Acyclic 4-choosability of planar graphs

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\textbf{A B S T R A C T}

A proper vertex coloring of a graph $G = (V, E)$ is acyclic if $G$ contains no bicolored cycle. Given a list assignment $L = \{L(v) \mid v \in V\}$ of $G$, we say $G$ is acyclically $L$-list colorable if there exists a proper acyclic coloring $\pi$ of $G$ such that $\pi(v) \in L(v)$ for all $v \in V$. If $G$ is acyclically $L$-list colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V$, then $G$ is acyclically $k$-choosable. In this paper we prove that planar graphs without 4, 7, and 8-cycles are acyclically 4-choosable.

\section{1. Introduction}

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A proper vertex coloring of $G$ is an assignment $\pi$ of integers (as colors) to the vertices of $G$ such that $\pi(u) \neq \pi(v)$ if the vertices $u$ and $v$ are adjacent in $G$. A $k$-coloring is a proper vertex coloring using $k$ colors. A proper vertex coloring of a graph is acyclic if there is no bicolored cycle in $G$. The acyclic chromatic number of a graph $G$, denoted by $\chi_a(G)$, is the smallest integer $k$ such that $G$ has an acyclic $k$-coloring.

Acyclic coloring of graphs was introduced by Grünbaum in [13] and studied by Mitchem [17], Albertson and Berman [1] and Kostochka [15]. In 1979, Borodin [2] proved Grünbaum’s conjecture that every planar graph is acyclically 5-colorable. This bound is best possible. In 1973, Grünbaum [13] gave an example of a 4-regular planar graph which is not acyclically 4-colorable. Furthermore, bipartite planar graphs which are not acyclically 4-colorable were constructed in [16]. The girth of a graph $G$, denoted by $g(G)$, is the length of a shortest cycle.

In 1999, Borodin et al. [10] considered planar graphs with large girth. More specifically, they proved the following theorem.

\textbf{Theorem 1.} (1) If $G$ is planar with $g(G) \geq 5$, then $\chi_a(G) \leq 4$;
(2) If $G$ is planar with $g(G) \geq 7$, then $\chi_a(G) \leq 3$.

Given a list assignment $L = \{L(v) \mid v \in V\}$ of a graph $G$, we say $G$ is acyclically $L$-list colorable if there is an acyclic coloring $\pi$ of the vertices such that $\pi(v) \in L(v)$ for every vertex $v$. The coloring $\pi$ is called an acyclic $L$-coloring of $G$. If $G$ is acyclically $L$-list colorable for any list assignment $L$ with $|L(v)| \geq k$ for all $v \in V$, then $G$ is acyclically $k$-choosable. The acyclic list chromatic number or acyclic choosability of $G$, denoted by $\chi'_a(G)$, is the smallest integer $k$ such that $G$ is acyclically $k$-choosable.

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Borodin et al. [6] first investigated acyclic list coloring of planar graphs. They proved that every planar graph is acyclically 7-choosable. They also put forward the following challenging conjecture:

**Conjecture 1.** Every planar graph is acyclically 5-choosable.

This conjecture attracted much recent attention. Efforts are made to verify the conjecture for planar graphs with restrictions on the existence of short cycles. Wang and Chen [21] proved that every planar graph without 4-cycles is acyclically 6-choosable. Some sufficient conditions for a planar graph to be acyclically 5-choosable were established in [19,12,7,22]. In particular, in [7], Borodin and Ivanova proved that a planar graph $G$ is acyclically 5-choosable if $G$ does not contain an $i$-cycle adjacent to a $j$-cycle where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$. This result absorbs most of the previous work in this direction, including [19].

Let $G$ be a planar graph. Recently, $\chi'_5(G) \leq 3$ was proved if $g(G) \geq 7$ by Borodin et al. [5]; or if $G$ contains no cycles of lengths from 4 to 12 by Hocquard and Montassier [14], which was strengthened to the absence of 4- to 11-cycles by Borodin and Ivanova [8].

It is proved in [3] that $\chi'_5(G) \leq 4$ if $G$ contains neither 4-cycles nor 5-cycles. Moreover, $\chi'_5(G) \leq 4$ was obtained in the following cases: $g(G) \geq 5$ by Montassier [18], which extends the conclusion (2) of Theorem 1; or if $G$ has no $\{4, 5, 6\}$-cycles, or without $\{4, 5, 7\}$-cycles, or without $\{4, 5\}$-cycles and intersecting $3$-cycles by Montassier et al. [20]; or neither $\{4, 5\}$-cycles nor 8-cycles having a triangular chord by Chen and Raspaud [11]; or neither 4-cycles nor 6-cycles adjacent to a triangle by Borodin et al. [9].

The purpose of this paper is to give a sufficient condition for planar graphs to be acyclically 4-choosable. More precisely, we prove the following theorem.

