Orthonormal polynomials with exponential-type weights

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Abstract

Let $R=(-\infty, \infty)$ and let $w_\rho(x) := |x|^\rho \exp(-Q(x))$, where $\rho > -\frac{1}{2}$ and $Q(x) \in C^2 : \mathbb{R} \rightarrow \mathbb{R}^+ = [0, \infty)$ is an even function. In this paper we consider the properties of the orthonormal polynomials with respect to the weight $w_\rho^2(x)$, obtaining bounds on the orthonormal polynomials and spacing on their zeros. Moreover, we estimate $A_n(x)$ and $B_n(x)$ defined in Section 4, which are used in representing the derivative of the orthonormal polynomials with respect to the weight $w_\rho^2(x)$.

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1. Introduction and preliminaries

Let $R=(-\infty, \infty)$. Let $Q(x) \in C^2 : \mathbb{R} \rightarrow \mathbb{R}^+ = [0, \infty)$ be an even function and $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x) \, dx < \infty$ for all $n = 0, 1, 2, \ldots$. For $\rho > -\frac{1}{2}$, we set

$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}.$

Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree $n$ with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) \, dx = \delta_{mn} \quad \text{ (Kronecker’s delta)}$$

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and
\[ p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0. \]

Moreover, we denote the zeros of \( p_{n,\rho}(x) \) by
\[ -\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \]

A function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be quasi-increasing if there exists \( C > 0 \) such that \( f(x) \leq Cf(y) \) for \( 0 < x < y \). For any two sequences \( \{b_n\}_{n=1}^{\infty} \) and \( \{c_n\}_{n=1}^{\infty} \) of non-zero real numbers (or functions), we write \( b_n \prec c_n \) if there exists a constant \( C > 0 \) independent of \( n \) (or \( x \)) such that \( b_n \leq Cc_n \) for \( n \) large enough. We write \( b_n \sim c_n \) if \( b_n \prec c_n \) and \( c_n \prec b_n \). We denote the class of polynomials of degree at most \( n \) by \( \mathcal{P}_n \).

Throughout \( C, C_1, C_2, \ldots \) denote positive constants independent of \( n, x, t \), and polynomials of degree at most \( n \). The same symbol does not necessarily denote the same constant in different occurrences.

We shall be interested in the following subclass of weights from [3].

**Definition 1.1.** Let \( Q(x) : \mathbb{R} \to \mathbb{R}^+ \) be even continuous function and satisfies the following properties:

(a) \( Q'(x) \) is continuous in \( \mathbb{R} \), with \( Q(0) = 0 \).
(b) \( Q''(x) \) exists and is positive in \( \mathbb{R}\{0\} \).
(c) \( \lim_{x \to \infty} Q(x) = \infty. \)
(d) The function
\[ T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \]
is quasi-increasing in \((0, \infty)\) with
\[ T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \{0\}. \]
(e) There exists \( C_1 > 0 \) such that
\[ \frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)} \quad \text{a.e. } x \in \mathbb{R}\{0\}. \]

Then we write \( w(x) \in \mathcal{F}(C^2) \). If there also exist a compact subinterval \( J(\ni 0) \) of \( \mathbb{R} \), and \( C_2 > 0 \) such that
\[ \frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)} \quad \text{a.e. } x \in \mathbb{R}\{J, \}
\]
then we write \( w(x) \in \mathcal{F}(C^2+) \).

Here are some typical examples of \( \mathcal{F}(C^2+) \). Define for \( x + m > 1, m \geq 0, \) and \( \nu \geq 0 \)
\[ Q_{l, \nu, m}(x) := |x|^m \left( \exp_{l}(|x|^\nu) - \nu^* \exp_{l}(0) \right), \]
where \( x^* = 0 \) if \( x = 0 \), otherwise \( x^* = 1 \) and \( \exp_l(x) := \exp(\exp(\ldots(\exp(x))\ldots)) \) denotes the \( l \)th iterated exponential. We also define

\[
Q(x) := (1 + |x|)^{|x|^\tau} - 1, \quad \tau > 1.
\]

Then the exponents \( \exp(-Q_{l,x,m}(x)) \) and \( \exp(-Q_{x}(x)) \) belong to \( \mathcal{F}(C^2+) \).

In \( \mathbb{R}^+ \), we consider another exponential weights.

**Definition 1.2.** Let \( \tilde{w}(t) = e^{-R(t)} \) where \( R : \mathbb{R}^+ \to \mathbb{R}^+ \). Let \( Q(x) = R(t) \), \( t = x^2 \) and satisfies the following properties:

(a) \( t^{1/2}R'(t) \) is continuous in \( \mathbb{R}^+ \) with limit 0 at 0 and \( R(0) = 0 \).

(b) \( R''(t) \) exists in \( (0, \infty) \) while \( Q''(x) \) is positive \( (0, \infty) \).

(c) \( \lim_{t \to \infty} R(t) = \infty \).

(d) The function

\[
\tilde{T}(t) := \frac{tR'(t)}{R(t)}, \quad t \in (0, \infty)
\]

is quasi-increasing in \( (0, \infty) \) with

\[
\tilde{T}(t) \geq \tilde{\Lambda} > \frac{1}{2}, \quad t \in (0, \infty).
\]

(e) There exists \( C_1 > 0 \) such that

\[
\frac{|R''(t)|}{R'(t)} \leq C_1 \frac{R'(t)}{R(t)} \quad \text{a.e. } t \in (0, \infty).
\]

Then we write \( \tilde{w} \in \mathcal{L}(C^2) \). If there also exist a compact subinterval \( J(\ni 0) \) of \( \mathbb{R} \) and \( C_2 > 0 \) such that

\[
\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)} \quad \text{a.e. } x \in \mathbb{R} \setminus J,
\]

then we write \( \tilde{w} \in \mathcal{L}(C^2+) \).

Similarly for \( \rho > -\frac{1}{2} \), we set

\[
\tilde{w}_\rho(t) := t^\rho \tilde{w}(t), \quad t \in \mathbb{R}^+.
\]

Then the orthonormal polynomial of degree \( n \) with respect to \( \tilde{w}_\rho^2(t) \) is denoted by \( \tilde{p}_{n,\rho}(t) = \tilde{p}_n(\tilde{w}_\rho^2; t) \). More precisely, \( \tilde{p}_{n,\rho}(t) \) satisfies that

\[
\int_0^\infty \tilde{p}_{n,\rho}(t) \tilde{p}_{m,\rho}(t) \tilde{w}_\rho^2(t) \, dt = \delta_{mn}
\]

and

\[
\tilde{p}_{n,\rho}(t) = \tilde{\gamma}_n t^n + \cdots, \quad \tilde{\gamma}_n = \tilde{\gamma}_{n,\rho} > 0.
\]
The zeros of $\tilde{p}_{n,\rho}(t)$ are denoted by

$$0 < t_{n,\rho} < t_{n-1,\rho} < \cdots < t_{2,\rho} < t_{1,\rho}.$$ 

Let $0 < p < \infty$. The $L_p$ Christoffel functions $\lambda_{n,p}(w_\rho; x)$ with a weight $w_\rho(x)$ are defined by

$$\lambda_{n,p}(w_\rho; x) = \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} |Pw_\rho|^p(u) \, du / |P|^p(x).$$

Especially if $p = 2$, we have

$$\lambda_n(w_\rho^2; x) := \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} (Pw_\rho)^2(u) \, du / P^2(x) = \frac{1}{\sum_{j=0}^{n-1} p_{j,\rho}^2(x)}.$$

The corresponding Christoffel functions $\tilde{\lambda}_{n,p}(\tilde{w}_\rho; t)$ with a weight $\tilde{w}_\rho$ are defined similarly. The numbers $\tilde{\lambda}_{n,j} = \tilde{\lambda}_n(x_{j,n},\rho)$ and $\tilde{\lambda}_{n,i} = \tilde{\lambda}_n(t_{j,n,\rho})$, $j = 1, 2, \ldots, n$ are called the Christoffel numbers.

Levin and Lubinsky [3] investigated the weight $w(x) \in \mathcal{F}(C^2)$ and the orthonormal polynomials with respect to $w^2(x)$. In $\mathbb{R}^+$, they [4,5] considered the weight functions $\tilde{w}_\rho(t) = t^p \tilde{w}(t)$ where $\tilde{w}(t) = e^{-R(t)} \in \mathcal{L}(C^2)$, $R(t) = Q(x)$, $t = x^2$ and estimate the orthonormal polynomials $\hat{p}_{n,\rho}(t)$ with respect to the weights $\tilde{w}_\rho(t)$. Furthermore, they proved

$$w \in \mathcal{F}(C^2), \quad w(x) = e^{-Q(x)} \iff \tilde{w} \in \mathcal{L}(C^2), \quad \tilde{w}(t) = e^{-R(t)},$$

in [4, Definition 1.1 and Lemma 2.2]. If $Q(x)$ satisfies

$$1 < \Lambda_1 \leq \frac{(xQ'(x))^2}{Q''(x)} \leq \Lambda_2,$$

where $\Lambda_i$, $i = 1, 2$ are constants, then we call exp$(-Q(x))$ the Freud-type weight. Then the class $\mathcal{F}(C^2)$ contains the Freud-type weights. For certain generalized Freud-type weight $w(x)$, Kasuga and Sakai [2] investigated the orthonormal polynomials associated with $w_\rho(x) = |x|^\rho w(x)$, obtaining bounds on the orthonormal polynomials, zeros, Christoffel functions, and the restricted range inequalities. In [1], we investigated the infinite–finite range inequality, an estimate for the Christoffel function and the Markov–Bernstein inequality with respect to the weights $w_\rho(x) = |x|^\rho w(x)$, $w(x) \in \mathcal{F}(C^2)$.

In this paper we consider the properties of the orthonormal polynomials with respect to the weight $w_\rho(x) = |x|^\rho w(x)$ on $\mathbb{R}$, $w(x) \in \mathcal{F}(C^2)$. First, we prove the relations between $p_{n,\rho}(x)$ and $\hat{p}_{n,\rho}(t)$, which are similar to the relation between Hermite polynomials and Laguerre polynomials. From this relations, we investigate the bounds on the orthonormal polynomials $p_{n,\rho}(x)$ and spacing on their zeros. Moreover, we estimate $A_n(x)$ and $B_n(x)$ defined in Section 4, which are used in representing the derivative of the orthonormal polynomials with respect to the weight $w_\rho^2(x)$.

In the following we introduce useful notations.

