of the assumption $\det F(\lambda) < 0$ for all $\lambda \leqslant 0$.

From the former proof (ii), we find that when $\lambda = \mu(Q_0) \leqslant 0$, a contradiction shows that $\det F(\lambda) > 0$ for all $\lambda \leqslant 0$ fails to work.

Hence we come to the conclusion that Eq. (1.1) has at least one bounded nonoscillatory solution.

The proof is complete. □

Theorem 3.2. *Let n be odd. Assume $\mu(d\eta(\theta)) \geqslant 0$ on $[-r,0]$. Then Eq.* (1.1) *has at least one nonoscillatory solution. Furthermore, if $\mu(Q_0) \leqslant -\mu(\int_{-r}^{0} e^{\mu(Q_0)\theta}\,d\eta(\theta))$, then Eq.* (1.1) *has at least one bounded nonoscillatory solution.*

Proof. From the similar reasoning with Theorem 3.1, it suffices to prove that there exists $\lambda \in R$ $(\lambda \leqslant 0)$, such that $\mu(-F(\lambda)) \leqslant 0$, and another $\lambda \in R$ $(\lambda \leqslant 0)$, such that $\mu(F(\lambda)) \leqslant 0$.

When $\lambda + \mu(-Q_0) \leqslant 0$, from the property (iii) of matrix measures and Lemma 2.4 we have

$$\begin{aligned}\mu\big(-F(\lambda)\big) &= \mu\left(\lambda I - Q_0 - \int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \leqslant \lambda + \mu(-Q_0) + \mu\left(-\int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right)\\ &\leqslant \lambda + \mu(-Q_0) + \mu\left(-\int_{-r}^{0} e^{(\lambda+\mu(-Q_0))\theta} e^{-\mu(-Q_0)\theta}\,d\eta(\theta)\right)\\ &\leqslant \lambda + \mu(-Q_0) + \mu\left(-\int_{-r}^{0} e^{-\mu(-Q_0)\theta}\,d\eta(\theta)\right)\\ &\leqslant \mu\left(-\int_{-r}^{0} e^{-\mu(-Q_0)\theta}\,d\eta(\theta)\right) \leqslant 0.\end{aligned}$$

When $\lambda \geqslant \mu(Q_0) + \mu(\int_{-r}^{0} e^{\mu(Q_0)\theta}\,d\eta(\theta))$, we have

$$\begin{aligned}\mu\big(F(\lambda)\big) &= \mu\left(-\lambda I + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right) \leqslant -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\lambda\theta}\,d\eta(\theta)\right)\\ &= -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{(\lambda-\mu(Q_0))\theta} e^{\mu(Q_0)\theta}\,d\eta(\theta)\right)\\ &\leqslant -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\mu(Q_0)\theta}\,d\eta(\theta)\right) \leqslant 0.\end{aligned}$$

Under the assumption $\mu(Q_0) \leqslant -\mu(\int_{-r}^{0} e^{\mu(Q_0)\theta}\,d\eta(\theta))$, it is obvious that if we, respectively, let $\lambda_1 = -\mu(-Q_0) \leqslant \mu(Q_0) \leqslant 0$ and $\lambda_2 = \mu(Q_0) + \mu(\int_{-r}^{0} e^{\mu(Q_0)\theta}\,d\eta(\theta)) \leqslant 0$; then we can, respectively, obtain $\mu(-F(\lambda_1)) \leqslant 0$ and $\mu(F(\lambda_2)) \leqslant 0$.

Then the above discussion shows that Eq. (1.1) is nonoscillatory, and the assumption $\mu(Q_0) \leqslant -\mu(\int_{-r}^{0} e^{\mu(Q_0)\theta}\, d\eta(\theta))$ implies that Eq. (1.1) has at least one bounded nonoscillatory solution. $\square$

Corollary 3.1. *Let n be odd. If $\mu(Q_j) \leqslant 0$, $j = 1, 2, \ldots, m$, and*

$$r \cdot \mu\left(-\sum_{j=1}^{m} e^{\mu(-Q_0)\tau_j} Q_j\right) \leqslant \frac{1}{e}, \tag{3.4}$$

where $r = \max_{1\leqslant j\leqslant m} \tau_j$, then Eq. (1.3) is nonoscillatory. If we further assume $\mu(Q_0) \leqslant 0$, then Eq. (1.3) has at least one bounded nonoscillatory solution.