**Theorem 2.** Every planar graph without $\{4, 7, 8\}$-cycles is acyclically 4-choosable.

2. Notation

Only simple graphs are considered in this paper. A planar graph is a particular drawing of a planar graph in the Euclidean plane. For a planar graph $G$, we denote its face set by $F(G)$. For an integer $k$, we denote by $k^+$ (respectively, $k^-$) any integer which is at least (respectively, at most) $k$. A $k$-vertex is a vertex of degree $k$, and a $k^+$-vertex and $k^-$-vertex is a vertex of degree at least $k$ and at most $k$, respectively. Similarly, we define a $k$-face, $k^+$-face, $k^-$-face, etc. We say that two cycles (or faces) are adjacent if they have at least one common edge. We say cycles (or faces) $C_1$ and $C_2$ are adjacent with crossing edge $e$ if $e$ is a common edge of $C_1$ and $C_2$. A triangle is synonymous with a 3-cycle. For $k \in V(G) \cup F(G)$, let $n_k(x)$, and $t(x)$ denote the number of $2$-vertices, and $3$-faces adjacent or incident to $x$, respectively. For a vertex $v \in V(G)$, let $m_5(v)$ denote the number of 5-faces incident to $v$. Let $N(v)$ denote the set of neighbors of a vertex $v$. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of $f$ and write $f = [u_1u_2 \cdots u_n]$ if $u_1$, $u_2$, $\ldots$, $u_n$ are the boundary vertices of $f$ in clockwise order. Sometimes, we write simply $V(f) = V(b(f))$. A 3-face $f = \{v_1v_2v_3\}$ is called an $(a_1, a_2, a_3)$-face if the degree of the vertex $v_i$ is $a_i$ for $i = 1, 2, 3$. An edge $uv$ is a $(b_1, b_2)$-edge if $d(u) = b_1$ and $d(v) = b_2$. A $3^+$-vertex $u$ is called a sponsor of a 3-face $f$ if $u$ is not incident to $f$ but adjacent to a 3-vertex $v$ which is incident to $f$. Denote by $s(u)$ the number of 3-faces sponsored by $u$.

Suppose $v$ is a 4-vertex. Let $v_1$, $v_2$, $v_3$, $v_4$ be the neighbors of $v$ in a cyclic order. Let $f_i$ be the face with $vv_i$ and $vv_{i+1}$ as two boundary edges for $i = 1, 2, 3, 4$, where indices are taken modulo 4. We call $v$ a weak vertex if the following conditions hold:

1. $d(v_1) = 2$;
2. $d(v_2) = 3$;
3. $d(v_3) = d(v_4) = 5$.

Fig. 1 shows a weak vertex $v$. A $4^+$-vertex $v$ is called a strong vertex if it is not weak. For all figures in this paper, a vertex is represented by a solid point when all of its incident edges are drawn (see Fig. 2); otherwise it is represented by a hollow point.

3. Structural properties

In order to complete the proof, we assume that $G$ is a counterexample to Theorem 2 with the least number of vertices. Thus $G$ is connected. We first study the structural properties of $G$, then use Euler’s formula and discharging technique to derive a contradiction.

First, we have Lemmas 1–3, whose proofs are provided in [20,18,11], respectively.

**Lemma 1** ([20]).

1. There are no 1-vertices.
2. No 2-vertex is incident to a 3-face.
3. No 2-vertex is adjacent to a 3$^-$-vertex.
4. A 3-vertex is adjacent to at most one 3-vertex.
5. A 4-vertex is adjacent to at most one 2-vertex.
6. No 3-face is incident to two 3-vertices and one 4-vertex.
Lemma 2 ([18]).

(F1) No 5-vertex is adjacent to three 2-vertices and one 3-vertex.

(F2) No 6-vertex is adjacent to five 2-vertices.

(F3) No 6-vertex is adjacent to four 2-vertices and one 3-vertex.

Lemma 3 ([11]). Let $v$ be a 5-vertex with $t(v) = 2$. If a 3-face incident to $v$ is a $(3, 3, 5)$-face, then $n_2(v) = 0$.

Lemma 4 ([9]). No 3-vertex can be a sponsor.

In what follows, let $L$ be a list assignment of $G$ with $|L(v)| = 4$ for all $v \in V(G)$.

Lemma 5. If $f = [x_1, x_2, \cdots, x_5]$ is a 5-face with $d(x_1) = d(x_4) = 2$ and $d(x_2) = d(x_3) = 4$, then $f$ is not adjacent to any 3-face.

Proof. Since a 2-vertex is not incident to a 3-face, it suffices to show that $f$ is not adjacent to a 3-face with crossing edge $x_2x_3$. Assume to the contrary that $f^* = [x_2x_3u]$ is a 3-face adjacent to $f$. Let $N(x_2) = \{x_1, x_3, u, y_1\}$ and $N(x_3) = \{x_2, x_4, u, y_2\}$. By the minimality of $G, G - \{x_1\}$ admits an acyclic $L$-coloring $\pi$. If $x_2$ and $x_3$ have different colors, then color $x_1$ properly, we obtain an acyclic $L$-coloring of $G$. Assume $\pi(x_2) = \pi(x_3) = 1$. If there is a color $c \in L(x_1) \setminus \{1, \pi(y_1), \pi(u), \pi(x_3)\}$, then color $x_1$ with color $c$, we again obtain an acyclic $L$-coloring of $G$. Thus we may assume that $L(x_1) = \{1, 2, 3, 4\}, \pi(x_3) = 2, \pi(y_1) = 3$, and $\pi(u) = 4$. If $\pi(y_2) \neq 1$, then color $x_1$ with color 2, we obtain an acyclic $L$-coloring of $G$. Thus we assume further that $\pi(y_2) = 1$. If $L(x_2) \neq L(x_1)$, we recolor $x_2$ with a color in $L(x_2) \setminus L(x_1)$ and then color $x_1$ properly. If $L(x_2) = L(x_1)$, then we recolor $x_2$ with a color $a \in L(x_1) \setminus \{1, 2, 4\}, x_3$ with a color different from 1, $a, \pi(x_4)$, and finally color $x_1$ with 3. This completes the proof of the lemma. \qed

4. Proof of Theorem 2

We define a weight function $\omega$ on the vertices and faces of $G$ by letting $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d(f) - 6$ if $f \in F(G)$. It follows from Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation $\sum_{v \in V(G)}d(v) = \sum_{f \in F(G)}d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces is equal to

$$\sum_{v \in V(G)}(2d(v) - 6) + \sum_{f \in F(G)}(d(f) - 6) = -12. \tag{1}$$
The following claims follow from the assumption that $G$ has no cycles of lengths 4, 7, 8.

