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# Seshadri constants and the generation of jets

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ABSTRACT

In this paper we explore the connection between Seshadri constants and the generation of jets. It is well known that one way to view Seshadri constants is to consider them as measuring the rate of growth of the number of jets that multiples of a line bundle generate. Here we ask, conversely, what we can say about the number of jets once the Seshadri constant is known. As an application of our results, we prove a characterization of projective space among all Fano varieties in terms of Seshadri constants.



## 0. Introduction

Consider a smooth projective variety $X$ and an ample line bundle $L$ on $X$. The *Seshadri constant* of $L$ at a given point $x \in X$ is the real number

$$\varepsilon(L, x) \stackrel{\text{def}}{=} \inf_C \frac{L \cdot C}{\operatorname{mult}_x C},$$

where the infimum is taken over all irreducible curves $C$ passing through $x$. It is well known that $\varepsilon(L, x)$ encodes asymptotic information about the linear series $|kL|$ for $k \gg 0$. Specifically, denote for $k \geqslant 1$ by $s(kL, x)$ the maximal integer $s$ such that the linear series $|kL|$ generates $s$-jets at $x$, i.e., the maximal integer $s$ such that the evaluation map

$$H^0(X, kL) \longrightarrow H^0(X, kL \otimes \mathcal{O}_X / \mathcal{I}_x^{s+1})$$

is onto. Then one has

$$\varepsilon(L, x) = \lim_{k\to\infty} \frac{s(kL, x)}{k}$$

(see [5, 6.3]). So if the numbers $s(kL, x)$ were known for $k \gg 0$, then one could compute the Seshadri constant as a limit of a sequence of rational numbers. In all situations that we know of, however, Seshadri constants have not been determined in this way, but rather by finding suitably singular curves. It is therefore interesting to ask:

*When the value $\varepsilon(L, x)$ is known, what can be said about the numbers $s(kL, x)$ for $k \geqslant 1$ ?*

It seems to us that this question is close in spirit to Demailly's original purpose when defining Seshadri constants in [5].

Our first result shows that under certain non-positivity assumptions on the canonical divisor of $X$ the range for the numbers $s(kL, x)$ is quite restrictive.

**Theorem 1.** *Let $X$ be a smooth projective variety of dimension $n$.*

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(a) *If $X$ is Fano, then for any integer $k \geqslant 1$*

$$\lfloor (k+1)\,\varepsilon(-K_X, x)\rfloor - (n+1) \leqslant s(k(-K_X), x) \leqslant \lfloor k\,\varepsilon(-K_X, x)\rfloor\,.$$

(b) *If $K_X = \mathcal{O}_X$, then for any ample line bundle $L$ on $X$, any point $x \in X$, and any integer $k \geqslant 1$ we have*

$$\lfloor k\,\varepsilon(L, x)\rfloor - (n+1) \leqslant s(kL, x) \leqslant \lfloor k\,\varepsilon(L, x)\rfloor\,.$$

Note that the upper bound – which in fact holds in both parts without any assumptions on $X$ – is well known and is stated here merely for the sake of completeness. We will show in Proposition 2.3 that $s(kL, x)$ can attain this upper bound only if $\varepsilon(L, x)$ is either computed by a smooth curve or if it is not computed by a curve at all. As for the lower bounds, we will prove somewhat stronger statements in Propositions 1.1 and 2.1 respectively.

The above bounds on the numbers $s(kL, x)$ lead in particular to the following new characterization of projective spaces.

**Theorem 2.** *Let $X$ be a smooth Fano variety of dimension $n$ such that there exists a point $x \in X$ with*

$$\varepsilon(-K_X, x) = n + 1.$$

*Then $X$ is the projective space $\mathbb{P}^n$.*

We will in fact show that on Fano varieties different from $\mathbb{P}^n$ one has $\varepsilon(-K_X, x) \leqslant n$ for all points $x$ (Theorem 1.7).

## 1. Fano varieties

We begin by considering jets of the anticanonical bundle on Fano varieties.