Corollary 3.2. *Let n be odd. If $\mu(Q_j) \geqslant 0$, $j = 1, 2, \ldots, m$, then Eq. (1.3) is nonoscillatory. Furthermore, if $\mu(Q_0) \leqslant -\mu(\sum_{j=1}^{m} e^{-\mu(Q_0)\tau_j} Q_j)$, then Eq. (1.3) has at least one bounded nonoscillatory solution.*

Remark 3.1. Under the assumption $\mu(d\eta(\theta)) \leqslant 0$ on $[-r, 0]$ and that n is odd, from the inequality (3.3), we have

$$\begin{aligned}\frac{1}{e} &\geqslant r \cdot \mu\left(-\int_{-r}^{0} e^{-\mu(-Q_0)\theta}\, d\eta(\theta)\right) \geqslant r \cdot \mu\left(-\int_{-r}^{0} e^{\mu(Q_0)\theta}\, d\eta(\theta)\right) \\ &\geqslant \mu\left(-\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right) \geqslant -\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right),\end{aligned}$$

that is,

$$\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right) \geqslant -\frac{1}{e}, \tag{3.5}$$

which is opposite to inequality (3.1). This reasoning shows that the sufficient condition for Eq. (1.1) to be nonoscillatory is contained in the necessary conditions (3.5). Therefore one can see that our conclusion in Theorem 3.1 is reasonable.

We next give two examples as applications of Corollaries 3.1 and 3.2.

Example 3.1. Consider the equation

$$x'(t) = Q_0 x(t) + Q x(t - \tau), \tag{3.6}$$

where

$$n = 3, \quad Q_0 = \begin{pmatrix} -5 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & \frac{1}{2e} & -\frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} -\frac{1}{4} & \frac{1}{10} & \frac{1}{4e} \\ \frac{1}{2e} & -\frac{1}{e} & \frac{1}{4e} \\ 0 & \frac{1}{5} & -\frac{1}{4} \end{pmatrix} \quad \text{and} \quad \tau = \frac{1}{6}.$$

By easy calculation, we have $\mu_1(Q) = -1/4 + 1/(2e) < 0$, $\mu_1(-Q) = 1/e + 3/10$, $\mu_1(Q_0) = -1/2 < 0$, $\mu_1(-Q_0) = 6$, and

$$\tau \cdot \mu_1(-e^{\mu_1(-Q_0)\tau} Q) = \frac{1}{6} \cdot e \cdot \mu_1(-Q) = \frac{e}{6}\left(\frac{1}{e} + \frac{3}{10}\right) = \frac{1}{6} + \frac{e}{20} < \frac{1}{e}.$$

Then Eq. (3.6) satisfies all the conditions in Corollary 3.1, and hence Eq. (3.6) has at least one bounded nonoscillatory solution.

Example 3.2. Consider the equation

$$x'(t) = Q_0 x(t) + Q_1 x(t-\tau_1) + Q_2 x(t-\tau_2), \tag{3.7}$$

where

$$n = 3, \qquad Q_0 = \begin{pmatrix} -4 & 0 & 0 \\ 1 & -3 & -1 \\ -1 & -\frac{1}{2} & -2 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -3 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} \frac{1}{5} & -\frac{1}{8} & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \qquad \tau_1 = \frac{1}{2} \quad \text{and} \quad \tau_2 = 1.$$

One can easily obtain that $\mu_1(Q_0) = -1$, $\mu_1(Q_1) = 0$, and $\mu_1(Q_2) = 1/4$, so the conditions $\mu(Q_j) \geqslant 0$, for $j = 1, 2$, are satisfied in Corollary 3.2. In addition,

$$\begin{aligned} -\mu_1\left(\sum_{j=1}^{2} e^{-\mu_1(Q_0)\tau_j} Q_j\right) &\geqslant -\mu_1(e^{-\mu_1(Q_0)\tau_1} Q_1) - \mu_1(e^{-\mu_1(Q_0)\tau_2} Q_2) \\ &= -\mu_1(e^{-\mu_1(Q_0)\tau_2} Q_2) = -\frac{e}{4} > \mu_1(Q_0) = -1. \end{aligned}$$

From Corollary 3.2, we conclude that Eq. (3.7) has at least one bounded nonoscillatory solution.