1. There is no 4-face and no 7-face.
2. If $f$ is a non-simple 6-face, then its boundary consists of two edge-disjoint triangles, as shown in Fig. 3.
3. If $f$ is an 8-face then the boundary of $f$ consists of either a 5-cycle and a 3-cycle, or two 3-cycles joined by a cut-edge, as depicted by Fig. 4.
4. No 6-cycle or a non-simple 6-face is adjacent to a 3-cycle.
5. No 5-cycle is adjacent to two 3-cycles.
6. For a vertex $v \in V(G)$, $t(v) \leq \lfloor \frac{d(v)}{2} \rfloor$.
7. A 5-cycle cannot share two consecutive edges with a 6-face.
8. If two 5-cycles are adjacent, then they share exactly two consecutive edges. In particular, if two 5-faces are adjacent, then their boundaries share a 2-vertex.
9. A 5-cycle is adjacent to at most one 5-cycle.
10. If two 5-cycles are adjacent, then none of them is adjacent to a 3-cycle.
11. If a 3-vertex $v$ is incident to a 3-face and a 5-face, then the other face incident to $v$ is a 9$^+$-face.
12. If $f$ is a face with $t(f) \geq 2$, then $f$ is a 9$^+$-face.
13. If $v$ is a weak vertex, then the other face incident to $v$ (the face different from the 3-face and the two 5-faces) is a 9$^+$-face.
14. For any vertex $v$, $m_5(v) \leq \lfloor 2d(v)/3 \rfloor$.

Our discharging rules are as follows:

**R0:** Every 4$^+$-vertex sends 1 to each adjacent 2-vertex and $\frac{1}{2}$ to each sponsored 3-face.

**R1:** Suppose that $f = \{v_1v_2v_3\}$ is a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. We use $(d(v_1), d(v_2), d(v_3)) \rightarrow (c_1, c_2, c_3)$ to denote that the vertex $v_i$ gives $f$ the amount of weight $c_i$ for $i = 1, 2, 3$.

**R1a:** If $m_8^+(f) = 0$, then $(4^+, 4^+, 4^+) \rightarrow (1, 1, 1)$.

**R1b:** If $m_8^+(f) = 1$, then
- $(3, 3, 5^+) \rightarrow (0, 0, \frac{5}{2})$;
- $(3, 4^+, 4^+) \rightarrow (0, \frac{13}{12}, \frac{13}{12})$;
- $(4^+, 4^+, 4^+) \rightarrow (\frac{8}{9}, \frac{8}{9}, \frac{8}{9})$.
R1c: If \( m_{d+}(f) = 2 \), then
- \((3, 3, 5^+ \rightarrow (0, 0, \frac{4}{3})\):
- \((3, 4^+, 4^+ \rightarrow (0, \frac{7}{12}, \frac{11}{12})\):
- \((4^+, 4^+, 4^+ \rightarrow \left(\frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right)\).

R1d: If \( m_{d+}(f) = 3 \), then
- \((3, 3, 5^+ \rightarrow (0, 0, 1)\):
- \((3, 4^+, 4^+ \rightarrow (0, \frac{2}{3}, \frac{2}{3})\):
- \((4^+, 4^+, 4^+ \rightarrow \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)\).

R2: Suppose \( f \) is a 5-face. If \( m_{d+}(f) \leq 2 \), then each strong \( 4^+ \)-vertex incident to \( f \) sends \( \frac{1 - m_{d+}(f)/3}{m(f)} \) to \( f \).

R3: Every \( 8^+ \)-face sends \( \frac{1}{2} \) to each adjacent 3-face and 5-face.

R4: If a \( 9^+ \)-face \( f \) is adjacent to a \( 6^+ \)-face \( f' \) by a common \((4, 2)\)-edge \( uv \), then \( \tau(f \rightarrow u) = \frac{1}{2} \).

In the following, we show that \( \omega^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \).

**Lemma 7.** For every face \( f \), \( \omega^*(f) \geq 0 \).

**Proof.** Depending on the degree of \( f \), we divide the proof into five cases.

**Case 1.** \( d(f) = 3 \).

The initial charge is \( \omega(f) = -3 \). Let \( f = [v_1v_2v_3] \) such that \( d(v_1) \leq d(v_2) \leq d(v_3) \). By (C2), (C4) and (C6), \( f \) is either a \((3, 3, 5^+)\)-face, or a \((3, 4^+, 4^+)\)-face, or a \((4^+, 4^+, 4^+)\)-face. Let \( f_i \) be the face adjacent to \( f \) with crossing edge \( v_iv_{i+1} \), where \( i \) is taken modulo 3.