**Proposition 1.1.** *Let $X$ be a smooth Fano variety of dimension $n$, and let $x \in X$. Let $\varepsilon = \varepsilon(-K_X, x)$.*
*If $\varepsilon < \sqrt[n]{(-K_X)^n}$ or if $\sqrt[n]{(-K_X)^n}$ is not an integer, then*

$$\lfloor (k+1)\,\varepsilon\rfloor - n \leqslant s(k(-K_X), x) \leqslant \lfloor k\,\varepsilon\rfloor\,.$$

*In the alternative case, one has*

$$(k+1)\varepsilon - (n+1) \leqslant s(k(-K_X), x) \leqslant k\varepsilon.$$

**Proof.** The upper bound – which in fact holds without any assumption on $K_X$ – follows from the fact that $\varepsilon(L, x)$ is not only the limit, but also the *supremum* of the numbers $\frac{1}{k}s(kL, x)$, see [5, (6.3)].

The lower bound is proven via vanishing as in [10, Proposition 5.1.19(i)]: One considers the blow-up $f : X' \to X$ of $X$ at $x$ with exceptional divisor $E$ over $x$. For $|k(-K_X)|$ to generate $s$-jets at $x$ it is enough to have $H^1(kf^*(-K_X) - (s+1)E) = 0$. As $K_{X'} = f^*(K_X) + (n-1)E$, this vanishing will follow if the line bundle $(k+1)f^*(-K_X) - (s+n)E$ is nef and big. Let us write $\varepsilon = \varepsilon(-K_X, x)$, and suppose first that $\varepsilon < \sqrt[n]{(-K_X)^n}$. In that case $(k+1)f^*(-K_X) - (s+n)E$ is nef and big as long as $\frac{s+n}{k+1} \leqslant \varepsilon$. Therefore

$$s(k(-K_X), x) \geqslant \lfloor (k+1)\varepsilon\rfloor - n. \tag{1}$$

Suppose then that $\varepsilon = \sqrt[n]{(-K_X)^n}$. In that case the line bundle in question is ample if $\frac{s+n}{k+1} < \varepsilon$, i.e, if $s < (k+1)\varepsilon - n$. Now, if $\varepsilon$ is not an integer, then this inequality is equivalent to $s \leqslant \lfloor (k+1)\varepsilon\rfloor - n$, so that we get again (1). Finally, if $\varepsilon = \sqrt[n]{(-K_X)^n}$ is an integer, then we get

$$s(k(-K_X), x) \geqslant \lfloor (k+1)\varepsilon\rfloor - (n+1) = (k+1)\varepsilon - (n+1),$$

as claimed. □

**Example 1.2** (*Projective Space*). For $X = P^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ one has $\varepsilon(L, x) = 1$ for all $x \in X$ and $s(kL, x) = k$ for all $k$. So here the value

$$s(k(-K_X), x) = k(n+1)$$

lies at the upper bound given by Proposition 1.1. We will show that projective spaces are the only Fano varieties where this happens (Theorem 1.7).

The proposition leads immediately to a surprising upper bound on Seshadri constants:

**Corollary 1.3.** *For any smooth Fano variety $X$, one has*

$$\varepsilon(-K_X, x) \leqslant n + 1$$

*for all $x \in X$. If $\varepsilon(-K_X, x) < \sqrt[n]{(-K_X)^n}$ or if $\sqrt[n]{(-K_X)^n}$ is not an integer, then the stronger inequality*

$$\varepsilon(-K_X, x) \leqslant n$$

*holds.*

**Proof of the Corollary.** This follows from the inequalities in the proposition. In fact, it is enough to show that if $\varepsilon$ is a positive real number and $b$ is an integer such that

$$\lfloor (k+1)\,\varepsilon \rfloor - b \leqslant \lfloor k\,\varepsilon \rfloor \tag{2}$$