(a) Mhaskar–Rahmanov–Saff (MRS) numbers $a_x$ and $\tilde{a}_t$ are defined as the positive roots of the following equations:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} \, du, \quad x > 0, \quad t = \frac{1}{\pi} \int_0^1 \frac{\tilde{a}_t v R'(\tilde{a}_t v)}{(v(1-v))^{1/2}} \, dv, \quad t > 0.$$
(b) Let
\[ \eta_x = (xT(a_x))^{-2/3}, \quad x > 0, \quad \tilde{\eta}_t = (tT(\tilde{a}_t))^{-2/3}, \quad t > 0. \]

(c) The functions \( \varphi_u(x) \) and \( \tilde{\varphi}_u(t) \) are defined as the following:
\[
\varphi_u(x) = \begin{cases} 
\frac{a_u^2 - x^2}{u[(a_u + x + a_u \eta_u)(a_u - x + a_u \eta_u)]^{1/2}}, & |x| \leq a_u, \\
\varphi_u(a_u), & a_u < |x|, 
\end{cases}
\]
\[
\tilde{\varphi}_u(t) = \begin{cases} 
(t + \tilde{a}_u u^{-2})^{1/2}(\tilde{a}_u - t) / u(\tilde{a}_u - t + \tilde{a}_u \tilde{\eta}_u)^{1/2}, & |t| \leq \tilde{a}_u, \\
\tilde{\varphi}_u(\tilde{a}_u), & \tilde{a}_u < t, \\
\tilde{\varphi}_u(0), & t < 0.
\end{cases}
\]

This paper is organized as follows. In Section 2, we state the relations between \( p_{n, \rho}(x) \) and \( \tilde{p}_{n, \rho}(t) \) and the bounds on the orthonormal polynomials and spacing on their zeros. In Section 3, we prove the results of Section 2. In Section 4, we estimate \( A_n(x) \) and \( B_n(x) \) defined in Section 4, which are used in representing the derivative of the orthonormal polynomials with respect to the weight \( \tilde{w}_\rho^2(x) \). Finally Section 5 is an appendix containing various estimates and well known theorems from \([1, 3–5]\).

2. Theorems

In the following theorem, we state the relations between \( p_{n, \rho}(x) \) and \( \tilde{p}_{n, \rho}(t) \).

**Theorem 2.1.** Let \( w(x) = \tilde{w}(t) \) with \( t = x^2 \). Then the orthonormal polynomials \( p_{n, \rho}(x) \) with \( \rho > -\frac{1}{2} \) on \( \mathbb{R} \) can be entirely reduced to the orthonormal polynomials in \( \mathbb{R}^+ \) as follows: For \( n = 0, 1, 2, \ldots, \)
\[
p_{2n, \rho}(x) = \tilde{p}_{n, 1/2(\rho-1/2)}(t) \quad \text{and} \quad p_{2n+1, \rho}(x) = x \tilde{p}_{n, 1/2(\rho+1/2)}(t).
\]

These formulas are in some respects the analogues of \([6, (5.6.1) \text{ and } (4.1.5)]\). Especially, \([6, (5.6.1)]\) shows that Hermite polynomials can be reduced to Laguerre polynomials with the parameter \( \alpha = \pm \frac{1}{2} \). Moreover, these formulas are used almost everywhere in proving the other results of Section 2.

For zeros, we prove:

**Theorem 2.2.** Let \( \rho > -\frac{1}{2} \) and \( w(x) \in \mathcal{F}(C^2+) \). Then

(a) For the minimum positive zero \( x_{[n/2], n, \rho} \),
\[
x_{[n/2], n, \rho} \sim a_n n^{-1}
\]
and for the maximum zero \( x_{1, n, \rho} \),
\[
1 - \frac{x_{1, n, \rho}}{a_n} \sim \eta_n.
\]

(b) For \( n \geq 1 \) and \( 1 \leq j \leq n - 1 \),
\[
x_{j, n, \rho} - x_{j+1, n, \rho} \sim \varphi_n(x_{j, n, \rho}). \tag{2.1}
\]
If we assume that \( w(x) \in \mathcal{F}(C^2) \) instead, then in (a), for some constant \( C > 0 \)
\[
x_{[n/2],n,\rho} \sim a_n n^{-1}, \quad a_n (1 - C \eta_n) \leq x_{1,n,\rho} < a_n + \rho/2, 
\]
and (b) holds with \( \sim \) replaced by \( \gtrsim \).

In the following theorems, we investigate the bounds on the orthonormal polynomials \( p_{n,\rho}(x) \).

**Theorem 2.3** (cf. Levin and Lubinsky [3, Theorem 1.17]). Let \( \rho > -\frac{1}{2} \) and let \( w(x) \in \mathcal{F}(C^2) \). Then uniformly for \( n \geq 1 \),
\[
\sup_{x \in \mathbb{R}} |p_{n,\rho}(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} |x^2 - a_n^2|^{1/4} \sim 1.
\]

**Theorem 2.4** (cf. Levin and Lubinsky [3, Theorem 1.18]). Let \( \rho > -\frac{1}{2} \) and let \( w(x) \in \mathcal{F}(C^2+) \). Then uniformly for \( n \geq 1 \) we have the following:
\[
\sup_{x \in \mathbb{R}} |p_{n,\rho}(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim a_n^{-1/2} (n T(a_n))^{1/6}.
\]

If \( w(x) \in \mathcal{F}(C^2) \), this estimate holds with \( \sim \) replaced by \( \gtrsim \).

Recall that the Lagrange fundamental polynomials at the zeros of \( p_{n,\rho}(x) \) are polynomials \( l_{j,n,\rho}(x) \in \mathcal{P}_{n-1} \), given by
\[
l_{j,n,\rho}(x) = \frac{p_{n,\rho}(x)}{(x - x_{j,n,\rho}) p'_{n,\rho}(x_{j,n,\rho})}.
\]

**Theorem 2.5** (cf. Levin and Lubinsky [3, Theorem 1.19 (a) and (b)]). Let \( w(x) \in \mathcal{F}(C^2+) \) and \( \rho > -\frac{1}{2} \). Then there exists \( n_0 \) such that uniformly for \( n \geq n_0 \) and \( 1 \leq j \leq n \),
(a) \[
|p'_{n,\rho}w|(x_{j,n,\rho}) \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim \varphi_n(x_{j,n,\rho})^{-1} [a_n^2 - x_{j,n,\rho}^2]^{-1/4}.
\]
(b) \[
|p_{n-1,\rho}(x_{j,n,\rho})|w(x_{j,n,\rho}) \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim a_n^{-1} [a_n^2 - x_{j,n,\rho}^2]^{1/4}.
\]
(c) \[
\max_{x \in \mathbb{R}} \left| l_{j,n,\rho}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right| w^{-1}(x_{j,n,\rho}) \left( |x| + \frac{a_n}{n} \right)^{-\rho} \sim 1.
\]
(d) For \( j \leq n - 1 \) and \( x \in [x_{j+1,n,\rho}, x_{j,n,\rho}] \),
\[
|p_{n,\rho}(x)|w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim \min \{ |x - x_{j,n,\rho}|, |x - x_{j+1,n,\rho}| \} \varphi_n(x_{j,n,\rho})^{-1} [a_n^2 - x_{j,n,\rho}^2]^{-1/4}. \tag{2.2}
\]

If we assume that \( w(x) \in \mathcal{F}(C^2) \) instead, then (a) holds with \( \sim \) replaced by \( \leq C \) and (b) holds with \( \sim \) replaced by \( \gtrsim \).
Theorem 2.6 (cf. Levin and Lubinsky [3, Theorem 13.6]). Let \( w(x) \in \mathcal{F}(C^2) \). Assume \( pp > -1/2 \) if \( 0 < p < \infty \) and \( \rho > 0 \) if \( p = \infty \). Then we have for \( n \geq 1 \),

\[
\left\| p_{n, \rho}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \sim a_n^{1/p-1/2} \begin{cases} 
1, & p < 4, \\
\left( \log(1 + nT(a_n)) \right)^{1/4}, & p = 4, \\
\left( nT(a_n) \right)^{2/(4-p)}, & p > 4.
\end{cases}
\]

Let

\[
\Phi_n(x) := \max \left\{ \eta_n, 1 - \frac{|x|}{a_n} \right\} \quad \text{and} \quad x^+ := \begin{cases} 
x, & x > 0, \\
0, & x \leq 0.
\end{cases}
\]

Theorem 2.7. Let \( w(x) \in \mathcal{F}(C^{2+}) \) and \( 0 \leq s \leq t < \infty \). Assume \( pp > -1/2 \) if \( 0 < p < \infty \) and \( \rho > 0 \) if \( p = \infty \). Then we have for \( n \geq 2 \),

\[
\left\| \Phi^{(t/4-1/p^+)}_n(x) \left( p_{n, \rho} w(x) \right) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \sim \left\| \Phi^{(t/4-1/p^+)}_n(x) \left( p_{n, \rho} w(x) \right) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(a_n/2 \leq |x| \leq a_n(1-\eta_n))}
\]

\[
\sim \begin{cases} 
a_n^{1/p-t/2} \log n & \text{if } s = t \text{ and } 4 \leq pt < \infty, \\
a_n^{1/p-s/2} & \text{if } s < t \text{ and } 4 \leq pt < \infty \text{ or if } pt < 4, \\
a_n^{-s/2} & \text{if } p = \infty.
\end{cases}
\]

3. Proofs of theorems

To prove our theorems we use the results of [3–5].

**Proof of Theorem 2.1.** For \( k = 0, 1, 2, \ldots, n - 1 \),

\[
\int_{-\infty}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}(x^2) x^{2k} w_\rho^2(x) \, dx = 2 \int_{0}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}(x^2) x^{2k} w_\rho^2(x) \, dx = \int_{0}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}(t) t^k \tilde{w}_{1/2(\rho-1/2)}^2(t) \, dt = 0.
\]

In the above equation, we have the first equality by the integration of even function, the second equality by the substitution \( t = x^2 \), and the final equality by the orthogonality for the polynomials of degree at most \( n - 1 \). For \( k = 0, 1, 2, \ldots, n - 1 \), we have

\[
\int_{-\infty}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}(x^2) x^{2k+1} w_\rho^2(x) \, dx = 0
\]

since the integrand is odd by the definitions. On the other hand, we have

\[
\int_{-\infty}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}^2(x^2) w_\rho^2(x) \, dx = 2 \int_{0}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}^2(x^2) w_\rho^2(x) \, dx = \int_{0}^{\infty} \tilde{p}_{n, 1/2(\rho-1/2)}^2(t) \tilde{w}_{1/2(\rho-1/2)}^2(t) \, dt = 1.
\]
Similarly, we have for \( k = 0, 1, 2, \ldots, 2n \)
\[
\int_{-\infty}^{\infty} x^k \tilde{P}_{n,1/2}(x^2) \frac{w^2}{\rho}(x) \, dx = 0
\]
and
\[
\int_{-\infty}^{\infty} \left( x^k \tilde{P}_{n,1/2}(x^2) \right)^2 \frac{w^2}{\rho}(x) \, dx = 1.
\]

Therefore, the result is proved. □

**Proof of Theorem 2.2.** From Theorem 2.1 we have the following:
\[
P_{2n,\rho}(x) = \tilde{P}_{n,1/2}(\rho-1/2)(t), \quad P_{2n+1,\rho}(x) = xP_{2n,\rho+1}(x) = t^{1/2} \tilde{P}_{n,1/2}(\rho+1/2)(t).
\]

We only give the proof of the case of \( P_{2n,\rho}(x) \), because for the case of \( P_{2n+1,\rho}(x) \) we see that \( \frac{1}{2}(\rho - \frac{1}{2}) \) may be replaced with \( \frac{1}{2}(\rho + \frac{1}{2}) \).