If \( m_{d+}(f) = 0 \), then each \( f_i \) is a 5-face for \( i = 1, 2, 3 \). If \( d(v_1) = 3 \), then by (C1), (C3) and Lemma 4, the other neighbor of \( v_1 \) is a \( 4^+ \)-vertex. So \( f_1 \) and \( f_2 \) have a common \((3, 4^+)\)-edge, which contradicts (8) of Lemma 6. So \( f \) is a \((4^+, 4^+, 4^+)\)-face and thus \( \omega^*(f) \geq 3 + 3 \times 3 = 0 \) by (R1a).

If \( m_{d+}(f) = 1 \), then \( f \) takes \( \frac{1}{2} \) from its adjacent \( 8^+ \)-face by (R3). If \( f \) is a \((3, 3, 5^+)\)-face, then for \( i = 1, 2 \), let \( u_i \) be the other neighbor of \( v_i \). By (C4), \( d(u_i) \geq 4 \) and hence is a sponsor of \( f \). As each sponsor sends \( \frac{1}{2} \) to \( f \), we have \( \omega^*(f) \geq -3 + \frac{1}{2} + \frac{2}{3} + \frac{3}{2} = 0 \) by (R1b). If \( f \) is a \((3, 4^+, 4^+)\)-face, then another neighbor of \( v_1 \) not on \( b(f) \) is a \( 4^+ \)-vertex and thus \( \omega^*(f) \geq -3 + \frac{1}{2} + \frac{2}{3} + \frac{3}{2} \times 2 = 0 \) by (R1b). If \( f \) is a \((4^+, 4^+, 4^+)\)-face, then each incident \( 4^+ \)-vertex sends \( \frac{2}{3} \) to \( f \). So \( \omega^*(f) \geq -3 + \frac{2}{3} + \frac{2}{3} = 0 \).

If \( m_{d+}(f) = 2 \), then \( f \) takes \( \frac{1}{3} \times 2 = \frac{2}{3} \) from its adjacent two \( 8^+ \)-faces by (R3). Similarly as in the previous paragraph, depending on \( f \) is a \((3, 3, 5^+)\)-face, or a \((3, 4^+, 4^+)\)-face, or a \((4^+, 4^+, 4^+)\)-face, by (R1c), we have \( \omega^*(f) \geq 3 + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{3}{2} = 0 \), or \( \omega^*(f) \geq -3 + \frac{2}{3} + \frac{1}{2} + \frac{11}{12} \times 2 = 0 \), or \( \omega^*(f) \geq -3 + \frac{2}{3} + \frac{3}{2} \times 3 = 0 \).

If \( m_{d+}(f) = 3 \), then \( f \) takes \( \frac{1}{3} \times 3 = 1 \) from all its adjacent \( 8^+ \)-faces by (R3). Again depending on \( f \) is a \((3, 3, 5^+)\)-face, or a \((3, 4^+, 4^+)\)-face, or a \((4^+, 4^+, 4^+)\)-face, by (R1d), we have \( \omega^*(f) \geq -3 + 1 + \frac{1}{2} \times 2 + \frac{3}{2} = 0 \), or \( \omega^*(f) \geq -3 + 1 + \frac{2}{3} \times 3 = 0 \).

**Case 2.** \( d(f) = 5 \).

By (R2) and (R3), to show that \( \omega^*(f) \geq 0 \), it suffices to show that \( n^*(f) \geq 1 \). Assume \( f = [v_1v_2v_3v_4v_5] \). Since every 3-vertex is adjacent to at most one 3-vertex (by (C4)) and every 2-vertex is adjacent to no 3-vertex, there exist non-constant indices \( i, j \) (i.e., \( i \neq j \pm 1 \bmod 5 \)) such that \( v_i \) and \( v_j \) are \( 4^+ \)-vertices. By definition, if a 4-vertex incident to \( f \) is weak, then it lies on the intersection of a 5-face and a 3-face. Since \( f \) is adjacent to at most one 3-face (by (5) of Lemma 6), one of \( v_i, v_j \) is a strong vertex. Thus \( n^*(f) \geq 1 \) and \( \omega^*(f) \geq 0 \).

**Case 3.** \( d(f) = 6 \).

The initial charge is 0 and no charge is sent out. So \( \omega^*(f) = \omega(f) = 0 \).

**Case 4.** \( d(f) = 8 \).

The initial charge is \( \omega(f) = 2 \). By (3) of Lemma 6, the boundary of \( f \) consists of either one 5-cycle and one 3-cycle, or two 3-cycles and a cut-edge.

If the boundary of \( f \) consists of one 5-cycle \( C = [v_1v_2v_3v_4v_5] \) and one 3-cycle \( C' = [v_1v_2v_3v_4] \), then by (5) of Lemma 6, \( f \) is adjacent to at most one 3-face. By (C3), there are at most two 2-vertices in \( V(C) \). Furthermore, (9) of Lemma 6 implies that \( C \) is incident to at most one 5-face which shares two common edges with \( f \). As \( C' \) is incident to at most three 5-faces, we conclude that \( m_3(f) \leq 5 \). Therefore, \( \omega^*(f) \geq 2 - \frac{1}{2} \times 5 - \frac{1}{2} = 0 \) by (R3).