Remark 3.2. When n is even, the following two examples suggest the invality of Theorems 3.1 and 3.2.

Example 3.3. Consider the equation

$$x'(t) = Q_0 x(t) + Q x(t-\tau), \tag{3.8}$$

where

$$Q_0 = 0, \qquad Q = \begin{pmatrix} -\frac{1}{e} & -\frac{1}{2e} \\ \frac{1}{2e} & -\frac{1}{e} \end{pmatrix} \quad \text{and} \quad \tau = \frac{2}{3}.$$

One can easily obtain $\mu_1(Q_0) = 0$, $\mu_1(Q) = -1/(2e) < 0$, $\mu_1(-Q) = 3/(2e)$ and

$$\tau \cdot \mu_1(-e^{\mu_1(-Q_0)\tau} Q) = \frac{2}{3}\mu_1(-Q) = \frac{1}{e}.$$

Hence Eq. (3.8) satisfies the conditions in Corollary 3.1 except the one that n is odd.

However, since for each $\lambda \in R$

$$\det F(\lambda) = \det\begin{pmatrix} -\lambda - \frac{1}{e}e^{-\lambda\tau} & -\frac{1}{2e}e^{-\lambda\tau} \\ \frac{1}{2e}e^{-\lambda\tau} & -\lambda - \frac{1}{e}e^{-\lambda\tau} \end{pmatrix} = \left(\lambda + \frac{1}{e}e^{-\lambda\tau}\right)^2 + \frac{1}{4e^2}e^{-2\lambda\tau} > 0,$$

that is, the characteristic equation of Eq. (3.8) has no real root, it follows from Lemma 1.1 that Eq. (3.8) is oscillatory.

Example 3.4. Consider the equation

$$x'(t) = Q_0 x(t) + Q x(t-\tau), \tag{3.9}$$

where

$$n = 2, \qquad Q_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \frac{1}{2}.$$

Obviously, $\mu_1(Q) = 2 > 0$, which satisfies the condition $\mu(Q_j) \geqslant 0$, for $j = 1, \ldots, m$, in Corollary 3.2.

However, from Lemma 1.1 and

$$\det F(\lambda) = \det\begin{pmatrix} -\lambda + 2 & -2e^{-\lambda\tau} \\ 2e^{-\lambda\tau} & -\lambda + 1 \end{pmatrix} = (\lambda - 1)(\lambda - 2) + 4e^{\lambda} > 0,$$

for each $\lambda \in R$, since $\min_{\lambda \in [1,2]} \det F(\lambda) > -1/4 + 4e^2 > 0$, we know that Eq. (3.9) is oscillatory.

The two examples imply that when n is even, Theorems 3.1 and 3.2 are invalid, and we need stronger conditions to guarantee the nonoscillation of Eq. (1.1).

4. The nonoscillation of Eq. (1.2)

There are few papers concerned with the oscillatory problem of Eq. (1.2), and much less about its nonoscillation. Paper [3] is one of the most recent papers about the oscillation of its discrete case with several neutral terms. Here, in this section, we will establish the nonoscillatory criteria of Eq. (1.2).

For a matter of completeness, we first give the oscillatory criteria of Eq. (1.2).

Lemma 4.1. *Let $A \in R^{n \times n}$. If $\mu(A) < 0$, then* $\det A \neq 0$.