holds for all $k \geqslant 1$, then $\varepsilon \leqslant b$. This latter assertion is obvious when $\varepsilon$ is an integer. If $\varepsilon = e + \delta$ with an integer $e$ and $0 < \delta < 1$, then there is an integer $k \geqslant 1$ such that $k\delta < 1$ and $(k+1)\delta \geqslant 1$. We then have

$$\lfloor (k+1)\varepsilon \rfloor - b = (k+1)e + 1 - b \quad \text{and} \quad \lfloor k\varepsilon \rfloor = ke$$

so that (2) implies $\varepsilon \leqslant b$, as claimed. □

**Remark 1.4.** The upper bound of $n+1$ in Corollary 1.3 follows also from the fact that for every point $x$ on a smooth Fano variety $X$ of dimension $n$ there is a rational curve $C$ passing through $x$ and such that $-K_X \cdot C \leqslant n+1$. This is a deep fact proved by Mori and Kollár (see [9, Theorem V.1.6]). By contrast, our argument is fairly elementary.

As a further consequence of Proposition 1.1, we obtain the following characterization of $\mathbb{P}^n$ via Seshadri constants. Its statement will be strengthened considerably in Theorem 1.7.

**Corollary 1.5.** *Let $X$ be a smooth Fano variety such that*

$$\varepsilon(-K_X, x) = n+1$$

*for all $x \in X$. Then $X \cong \mathbb{P}^n$.*

**Proof.** According to the previous result it is enough to show that the condition $\varepsilon(-K_X, x) = \sqrt[n]{(-K_X)^n} = n+1$ implies that $X = \mathbb{P}^n$. This can be seen as follows. From the assumption and Proposition 1.1 we get

$$s(k(-K_X), x) = k\,\varepsilon(-K_X, x) = k(n+1).$$

Putting $s = s(k(-K_X), x)$ a result of Beltrametti and Sommese [3, Theorem 3.1] implies then that

$$(k(-K_X))^n \geqslant s^n + s^{n-1} \tag{3}$$

unless $X \cong \mathbb{P}^n$. Bearing in mind that

$$(k(-K_X))^n = k^n(n+1)^n$$

we get a contradiction with (3), unless $X \cong \mathbb{P}^n$. (Note that the cited result in [3] assumes the line bundle in question to be *s-jet ample*, whereas in our situation we only know that the bundle generates $s$-jets at all points. The proof in [3] works, however, under this weaker assumption.) □

**Remark 1.6.** For the proof of Corollary 1.5 one could also invoke the following characterization of the projective space conjectured in [9, Conjecture V.1.7] and proved by Kebekus in [8]:

> *A smooth projective variety $X$ of dimension $n$ is isomorphic to the projective space $\mathbb{P}^n$ if and only if it is Fano and $-K_X \cdot C \geqslant n+1$ for every rational curve $C \subset X$.*

The assumptions of the previous corollary imply that $-K_X$ generates $n+1$ jets at every point $x \in X$ so that the inequality $-K_X \cdot C \geqslant n+1$ is fulfilled for an arbitrary curve $C$ on $X$.

Now we show a considerably stronger version of Corollary 1.5. Of course this result implies Theorem 2 stated in the introduction.

**Theorem 1.7.** *Let $X$ be a Fano variety such that $X \ncong \mathbb{P}^n$, then the inequality*

$$\varepsilon(-K_X, x) \leqslant n$$

*holds for all points $x \in X$.*

In view of Corollary 1.3 we need to show the following statement:

If $(-K_X)^n = (n+1)^n$ and if $\varepsilon(-K_X, x) = n+1$ for some point $x \in X$, then $X \cong \mathbb{P}^n$.

In the surface case it is easy to verify the above statement because we know exactly all Fano surfaces.

**Example 1.8** (*Del Pezzo Surfaces*). Let $X$ be a Del Pezzo surface, i.e., a smooth Fano variety of dimension two. Then

$$\varepsilon(-K_X, x) \leqslant 2 \tag{4}$$

for all $x \in X$, unless $X \cong \mathbb{P}^2$. In fact, $X$ is either

(i) $\mathbb{P}^1 \times \mathbb{P}^1$, or
(ii) the blow-up of $\mathbb{P}^2$ in $r$ points, with $0 \leqslant r \leqslant 8$.

