1. Introduction

For a directed graph $G$, a path $\mu$ is a directed path with no repeated vertex, and here $\mu$ denotes the set of the vertices encountered by $\mu$. Consider a partition $M=\{\mu_1, \mu_2, \ldots, \mu_p\}$ of the vertex-set into paths. Let $k$ be an integer, $k \geq 1, k \leq \max|\mu|$; put:

$$B_k(M) = \sum_{i=1}^{p} \min\{k, |\mu_i|\}.$$

A partition $M$ is $k$-optimal if $M$ minimizes $B_k(M)$. For instance, if $G$ has a Hamiltonian path $\mu_0$, then $M=\{\mu_0\}$ is a $k$-optimal partition. So the $k$-optimal partitions extend in some sense the concept of Hamiltonian paths. The theorem of Greene and Kleitman [10], which extends the Dilworth theorem [5], shows an important property of $k$-optimal partitions for the graph of a partially ordered set (in [1, 5], the $k$-optimal partitions are called "$k$-saturated"). This paper shows that similar properties also hold for several classes of graphs.

2. Strongly Coloured Paths

Let $G=(X, U)$ be a directed graph. For $k \geq 1$, a partial $k$-colouring is a family of $k$ disjoint stable sets $S_1, S_2, \ldots, S_k$. If a vertex $x$ belongs to $S_\mu$, we say that $x$ is coloured with $\mu$; some of the vertices may bear no colour.

Clearly, for a partial $k$-colouring $(S_1, S_2, \ldots, S_k)$, the number of different colours encountered by a path $\mu$ is $\leq \min\{k, |\mu|\}$. A path $\mu$ is strongly coloured if it meets exactly $\min\{k, |\mu|\}$ different colours. In this case we say also that the colouring is strong for $\mu$.

Conjecture A. Given a graph $G$ and an integer $k$, $1 \leq k \leq \max|\mu|$, for every $k$-optimal partition $M$ there exists a partial $k$-colouring $(S_1, S_2, \ldots, S_k)$ which is strong for every path $\mu \in M$.

Let $M$ be a path partition, let $\mu \in M, x \in \mu$; the vertex $x$ is said to be at level $k$ if the portion of $\mu$ which starts from $x$ contains $k$ vertices. So the vertices at level 1 are the terminal ends of paths. Let $L_k(M)$ be the set of vertices at level $k$. Clearly,

$$B_k(M) = |L_1(M)| + |L_2(M)| + \cdots + |L_k(M)|.$$

Theorem 1. Conjecture A is valid for $k = 1$. Furthermore, if there exists a partition $M_0$ with $L_1(M_0) = L_1$, then for every partition $M$ with $L_1(M) \subset L_1$ and $|M|$ minimum relative to this condition, there exists a stable set $S$ which meets each path of $M$ exactly once.

Gallai and Milgram [8] proved that if the maximum size of a stable set is $\alpha$, it is always possible to partition the vertex-set into $\alpha$ paths. By the same argument, one can prove a stronger version which is equivalent to Theorem 1 (Linial, [11]). In [2], we gave a proof for a stronger result which immediately yields Theorem 1.

Gallai [7] and Roy [13] have proved that all the vertices can be coloured with only $\max|\mu|$ colours. A slightly stronger result can be proved by the same argument.
Theorem 2. Conjecture A is valid for \( k = \max |\mu| \); furthermore, every partition is \( k \)-optimal.

Proof. Let \( G \) be a graph of order \( n \) with \( \max |\mu| = k \). Since \( B_k(M) \) is equal to \( n \) for all \( M \), every partition is \( k \)-optimal.

We shall define a partial graph \( H \) of \( G \) by adding successively, to the arcs which belong already to the paths of \( M \), some arcs of \( G \), provided they do not create circuits. When no more arcs can be added, we obtain a partial graph \( H \) which is acyclic. For \( x \in X \), put

\[
t(x) = \max \{|\mu|/\mu \text{ is a path of } H \text{ starting from } x\}.
\]

If \((x, y)\) is an arc of \( H \), then \( t(x) > t(y) \).
If \((x, y)\) is an arc of \( G - H \), then \( t(x) < t(y) \) because \( H + (x, y) \) contains a circuit, so \( H \) contains a path from \( y \) to \( x \).

Thus the function \( t(x) \) is a \( k \)-colouring of \( G \) (i.e. \( (x, y) \in U \Rightarrow t(x) \neq t(y) \)). Clearly, this \( k \)-colouring is strong for \( M \).