Theorem 4.1. *Assume $\mu(Q_0) \leqslant 0$, $\mu(d\eta(\theta)) \leqslant 0$ on $[-r, 0]$, $0 \leqslant -\mu(-A) \leqslant \mu(A) \leqslant 1$, and*

$$\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right) < -\frac{1}{e}. \tag{4.1}$$

Then Eq. (1.2) *is oscillatory.*

Proof. From Lemma 1.2, it suffices to prove that Eq. (1.6) has no real root. Let

$$G(\lambda) = -\lambda(I - Ae^{-\lambda\tau}) + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta).$$

From Lemma 4.1, we only need to show $\mu(G(\lambda)) < 0$ for all $\lambda \in R$. It follows from inequality (4.1) that

$$\mu\left(\int_{-r}^{0} d\eta(\theta)\right) \leqslant \frac{e^{\mu(Q_0)r}}{r}\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right) < -\frac{e^{\mu(Q_0)r}}{re} < 0.$$

We next show $\mu(G(\lambda)) < 0$, for all $\lambda \in R$, in three cases.

(1) $\lambda = 0$. It is obvious that

$$\mu\big(G(0)\big) = \mu\left(Q_0 + \int_{-r}^{0} d\eta(\theta)\right) < 0.$$

(2) $\lambda > 0$. From $\mu(A) \leqslant 1$, we have

$$\begin{aligned}\mu\big(G(\lambda)\big) &= \mu\left(-\lambda(I - Ae^{-\lambda\tau}) + Q_0 + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right)\\ &\leqslant -\lambda + \lambda e^{-\lambda\tau}\mu(A) + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right)\\ &\leqslant -\lambda + \lambda e^{-\lambda\tau} < 0.\end{aligned}$$

(3) $\lambda < 0$. From $-\mu(-A) \geqslant 0$, we know

$$\begin{aligned}\mu\big(G(\lambda)\big) &\leqslant -\lambda - \lambda e^{-\lambda\tau}\mu(-A) + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right)\\ &\leqslant -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right).\end{aligned}$$

If $-\lambda + \mu(Q_0) \leqslant 0$, then

$$\mu\big(G(\lambda)\big) \leqslant \mu\left(\int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right) < 0.$$

If $-\lambda + \mu(Q_0) > 0$, from inequality (4.1) and $e^x \geqslant ex$, for $x \in R$, we have

$$\begin{aligned}\mu\big(G(\lambda)\big) &\leqslant -\lambda + \mu(Q_0) + \mu\left(\int_{-r}^{0} e^{(\lambda-\mu(Q_0))\theta} e^{\mu(Q_0)\theta}\, d\eta(\theta)\right)\\ &\leqslant -\lambda + \mu(Q_0) + \big(-\lambda + \mu(Q_0)\big) e\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right)\\ &= \big(-\lambda + \mu(Q_0)\big)\left[1 + e\mu\left(\int_{-r}^{0} |\theta| e^{\mu(Q_0)\theta}\, d\eta(\theta)\right)\right] < 0.\end{aligned}$$

The above discussion shows that Eq. (1.2) is oscillatory. □

Remark 4.1. One can see that Theorem 4.1 extends the Proposition 3.1 and that Eq. (1.2) is oscillatory independently on the delay τ under the condition $0 \leqslant -\mu(-A) \leqslant \mu(A) \leqslant 1$.

For convenience, we let $Q_0 = 0$ in Eq. (1.2), and consider the following functional differential system of neutral type:

$$\big(x(t) - Ax(t-\tau)\big)' = \int_{-r}^{0} d\eta(\theta) x(t+\theta). \tag{4.2}$$

In addition, we let $G(\lambda) = -\lambda I + \lambda e^{-\lambda\tau} A + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)$ and denote the eigenvalues of $G(\lambda)$ by $\bar{\lambda}_{G(\lambda)}$.