(a) Since \( x_{2n,\rho}^2 = t_{2n,1/(\rho-1/2)} \) and \( x_{2n,\rho}^2 = t_{1,n,1/2(\rho-1/2)} \), we have the results by (A.1) and (a) of Theorem A.3.

(b) By (A.9), (A.2), and (A.1), we have for \( j = 1, 2, \ldots, n-1 \),
\[
x_{j+1,2n,\rho}^2 - x_{j,2n,\rho}^2 = t_{j,n,1/2(\rho-1/2)} - t_{j+1,n,1/2(\rho-1/2)} = \tilde{\varphi}_n(t_{j,n,1/2(\rho-1/2)}) \sim \varphi_{2n}(x_{j,2n,\rho}) \left( x_{j,2n,\rho}^2 + \frac{a_n^2}{n^2} \right)^{1/2}.
\]

Since for \( j = 1, 2, \ldots, n-1 \) by (a)
\[
\left( x_{j,2n,\rho} + \frac{a_n}{n} \right) \sim x_{j,2n,\rho} \sim \left( x_{j,2n,\rho} + x_{j+1,2n,\rho} \right)
\]
and
\[
x_{n,2n,\rho} - x_{n+1,2n,\rho} = 2x_{n,2n,\rho} \sim \frac{a_n}{n} \sim \varphi_{2n}(x_{n,2n,\rho}),
\]
we have for \( j = 1, 2, \ldots, n, \)
\[
x_{j,2n,\rho} - x_{j+1,2n,\rho} \sim \varphi_{2n}(x_{j,2n,\rho}). \tag{3.1}
\]

For \( n + 1, n + 2, \ldots, 2n \) we also obtain (3.1) by the symmetry of \( x_{j,2n,\rho} \). For the case \( w(x) \in \mathcal{F}(C^2) \), the proof is the same as the above. □

**Proof of Theorem 2.3.** First we prove the result for the even case. Let \( \rho > -\frac{1}{2} \) and \( \gamma = \frac{1}{2}(\rho - \frac{1}{2}) \). Then \( \gamma > -\frac{1}{2} \) and we have by (A.1)
\[
|P_{2n,\rho}(x)|w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} |x^2 - a_{2n}^2|^{1/4}
\sim |P_{2n,\rho}(x)|w(x) \left( x^2 + \frac{a_{2n}^2}{n^2} \right)^{\rho/2} |x^2 - a_{2n}^2|^{1/4}
= |\tilde{P}_{n,\gamma}(t)|\tilde{w}(t) \left( t + \frac{a_n}{n^2} \right)^{\gamma} \left( t + \frac{a_n}{n^2} \right)^{1/4} |t - \tilde{a}_n|^{1/4}.
\]
Therefore, we have by Theorem A.4
\[
\sup_{x \in \mathbb{R}} |p_{2n, \rho}(x)| w(x) \left( |x| + \frac{a_n}{n} \right)^\rho |x^2 - a_{2n}^2|^{1/4} \sim 1.
\]

Now, we show the result for the odd case. Let \( \gamma = \frac{1}{2} (\rho + \frac{1}{2}) \). Then since
\[
p_{2n+1, \rho}(x) = x p_{2n, \rho+1}(x) \quad \text{and} \quad \left( |x| + \frac{a_{2n+1}}{2n+1} \right)^\rho \sim \left( |x| + \frac{a_{2n}}{2n} \right),
\]
we obtain
\[
|p_{2n+1, \rho}(x)| w(x) \left( |x| + \frac{a_{2n+1}}{2n+1} \right)^\rho |x^2 - a_{2n+1}^2|^{1/4} 
\lesssim |p_{2n, \rho+1}(x)| w(x) \left( |x| + \frac{a_{2n}}{2n} \right)^{\rho+1} \left( |x^2 - a_{2n}^2|^{1/4} + |a_{2n+2} - a_{2n}^2|^{1/4} \right)
\sim |\tilde{p}_{n, \gamma}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^{\gamma+1/4} \left( |t - \tilde{a}_n|^{1/4} + |\tilde{a}_{n+1} - \tilde{a}_n|^{1/4} \right).
\]

By Theorems A.4 and A.5
\[
|\tilde{p}_{n, \gamma}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^{\gamma+1/4} |t - \tilde{a}_n|^{1/4} \lesssim 1
\]
and
\[
|\tilde{p}_{n, \gamma}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^{\gamma+1/4} |\tilde{a}_{n+1} - \tilde{a}_n|^{1/4}
\lesssim \begin{cases} 
\tilde{a}_n - t^{-1/4} \tilde{a}_{n+1} - \tilde{a}_n |^{1/4}, & 0 < t \leq \tilde{a}_{n/2}, t > \tilde{2}_n, \\
\tilde{a}_n^{-1/2} (n \tilde{T}(\tilde{a}_n))^{1/6} \tilde{a}_{n+1} - \tilde{a}_n |^{1/4}, & \tilde{a}_{n/2} < t < \tilde{a}_2n, \end{cases}
\lesssim 1,
\]
because \( \tilde{a}_{n+1} - \tilde{a}_n \lesssim \tilde{a}_n/(n \tilde{T}(\tilde{a}_n)) \) by (A.8). On the other hand, from (A.11) we have for \( x_{j,2n+2,\rho+1} \) with \( \varepsilon_1 a_{2n} < x_{j,2n+2,\rho+1} < \varepsilon_2 a_{2n}, 0 < \varepsilon_1 < \varepsilon_2 < 1, \)
\[
|(p_{2n+1, \rho} w)(x_{j,2n+2,\rho+1})| \left( |x_{j,2n+2,\rho+1}| + \frac{a_{2n+1}}{2n+1} \right)^\rho \left| x_{j,2n+2,\rho+1} - a_{2n+1}^2 \right|^{1/4}
\sim a_n^{3/2+\rho} |(p_{2n, \rho+1} w)(x_{j,2n+2,\rho+1})|
\sim a_n^{3/2+\rho} |\tilde{p}_{n, \gamma}(t_{j,n+1, \gamma})| \gtrsim a_n^{3/2+\rho} a_n^{-1/2-\gamma} \sim 1,
\]
because \( \varepsilon_1^2 \tilde{a}_n < t_{j,n+1, \gamma} < \varepsilon_2^2 \tilde{a}_2n \) by (b) of Theorem A.6 and (A.1). Therefore, we have
\[
\sup_{x \in \mathbb{R}} |p_{2n+1, \rho}(x)| w(x) \left( |x| + \frac{a_{2n+1}}{2n+1} \right)^\rho \left| x^2 - a_{2n+1}^2 \right|^{1/4} \sim 1.
\]

**Proof of Theorem 2.4.** Let \( \gamma = \frac{1}{2} (\rho - \frac{1}{2}) \). Then
\[
A := \sup_{x \in \mathbb{R}} |p_{2n, \rho}(x)| w(x) \left( |x| + \frac{a_{2n}}{2n} \right)^\rho \sim \sup_{t \in \mathbb{R}^+} |\tilde{p}_{n, \gamma}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^{\gamma+1/4}.
\]
Now we estimate $A$. By Theorem A.2, Theorem A.4, (A.1), and (A.2), there exists some constant $L > 0$ such that

$$A \lesssim \sup_{t \leq \bar{a}_n(1 - L\bar{n})} |\tilde{p}_{n,\gamma}(t)| \tilde{w}(t) \left( t + \frac{\bar{a}_n}{n^2} \right)^{\gamma + 1/4}$$

$$\lesssim \max_{t \leq \bar{a}_n(1 - L\bar{n})} (\bar{a}_n - t)^{-1/4} \lesssim (\bar{a}_n \bar{n})^{-1/4} \lesssim \bar{a}_n^{-1/4} \left( n \bar{T}(\bar{a}_n) \right)^{1/6}$$

$$\lesssim a_n^{-1/2} (2nT(a_n))^{1/6}.$$  

For the lower bounds, we use the Bernstein inequality [5, Theorem 1.5(1.20)]. Then

$$\left| t_{1,n,\gamma}^{1/4} \left( \tilde{p}_{n,\gamma}'(t_{1,n,\gamma}) \tilde{w}_{\gamma}(t_{1,n,\gamma}) \tilde{\phi}_n(t_{1,n,\gamma}) \right) \right| = \left| \left( \tilde{p}_{n,\gamma}' \tilde{w}_{\gamma} \right)'(t_{1,n,\gamma}) \tilde{\phi}_n(t_{1,n,\gamma}) t_{1,n,\gamma}^{\gamma + 1/4} \right|$$

$$\lesssim \sup_{t \in \mathbb{R}^+} \left| \tilde{p}_{n,\gamma}' \tilde{w}_{\gamma}(t) \left( t + \frac{\bar{a}_n}{n^2} \right)^{\gamma + 1/4} \right| \sim A.$$

On the other hand, we have by (A.10) and (a) of Theorem A.3

$$\left| t_{1,n,\gamma}^{1/4} \tilde{p}_{n,\gamma}'(t_{1,n,\gamma}) \tilde{w}_{\gamma}(t_{1,n,\gamma}) \tilde{\phi}_n(t_{1,n,\gamma}) \right| \sim (\bar{a}_n - t_{1,n,\gamma})^{1/4} \sim (\bar{a}_n \bar{n})^{-1/4}$$

$$\sim \bar{a}_n^{-1/4} \left( n \bar{T}(\bar{a}_n) \right)^{1/6} \sim a_n^{-1/2} \left( nT(a_n) \right)^{1/6}.$$  

Consequently, we have $A \sim a_n^{-1/2} \left( nT(a_n) \right)^{1/6}$. The case of $p_{2n+1,\rho}(x)$ is similar. □