In Case (i) we have $K_X = \mathcal{O}(-2,-2)$ so that $\varepsilon(-K_X, x) = 2$ for all $x \in X$. In Case (ii) we have $(-K_X)^2 = 9 - r$, and therefore the number $\sqrt{(-K_X)^2}$ is an integer only if $r = 0$, $r = 5$, or $r = 8$. If $r = 0$ then $X = \mathbb{P}^2$, and if $r = 5$ or $r = 8$ then $\sqrt{(-K_X)^2} = 2$ or $\sqrt{(-K_X)^2} = 1$ respectively. So we get the inequality (4) by applying Corollary 1.3. Broustet [4] has recently determined the precise values of $\varepsilon(-K_X, x)$ for $0 \leqslant r \leqslant 8$.

Now we give a proof of Theorem 1.7 valid in arbitrary dimension. This proof is motivated by the methods of [6].

**Proof of Theorem 1.7.** As in the proof of 1.5 we get

$$s((-K_X), x) = \varepsilon(-K_X, x) = n + 1$$

in the fixed point $x$. Looking at the exact sequences defining bundles of $k$ jets of $-K_X$

$$0 \to \mathrm{Sym}^k \Omega_X \otimes (-K_X) \to J_k(-K_X) \to J_{k-1}(-K_X) \to 0$$

for $k = 1, \ldots, n+1$ and computing inductively we obtain

$$c_1(J_{n+1}(-K_X)) = \mathcal{O}_X.$$

Since by assumption the vector bundle $J_{n+1}(-K_X)$ is globally generated at the point $x$ and its determinant is trivial, it follows that the bundle itself is trivial. (This is because the determinant of global sections generating at $x$ does not vanish anywhere.)

Now, let $f : \mathbb{P}^1 \to X$ be a rational curve on $X$ (i.e., the map $f$ is non-constant). Let

$$f^*(T_X) = \bigoplus_{i=1}^{n} \mathcal{O}(a_i) \quad \text{and} \quad f^*(-K_X) = \mathcal{O}(b).$$

Note that $b > 0$ since $-K_X$ is ample. Dualizing the defining exact sequence for $(n+1)$th jets we have

$$0 \to J_n(-K_X)^* \to J_{n+1}(-K_X)^* \to \mathrm{Sym}^{n+1} T_X \otimes K_X \to 0.$$

The bundle in the middle is trivial, it is in particular globally generated, hence the same is true for its quotient on the right. We write

$$f^*(\mathrm{Sym}^{n+1} T_X) = \left( \bigoplus_{i=1}^{n} \mathcal{O}((n+1)a_i) \right) \oplus P,$$

where $P$ abbreviates the remaining summands. Thus

$$f^*(\mathrm{Sym}^{n+1} T_X \otimes K_X) = \left( \bigoplus_{i=1}^{n} \mathcal{O}((n+1)a_i - b) \right) \oplus P(-b).$$

It follows that

$$(n+1)a_i - b \geqslant 0,$$

which in view of $b > 0$ implies $a_i > 0$ for all $i = 1, \ldots, n$ and we conclude by the Mori characterization of projective space [9, Theorem V.3.2]. □

## 2. Varieties with trivial canonical bundle

We consider now varieties whose canonical bundle is trivial. A straightforward modification of the proof of Proposition 1.1 yields the following statement:

**Proposition 2.1.** *Let $X$ be a smooth projective variety of dimension $n$ such that $K_X = \mathcal{O}_X$, let $L$ be an ample line bundle on $X$ and $x \in X$ a point.*