Theorem 4.2. *Let n be odd. Assume $\mu(d\eta(\theta)) \leqslant 0$ on $[-r, 0]$ and*

$$\mu(A) \leqslant \max\left\{\frac{1}{e} - \frac{\tau}{e} e^{r/\tau} \mu\left(-\int_{-r}^{0} d\eta(\theta)\right), r e^{-\tau/r}\left(\frac{1}{r} - e\mu\left(-\int_{-r}^{0} d\eta(\theta)\right)\right)\right\}.$$

Then Eq. (4.2) *is nonoscillatory and has at least one bounded nonoscillatory solution.*

Proof. From Lemma 1.2, it suffices to show there exists $\lambda_0 \leqslant 0$, such that $\det G(\lambda_0) = 0$. Assume $\det G(\lambda) \neq 0$ for all $\lambda \leqslant 0$, then either $\det G(\lambda) < 0$ for all $\lambda \leqslant 0$, or $\det G(\lambda) > 0$ for all $\lambda \leqslant 0$, since $\det G(\lambda)$ is continuous about λ.

(i) $\det G(\lambda) < 0$ for all $\lambda \leqslant 0$. We know that $G(\lambda)$ has at least one negative real eigenvalue for each $\lambda \leqslant 0$. Then from $-\mu(-G(\lambda)) \leqslant \operatorname{Re} \bar{\lambda}_{G(\lambda)} \leqslant \mu(G(\lambda))$, it follows that $\mu(-G(\lambda)) > 0$, $\forall \lambda \leqslant 0$.

On the other hand, when $\lambda \leqslant 0$,

$$\begin{aligned}\mu\big(-G(\lambda)\big) &= \mu\left(\lambda(I - Ae^{-\lambda\tau}) - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right)\\ &\leqslant \lambda - \lambda e^{-\lambda\tau}\mu(A) + e^{-\lambda r}\mu\left(-\int_{-r}^{0} d\eta(\theta)\right).\end{aligned}$$

If $\mu(A) \leqslant re^{-\tau/r}(1/r - e\mu(-\int_{-r}^{0} d\eta(\theta)))$, we let $\lambda = -1/r$, then

$$\mu\big(-G(\lambda)\big) \leqslant -\frac{1}{r} + \frac{1}{r}e^{\tau/r}\mu(A) + e\mu\left(-\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0.$$

If $\mu(A) \leqslant 1/e - (\tau/e)e^{r/\tau}\mu(-\int_{-r}^{0} d\eta(\theta))$, we let $\lambda = -1/\tau$, then

$$\mu\big(-G(\lambda)\big) \leqslant -\frac{1}{\tau} + \frac{1}{\tau}e\mu(A) + e^{r/\tau}\mu\left(-\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0.$$

We get a contradiction.

(ii) $\det G(\lambda) > 0$ for all $\lambda \leqslant 0$. It is known that $\det(-G(\lambda)) < 0$, and then $-G(\lambda)$ has at least one negative real root for each $\lambda \leqslant 0$. So $\mu(G(\lambda)) > 0$ for all $\lambda \leqslant 0$. However, when we let $\lambda = 0$,

$$\mu\big(G(\lambda)\big) = \mu\left(-\lambda(I - Ae^{-\lambda\tau}) + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right) = \mu\left(\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0.$$

It is also a contradiction.

Hence Eq. (4.2) is nonoscillatory, and has at least one bounded nonoscillatory solution. □

Theorem 4.3. *Let n be odd. Assume $\mu(d\eta(\theta)) \geqslant 0$ on $[-r, 0]$. Then Eq.* (4.2) *is nonoscillatory.*

Proof. From the similar reasoning with the proof of Theorem 3.2, it suffices to prove that there exists $\lambda \in R$ such that $\mu(-G(\lambda)) \leqslant 0$, and another $\lambda \in R$ such that $\mu(G(\lambda)) \leqslant 0$.

