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The Positivity of a Sequence of Numbers and the Riemann Hypothesis

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In this note, we prove that the Riemann hypothesis for the Dedekind zeta

1. THE RIEMANN ZETA FUNCTION

Let $\{\lambda_n\}$ be a sequence of numbers given by

$$(n-1)!\,\lambda_n = \frac{d^n}{ds^n}\,[\,s^{n-1}\log \xi(s)\,]_{s=1} \tag{1.1}$$

for all positive integers n, where

$$\xi(s) = s(s-1)\,\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

with $\zeta(s)$ being the Riemann zeta function.

THEOREM 1. *A necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line is that λ_n is non-negative for every positive integer n.*

Proof. Define

$$\varphi(z) = \xi\left(\frac{1}{1-z}\right) = 4\int_1^\infty [\,x^{3/2}\psi'(x)\,]'\,(x^{-1/2}x^{1/2(1-z)} + x^{-1/2(1-z)})\,dx \tag{1.2}$$

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for z in the unit disk, where

$$\psi(x)=\sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

Write

$$\xi(s)=\prod_{\rho}\left(1-\frac{s}{\rho}\right) \tag{1.3}$$

where the product is taken over all nontrivial zeros of the Riemann zeta function with ρ and $1-\rho$ being paired together. It follows that

$$\varphi(z)=\prod_{\rho}\frac{1-(1-(1/\rho))z}{1-z}.$$

A necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line is that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. Put

$$\frac{\varphi'(z)}{\varphi(z)}=\sum_{n=0}^{\infty}\lambda_{n+1}z^n$$

for $|z|<\frac{1}{4}$, where

$$\lambda_n=\sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^n\right] \tag{1.4}$$

for every positive integer n. On the other hand, by (1.3) we have

$$\begin{aligned}\frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s)]_{s=1}&=-\sum_{\rho}\sum_{k=0}^{n-1}\binom{n}{k}(\rho-1)^{k-n}\\&=\sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^n\right],\end{aligned}$$

and hence λ_n is also given by the expression (1.1). Let

$$\varphi(z)=1+\sum_{j=1}^{\infty}a_j z^j. \tag{1.5}$$

We find that

$$\lambda_n=n\sum_{l=1}^{n}\frac{(-1)^{l-1}}{l}\sum_{\substack{1\leqslant k_1,\ldots,k_l\leqslant n\\ k_1+\cdots+k_l=n}} a_{k_1}\cdots a_{k_l}$$

for every positive integer n. Expanding the right side of (1.2) in power series (1.5), we find that

$$a_j = 2 \sum_{n=0}^{\infty} \frac{(j+n)\cdots(j+1)}{n!\,(n+1)!\,2^n} \int_1^\infty [x^{3/2}\psi'(x)]'\,(x^{-1/2} + (-1)^{n+1})(\log x)^{n+1}\,dx \tag{1.6}$$

for every positive integer j. By (1.6) we can write

$$\begin{aligned}
a_j &= 4 \sum_{n=0}^{\infty} \frac{(n+j)\cdots(n+1)}{j!\,(n+1)!\,2^{n+1}} \int_1^\infty [x^{3/2}\psi'(x)]'\,(x^{-1/2} + (-1)^{n+1})(\log x)^{n+1}\,dx \\
&= \frac{4}{j!}\frac{d^j}{dt^j}\Bigg\{ t^{j-1} \sum_{n=0}^{\infty} \frac{(t/2)^{n+1}}{(n+1)!} \int_1^\infty [x^{3/2}\psi'(x)]' \\
&\qquad \times (x^{-1/2} + (-1)^{n+1})(\log x)^{n+1}\,dx \Bigg\}_{t=1} \\
&= \frac{4}{j!}\frac{d^j}{dt^j}\left\{ t^{j-1} \int_1^\infty [x^{3/2}\psi'(x)]'\,(x^{-1/2}[e^{(t/2)\ln x} - 1] + [e^{-(t/2)\ln x} - 1]) \right\}_{t=1} \\
&= \frac{4}{j!}\frac{d^j}{dt^j}\left\{ t^{j-1} \int_1^\infty [x^{3/2}\psi'(x)]'\,(x^{-1/2}e^{(t/2)\ln x} + e^{-(t/2)\ln x}) \right\}_{t=1} \\
&= 4 \sum_{l=1}^{j} \binom{j-1}{j-l} \frac{1}{l!} \int_1^\infty [x^{3/2}\psi'(x)]' \left(\frac{1}{2}\log x\right)^l [1 + (-1)^l x^{-1/2}]\,dx.
\end{aligned}$$

This expression implies that a_j is a positive real number for every positive integer j. Since the identity

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \left(\sum_{i=0}^{\infty} a_i z^i\right)\left(\sum_{j=0}^{\infty} \lambda_{j+1} z^j\right)$$

holds, we have the recurrence relation

$$\lambda_n = n a_n - \sum_{j=1}^{n-1} \lambda_j a_{n-j}$$

for every positive integer n.