**Lemma 3.1.** Let $\tilde{w} \in \mathcal{L}(C^2)$ and $\rho > -\frac{1}{2}$. Then uniformly for $n \geq 1$

$$|\tilde{p}_{n,\rho}(0)| \sim \left( \frac{n^2}{\bar{a}_n} \right)^{\rho} \left( \frac{n}{\bar{a}_n} \right)^{1/2} \quad (3.2)$$

and for $t \in [0, t_{n,n,\rho}]$,

$$|\tilde{p}_{n,\rho}(t)| \sim |t - t_{n,n,\rho}| \left( \frac{n^2}{\bar{a}_n} \right)^{\rho + 1} \left( \frac{n}{\bar{a}_n} \right)^{1/2} \quad (3.3)$$

**Proof.** By Theorem A.5, we know that

$$|\tilde{p}_{n,\rho}(0)| \lesssim \left( \frac{n^2}{\bar{a}_n} \right)^{\rho} \left( \frac{n}{\bar{a}_n} \right)^{1/2}.$$  

On the other hand, since $|\tilde{p}_{n,\rho}(t)|$ is decreasing on $[0, t_{n,n,\rho}]$ we have by the mean value property, (A.10), and (a) of Theorem A.3,

$$|\tilde{p}_{n,\rho}(0)| \geq |\tilde{p}_{n,\rho}'(t_{n,n,\rho})| t_{n,n,\rho} \sim \left( \frac{n^2}{\bar{a}_n} \right)^{\rho} \left( \frac{n}{\bar{a}_n} \right)^{1/2}.$$  

Therefore, (3.2) is proved. Similarly for $t \in [0, t_{n,n,\rho}]$, we have by the mean value property

$$|\tilde{p}_{n,\rho}(t)| \geq |t - t_{n,n,\rho}| |\tilde{p}_{n,\rho}'(t_{n,n,\rho})| \sim |t - t_{n,n,\rho}| \left( \frac{n^2}{\bar{a}_n} \right)^{\rho + 1} \left( \frac{n}{\bar{a}_n} \right)^{1/2}.$$
and we have by (A.12),
\[
|\tilde{p}_{n,\rho}(t)| \lesssim |t - t_{n,n,\rho}| \tilde{p}_{n,\rho}'(t_{n,n,\rho})| \sim |t - t_{n,n,\rho}| \left(\frac{n^2}{a_n}\right)^{\rho+1} \left(\frac{n}{a_n}\right)^{1/2}.
\]
Therefore, (3.3) is also proved. \(\square\)

**Proof of Theorem 2.5.** (a) To prove (a), we use (A.10). Let \(\gamma = \frac{1}{2}(\rho - \frac{1}{2})\). Then for \(j = 1, 2, \ldots, 2n\)
\[
|p'_{2n,\rho}(x_j,2n,\rho)|w(x_j,2n,\rho) \left(\left|x_j,2n,\rho\right| + \frac{a_{2n}}{2n}\right)^{\rho}
\]
\[
\sim |p'_{2n,\rho}(x_j,2n,\rho)|w(x_j,2n,\rho)|x_j,2n,\rho|^\rho = 2|\tilde{p}_{n,\gamma}'(t_{j,n,\gamma})|^3/4
\]
\[
\sim \phi_{2n}(x_j,2n,\rho)^{-1}x_j^{1/2}2n,\rho(a_{2n}^2 - x_j^{2},2n,\rho)^{-1/2}x_j^{3/2}
\]
\[
\sim \phi_{2n}(x_j,2n,\rho)^{-1}(a_{2n}^2 - x_j^{2},2n,\rho)^{-1/2}.
\]

Here, we used (a) of Theorems 2.2, 2.1, (A.10), (A.1)–(A.3), and so on. Now, let \(\gamma = \frac{1}{2}(\rho + \frac{1}{2})\). Then since \(x_j,2n+1,\rho = x_j,2n,\rho+1\) for \(j = 1, 2, \ldots, n\) and
\[
p'_{2n+1,\rho}(x) = xp'_{2n,\rho+1}(x) + p_{2n,\rho+1}(x),
\]
we have for \(j = 1, 2, \ldots, n, n+2, \ldots, 2n+1\) by the even case
\[
|p'_{2n+1,\rho}w(x_j,2n+1,\rho)\left(\left|x_j,2n+1,\rho\right| + \frac{a_{2n+1}}{2n+1}\right)^{\rho}|
\]
\[
\sim \left|p'_{2n,\rho+1}w(x_j,2n,\rho+1)\left(\left|x_j,2n,\rho+1\right| + \frac{a_{2n+1}}{2n+1}\right)^{\rho+1}\right|
\]
\[
\sim \phi_{2n+1}(x_j,2n+1,\rho)^{-1}(a_{2n+1}^2 - x_j^{2},2n+1,\rho)^{-1/4}.
\]

For \(x_{n+1,2n+1,\rho} = 0\), we obtain by Lemma 3.1
\[
|p'_{2n+1,\rho}w(0)\left(\frac{a_{2n+1}}{2n+1}\right)^{\rho}||p_{2n,\rho+1}(0)|\left(\frac{a_n}{n}\right)^{\rho} = |\tilde{p}_{n,\gamma}(0)|\left(\frac{a_n}{n}\right)^{\rho}
\]
\[
\sim \frac{n}{a_n^{3/2}} \sim \phi_{2n+1}(0)^{-1}(a_{2n+1}^2)^{-1/4}.
\]

Therefore, (a) is proved.

(b) By the definition of \(\lambda_n(x) = \lambda_n(w^2_\rho, x)\) we have
\[
\lambda_n^{-1}(x_j,n,\rho) = \frac{\gamma_n-1,\rho}{\gamma_n,\rho}p'_{n,\rho}(x_j,n,\rho)p_{n-1,\rho}(x_j,n,\rho).
\]
So, we have

\[ p_{n-1,\rho}(x_{j,n,\rho}) = \frac{\gamma_{n,\rho} - \lambda_{n-1}(x_{j,n,\rho})}{\gamma_{n-1,\rho}} (p'_{n,\rho}(x_{j,n,\rho}))^{-1}. \]

Then we have by (a) and Theorem A.9

\[
\left| (p_{n-1,\rho} w)(x_{j,n,\rho}) \left| x_{j,n,\rho} + \frac{a_n}{n} \right|^{\rho} \right|
\]

\[
= \frac{\gamma_{n,\rho} - \lambda_{n-1}(x_{j,n,\rho})}{\gamma_{n-1,\rho}} (x_{j,n,\rho})^{2\rho} \left| x_{j,n,\rho} + \frac{a_n}{n} \right|^{\rho-1}
\]

\[
\sim \frac{\gamma_{n,\rho}}{\gamma_{n-1,\rho}} \left( \frac{a_n^2 - x_{j,n}^2}{\rho} \right)^{1/4}.
\]

Here \( \gamma_{n-1,\rho}/\gamma_{n,\rho} \sim a_n \) (see Lemma 4.7 in Section 4, and note that its proof is independent of Theorem 2.5). Therefore, (b) is proved.

To prove (c) and (d), let

\[
\gamma := \begin{cases} 
\frac{1}{2} \left( \rho - \frac{1}{2} \right) & \text{if } n \text{ is even} \\
\frac{1}{2} \left( \rho + \frac{1}{2} \right) & \text{if } n \text{ is odd}
\end{cases}
\quad \text{and } m := \left[ \frac{n}{2} \right].
\]

(c) First, we easily have

\[
\max_{x \in \mathbb{R}} \left( l_{j,n,\rho} w \right)(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \left( |x_{j,n,\rho}| + \frac{a_n}{n} \right)^{-\rho} \geq \left| (l_{j,n,\rho} w)(x_{j,n,\rho}) \left| x_{j,n,\rho} + \frac{a_n}{n} \right|^{\rho} \left( |x_{j,n,\rho}| + \frac{a_n}{n} \right)^{-\rho} \right| = 1.
\]

Therefore, it remains to obtain the upper bounds. By Theorem A.8, it suffices to prove the upper bounds for \( |x| \leq a_n \). First, suppose \( 1 \leq j \leq m \). Then we have by Theorem 2.1

\[
B := |l_{j,n,\rho}(x)| w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \left( |x_{j,n,\rho}| + \frac{a_n}{n} \right)^{-\rho}
\]

\[
\lesssim \left[ \tilde{l}_{j,m,\gamma}(t) \right] \tilde{w}(t) \left( t + \frac{\tilde{a}_m}{m^2} \right)^{\gamma} \tilde{w}^{-1}(t_{j,m,\gamma}) \left( t_{j,m,\gamma} + \frac{\tilde{a}_m}{m^2} \right)^{-\gamma}
\]

\[
\times \begin{cases} 
\frac{(|x| + |x_{j,n,\rho}|)(|x| + a_n/n)^{1/2}}{|x|^{3/2}} & \text{if } n = 2m, \\
\frac{|x||x| + |x_{j,n,\rho}|)(|x| + a_n/n)^{1/2}}{|x|^{3/2}} & \text{if } n = 2m + 1.
\end{cases}
\]

For \( |x| \leq a_n \) if \( |x| \leq 2|x_{j,n,\rho}| \) or \( |x_{j,n,\rho}| \geq a_n/2 \), then \( |x|/|x_{j,n,\rho}| \lesssim 1 \). Therefore, for these cases we have \( B \lesssim 1 \) by (A.12). Now, suppose \( 2|x_{j,n,\rho}| \leq |x| \leq a_n \) and \( |x_{j,n,\rho}| \leq a_n/2 \). Then
since $|x - x_{j,n,\rho}| \geq |x_{j,n,\rho}|$ and $|x - x_{j,n,\rho}| \sim |x|$, we also have (a) and (A.4)

$$B \approx \frac{p_{n,\rho}(x)}{|x - x_{j,n,\rho}|} \frac{w(x)\left(|x| + \frac{a_n}{n}\right)^\rho}{p_{n,\rho}(x_{j,n,\rho})} w^{-1}(x_{j,n,\rho})\left(|x_{j,n,\rho}| + \frac{a_n}{n}\right)^{-\rho}$$

$$\approx \frac{p_{n,\rho}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho}{x - x_{j,n,\rho}} \varphi_n(x_{j,n,\rho})(a_n^2 - x_{j,n,\rho}^2)^{1/4}$$

$$\approx \frac{a_n^{3/2}}{n} \frac{p_{n,\rho}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho}{x - x_{j,n,\rho}}.$$

Then for $|x| \leq a_n/2$, we have by Theorem 2.3 and (a) of Theorem 2.2

$$B \approx \frac{a_n^{3/2}}{n} \frac{(a_n^2 - x_{j,n,\rho}^2)^{-1/4}}{|x_{j,n,\rho}|} \approx 1$$

and for $|x| \geq a_n/2$, we have by Theorem 2.4 and (A.5),

$$B \approx \frac{a_n^{3/2}}{n} \frac{a_n^{-1/2}(nT(a_n))^{1/6}}{|x|} \approx 1.$$

Therefore, we also have $B \approx 1$ for these cases. Thus, we proved for $1 \leq j \leq m$

$$\sup_{|x| \leq a_n} \left|l_{j,n,\rho}w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right| w^{-1}(x_{j,n,\rho})\left(|x_{j,n,\rho}| + \frac{a_n}{n}\right)^{-\rho} \approx 1.$$

Moreover, when $n = 2m + 1$, we have by (b), (A.6), Theorems 2.3 and 2.4

$$|l_{m+1,2m+1,\rho}(x)|w(x)\left(|x| + \frac{a_{2m+1}}{2m+1}\right)^\rho$$

$$\times w^{-1}(x_{m+1,2m+1,\rho})\left(|x_{m+1,2m+1,\rho}| + \frac{a_m}{m}\right)^{-\rho}$$

$$= \left|p_{2m,\rho+1}(x)w(x)\left(|x| + \frac{a_{2m+1}}{2m+1}\right)^\rho (p_{2m,\rho+1}w)^{-1}(0)\left(\frac{a_m}{m}\right)^{-\rho}\right|$$

$$\approx \frac{a_m^{3/2}}{m} \left\{ \begin{array}{ll}
\frac{m}{a_m^{3/2}}, & |x| \leq a_m/2 \\
\frac{a_m^{3/2}}{m^{3/2}}, & |x| \geq a_m/2
\end{array} \right\} \approx 1.$$

If we use the symmetry of the zeros, then (c) can be proved for all $j(1 \leq j \leq n)$.