If we let $\lambda = 0$, then

$$\mu\big(-G(\lambda)\big) = \mu\left(\lambda(I - Ae^{-\lambda\tau}) - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right) = \mu\left(-\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0.$$

It is obvious that if $\lambda > 0$ is large enough, we have

$$\begin{aligned}\mu\big(G(\lambda)\big) &= \mu\left(-\lambda(I - Ae^{-\lambda\tau}) + \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\right)\\ &\leqslant -\lambda + \lambda e^{\lambda\tau}\mu(A) + \mu\left(\int_{-r}^{0} d\eta(\theta)\right)\\ &\leqslant \lambda(e^{-\lambda\tau} - 1) + \mu\left(\int_{-r}^{0} d\eta(\theta)\right) \leqslant 0.\end{aligned}$$

Therefore, Eq. (4.2) is nonoscillatory. □

Corollary 4.1. *Assume* $\mu(Q_j) \leqslant 0$, $j = 0, 1, \ldots, m$, $0 \leqslant -\mu(-A) \leqslant \mu(A) \leqslant 1$ *and*

$$\mu\left(\sum_{j=1}^{m} \tau_j e^{-\mu(Q_0)\tau_j} Q_j\right) < -\frac{1}{e}.$$

Then Eq. (1.4) *is oscillatory.*

Corollary 4.2. *Let* n *be odd, and* $Q_0 = 0$. *Assume* $\mu(Q_j) \leqslant 0$, $j = 1, 2, \ldots, m$, *and*

$$\mu(A) \leqslant \max\left\{\frac{1}{e} - \frac{\tau}{e} e^{r/\tau} \mu\left(-\sum_{j=1}^{m} Q_j\right), r e^{-\tau/r}\left(\frac{1}{r} - e\mu\left(-\sum_{j=1}^{m} Q_j\right)\right)\right\},$$

where $r = \max_{1 \leqslant j \leqslant m}\{\tau_j\}$. *Then Eq.* (1.4) *is nonoscillatory, and has at least one bounded nonoscillatory solution.*

Corollary 4.3. *Let* n *be odd, and* $Q_0 = 0$. *Assume* $\mu(Q_j) \geqslant 0$, $j = 1, 2, \ldots, m$. *Then Eq.* (1.4) *is nonoscillatory.*

The following examples are given as applications of Corollaries 4.1–4.3.

Example 4.1. Consider the equation

$$\big(x(t) - Ax(t-\tau)\big)' = Q_0 x(t) + \sum_{j=1}^{2} Q_j x(t - \tau_j), \tag{4.3}$$

where

$$n = 3, \qquad A = I, \qquad Q_0 = 0, \qquad \tau = 1,$$

$$Q_1 = \begin{pmatrix} -2 & 3 & 1 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -2 & \frac{1}{2} \\ -\frac{1}{2} & 0 & -1 \end{pmatrix}, \qquad \tau_1 = \frac{3}{2}, \qquad \tau_2 = 3.$$

Obviously, $\mu_1(Q_0) = \mu_1(Q_1) = \mu_1(Q_2) = 0$, $-\mu_1(-A) = \mu_1(A) = 1$,

$$\begin{aligned} \mu_1\left(\sum_{j=1}^{2} \tau_j e^{-\mu_1(Q_0)\tau_j} Q_j\right) &= \mu_1\left(\frac{3}{2} Q_1 + 3 Q_2\right) \\ &= \mu_1\left(\frac{3}{2} \cdot \begin{pmatrix} -3 & 3 & 1 \\ 0 & -8 & 1 \\ 1 & 0 & -4 \end{pmatrix}\right) = -3 < -\frac{1}{e}. \end{aligned}$$

Then Corollary 4.1 shows that Eq. (4.3) is oscillatory.

Example 4.2. Consider the equation

$$\big(x(t) - Ax(t-\tau)\big)' = Q_0 x(t) + \sum_{j=1}^{2} Q_j x(t - \tau_j), \tag{4.4}$$

where

$$n=3,\qquad Q_0=0,\qquad A=\begin{pmatrix}-\frac{1}{e} & 0 & -1\\ \frac{1}{e^2} & \frac{1}{4}e^{-\frac{3}{2}} & 0\\ 0 & -\frac{1}{4}e^{-\frac{3}{2}} & -2\end{pmatrix},$$