By (1.1), λ_n is a real number for every positive integer n. If the nontrivial zeros of $\zeta(s)$ lie on the critical line, then $|1-(1/\rho)| = 1$ for every nontrivial zero ρ of $\zeta(s)$. Put $1-(1/\rho)=\exp(i\theta_\rho)$ for some real number θ_ρ. Then by (1.4) we have

$$\lambda_n = \sum_\rho (1-e^{in\theta_\rho}) = \sum_\rho (1-\cos n\theta_\rho).$$

This implies that the number λ_n is nonnegative for every positive integer n.

Conversely, if the number λ_n is nonnegative for every positive integer n, then

$$\lambda_n \leqslant n a_n$$

for every positive integer n. It follows that

$$\sum_{n=1}^{\infty} |\lambda_n z^{n-1}| \leqslant \sum_{n=1}^{\infty} n a_n \, |z|^{n-1} = \varphi'(|z|) < \infty$$

for z in the unit disk. This implies that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk.

This completes the proof of the theorem.

2. THE DEDEKIND ZETA FUNCTION

Let k be an algebraic number field with r_1 real places and r_2 imaginary places. The Dedekind zeta function $\zeta_k(s)$ of k is defined by

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1-N\mathfrak{p}^{-s})^{-1}$$

for $\operatorname{Re} s > 1$, where the product is taken over all the finite prime divisors of k. Put $G_1(s)=\pi^{-s/2}\Gamma(s/2)$ and $G_2(s)=(2\pi)^{1-s}\,\Gamma(s)$. Define

$$Z_k(s) = G_1(s)^{r_1}\, G_2(s)^{r_2}\, \zeta_k(s).$$

By Theorem 3 of Chapter VII, Section 6, of [4], the function $Z_k(s)$ is analytic in the complex plane except for simple poles at $s=0$ and $s=1$, and satisfies the functional identity

$$Z_k(s) = |\mathfrak{d}|^{(1/2)-s}\, Z_k(1-s)$$

where $\mathfrak{d}$ is the discriminant of k. Its residues at $s=0$ and $s=1$ are respectively $-c_k$ and $|\mathfrak{d}|^{-1/2}\, c_k$ with $c_k = 2^{r_1}(2\pi)^{r_2}\, hR/e$, where h, R, and e are respectively the number of ideal classes of k, the regulator of k, and the number of roots of unity in k. Let $\xi_k(s) = c_k^{-1} s(s-1)\, |\mathfrak{d}|^{s/2}\, Z_k(s)$. Then $\xi_k(s)$ is an entire function and $\xi_k(0)=1$.

Let $\{\lambda_n\}$ be a sequence of numbers given by

$$(n-1)!\,\lambda_n = \frac{d^n}{ds^n}\,[s^{n-1} \log \xi_k(s)]_{s=1}$$

for all positive integers n. The aim now is to prove the following theorem.

THEOREM 2. *A necessary and sufficient condition for the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ to lie on the critical line is that λ_n is non-negative for every positive integer n.*

3. PROOF OF THE THEOREM 2

LEMMA 3.1. *The identity*

$$\lambda_n = \sum_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right)$$

holds for every positive integer n, where summation is taken over all non-trivial zeros of the Dedekind zeta function $\zeta_k(s)$ with ρ and $1-\rho$ being paired together.

Proof. By Theorem 2 of Barner [1], we have the formula (cf. Chapter 2 of [2])

$$\xi_k(s) = \prod_\rho \left(1 - \frac{s}{\rho}\right), \tag{3.1}$$

where the product is taken over all zeros of $\xi_k(s)$ with ρ and $1-\rho$ being always paired together. An argument similar to that made for the Riemann zeta function in Chapter 2 of [2] shows that the convergence of the product (3.1) is uniform on compact subsets of the complex plane.