(d) To prove (d), we use (A.13), Lemmas A.1 and 3.1. Suppose $1 \leq j \leq m - 1$ and let $x \in [x_{j+1,n,\rho}, x_{j,n,\rho}]$. Then we have by (A.13)

$$|p_{n,\rho}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho$$

$$\approx |\tilde{p}_{m,\gamma}(t)|\tilde{w}_\gamma(t)\tilde{\omega}_\gamma(t)^{1/4}$$

$$\approx \min\{|t - t_{j,m,\gamma}|, |t - t_{j+1,m,\gamma}||\tilde{p}_{m,\gamma}(t_{j,m,\gamma})\tilde{\omega}_m - t_{j,m,\gamma})\tilde{\omega}_m^{-1}t_{j,m,\gamma})^{-1/4}$$

$$\approx \min\{|x^2 - x_{j,n,\rho}^2|, |x^2 - x_{j+1,n,\rho}^2|\varphi_n^{-1}(x_{j,n,\rho})|x_{j,n,\rho}|^{-1}(a_n^2 - x_{j,n,\rho}^2)^{-1/4}$$

$$\approx \min\{|x - x_{j,n,\rho}|, |x - x_{j+1,n,\rho}|\varphi_n^{-1}(x_{j,n,\rho})(a_n^2 - x_{j,n,\rho}^2)|^{-1/4}.$$
Moreover, when \( n = 2m + 1 \), from (3.3) we have for \( x \in [x_{m+1,2m+1,\rho}, x_{m,2m+1,\rho}] \),
\[
|p_{2m+1,\rho}(x)| w(x) \left( |x| + \frac{a_{2m+1}}{2m+1} \right)^\rho
\]
\[
= |xp_{2m,\rho+1}(x)| w(x) \left( |x| + \frac{a_{2m+1}}{2m+1} \right)^\rho
\]
\[
\sim |t^{1/2} \tilde{p}_{m,\gamma}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_m}{m^2} \right)^{\rho/2}
\]
\[
\sim |t - t_{m,m,\gamma}| \left( \frac{m^2}{\tilde{a}_m} \right)^{3/4} \left( \frac{m}{\tilde{a}_m} \right)^{1/2}
\]
\[
\sim \frac{m}{a_m^{3/2}} \min\{|x|, |x - x_{m,2m+1,\rho}|\} \sim \frac{m}{a_m^{3/2}} \min\{|x - x_{m+1,2m+1,\rho}|, |x - x_{m,2m+1,\rho}|\}.
\]

When \( n = 2m \), since from (3.3) we have for \( x \in [0, x_{m,2m,\rho}] \),
\[
|p_{2m,\rho}(x)| w(x) \left( |x| + \frac{a_{2m}}{2m} \right)^\rho \sim |\tilde{p}_{m,1/2(\rho-1/2)}(t)| \tilde{w}(t) \left( t + \frac{\tilde{a}_m}{m^2} \right)^{\rho/2}
\]
\[
\sim |t - t_{m,m,1/2(\rho-1/2)}| \left( \frac{m^2}{\tilde{a}_m} \right)^{3/4} \left( \frac{m}{\tilde{a}_m} \right)^{1/2}
\]
\[
\sim \frac{m}{a_m^{3/2}} \|x| - |x_{m,2m,\rho}|\,
\]
we obtain in this case that for \( x \in [x_{m+1,2m,\rho}, x_{m,2m,\rho}] \),
\[
|p_{2m,\rho}(x)| w(x) \left( |x| + \frac{a_{2m}}{2m} \right)^{\rho} \sim \frac{m}{a_m^{3/2}} \min\{|x - x_{m+1,2m,\rho}|, |x - x_{m,2m,\rho}|\}.
\]

If we use the symmetric property (d) can be proved for \( m+1 \leq j \leq n \). Therefore, (d) is proved completely. \( \square \)

**Proof of Theorem 2.6.** From Theorems A.8 and 2.3 we have
\[
\left\| (p_{n,\rho} w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \lesssim \left\| (p_{n,\rho} w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R}; L_{a_n/n} \leq |x| \leq a_n(1 - L_{a_n}))}
\]
\[
\lesssim a_n^{-1/2} \left\| \left( 1 - \frac{|x|}{a_n} \right)^{-1/4} \right\|_{L_p(\mathbb{R}; L_{a_n/n} \leq |x| \leq a_n(1 - L_{a_n}))}
\]
\[
\lesssim a_n^{-1/2} \left\{ \begin{array}{ll}
1, & p < 4, \\
(\log(1 + nT(a_n)))^{1/4}, & p = 4, \\
(nT(a_n))^{\frac{2}{9}\left( \frac{1}{4} - \frac{1}{p} \right)}, & p > 4.
\end{array} \right.
\]

Now, we estimate the lower bounds of \( \left\| (p_{n,\rho} w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \). Since by (2.2), (2.1), and Lemma A.1
\[
\int_{x_{j+1,\rho}}^{x_j,\rho} \left| (p_{n,\rho} w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right|^p dx
\]
\[
\sim \int_{x_{j+1,\rho}}^{x_j,\rho} \min\{|x - x_{j,\rho}|, |x - x_{j+1,\rho}|\}^p dx \phi_n(x_j,\rho)^{-p} \left( a_n^2 - x_{j,\rho}^2 \right)^{-p/4}
\]
\[
\sim |x_{j,\rho} - x_{j+1,\rho}| \phi_n(x_{j,\rho})^{-p} \left( a_n^2 - x_{j,\rho}^2 \right)^{-p/4}
\]
\[
\sim |x_{j,\rho} - x_{j+1,\rho}| \left( a_n^2 - x_{j,\rho}^2 \right)^{-p/4}.
\]
we have from (a) of Theorem 2.2
\[ \left\| (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \sim \left( \int_{x_n}^{x_{1,n}} \left( a_n^2 x^2 \right)^{-p/4} \, dx \right)^{1/p} \]
\[ \sim \left( \int_{-a_n(1-\eta_n)}^{a_n(1-\eta_n)} \left( a_n^2 x^2 \right)^{-p/4} \, dx \right)^{1/p} \]
\[ \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log(1 + nT(a_n)))^{(1/4)}, & p = 4, \\ (nT(a_n))^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{p} \right)}, & p > 4. \end{cases} \]

Therefore, we have the result. \( \square \)

**Proof of Theorem 2.7.** First, assume \( 4 \leq pt < \infty \). Since by Theorem A.8
\[ \left\| (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \sim \left( \int_{x_n}^{x_{1,n}} \left( a_n^2 x^2 \right)^{-p/4} \, dx \right)^{1/p} \]
\[ \sim \left( \int_{-a_n(1-\eta_n)}^{a_n(1-\eta_n)} \left( a_n^2 x^2 \right)^{-p/4} \, dx \right)^{1/p} \]
\[ \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log(1 + nT(a_n)))^{(1/4)}, & p = 4, \\ (nT(a_n))^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{p} \right)}, & p > 4. \end{cases} \]
and from Theorem 2.3 for \( |x| \leq a_n(1 - \eta_n) \),
\[ (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \rightarrow a_n^{-s/2} \left( 1 + \frac{|x|}{a_n} \right)^{(t-s)/4-1/p}, \]
we have
\[ \left\| (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \]
\[ \sim a_n^{1/p-s/2} \left\| 1 - u \{ (t-s)/4-1/p \} \right\|_{L_p(|u| \leq 1-\eta_n)} \sim \left\{ \begin{array}{ll} a_n^{1/p-t/2} \log n, & s = t, \\ a_n^{1/p-s/2}, & s < t. \end{array} \right. \]

Now, we estimate the lower bounds. Similarly to the proof of Theorem 2.6, since by (2.2), (2.1), and Lemma A.1,
\[ \int_{x_{j,n}}^{x_{j+1,n}} \left\| (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \]
\[ \sim |x_{j,n} - x_{j+1,n}| a_n^{-sp/2} \left( 1 + \frac{|x_{j,n}|}{a_n} \right)^{(t-s)p/4-1}, \]
we have from (a) of Theorem 2.2,
\[ \left\| (p_{n,p}w)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_p(\mathbb{R})} \]
\[ \sim a_n^{-s/2} \left( \log n \right)^{(t-s)p/4-1}, \]
\[ \sim \left\{ \begin{array}{ll} a_n^{1/p-t/2} \log n, & s = t, \\ a_n^{1/p-s/2}, & s < t. \end{array} \right. \]
For the case $0 \leq pt < 4$, the result can be proved similarly. Especially, when $p = \infty$, we know easily the result from Theorem 2.3. Consequently, we proved the result. □

4. Further properties of $p_{n, \rho}(x)$

In the rest of this paper we let $p_{n}(x) = p_{n, \rho}(x)$ simply and we assume that $\rho > -\frac{1}{2}$.