If the boundary of \( f \) consists of two 3-cycles \( C = [v_1v_2v_5v_5] \) and \( C' = [v_2v_5v_5] \) and a cut-edge \( v_2v_3 \), then \( f \) is not adjacent to any 3-faces and is adjacent to at most six 5-faces. Therefore, by (R3), \( \omega^*(f) \geq 2 - \frac{1}{2} \times 6 = 0 \).

**Case 5.** \( d(f) \geq 9 \).

Let \( m_{d}(f) \) denote the number of \( d \)-faces adjacent to \( f \) by a common \((4, 2)\)-edge. Let \( m_5(f) \) denote the number of 5-faces adjacent to \( f \). Since a 2-vertex is not adjacent to any 3-vertex, \( \tau(f) + m_5(f) + 2m_{d}(f) \leq d(f) \). By (R3) and (R4), we conclude that \( \omega^*(f) \geq d(f) - \frac{1}{2}(\tau(f) + m_5(f) + 2m_{d}(f)) \geq d(f) - \frac{1}{2}d(f) = \frac{1}{2}d(f) \geq 6 \geq 0 \). This completes the proof of Lemma 7. \( \square \)
It remains to show that for each vertex \( v \), \( \omega^*(v) \geq 0 \). Let \( v \in V(G) \). By (C1), \( d(v) \geq 2 \). In the following, let \( v_1, v_2, \ldots, v_{d(v)} \) denote the neighbors of \( v \) in a cyclic order, and let \( f_i \) denote the incident face of \( v \) with \( v_{2i} \) and \( v_{2i+1} \) as two boundary edges for \( i = 1, 2, \ldots, d(v) \), where indices are taken modulo \( d(v) \).

If \( d(v) = 2 \), then the initial charge is \( \omega(v) = -2 \). By (C3), \( v \) is adjacent to two 4\(^+\)-vertices. Therefore, \( \omega^*(v) \geq -2 + 1 \times 2 = 0 \) by (R0). If \( d(v) = 3 \), then the initial charge is 0 and no charge is sent out, since \( v \) cannot be a sponsor by Lemma 4. So the final charge is also 0.

In the following, we consider the charge of 4\(^+\)-vertices. The following observation follows easily from (R1).

**Observation 1.** Assume \( f \) is a 3-face incident to \( v \).

1. If \( d(v) = 4 \), then \( \tau(v \to f) \leq \frac{12}{17} \).
2. If \( d(v) \geq 5 \), then \( \tau(v \to f) \leq \frac{5}{3} \).

The amount of charge sent from \( v \) to incident 5-faces is more complicated. To estimate the amount of charge sent from \( v \) to a 5-face \( f \), we divide the 5-faces incident to \( v \) into four types.

Suppose \( f = [v v_1 w_1 w_2 v_2] \) is incident to \( v \).

- If \( d(v_1) = d(v_2) = 2 \), then \( f \) is of type 1 with respect to \( v \).
- If \( d(v_1) = 2 \) and \( v_2 \) is incident to a 3-face sponsored by \( v \), then \( f \) is of type 2 with respect to \( v \).
- If \( d(v_1) = 2 \) and \( v_2 \) is adjacent to a 3-face, then \( f \) is of type 3 with respect to \( v \).
- Otherwise, \( f \) is of type 4 with respect to \( v \).

For \( i \in \{1, 2, 3, 4\} \), let \( T_i(v) \) denote the set of 5-faces of type \( i \) with respect to \( v \) and let \( m_5^i(v) = |T_i(v)| \). So \( \sum_{i=1}^{4} m_5^i(v) = m_5(v) \).

**Observation 2.** If \( f = [v v_1 w_1 w_2 v_2] \) is a 5-face incident to \( v \), then \( \tau(v \to f) \leq \frac{1}{2} \). Moreover,

1. If \( f \in T_1(v) \cup T_2(v) \), then \( \tau(v \to f) \leq \frac{1}{6} \).
2. If \( f \in T_2(v) \), then \( \tau(v \to f) = 0 \).

**Proof.** If \( v \) is not strong, then \( \tau(v \to f) = 0 \). In the following, we assume that \( v \) is a strong vertex.

Let \( f' \) and \( f'' \) be the faces adjacent to \( f \) with crossing edge \( v v_1 \) and \( v v_2 \), respectively.

If \( f \) is of type 1, then by (C2) and (7) of Lemma 6, each of \( f', f'' \) is either a 5-face or an 8\(^+\)-face. Moreover, by (9) of Lemma 6, at least one of \( f', f'' \) is a 9\(^+\)-face. So \( m_5(f) \geq 2 \). By (C5), \( v \) is a 5\(^+\)-vertex, and \( v_1, w_1, w_2 \) are 4\(^+\)-vertices. If both \( w_1, w_2 \) are 4\(^-\)-vertices, then by Lemma 5, \( f \) is not adjacent to any 3-face and hence both \( w_1, w_2 \) are strong. Otherwise, at least one of \( w_1, w_2 \) is a 5\(^+\)-vertex. In any case, \( n^*(f) \geq 2 \) and hence \( \tau(v \to f) \leq \frac{1-\frac{1}{2}}{4} = \frac{1}{6} \) by (R2).