*If $\varepsilon(L, x) < \sqrt[n]{L^n}$ or if $\sqrt[n]{L^n}$ is not an integer, then one has for every integer $k \geqslant 1$ the inequalities*

$$\lfloor k\,\varepsilon(L, x) \rfloor - n \leqslant s(kL, x) \leqslant \lfloor k\,\varepsilon(L, x) \rfloor .$$

*In the alternative case (where $\varepsilon(L, x) = \sqrt[n]{L^n}$ is an integer) one has*

$$k\sqrt[n]{L^n} - (n+1) \leqslant s(kL, x) \leqslant k\sqrt[n]{L^n}.$$

So there are only $n+1$ potential values of $s(kL, x)$ in the first case, and $n+2$ potential values in the second case. This means that there is surprisingly little room for the numbers $s(kL, x)$.

**Example 2.2.** Consider a smooth quartic surface $X \subset \mathbb{P}^3$ containing a line $\ell$. Then for $L = \mathcal{O}_X(1)$ and $x \in \ell$ one has $\varepsilon(L, x) = 1$, and

$$s(kL, x) = k = k\,\varepsilon(L, x).$$

So in this case $s(kL, x)$ has its maximal possible value for all $k \geqslant 1$.

The following proposition gives interesting constraints on maximal (in the sense of Proposition 2.1) values of the numbers $s(kL, x)$.

**Proposition 2.3.** *Let $X$ be a smooth projective variety, let $L$ be an ample line bundle on $X$, and let $x \in X$ be any point. If $\varepsilon(L, x)$ is computed by a curve, i.e., if there is a curve $C \subset X$ such that*

$$\varepsilon(L, x) = \frac{L \cdot C}{\mathrm{mult}_x(C)},$$

*then*

(i) $s(kL, x) < k\,\varepsilon(L, x)$ *for all* $k \geqslant 1$, *or*
(ii) *$C$ is smooth at $x$.*

*If* $\dim(X) = 2$, *then in Case* (ii) *one has* $C^2 \leqslant 1$. *If in addition* $\varepsilon(L, x) < \sqrt{L^2}$, *then* $C^2 \leqslant 0$.

In other words, if one has $s(kL, x) = k\,\varepsilon(L, x)$ for *some* $k$, then the Seshadri constant $\varepsilon(L, x)$ cannot be computed by a curve that is singular at $x$.

Let us point out the following sample application of Proposition 2.3.

**Corollary 2.4.** *Let $X$ be an abelian surface of Picard number one. Then for any ample line bundle $L$ on $X$, any $x \in X$, and any integer $k \geqslant 1$, one has*

$$s(kL, x) < k\,\varepsilon(L, x).$$

This follows from the fact that in the situation of the corollary one knows from [1, Sect. 6] that $\varepsilon(L, x)$ is computed by a singular curve.

**Proof of Proposition 2.3.** Let $k \geqslant 1$, and write $s = s(kL, x)$, $\varepsilon = \varepsilon(L, x)$, and $m = \mathrm{mult}_x(C)$. As $kL$ generates $s$-jets at $x$, there is a divisor $D \in |kL|$ with $\mathrm{mult}_x(D) = s$ and with prescribed tangent cone at the point $x$. So if $C$ is not smooth at $x$, then we can arrange that the projective tangent cones

$$\mathbb{P}TC_x(D) \quad \text{and} \quad \mathbb{P}TC_x(C)$$

intersect, while still $C \not\subset D$. Then by the intersection inequality [7, Corollary 12.4] we have

$$D \cdot C \geqslant s \cdot m + 1,$$

and hence

$$\varepsilon = \frac{L \cdot C}{m} = \frac{1}{k}\frac{D \cdot C}{m} \geqslant \frac{s}{k} + \frac{1}{km},$$

which implies

$$s \leqslant \varepsilon k - \frac{1}{m} < \varepsilon k.$$

This proves the first assertion.