Theorem 4.1. We have a representation:

$$p'_{n}(x) = A_{n}(x)p_{n-1}(x) - B_{n}(x)p_{n}(x) - 2\rho_{n}\frac{p_{n}(x)}{x}. \quad (4.1)$$

Here

$$A_{n}(x) = 2b_{n}\int_{-\infty}^{\infty} p_{n}^{2}(u)\overline{Q}(x, u)w^{2}_{\rho}(u)\,du,$$

$$B_{n}(x) = 2b_{n}\int_{-\infty}^{\infty} p_{n}(u)p_{n-1}(u)\overline{Q}(x, u)w^{2}_{\rho}(u)\,du,$$

where

$$\overline{Q}(x, t) = \frac{Q'(x) - Q'(u)}{x - u}, \quad b_{n} = \frac{\gamma_{n-1}}{\gamma_{n}} \quad \text{and} \quad \rho_{n} = \begin{cases} \rho, & \text{n is odd}, \\ 0, & \text{n is even}. \end{cases}$$

Proof. Using the reproducing kernel

$$K_{n}(x, u) = b_{n}\left\{ p_{n}(x)p_{n-1}(u) - p_{n}(u)p_{n-1}(x) \right\} / (x - u)$$

we have easily the results by the same method as [2, Theorem 1.6]. □

Theorem 4.2 (cf. Levin and Lubinsky [3, Theorem 13.7]). Let $w(x) \in \mathcal{F}(C^{2})$, $L > 0$, and $\gamma > 0$. Then there exist $C, n_{0} > 0$ such that for $n \geq n_{0}$ and $\gamma\frac{a_{n}}{n} \leq |x| \leq a_{n}(1 + Ln_{n})$,

$$\frac{A_{n}(x)}{2b_{n}} \sim \varphi_{n}(x)^{-1}a_{n}^{2}(1 + 2Ln_{n})^{-2} - x^{2} - \frac{1}{2}, \quad |B_{n}(x)| \lesssim A_{n}(x), \quad (4.2)$$

and for $\gamma\frac{a_{n}}{n} \leq |x| \leq \varepsilon a_{n}$ with $0 < \varepsilon < 1$ small enough, there exists $0 < \lambda(\varepsilon) < 1$ such that

$$|B_{n}(x)| < \lambda(\varepsilon)A_{n}(x). \quad (4.3)$$

Remark 4.3. Let $w(x) \in \mathcal{F}(C^{2})$. If $\rho \geq 0$ or $\Lambda \geq 2$ then (4.2) and (4.3) holds for $|x| \leq a_{n}(1 + Ln_{n})$. For $1 < \Lambda < 2$, if $Q''(x)$ is bounded on a certain interval $(0, c]$, or if there exist $0 < \delta \leq 2$ and a positive constant $C$ such that $Q''(x) \leq Cx^{-\delta}$ on a certain interval $(0, c]$ and $\rho \geq \frac{\delta - 1}{2}$ then (4.2) and (4.3) holds for $|x| \leq a_{n}(1 + Ln_{n})$. Moreover, for $1 < \Lambda < 2$, if $Q''(x)$ is non-increasing a certain interval $(0, c]$ and $\Lambda + 2\rho - 1 \geq 0$, then (4.2) and (4.3) holds for $|x| \leq a_{n}(1 + Ln_{n})$.

Proof of Theorem 4.2. First we can show (4.2) by repeating the methods of the proof of [3, Theorem 13.7; 5, Theorem 4.1]. The second inequality in (4.2) is shown by the Cauchy–Schwarz inequality. Formula (4.2) is proved by dividing into two parts, that is, the upper bounds part and the lower bounds part. Let us define

$$\Theta_{n, \rho}(x) = \frac{A_{n}(x)}{2b_{n}}\varphi_{n}(x)|a_{n}^{2} - x^{2}|^{1/2}.$$
To prove Theorem 4.2 we need some lemmas.

**Lemma 4.4.** Assume that \( w(x) \in \mathcal{F}(C^2) \). Then

(a) \[
\int_{-\infty}^{\infty} (p_n w_p)^2(u) u' Q(u) \, du = n + \rho + \frac{1}{2}.
\]

(b) \[
\int_{-\infty}^{0} (p_n w_p)^2(u) |Q'(u)| \, du = \int_{0}^{\infty} (p_n w_p)^2(u) Q'(u) \, du \sim \frac{n}{a_n}.
\]

(c) \[
\int_{-\infty}^{\infty} (p_n w)^2(u) \left[ \left| u \right| + \frac{a_n}{n} \right]^{2\rho} u Q'(u) \, du \sim n.
\]

(d) \[
\int_{0}^{\infty} (p_n w)^2(u) \left[ \left| u \right| + \frac{a_n}{n} \right]^{2\rho} Q'(u) \, du \sim \frac{n}{a_n}.
\]

**Proof.** Since (a) and (b) are similar results to [3, Lemma 12.7], we can prove them by repeating the methods of the proof of [3, Lemma 12.7; 5, Theorem 4.2]. Moreover, (c) and (d) can be obtained easily, using (a), (b), and their proofs. \( \square \)

**Lemma 4.5** (cf. Levin and Lubinsky [3, Theorem 12.11]). Let \( w(x) \in \mathcal{F}(C^2) \). For \( \gamma a_n \leq |x| \leq \Gamma_n \) we have

\[
\Theta_{n, \rho}(x) \sim 1,
\]

where \( \Gamma_n = a_n (1 - M \eta_n) \) and \( M > 0 \) is chosen such that \( x_{1,n, \rho} > a_n \left( 1 - (M/2) \eta_n \right) \).

**Proof.** We split \( \Theta_{n, \rho}(x) \) into two parts as the following:

\[
\Theta_{n, \rho}(x) = \left[ \int_{|u| \leq \gamma a_n/2n} + \int_{|u| \geq \gamma a_n/2n} \right] (p_n w_p)^2(u) \frac{Q(x,u)}{u} \, du \varphi_n(x) |a_n^2 - x^2|^{1/2}
\]

\[
=: A + B.
\]

Here, if we use the method of [3, Chapter 12] with Lemma 4.4 and Theorem 2.3, then we can show \( B \sim 1 \) by replacing \( \psi_n(x) \) with \( (p_n w)^2(x) \left( |x| + \frac{a_n}{n} \right)^{2\rho} |a_n^2 - x^2|^{1/2} \). On the other hand, we know by Theorem 2.3 that

\[
A \sim \frac{1}{a_n} \int_{|u| \leq \gamma a_n/2n} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}} \frac{Q(x,u)}{u} \, du \varphi_n(x) |a_n^2 - x^2|^{1/2}.
\]

Then since for \( \gamma a_n \leq |x| \leq a_n/2 \) and \( |u| \leq \frac{\gamma a_n}{2n} \)

\[
\frac{Q(x,u)}{u} \sim \frac{n}{a_n}
\]

and for \( a_n/2 < |x| \leq \Gamma_n \) and \( |u| \leq \frac{\gamma a_n}{2n} \)

\[
\frac{Q(x,u)}{u} \sim \frac{Q'(a_{2n})}{a_n} \sim \frac{n \sqrt{T(a_n)}}{a_n^2}.
\]

we have \( A \sim 1 \). So, the lemma is proved. \( \square \)

Now we prove (4.2).
Proof of the upper bounds of \( A_n(x) \). Let \( \frac{x}{n} \leq |x| \leq a_n(1+L_\eta_n) \). If we distinguish three ranges of \( x \); (i) \( \frac{x}{n} \leq |x| \leq \Gamma_n \), (ii) \( \Gamma_n \leq x \leq a_n(1 + L_\eta_n) \), and (iii) \(-a_n(1 + L_\eta_n) \leq x \leq -\Gamma_n \), then the upper bounds for \( A_n(x) \) can be proved easily by repeating methods of proofs of [3, Theorem 13.7; 5, Theorem 4.1] for upper bounds. □

Proof of the lower bounds of \( A_n(x) \). Similarly to the methods of [3, Lemma 13.8; 5, Theorem 4.2], we can choose the numbers \( x \) satisfying that uniformly for \( r \in [0, 2n] \),

\[
\int_{[a_n, a_n]} (p_n w_p)^2(u) Q(u) \, du \sim \frac{n}{a_n}.
\]

If we use these numbers \( x \) satisfying that \( \frac{x}{n} \leq |x| \leq a_n(1+L_\eta_n) \), then the lower bounds for \( A_n(x) \) can be proved easily by repeating methods of [3, Theorem 13.7; 5, Theorem 4.1] for lower bounds. □

In the following, we will prove (4.3). Let \( \frac{x}{n} \leq |x| \leq \varepsilon a_n \) for \( 0 < \varepsilon < \frac{1}{2} \) small enough. By (4.4) we obtain

\[
\int_{|u| \leq \frac{a_n}{2n}} \frac{|u|^{2\rho}}{(|u| + a_n/n)^2} \frac{Q(x, u)}{(a_n^2 - u^2)^{1/2}} \, du \sim \frac{n}{a_n^2} \int_{|u| \leq \frac{a_n}{2n}} \frac{|u|^{2\rho}}{(|u| + a_n/n)^2} \, du \\
\sim O \left( \frac{a_n}{n} \right) \frac{n}{a_n^2}.
\]

Choose \( 0 < \theta < 1 \) satisfying that

\[
\int_{|u| \leq a_0} \frac{Q(x, u)}{|a_0^2 - u^2|^{1/2}} \, du \leq C \frac{\sigma_0(x)}{|a_0^2 - x^2|^{1/2}} \leq \frac{C \theta n}{a_0^2} \leq \frac{C \theta n}{a_n^2}.
\]

Here,

\[
\sigma_0(x) = \frac{1}{\pi^2} \frac{(a_0^2 - x^2)^{1/2}}{(a_0^2 - s^2)^{1/2}} \int_{-a_0}^{a_0} \frac{Q(x, s)}{(a_0^2 - s^2)^{1/2}} \, ds, \quad x \in [-a_0, a_0]
\]

is the density of the equilibrium measure of total mass for the field \( Q \). It is shown in [3, Theorem 5.3] that for \( L > 1 \)

\[
\sigma_t(x) \sim \frac{1}{\pi^2} \frac{(a_t^2 - x^2)^{1/2}}{(a_t^2 - s^2)^{1/2}}, \quad |x| \leq a_t.
\]

Then using Theorem 2.3,

\[
\int_{|u| \leq a_0} (p_n w_p)^2(u) Q(x, u) \, du \sim \int_{|u| \leq \frac{a_n}{2n}} \frac{|u|^{2\rho}}{(|u| + a_n/n)^2} \frac{Q(x, u)}{(a_n^2 - u^2)^{1/2}} \, du \\
+ \int_{\frac{a_n}{2n} \leq |u| \leq a_0} \frac{|u|^{2\rho}}{(|u| + a_n/n)^2} \frac{Q(x, u)}{(a_n^2 - u^2)^{1/2}} \, du \\
\leq O \left( \frac{a_n}{n} \right) \frac{n^2}{a_n^2} + C \frac{n}{a_n^2}.
\]

(4.5)