If \( f \) is of type 3, then \( v_1 \) is a 2-vertex and \( f'' \) is a 3-face. By (1), (7) and (10) of Lemma 6, \( f' \) is a 9\(^+\)-face. By (C3), \( w_1 \) is a 4\(^+\)-vertex. As \( f \) is not adjacent to any other 3-faces (by (5) of Lemma 6), \( w_1 \) is a strong vertex. If \( v_2 \) is a strong vertex, then \( n^*(f) \geq 2 \) and hence \( \tau(v \to f) \leq \frac{1-\frac{1}{2}}{4} = \frac{1}{6} \) by (R2). Otherwise, assume \( v_2 \) is a weak 4-vertex. By (13) of Lemma 6, the face adjacent to \( f \) by crossing edge \( v_2 w_2 \) is a 9\(^+\)-face. It follows immediately that \( m_9^+(f) \geq 3 \) and thus \( v \) sends nothing to \( f \) by (R2).

If \( f \) is of type 2, then by (1), (7), (10) and (11) of Lemma 6, \( f' \) and \( f'' \) are 9\(^+\)-faces. Hence \( m_9^+(f_i) \geq 3 \) and \( \tau(v \to f) = 0 \) by (R2).

Assume \( f \) is of type 4. If \( n^*(f) \geq 2 \), then \( \tau(v \to f) \leq \frac{1}{2} \) by (R2).

Assume \( n^*(f) = 1 \), i.e., \( v \) is the only strong vertex incident to \( f \). As \( f \) is adjacent to at most one 3-face and any weak 4\(^-\)-vertex incident to \( f \) is contained in the intersection of \( f \) and a 3-face, we conclude that, including \( v, f \) is incident to at most three 4\(^+\)-vertices. On the other hand, by (C4), \( f \) is incident to at least two 4\(^+\)-vertices.

If \( f \) is incident to exactly two 4\(^+\)-vertices, then these two 4\(^+\)-vertices are not consecutive. We may assume that \( w_1 \) is a weak 4\(-\)vertex and \( v_1, v_2, w_2 \) are 3\(^-\)-vertices. By definition of a weak vertex, \( w_1 \) is adjacent to a 2-vertex and either \( w_1 v_1 \) or \( w_1 w_2 \) is incident to a 3-face, contrary to (C10).

Assume \( f \) is incident to three 4\(^+\)-vertices, say \( v, x, y, \) and \( x, y \) are weak 4\(-\)vertices, where \( \{x, y\} \subseteq \{v_1, v_2, w_1, w_2\} \). As \( f \) is adjacent to at most one 3-face, by definition of weak vertices, we conclude that \( f \) is adjacent to a 3-face with crossing edge \( xy \). Let \( x' \) and \( y' \) be the other neighbors of \( x, y \), respectively on the boundary of \( f \). By (13) of Lemma 6, the face adjacent to \( f \) with crossing edges \( xx' \) and \( yy' \) are 9\(^+\)-faces. Hence \( m_9^+(f) \geq 2 \) and \( \tau(v \to f) \leq 1 - \frac{2}{3} = \frac{1}{3} \) by (R2).

The calculation of the new charge of 4\(^+\)-vertices is more complicated. We use three lemmas to take care of 4\(-\)vertices, 5\(-\)vertices and 6\(^+\)-vertices separately.

**Lemma 8.** If \( d(v) = 4 \), then \( \omega^*(v) \geq 0 \).
Lemma 9. If $d(v) = 5$, then $\omega^*(v) \geq 0$.

Proof. The initial charge is $\omega(v) = 4$. By (6) of Lemma 6 and (C7), $t(v) \leq 2$ and $n_2(v) \leq 3$. Depending on the value of $t(v)$, the proof is divided into three cases.

Case 1. $t(v) = 2$.

Without loss of generality, assume $f_1 = [vv_1v_2]$ and $f_4 = [vv_4v_5]$ are 3-faces. It follows easily from (5) of Lemma 6 that $s(v) + m_5(v) \leq 1$, implying that the charge sent from $v$ to incident 5-faces and sponsored 3-faces is at most $1/2$ in total. If $v_3$ is a 3-vertex, then $\omega^*(v) \geq 4 - \frac{3}{2} \times 2 - \frac{1}{2} = \frac{1}{2}$. If $d(v_1) = 2$, then by Lemma 3, for each $i = 1, 4, f_i$ cannot be a $(3, 3, 5)$-face and thus $t(v \rightarrow f_i) \leq \frac{13}{12}$ by (R1). Hence $\omega^*(v) \geq 4 - \frac{13}{12} \times 2 - 1 - \frac{1}{2} = \frac{2}{3}$.

Case 2. $t(v) = 1$.

Let $f_1 = [vv_1v_2]$ be the 3-face incident to $v$. In this case, it follows easily from (5) of Lemma 6 that $s(v) + m_5(v) \leq 3$. So the charge sent from $v$ to incident 5-faces and sponsored 3-faces is at most $3/2$ in total. By (C8), $n_2(v) \leq 2$. Depending on the value of $n_2(v)$, we consider three subcases below.
Lemma 6

Observation 1.

By Observation 1.

If \( n_2(v) = 0 \), then \( \omega^*(v) \geq 4 - \frac{5}{3} = \frac{5}{3} \).

If \( n_2(v) = 1 \), then one of the following holds:

1. \( s(v) + m_3(v) \leq 2 \), implying that the charge sent from \( v \) to incident 5-faces and sponsored 3-faces is at most 1 in total and hence \( \omega^*(v) \geq 4 - \frac{3}{2} = \frac{1}{2} \).