$$Q_1=\frac{1}{e^2}\begin{pmatrix}-\frac{1}{3} & 0 & -\frac{1}{24}\\ \frac{2}{3e} & -\frac{1}{4} & \frac{1}{12}\\ 0 & -\frac{1}{8} & -\frac{1}{8}\end{pmatrix},\qquad Q_2=\frac{1}{e^2}\begin{pmatrix}-\frac{1}{6} & 0 & -\frac{1}{8}\\ \frac{1}{3e} & -\frac{1}{8} & \frac{1}{4}\\ 0 & \frac{1}{8} & -\frac{3}{8}\end{pmatrix},$$

$$\tau_1=1,\qquad \tau_2=2,\qquad \tau=1.$$

One can easily obtain that $\mu_1(A)=e^{-3/2}/2$, $\mu_1(Q_1)=\mu_1(Q_2)=0$,

$$\mu_1\left(-\sum_{j=1}^{2}Q_j\right)=\mu_1\left(\frac{1}{e^2}\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{6}\\ -\frac{1}{e} & \frac{3}{8} & -\frac{1}{3}\\ 0 & -\frac{1}{4} & \frac{1}{2}\end{pmatrix}\right)=\frac{1}{e^2},$$

$$\frac{1}{e}-\frac{\tau}{e}e^{r/\tau}\mu_1\left(-\sum_{j=1}^{2}Q_j\right)=\frac{1}{e}-\frac{1}{e}=0,$$

$$re^{-\tau/r}\left(\frac{1}{r}-e\mu_1\left(-\sum_{j=1}^{m}Q_j\right)\right)=2e^{-1/2}\left(\frac{1}{2}-\frac{1}{e}\right)=e^{-3/2}(e-2)>0.$$

Since $\mu_1(A)=e^{-3/2}/2<e^{-3/2}(e-2)$, we have that Eq. (4.4) has at least one bounded nonoscillatory solution from Corollary 4.2.

Example 4.3. Consider the equation

$$\big(x(t)-Ax(t-\tau)\big)'=Q_0x(t)+Q_1x(t-\tau_1), \tag{4.5}$$

where

$$n=3,\qquad Q_0=0,\qquad A=I,\qquad Q_1=\begin{pmatrix}-3 & 1 & 0\\ 1 & 0 & 4\\ 5 & 2 & -2\end{pmatrix},$$

$$\tau_1=\frac{1}{2},\qquad \tau=3.$$

Obviously, $\mu_1(Q_1)=3>0$, then Corollary 4.3 shows that Eq. (4.5) is nonoscillatory.

References

[1] Q. Kong, Oscillation for systems of functional differential equations, J. Math. Anal. Appl. 198 (1996) 608–619.

[2] C.J. Tian, B. Yang, B.G. Zhang, Oscillation for certain systems of functional differential equations, Dynam. Systems Appl. 8 (1999) 559–563.

[3] J.M. Ferreira, A.M. Pedro, Oscillations of differential–difference systems of neutral type, J. Math. Anal. Appl. 253 (2001) 274–289.
[4] J.M. Ferreira, I. Györi, Oscillatory behavior in linear retarded differential functional equations, J. Math. Anal. Appl. 128 (1987) 332–346.
[5] S. Chen, Q. Huang, Necessary and sufficient conditions for oscillations of solutions to systems of neutral functional differential equations, Funkcial. Ekvac. 33 (1990) 427–440.
[6] L.H. Erbe, Q. Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations, Dekker, New York, 1995.
[7] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Dekker, New York, 1987.
[8] O. Arino, I. Györi, Necessary and sufficient condition for oscillation of neutral differential system with several delays, J. Differential Equations 81 (1989) 98–105.
[9] Q. Kong, H.L. Freedman, Oscillation in delay differential systems, Differential Integral Equations 6 (1993) 1325–1336.
[10] J.P. Richard, M. Dambrine, F. Gouaisbaut, W. Perruquetti, Systems with delays: an overview of some recent advances, Stability Control Theory Appl. 3 (2000) 3–23.
[11] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, Dordrecht, 2000.
[12] T. Krisztin, Oscillation in linear functional differential systems, Differential Equations Dynam. Systems 2 (1994) 99–112.
[13] M. Vidyasagar, Nonlinear Systems Analysis, Prentice Hall, Englewood Cliffs, NJ, 1978.
[14] W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1965.