Since for \( x \in [0, a_n/2] \),

\[
Q'(x) \leq C \frac{n}{a_n} \left( \frac{x}{a_n} \right)^{\Lambda - 1}
\]
(see [3, Lemma 3.8 (3.12)]), we have for \(a_0n \leq |u| \leq a_{2n},\)
\[
\left| Q(x, u) - Q(x, -u) \right| = 2 \left| \frac{u Q'(x) - x Q'(u)}{x^2 - u^2} \right| \\
\lesssim \left| \frac{Q'(\varepsilon a_n) + \varepsilon |Q'(u)|}{a_n} \right| \lesssim \varepsilon^{\Lambda - 1} \frac{n}{a_n^2} + \frac{\varepsilon}{a_n} |Q'(u)|. \tag{4.6}
\]

Then using Cauchy–Schwartz inequality, (4.6), and (b) of Lemma 4.4, we have
\[
\left| \int_{a_0n \leq |u| \leq a_{2n}} p_n(u) p_{n-1}(u) w_\rho^2(u) Q(x, u) \, du \right| \\
\lesssim \frac{\varepsilon}{a_n} \int_0^\infty |p_n(u) p_{n-1}(u) w_\rho^2(u)| |Q'(u)| \, du \\
+ \varepsilon^{\Lambda - 1} \frac{n}{a_n^2} \int_0^\infty |p_n(u) p_{n-1}(u) w_\rho^2(u)\, du \\
\lesssim \varepsilon^{\Lambda - 1} \frac{n}{a_n^2} + \varepsilon^{\Lambda - 1} \frac{n}{a_n^2}. \tag{4.7}
\]

On the other hand, since \(|Q(x, u)| \lesssim |Q'(u)|\) for \(|u| \geq a_{2n},\) using Cauchy–Schwartz inequality and Theorem A.7 we have for some constant \(C > 0\)
\[
\left| \int_{|u| \geq a_{2n}} p_n(u) p_{n-1}(u) w_\rho^2(u) \overline{Q(x, u)} \, du \right| \lesssim O(e^{-nC}).
\]

Here we note that if \(\gamma \frac{a_n}{n} \leq |x| \leq \frac{1}{2} a_n,\) then
\[
\frac{A_n(x)}{b_n} \sim \frac{n}{a_n^2}.
\]

Therefore, if we take \(\theta > 0\) in (4.5) and \(\varepsilon > 0\) in (4.7) small enough then by (4.5), (4.7), and (4.8), we know that there exists \(0 < \lambda(\varepsilon) < 1\) such that
\[
\frac{|B_n(x)|}{b_n} \leq \lambda(\varepsilon) \frac{A_n(x)}{b_n}.
\]

Since \(b_n > 0,\) (4.3) follows. Therefore, Theorem 4.2 is proved completely. \(\square\)

To prove Remark 4.3, we restate it in the following:

**Remark 4.6.** Let \(w(x) \in \mathcal{F}(C^2).\) Then (4.2) and (4.3) holds for \(|x| \leq a_n(1 + L\eta_n)\) if one of the following conditions is satisfied:

(a) \(\rho \geq 0;\)
(b) \(\Lambda \geq 2;\)
(c) \(1 < \Lambda < 2,\) \(Q''(x)\) is bounded on a certain interval \((0, c);\)
(d) \(1 < \Lambda < 2,\) there exist \(0 < \delta \leq 2\) and a positive constant \(C\) such that \(Q''(x) \leq Cx^{-\delta}\) on a certain interval \((0, c)\) and \(\rho \geq \frac{\delta - 1}{2};\)
(e) \(1 < \Lambda < 2,\) \(Q''(x)\) is non-increasing a certain interval \((0, c)\) and \(\Lambda + 2\rho - 1 \geq 0.\)
**Proof.** It is sufficient to prove Lemma 4.5 for \(|x| \leq \gamma \frac{a_n}{n}\).

(a) Since \(|u|^{2p} \leq (|u| + \frac{a_n}{n})^{2p}\), we have similarly to the estimate of \(B\) in Lemma 4.5

\[
\Theta_{n, \rho}(x) \lesssim \int_{-\infty}^{\infty} (p_n w)^2(u) \left(\left|u| + \frac{a_n}{n}\right)^{2p} Q(x, u) \, du\right) \varphi_n(x) |a_n^2 - x^2|^{1/2} \lesssim 1.
\]

For the other cases we will prove that for \(|x| \leq \gamma \frac{a_n}{n}\)

\[
\int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) Q(x, u) \, du \lesssim \frac{n}{a_n^2}.
\]

This suffices for proving Lemma 4.5.

(b) For \(|x| \leq \gamma \frac{a_n}{n}\) and \(|u| \leq \frac{a_n}{2n}\) there exists a certain \(\xi\) between \(x\) and \(u\) such that we have by (d), (e) of Definition 1.1 and [3, Lemma 3.8 (3.42)]

\[
Q(x, u) = Q''(\xi) \leq C \left(\frac{Q'(\xi)}{Q(\xi)}\right)^2 \sim T(\xi) \frac{Q'(\xi)}{\xi} \\
\lesssim \frac{T(\xi)}{\xi} \frac{n}{a_n^2} \xi^{\Lambda-1} \lesssim \frac{T(\xi)}{a_n^2} \xi^{\Lambda-2} \lesssim \frac{n}{a_n^2} \lesssim \frac{n}{a_n^2}.
\]

Therefore, we have

\[
\int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) Q(x, u) \, du \lesssim \frac{n}{a_n^2} \int_{-\infty}^{\infty} (p_n w)^2(u) \, du \lesssim \frac{n}{a_n^2}.
\]

(c) Then we have by Theorem 2.3

\[
\int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) Q(x, u) \, du \lesssim \int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) \, du
\]

\[
\lesssim \frac{1}{a_n} \int |u| \lesssim |\gamma a_n / 2n (|u| + a_n / n)^{2p} \, du \lesssim \frac{1}{n} \lesssim \frac{n}{a_n^2}.
\]

(d) Let \(x, u \geq 0\). First, we obtain

\[
Q(x, u) = \frac{1}{x - u} \int_{u}^{x} Q''(s) \, ds \leq \frac{C}{x - u} \int_{u}^{x} s^{-2} \, ds = C \frac{x^{1-\delta} - u^{1-\delta}}{x - u} \leq C \frac{1}{u^{\delta}}.
\]

Then by Theorem 2.3,

\[
\int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) Q(x, u) \, du \lesssim \int |u| \lesssim |\gamma a_n / 2n \int (p_n w)^2(u) \frac{1}{u^{\delta}} \, du
\]

\[
\lesssim \frac{1}{a_n} \int |u| \lesssim |\gamma a_n / 2n (|u| + a_n / n)^{2p - \delta} \, du \lesssim \frac{1}{a_n} \left(\frac{a_n}{n}\right)^{1-\delta}
\]

\[
= \frac{n}{a_n^2} \left(\frac{a_n}{n}\right)^{2-\delta} \lesssim \frac{n}{a_n^2}.
\]
(e) Since \( \left( Q'(u) \right)' = Q''(u) - Q'(u)/u \leq 0 \) for a certain \( \zeta \in (0, u) \), we have

\[
\frac{Q'(u)}{u} - \frac{Q'(x) - Q'(u)}{x - u} = \frac{x}{u} \left( \frac{Q'(x)}{x} - \frac{Q'(u)}{u} \right) \geq 0.
\]

Then we have by Theorem 2.3 and \([3, \text{Lemma 3.8 (3.42)}]\)

\[
\int_{|u| \leq \gamma a_n/2n} \left( p_n w_\rho \right)^2 (u) Q(x, u) \, du
\leq \int_{|u| \leq \gamma a_n/2n} \left( p_n w_\rho \right)^2 (u) \frac{Q'(u)}{u} \, du
\lesssim \frac{1}{a_n} \int_{|u| \leq \gamma a_n/2n} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}} \frac{Q'(u)}{u} \, du
\lesssim \frac{1}{a_n} \frac{(n/a_n)^2}{a_n} \int_{|u| \leq \gamma a_n/2n} |u|^{\Lambda + 2\rho - 2} \, du
\lesssim \frac{n^{2-\Lambda}}{a_n^2} \lesssim \frac{n}{a_n^2}.
\]

Therefore, Lemma 4.5 for \( |x| \leq \gamma a_n/\bar{n} \) is proved. \( \square \)

The following lemma is useful.

**Lemma 4.7 (cf. Levin and Lubinsky [5, Lemma 5.2(b)]).** Let \( w(x) \in \mathcal{F}(C^2+) \). Then

\[ b_n \sim a_n. \]

**Proof.** This is similar to the proof of [5, Lemma 5.2]. By Cauchy–Schwarz inequality, Theorems A.8 and 2.3

\[ b_n := \gamma_{n-1, \rho} \gamma_{n, \rho} = \int_{-\infty}^{\infty} x p_n(x) p_{n-1}(x) w_\rho^2(x) \, dx \lesssim a_n. \]

For \( x_{jn} = x_{j,n,\rho} \neq 0 \), we know that by (4.1) and the Christoffel–Darboux formula

\[ b_n^{-1} = \lambda_{jn} \{ p_{n-1}(x_{jn}) \}^2 A_n(x_{jn}). \]

Since we know from (2.1) and (A.4) that the number of zeros of \( p_n(x) \) lying in \([a_n/4, a_n/2]\) is at least \( \lesssim n \), we have by (4.2) and Gauss-quadrature formula,

\[
\begin{align*}
\frac{nb_n^{-2}}{x_{jn} \in [a_n/4, a_n/2]} & \lesssim \sum_{x_{jn} \in [a_n/4, a_n/2]} \lambda_{jn} \{ p_{n-1}(x_{jn}) \}^2 \frac{A_n(x_{jn})}{b_n} \\
& \lesssim \frac{n}{a_n^2} \sum_{x_{jn} \in [a_n/4, a_n/2]} \lambda_{jn} \{ p_{n-1}(x_{jn}) \}^2 \\
& \lesssim \frac{n}{a_n^2} \int_{-\infty}^{\infty} p_{n-1}^2(x) w_\rho^2(x) \, dx \lesssim \frac{n}{a_n^2}.
\end{align*}
\]

Therefore, we obtain the lemma. \( \square \)
From Lemma 4.7 we obtain the following.

**Corollary 4.8.** Let \( w(x) \in \mathcal{F}(C^2+) \), \( L > 0 \) and \( \gamma > 0 \). Then there exist \( C, n_0 > 0 \) such that for \( n \geq n_0 \) and \( \gamma \frac{a_n}{n} \leq |x| \leq a_n(1 + L \eta_n) \),
\[
A_n(x) \sim a_n \varphi_n(x)^{-1} \left( a_n^2 (1 + 2 L \eta_n)^2 - x^2 \right)^{-1/2}.
\] (4.8)

Moreover, if either of the conditions in Remark 4.6 is satisfied, then (4.8) holds for \( |x| \leq a_n (1 + L \eta_n) \).