Now we consider the class of graphs satisfying the following property:

Property B. The maximum number of vertex-disjoint paths of cardinality \( k = \max |\mu| \) is equal to the minimum size of a set \( A \subset X \) which meets every path of cardinality \( k \).

Not all the graphs satisfy this property (see for instance the hypotraceable graph of Thomasen in [3, p. 240]).

Lemma. Every graph without circuits satisfies Property B.

For a vertex \( x \) of \( G \), let \( \lambda(x) \) be the maximum cardinality of a path issuing from \( x \). We construct a transportation network \( R \) as follows: add to the vertices of \( G \) a source \( a \) and a sink \( z \); for every vertex \( x \) with \( \lambda(x) = k \), draw the arc \((a, x)\); for every vertex \( y \) with \( \lambda(y) = 1 \), draw the arc \((y, z)\). Finally, draw every arc \((x', x'')\) of \( G \) which satisfies \( \lambda(x') = \lambda(x'') + 1 \). In \( R \), every path from the source to the sink is a path with \( k \) inner vertices, and, by the Lemma of Menger's Theorem (see [1, p. 161]), the maximum number of disjoint paths from \( a \) to \( z \) is equal to the minimum size of a separating set. The theorem follows.

Remark. For \( k = 2 \), this proposition is equivalent to the Theorem of König for bipartite graphs.

Theorem 3. Let \( G \) be a graph with no circuits—or, more generally, a graph which satisfies Property B. Then Conjecture A is valid for \( k = \max |\mu| - 1 \).

Proof. Let \( M \) be a path-partition which minimizes \( B_k(M) \), i.e. which maximizes \( |L_{k+1}(M)| \). The paths of \( M \) issuing from \( L_{k+1}(M) \) constitute a maximum set of pairwise disjoint maximum paths (otherwise, there exists a partition \( M' \) with \( |L_{k+1}(M')| > |L_{k+1}(M)| \), a contradiction). So, by the Lemma, there exists a set \( A \) with \( |A| = |L_{k+1}(M)| \) such that each path of \( M \) issuing from \( L_{k+1}(M) \) has exactly one point in \( A \). Let \( \tilde{G} \) be the graph obtained from \( G \) by removing the points of \( A \) and by adding, for every \( a \in A \), an arc \((z, z')\) where \( z \) (resp. \( z' \)) is the point which precedes (resp. follows) \( a \) in some \( \mu \in M \). There exists in \( \tilde{G} \) no path \( \tilde{\mu} \) of cardinality \( k + 1 \) (otherwise \( \tilde{\mu} \), containing an arc \((z, z')\)
as defined above, induces in $G$ a path $\mu$ of cardinality $k + 1$, which is a contradiction).

By Theorem 2, $\tilde{G}$ has a $k$-colouring which is strong for the paths $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_p \in \tilde{M}$. This $k$-colouring is also a partial $k$-colouring of $G$ which is strong for the paths $\mu_1, \mu_2, \ldots, \mu_p$.

**Corollary.** Let $G$ be a graph with $\max |\mu| = 3$. Then Conjecture A is valid for every $k$.

**Proof.** If $k = 1$, the result follows from Theorem 1; if $k = 3$, the result follows from Theorem 2; if $k = 2$, we may assume that $G$ has no parallel arcs, because a $k$-optimal partition $M$ is also a $k$-optimal partition for a graph $G'$ obtained from $G$ by removing some half of the double edges; if the result is true for $G'$, then there exists a partial $k$-colouring of $G$ which is strong for $M$. Also, we may assume that $G$ has no circuit of length 3: if such a circuit exists, it constitutes a connected component of $G$ for which Conjecture A is trivially true. So we may assume that $G$ has no circuits, in which case the result follows from Theorem 3.

**Theorem 4.** (Greene–Kleitman). Let $G = (X, U)$ be a transitive graph: i.e. $(x, y) \in U$ and $(y, z) \in U$ implies $(x, z) \in U$. Then Conjecture A is valid for all $k$.

**Proof.** For the graph of a partially ordered set, Greene and Kleitman [10] proved that if we denote by $\alpha_k$ the maximum number of vertices which can be coloured in a partial $k$-colouring, then $\min_M B_k(M) = \alpha_k$. This result extends Dilworth's theorem (case $k = 1$). A shorter proof has been given by M. Saks [14], and an extension has also been given by A. Frank [6].