Suppose now that $\dim(X) = 2$, and that we are in Case (ii). Then by the index theorem

$$L^2C^2 \leqslant (L \cdot C)^2 = \varepsilon^2 \leqslant L^2\,,$$

and hence $C^2 \leqslant 1$. If $\varepsilon < \sqrt{L^2}$, then the last inequality is strict, and we get $C^2 \leqslant 0$. □

While the upper bound in Proposition 2.1 holds without any assumptions, the following example shows that one cannot expect a lower bound valid for all $k \geqslant 1$ without additional assumptions on the underlying variety.

**Example 2.5.** Let $k_0$ be a positive integer. We show that there exist a smooth projective surface $X$ and an ample line bundle $L$ on $X$ such that for all $k = 1, \ldots, k_0$ we have $s(kL, x) = -1$ for every point $x \in X$, whereas $\varepsilon(L, x) = 1$.

To this end let $C$ be a curve of genus $g > k_0$ and let $D$ be a general divisor of degree 1 on $C$. Then $h^0(C, D) = h^0(C, 2D) = \cdots = h^0(C, k_0D) = 0$. Now let $X = C \times C$ be the product with projections $\pi_1$ and $\pi_2$. We set $L = \pi_1^*(D) \otimes \pi_2^*(D)$. The line bundle $L$ is ample and $\varepsilon(L, x) = 1$. By the Künneth formula we see that $h^0(X, L) = \cdots = h^0(X, k_0L) = 0$ and hence $s(kL, x) = -1$ for $k \leqslant k_0$, as claimed.

The next example shows that even in the surface case one cannot expect lower bounds on $s(kL, x)$ as in Proposition 2.1 without assumptions on $K_X$: The numbers $s(kL, x)$ may in fact be smaller than the lower bound of Proposition 2.1 for *all* values of $k$.

**Example 2.6.** Let $X$ be a smooth surface of degree 9 in $\mathbb{P}^3$ with $\rho(X) = 1$. For the line bundle $L = \mathcal{O}_X(1)$ one has $\varepsilon(L, x) = \left\lfloor \sqrt{L^2} \right\rfloor = 3$ for very general $x \in X$, by Steffens result [11]. We assert that

$$s(kL, x) < k\varepsilon(L, x) - 4.$$

for $k \gg 0$. In fact, if a line bundle generates $s$-jets at some point, then it must have at least $\binom{s+2}{2}$ independent global sections. But by Riemann–Roch we have

$$h^0(kL) = \chi(\mathcal{O}_X) + \frac{9}{2}k(k-5),$$

and this is for $k \gg 0$ not enough in order to generate jets of order $3k - 1 = k\,\varepsilon(L, x) - 4$. Note that the same argument works for any smooth surface $X \subset \mathbb{P}^3$ with $\rho(X) = 1$, whose degree is a square number $\geqslant 9$.

We now consider concrete applications of Proposition 2.1.

**Example 2.7** (*Jets of Theta Functions*). Consider an irreducible principally polarized abelian surface $(X, \Theta)$. One knows that $\varepsilon(\Theta, x) = \frac{4}{3}$ for every point $x \in X$ (see [11] and [1, Sect. 6]), hence Proposition 2.1 tells us that

$$\left\lfloor \frac{4}{3}k \right\rfloor - 2 \leqslant s(k\Theta, x) \leqslant \left\lfloor \frac{4}{3}k \right\rfloor$$

for every $k \geqslant 1$. So there are for each $k$ only three possible values that $s(k\Theta, x)$ can have. Here the possibilities are tabulated for small values of $k$:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s(kL, x)$ | 1 | 2 | 4 | 5 | 6 | 8 | 9 | 10 | 12 | 13 |
| | 0 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| | −1 | 0 | 2 | 3 | 4 | 6 | 7 | 8 | 10 | 11 |

Of course it is very hard to determine what the exact values are – especially as, contrary to the value of $\varepsilon(\Theta, x)$, they might and do depend on the point $x$. Specifically, as $|\Theta| = \{\Theta\}$, one has $s(\Theta, x) = 0$ if $x \notin \Theta$, and $s(\Theta, x) = -1$ if $x \in \Theta$. As for $s(2\Theta, x)$, one knows that the linear series $|2\Theta|$ defines a map $X \to \mathbb{P}^3$ of degree 2 onto the Kummer surface, hence $s(2\Theta, x) \geqslant 1$ for generic $x$, but $s(2\Theta, x) = 0$ for points mapped onto double points of the Kummer quartic in $\mathbb{P}^3$.