2. \( s(v) + m_3(v) = 3 \) and there is a 5-face, say \( f' \), incident to \( v \) which is of type 2 or type 3 with respect to \( v \). By Observation 2, \( \tau(v \to f') \leq \frac{1}{6} \) and hence \( \omega^*(v) \geq 4 - \frac{2}{3} - 1 - \frac{1}{6} = \frac{5}{6} \).

In the following of the proof of Case 2, we assume that \( n_2(v) = 2 \). It is obvious that \( s(v) \leq 1 \). If \( s(v) = 1 \), then by (C12), \( f_1 \) is a \((5, 4^*, 4^*)\)-face. By (R1), \( \tau(v \to f_1) \leq \frac{1}{6} \). Moreover, \( m_3(v) \leq 2 \) and any 5-face incident to \( v \) is of type 1, 2, or 3 with respect to \( v \). By Observation 2, \( \omega^*(v) \geq 4 \). Thus we also assume that \( \omega^*(v) = \frac{5}{6} \).

If \( f_1 \) is not a \((5, 3, 3)\)-face, then by (R1), \( \tau(v \to f_1) \leq \frac{1}{12} \). In this case, one of the following holds:

1. \( m_5(v) \leq 1 \), and hence \( \omega^*(v) \geq 4 - \frac{13}{12} - 2 - \frac{1}{2} = \frac{5}{12} \).

2. \( m_5(v) = 2 \) and one of the 5-facets incident to \( v \) is not of type 4 with respect to \( v \). In this case, by Observation 2, \( v \) sends at most \( \frac{1}{6} + \frac{1}{3} \) to the two 5-faces and hence \( \omega^*(v) \geq 4 - \frac{13}{12} - 2 - \frac{1}{6} = \frac{1}{4} \).

In the following, we assume that \( f_1 \) is a \((5, 3, 3)\)-face. Let \( f' \) be the face adjacent to \( f \) with crossing edge \( v_1v_2 \). Since \( v_1, v_2 \) are 3-vertices, \( f' \) is adjacent to \( f_1, f_2, f_3, f_4 \) as well. If \( f' \) is a 5-face or an 8-face, then by (11) of Lemma 6, \( f_2, f_3, f_4 \) are 9\(^*\)-faces. Hence \( \tau(v \to f_1) \leq \frac{1}{3} \). If \( m_5(v) \leq 1 \), then \( \omega^*(v) \geq 4 - \frac{4}{3} - 2 - \frac{1}{3} = \frac{1}{3} \). Otherwise, \( f_1, f_2, f_3, f_4 \) are both 5-faces and \( d(v_4) = 2 \). This implies that at least one of \( f_1, f_2, f_3, f_4 \) is a 5-face of type 1. Let \( \omega^*(v) \geq 4 - \frac{4}{3} - 2 - \frac{1}{3} = \frac{1}{3} \). Now, assume that \( f' \) is a 9\(^*\)-face. If both \( f_1, f_2, f_3, f_4 \) are 9\(^*\)-faces, then \( \tau(v \to f_1) \leq \frac{1}{3} \) and \( \omega^*(v) \geq 4 - \frac{13}{12} - 2 - \frac{1}{6} = \frac{1}{4} \). Moreover, \( n_2(v) \geq 3 \). It means that \( v \) sends nothing to \( f_2 \) by (R2). On the other hand, we see that \( \omega(v) \geq 4 \) by (C4) and \( v \) is a strong vertex by (5) of Lemma 6. Therefore \( \tau(v \to f_2) \leq \frac{1}{6} \) and hence \( \omega^*(v) \geq 4 - \frac{4}{3} - 2 - \frac{1}{6} = 0 \).

Assume exactly one of \( f_1, f_2, f_3, f_4 \) is a 5-face, say \( f_2, f_3, f_4 \). Then \( f_4 \) is a 9\(^*\)-face. As \( m_5(v) \leq 1 \), we have \( \tau(v \to f_1) = \frac{1}{6} \). If \( v_3 \) is a 2-vertex, then \( f_2 \) is of type 3 with respect to \( v \) and \( \tau(v \to f_2) \leq \frac{1}{6} \). Therefore \( \omega^*(v) \geq 4 - \frac{4}{3} - 2 - \frac{1}{6} = 0 \). Assume \( v_3 \) is a 3-vertex. Then \( v_4 = 2 \)-vertices and \( f_1 \) is a 6\(^*\)-face by (10) of Lemma 6. So, by (R0), \( \tau(v \to f_1) \leq \frac{1}{6} \)

Therefore \( \omega^*(v) \geq 4 - \frac{4}{3} - 2 - \frac{1}{6} = 0 \).

Next we consider the case that \( s(v) = 2 \). In this case, \( \omega^*(v) \geq 4 \). So \( s(v) + m_5(v) \leq 4 \), then \( \omega^*(v) \geq 4 - \frac{1}{2} = 3 \). Therefore \( \omega^*(v) \geq 4 - \frac{1}{2} \times 3 = \frac{5}{2} \).

Finally, we consider the case that \( s(v) = 1 \). It is obvious that \( s(v) + m_3(v) \leq 5 \). So \( \omega^*(v) \geq 4 \). Then \( m_5(v) \leq 1 \) by (10) of Lemma 6. Thus, by (R0), \( \omega^*(v) \geq 4 - \frac{1}{2} \times 4 = \frac{7}{2} \).

This completes the proof of Lemma 9. \( \square \)

It remains to consider 6\(^*\)-vertices. First we have the following observation.