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**Appendix A.**

**Lemma A.1.** (a) For \( u > 0 \), we have
\[
a^2_u = \tilde{a}_u/2, \quad T(a_u) = 2 \tilde{T}(\tilde{a}_u/2) \quad \text{and} \quad \eta_u = 2^{-1/3} \tilde{\eta}_u/2.
\] (A.1)

(b) Uniformly for \( t \in [0, \tilde{a}_n] \) and \( u > 0 \), we have
\[
\varphi_{2u}(\sqrt{t}) \sim \frac{\tilde{\varphi}_u(t)}{(t + \tilde{a}_u u^{-2})^{1/2}}.
\] (A.2)

(c) Uniformly for \( n \) and \( x \)
\[
\varphi_{n+M}(x), \quad M > 0, \quad \varphi_n(x_{jn}) \sim \varphi_n(x_{j+1,n}), \quad 1 \leq j \leq n - 1,
\] and uniformly for \( |x| \leq \varepsilon a_n, 0 < \varepsilon < 1 \), we have
\[
\varphi_{2n}(x) \sim \varphi_n(x) \sim \frac{a_n}{n}.
\] (A.3)

(d) For some \( \varepsilon > 0 \) and for large enough \( u \),
\[
T(a_u) \leq Cu^{2-\varepsilon}.
\] (A.4)

(e) There exists \( L_0 \) such that for any fixed \( L > L_0 \) and uniformly for \( u > 0 \),
\[
1 - \frac{a_u}{a_L u} \sim \frac{1}{T(a_u)}
\] (A.5)

and
\[
T(a_L u) \sim T(a_u).
\] (A.6)

(f) For \( t > 0 \) and \( \frac{s}{2} \leq t \leq 2s \), uniformly for \( u > 0 \),
\[
\left| 1 - \frac{a_x}{a_t} \right| \sim \left| 1 - \frac{s}{t} \right| \frac{1}{T(a_s)}.
\] (A.7)

**Proof.** (a) They are [4, (2.6), (2.5), (2.9)]. (b) This is [5, (2.13)]. (c) This is easily proved by the definition of \( \varphi_n(x) \). (d) This is [3, 3, (3.38)]. (e) This is [3, Lemma 3.6(3.35), Lemma 3.5(3.29)].

(f) This is [3, (3.50)]. □
Theorem A.2 (Levin and Lubinsky [4, Theorem 1.5]). Let \( \tilde{w} \in \mathcal{L}(C^2) \). Let \( 0 < p \leq \infty \) and \( L, \lambda > 0 \). Let \( \beta > -\frac{1}{p} \) if \( p < \infty \), and \( \beta \geq 0 \) if \( p = \infty \).

(a) There exist \( C, n_0 > 0 \) such that for \( n \geq n_0 \) and \( P \in \mathcal{P}_n \),

\[ \| (P \tilde{w})(t) t^\beta \|_{L_p(R^+)} \sim \| (P \tilde{w})(t) t^\beta \|_{L_p([L\tilde{a}_n^{-2}, \tilde{d}_n(1-\tilde{d}_n)])}. \]

(b) Given \( r > 1 \), there exist \( C, n_0 > 0 \) and \( \alpha > 0 \) such that for \( n > n_0 \) and \( \alpha \in \tilde{P}_n \),

\[ \| (P \tilde{w})(t) t^\beta \|_{L_p(\mathcal{S}_{rn}, r\mathcal{S})} \leq \exp(-Cn^2) \| (P \tilde{w})(t) t^\beta \|_{L_p([0, \tilde{d}_n])}. \]

Theorem A.3 (Levin and Lubinsky [4, Theorem 1.4; 5, Theorem 1.4]). Let \( \rho > -\frac{1}{2} \) and let \( w(x) \in \mathcal{L}(C^2+) \). Then

(a) For the minimum zero \( t_{n,n,\rho} \) we have

\[ t_{n,n,\rho} \sim \tilde{a}_n n^{-2} \]

and the maximum zero \( t_{1,n,\rho} \), we have for some \( C > 0 \)

\[ 1 - \frac{t_{1,n,\rho}}{\tilde{a}_n} \sim \tilde{n}_n. \]

(b) For \( n \geq 1 \) and \( 1 \leq j \leq n - 1 \),

\[ t_{j,n,\rho} - t_{j+1,n,\rho} \sim \tilde{d}_n(t_{j,n,\rho}). \quad (A.9) \]

If we assume that \( w(x) \in \mathcal{L}(C^2) \) instead, then in (a) for some constant \( C > 0 \)

\[ t_{n,n,\rho} \sim \tilde{a}_n n^{-2}, \quad \tilde{a}_n(1 - C\tilde{n}_n) \leq t_{1,n,\rho} < \tilde{a}_n + \rho + 1/4, \]

and (b) holds with \( \sim \) replaced by \( \lesssim \).

Theorem A.4 (Levin and Lubinsky [4, Theorem 1.2]). Let \( \rho > -\frac{1}{2} \) and let \( \tilde{w} \in \mathcal{L}(C^2) \). Let \( \tilde{p}_{n,\rho}(t) \) be the \( n \)th orthonormal polynomial for the weight \( \tilde{w}^2_\rho \). Then uniformly for \( n \geq 1 \),

\[ \sup_{t \in \mathbb{R}^+} \left[ \left| \tilde{p}_{n,\rho}(t) \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^\rho \left( t + \tilde{a}_n n^{-2} \right) (\tilde{a}_n - t) \right|^{1/4} \right] \sim 1. \]

Theorem A.5 (Levin and Lubinsky [5, Theorem 1.2]). Let \( \rho > -\frac{1}{2} \) and let \( \tilde{w} \in \mathcal{L}(C^2+) \). Let \( \tilde{p}_{n,\rho}(t) \) be the \( n \)th orthonormal polynomial for the weight \( \tilde{w}^2_\rho \). Then uniformly for \( n \geq 1 \),

\[ \sup_{t \in \mathbb{R}^+} \left| \tilde{p}_{n,\rho}(t) \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^\rho \sim \left( \frac{n}{\tilde{a}_n} \right)^{1/2} \right. \]

and

\[ \sup_{t \geq \tilde{a}_n} \left| \tilde{p}_{n,\rho}(t) \tilde{w}(t) \left( t + \frac{\tilde{a}_n}{n^2} \right)^\rho \sim \tilde{a}_n^{-1/2} (nT(\tilde{a}_n))^{1/6}. \]

If \( \tilde{w} \in \mathcal{L}(C^2) \), these estimates hold with \( \sim \) replaced by \( \lesssim \).
Theorem A.6 (Levin and Lubinsky [5, Theorem 1.3]). Let \( \tilde{w} \in \mathcal{L}(C^2+) \) and \( \rho > -\frac{1}{2} \). There exists \( n_0 \) such that uniformly for \( n \geq n_0, 1 \leq j \leq n \),

(a) \[
|\tilde{p}_{n,\rho}'\tilde{w}_\rho|(t_{j,n,\rho}) \sim \tilde{\varphi}(t_{j,n,\rho})^{-1}[t_{j,n,\rho}(\bar{a}_n - t_{j,n,\rho})]^{-1/4}.
\]

(b) \[
|\tilde{p}_{n-1,\rho}\tilde{w}_\rho|(t_{j,n,\rho}) \sim \bar{a}_n^{-1}[t_{j,n,\rho}(\bar{a}_n - t_{j,n,\rho})]^{1/4}.
\]

(c) \[
\max_{t \in \mathbb{R}^+} \left| \tilde{y}_{j,n,\rho}(t)\tilde{w}(t) \left( t + \frac{\bar{a}_n}{n^2} \right) \right| \tilde{w}^{-1}_\rho(t_{j,n,\rho}) \sim 1.
\]

(d) For \( j \leq n - 1 \) and \( t \in [t_{j+1,n,\rho}, t_{j,n,\rho}] \),

\[
|\tilde{p}_{n,\rho}\tilde{w}_\rho|(t) \sim \min\{ |t - t_{j,n,\rho}|, |t - t_{j+1,n,\rho}| \} \tilde{\varphi}(t_{j,n,\rho})^{-1}[t_{j,n,\rho}(\bar{a}_n - t_{j,n,\rho})]^{-1/4}.
\]

If we assume instead that \( \tilde{w} \in \mathcal{L}(C^2) \), then (a) holds with \( \sim \) replaced by \( \approx \) and (b) holds with \( \sim \) replaced by \( \approx \).

Theorem A.7 (Jung and Sakai [1, Theorem 2.4]). Let \( w(x) \in \mathcal{F}(C^2), 0 < p < \infty \) and \( L \geq 0 \). Let \( \beta \in \mathbb{R} \). Then given \( r > 1 \), there exists a positive constant \( C_2 \) such that for any polynomial \( P \in \mathcal{P}_n \)

\[
\| (Pw_\beta)(x) \|_{L_p(\alpha_n \leq |x|)} \leq \exp(-C_2n^p) \| (Pw_\beta)(x) \|_{L_p(L_{an/n} \leq |x| \leq \alpha_n(1 - L\eta_n))}.
\]

Theorem A.8 (Jung and Sakai [1, Theorem 2.5]). Let \( w(x) \in \mathcal{F}(C^2), 0 < p < \infty, \beta \in \mathbb{R}, \) and \( L \geq 0 \). Then we have for any polynomial \( P \in \mathcal{P}_n \),

\[
\left\| (Pw)(x) \left( |x| + \frac{\alpha_n}{n} \right)^\beta \right\|_{L_p(\mathbb{R})} \lesssim \left\| (Pw)(x) \left( |x| + \frac{\alpha_n}{n} \right)^\beta \right\|_{L_p(L_{an/n} \leq |x| \leq \alpha_n(1 - L\eta_n))}.
\]

Theorem A.9 (Jung and Sakai [1, Theorem 2.7]). Let \( \rho > -1/p, 0 < p < \infty \) and let \( w(x) \in \mathcal{F}(C^2) \).

(a) Let \( L > 0 \). Then uniformly for \( n \geq 1 \) and \( |x| \leq \alpha_n(1 + L\eta_n) \), we have

\[
\lambda_{np}(w_\rho; x) \sim \varphi_n(x)w^\rho(x) \left( |x| + \frac{\alpha_n}{n} \right)^{\rho p}.
\]

(b) Moreover, uniformly for \( n \geq 1 \) and \( x \in \mathbb{R} \),

\[
\lambda_{np}(w_\rho; x) \gtrsim \varphi_n(x)w^\rho(x) \left( |x| + \frac{\alpha_n}{n} \right)^{\rho p}.
\]

References

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