Counting sections carefully we can in fact rule out several values in the above table. Below we present the remaining possibilities. Values marked in bold are actually taken on.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s(kL, x)$ | **0** | **1** | | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| | **-1** | **0** | **2** | 3 | 4 | 6 | 7 | 8 | 10 | 11 |

There are (at least) two things that would be interesting to know in this context:

- Is $s(k\Theta, x)$ independent of $x$ when $k \gg 0$ ?
- Is the sequence of numbers $s(k\Theta, x)$ (for large $k$) the same for every principally polarized abelian surface $(X, \Theta)$ or does it depend on the moduli?

In the following two examples, yet more precise statement about the numbers $s(kL, x)$ can be obtained.

**Example 2.8** (*Abelian Surfaces of Type* $(1, 2)$). It can happen for certain line bundles that $s(kL, x)$ is not only below $k\,\varepsilon(L, x)$, as predicted by Proposition 2.3, but even below $k\,\varepsilon(L, x) - 1$ for *all* values of $k$. Consider for instance an abelian surface $X$ of Picard number 1 carrying a polarization $L$ of type $(1, 2)$. Then $L^2 = 4$ and, by [1, Sect. 6], $\varepsilon(L) = 2$. A line bundle that generates $s$-jets at some point must have at least $\binom{s+2}{2}$ independent global sections. As we have by Riemann–Roch $h^0(kL) = 2k^2$, this implies that $kL$ cannot generate $2k$-jets or $(2k - 1)$-jets at any point. So

$$s(kL, x) = 2k - 3 \quad \text{or} \quad s(kL, x) = 2k - 2$$

for every $x \in X$.

**Example 2.9** (*Quartic Surfaces*). Let $X$ be a smooth quartic surface in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$. For general $x \in X$ one has $\varepsilon(L, x) = 2$ by [2, Theorem]. As $h^0(kL) = 2 + 2k^2$, the dimension argument of the previous example gives the same conclusion as there:

$$s(kL, x) = 2k - 3 \quad \text{or} \quad s(kL, x) = 2k - 2.$$

The two previous examples are special cases of:

**Proposition 2.10.** *Let $X$ be a smooth projective surface with $K_X = \mathcal{O}_X$, and let $L$ be an ample line bundle on $X$. Suppose that $\varepsilon(L, x) = \sqrt{L^2}$ at a fixed point $x \in X$.*

*If $\sqrt{L^2}$ is an integer, then we have for $k \gg 0$*

$$s(kL, x) = k\sqrt{L^2} - 3 \quad \textit{or} \quad s(kL, x) = k\sqrt{L^2} - 2.$$

*and in the alternative case we have*

$$s(kL, x) = \left\lfloor k\sqrt{L^2} \right\rfloor - 2,$$

*for infinitely many values of $k$.*

**Proof.** For the linear series $|kL|$ to generate $s$-jets at $x$, the line bundle $kL$ needs to have at least $\binom{s+2}{2}$ independent global sections. By the Riemann–Roch theorem we have for $k \gg 0$

$$h^0(kL) = \chi(\mathcal{O}_X) + \frac{1}{2}L^2k^2.$$

If $\varepsilon(L, x) = \sqrt[n]{L^n}$ is an integer, then one finds with elementary calculations that the inequality $h^0(kL) \geqslant \binom{s+2}{2}$ cannot hold for $k \gg 0$ when $s \geqslant \lfloor k\,\varepsilon(L, x) \rfloor - 1$. The assertion follows then from Proposition 2.1. In the remaining case one has to work with the rounddown and therefore the assertion is somewhat weaker. $\square$

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