Observation 3. For any vertex \( v \), we have that \( \omega^*(v) \leq \omega^*(v) + 2\tau(v) + m_5(v) \leq d(v) \).

Proof. Suppose that \( v \) is a 6\(^*\)-vertex. Let

\[ A = \{ u \in N(v) : d(u) = 2 \}, \]

\[ B = \{ u \in N(v) : u \text{ is incident to a 3-face sponsored by } v \}, \]

\[ C = \{ u \in N(v) : v \in u \text{ is contained in a triangle} \}. \]

It follows from the definition that \( A, B, C \) are disjoint and \( n_2(v) = |A|, s(v) = |B| \) and \( 2\tau(v) = |C| \). Moreover, if \( f_1 = [v_1 v_2 v_3 v_4 v_5] \) is a 5-face of type 4 with respect to \( v \), then it follows from (10) of Lemma 6 that one of \( v_i \) and \( v_{i+1} \), say \( v_i \),
is a 3*-vertex not incident to a 3-face sponsored by v and moreover \( v v_1 \) is not contained in a triangle and \( v_i, i.e., v_i \notin A \cup B \cup C \). This implies that \( m^2(v) \leq d(v) - |A \cup B \cup C| \) and hence \( p_2(v) + s(v) + 2t(v) + m^3(v) \leq d(v). \)

**Lemma 10.** If \( d(v) \geq 6 \), then \( \omega^*(v) \geq 0. \)

**Proof.** By Observation 1, each 6*-vertex sends at most \( \frac{5}{3} \) to each incident 3-face. Therefore, by (R0), Lemma 6 and Observations 2 and 3, we obtain

\[
\omega^*(v) \geq 2d(v) - 6 - \frac{5}{3}t(v) - m_2(v) - \frac{1}{2}s(v) - \frac{1}{6}m_3(v) - 0 \cdot m_4(v) - \frac{1}{2} m^2(v) - \frac{1}{6} m^3(v) - \frac{1}{2} m^4(v) \\
\geq 2d(v) - 6 - \frac{5}{3}t(v) - (d(v) - s(v) - 2t(v) - m^4(v)) - \frac{1}{2}s(v) - \frac{1}{6}m_3(v) - \frac{1}{6}m^3(v) - \frac{1}{2}m^4(v) \\
= d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{2}s(v) - \frac{1}{6}m_2(v) - \frac{1}{6}m^3(v) - \frac{1}{6}m^3(v) \\
\geq d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v) - \frac{1}{6} (m_2(v) + m^3(v)) \\
\geq d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v) - \frac{1}{6} \cdot \left[ \frac{2d(v)}{3} \right] \\
\geq d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v) - \frac{1}{6} \cdot \frac{2d(v)}{3} \\
= \frac{8}{9}d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v).
\]

If \( \frac{8}{9}d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v) \geq 0 \), then we are done. Assume \( \frac{8}{9}d(v) - 6 + \frac{1}{3}t(v) + \frac{1}{3}s(v) < 0 \). Then \( d(v) = 6 \) and \( 4 > 2t(v) + 3s(v) \), which implies that \( s(v) + t(v) \leq 1 \).

First we consider the case that \( t(v) = 0 \). By (F2), \( n_2(v) \leq 4 \). By (14) of Lemma 6, \( m_2(v) \leq 4 \). If \( m_2(v) = 4 \), then w.l.o.g., we may assume that \( d(f_i) = d(f_j) = d(f_k) = 5 \). By (10) of Lemma 6, \( s(v) = 0 \). Therefore \( \omega^*(v) \geq 6 - 1 \times 4 - \frac{1}{2} \times 4 = 0 \).

Assume \( m_2(v) \leq 3 \). As \( n_2(v) \leq 4 \) and \( s(v) \leq 1 \), we have \( \omega^*(v) \geq 6 - 1 \times 4 - \frac{1}{2} \times 3 = 0 \).

Next we assume \( t(v) = 1 \), say \( f_1 = [v v_1 v_2] \) is a 3-face. Then \( s(v) = 0 \). It is easy to see that \( n_2(v) \leq 4 \) and \( m_2(v) \leq 3 \).

If \( n_2(v) \leq 2 \), then \( \omega^*(v) \geq 6 - \frac{5}{3} - 1 \times 2 - \frac{1}{2} \times 3 = \frac{5}{3} \) by (R0) and Observation 2.

If \( n_2(v) = 3 \), then one of the following holds:

- \( m_2(v) \leq 2 \), and hence \( \omega^*(v) \geq 6 - \frac{5}{3} - 1 \times 3 - \frac{1}{2} \times 2 = \frac{1}{3} \).
- \( m_2(v) = 3 \) and at least two of the 5-faces incident to \( v \) are of type 1 or 3 with respect to \( v \). Hence \( \omega^*(v) \geq 6 - \frac{5}{3} - 1 \times 3 - \frac{1}{2} \times 2 = \frac{5}{3} \).

If \( n_2(v) = 4 \), then by (F3), \( [v v_1 v_2] \) is \( (6, 4^+, 4^+) \)-face. It follows from (R1) that \( v \) sends at most 1 to \( f_i \). As \( m_2(v) \leq 3 \) and each incident 5-face is adjacent to at least two 9*-faces by (5) and (7) of Lemma 6, we conclude that \( \omega^*(v) \geq 6 - 1 \times 4 - \frac{1}{2} \times 3 = 0 \). \( \Box \)

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