Now, consider a partition $M = \{\mu_1, \mu_2, \ldots\}$ which minimizes $B_k(M)$, and an optimal partial $k$-colouring $(S_1, S_2, \ldots, S_k)$. Each $\mu_i \in M$ induces a clique and therefore meets at most once each colour. Then

$$\alpha_k = \left| \bigcup_{i=1}^{k} S_i \right| = \sum |\mu_i \cap \bigcup_{i=1}^{k} S_i| \leq \sum_{i} \min \{k, |\mu_i|\} = B_k(M) = \alpha_k$$

(by the theorem of Greene and Kleitman).

Hence the number of coloured vertices encountered by $\mu_i$ is exactly $\min \{k, |\mu_i|\}$, and these vertices have different colours.

**Corollary.** Every transitive graph satisfies Property $B$.

**Proof.** Let $M$ be a $(k - 1)$-optimal partition, with $k = \max |\mu|$. By Theorem 4, there exists a partial $(k - 1)$-colouring $(S_1, S_2, \ldots, S_{k-1})$ which is strong for $M$. So the set $T = X - \bigcup S_i$ meets every maximum clique, and therefore, meets every maximum path.

Also, $|L_k(M)|$ is the maximum number of disjoint maximum paths, and $|T| = L_k(M)$.

**Theorem 5.** Let $G = (X, U)$ be a graph containing a Hamiltonian path $\mu_0$. Then Conjecture A is valid for all $k$.

**Proof.** For $k = \max |\mu|$, Theorem 5 follows from Theorem 2.

For $k < \max |\mu|$, the only partition which minimizes $B_k(M)$ is $M_0 = \{\mu_0\}$ (since $B_k(M_0) = k$, and every other partition $M'$ satisfies $B_k(M') > k$). Clearly, any partial $k$-colouring (with no empty class) is strong for $M_0$. 
THEOREM 6. Let $G$ be a bipartite graph defined by two vertex-classes $X$ and $X'$. Then Conjecture A is valid for all $k$.

Consider first the case $k = 2p$ even. Let $M = (\mu_1, \ldots)$ be any partition. We define a partial $(2p)$-colouring by assigning successively a colour to the vertex of $\mu_i$ at level 1, then to the vertex of $\mu_i$ at level 2, etc. The colour assigned to $a \in \mu_i$ is the smallest integer (not yet used for $\mu_i$) in $\{1, 2, \ldots, p\}$ if $a \in X$, or in $\{1', 2', \ldots, p'\}$ if $a \in X'$. The vertex $a$ is left uncoloured if all of the colours $1, 2, \ldots, p, 1', 2', \ldots, p'$ have been used for $\mu_i$. Clearly, $\mu_i$ will be strongly coloured, and by processing separately each path $\mu_i$ of $M$, a partial $(2p)$-colouring of $G$ is obtained.

Now consider the case $k = 2p + 1$ odd. Let $M$ be a partition which minimizes $B_k(M)$. Let $\hat{G}$ be the subgraph of $G$ induced by $(X \cup X') - L_1(M) - L_2(M) - \cdots - L_{k-1}(M)$. Let $\bar{M}$ be the trace of $M$ on $\hat{G}$. Then $\bar{M}$ is a partition of $\hat{G}$ with $L_i(\bar{M}) \leq L_i(M)$ which minimizes $|\bar{M}|$; otherwise, we can obtain for $G$ another partition $M'$ with $L_i(M') = L_i(M)$ for $i < k$ and $|L_k(M')| < |L_k(M)|$, which contradicts the minimality of $|L_1(M)| + |L_2(M)| + \cdots + |L_k(M)|$.

By Theorem 1, there exists a stable set $S$ of $\hat{G}$ which meets every path of $\bar{M}$. Assign the colour 0 to all the vertices in $S$. Then colour successively the paths of $M$ with the colours $1, 2, \ldots, p, 1', 2', \ldots, p'$ according to the rules defined above (for the case $k = 2p$). Clearly, the colours $0, 1, 2, \ldots, p, 1', 2', \ldots, p'$ define a partial $k$-colouring of $G$, and each path of $M$ is strongly coloured.

The problems which remain open are:

PROBLEM 1. Is Conjecture A valid for every $k$?

PROBLEM 2. Is it true that every graph has a path partition $M$ and a partial $k$-colouring $(S_1, S_2, \ldots, S_k)$ such that $|\mu \cap \bigcup S_i| = \min\{k, |\mu|\}$ for every $\mu \in M$?

PROBLEM 3. Is it true that for every $k$, there exists a path partition $M$ such that $B_k(M) \leq \alpha_k$?

In fact, similar properties (weaker than Conjecture A) have been proved for acyclic graphs in [4, 12, 15] (see also some hints in [9]).

REFERENCES


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