Since $\xi_k(s) = \xi_k(1-s)$, we have

$$\frac{d^n}{ds^n}\,[s^{n-1} \log \xi_k(s)]_{s=1} = (-1)^n \frac{d^n}{ds^n}\,[(1-s)^{n-1} \log \xi_k(s)]_{s=0}. \tag{3.2}$$

Since $\zeta_k(s)$ does not vanish at $s=0$, we can write

$$\log \xi_k(s) = -\sum_\rho \sum_{m=1}^{\infty} \frac{\rho^{-m}}{m} s^m \tag{3.3}$$

where $|s| < \varepsilon$ for a sufficiently small positive number ε, where ρ and $1-\rho$ are paired together in the summation over ρ. Since the product (3.1) converges uniformly, the series (3.3) converges uniformly for $|s| < \varepsilon$. It follows that

$$\frac{1}{(n-1)!}\frac{d^n}{ds^n}[(1-s)^{n-1}\log \xi_k(s)]_{s=0} = -\sum_{\rho}\sum_{m=1}^{n}(-1)^{n-m}\binom{n}{m}\rho^{-m}.$$

This formula together with (3.2) implies the stated identity. ∎

Define

$$\varphi(z) = \xi_k\left(\frac{1}{1-z}\right)$$

for z in the unit disk. Since the function $\xi_k(s)$ is analytic in the complex plane of s, the function $\varphi(z)$ is analytic in the unit disk.

LEMMA 3.2. *Let*

$$\varphi(z) = 1 + \sum_{j=1}^{\infty} a_j z^j.$$

Then the coefficient a_j is a positive real number for every positive integer j.

Proof. Define ε_v to be one when v is a real place of k and to be two when v is an imaginary place of k. Let $x = \prod x_v$ be the variable in the half space $\mathbb{R}_+^{r_1+r_2}$. Denote by $|x|$ the product $\prod x_v^{\varepsilon_v}$, which is taken over all infinite places of k. If $N = r_1 + 2r_2$, then the Hecke theta function $\Theta_k(x)$ is defined by

$$\Theta_k(x) = \sum_{\mathfrak{b}} \exp\left(-\pi\,|\mathfrak{d}|^{-1/N}(N\mathfrak{b})^{2/N}\sum_v \varepsilon_v x_v\right)$$

where the summation over $\mathfrak{b}$ is taken over all nonzero integral ideals of k and where the summation over v is taken over all infinite places of k. Put $dx = \prod dx_v$. It follows from Theorem 3 of Chapter XIII, Section 3, in [3] that

$$\xi_k(s) = 1 + c_k^{-1} s(s-1)\int_{|x|\geqslant 1}\Theta_k(x)(|x|^{s/2} + |x|^{(1-s)/2})\frac{dx}{x}. \tag{3.4}$$

Let

$$\int_{|x|\geqslant 1}\Theta_k(x)(|x|^{1/2(1-z)} + |x|^{1/2}\,|x|^{-1/2(1-z)})\frac{dx}{x} = \sum_{m=0}^{\infty} b_m z^m. \tag{3.5}$$

It is clear that b_0 is a positive number. We have

$$b_m = \sum_{n=0}^{\infty} \frac{(m+n)\cdots(m+1)}{n!\,(n+1)!\,2^{n+1}} \int_{|x|\geqslant 1} \Theta_k(x)(1+|x|^{1/2}(-1)^{n+1})(\log|x|)^{n+1}\,\frac{dx}{x}$$

for every positive integer m. By computation, we find that

$$\begin{aligned} b_m &= \frac{1}{m!} \sum_{n=0}^{\infty} \frac{(n+m)\cdots(n+1)}{(n+1)!\,2^{n+1}} \\ &\quad \times \int_{|x|\geqslant 1} \Theta_k(x)(1+|x|^{1/2}(-1)^{n+1})(\log|x|)^{n+1}\,\frac{dx}{x} \\ &= \frac{1}{m!}\,\frac{d^m}{dt^m}\left(t^{m-1}\int_{|x|\geqslant 1} \Theta_k(x)(e^{(t/2)\log|x|}+|x|^{1/2}\,e^{-(t/2)\log|x|})\,\frac{dx}{x}\right)_{t=1}. \end{aligned}$$

It follows that

$$b_m = \sum_{l=1}^{m} \binom{m-1}{m-l}\frac{1}{l!}\int_{|x|\geqslant 1} \Theta_k(x)\left(\frac{1}{2}\log|x|\right)^l (|x|^{1/2}+(-1)^l)\,\frac{dx}{x} \tag{3.6}$$

for every positive integer m. Since $\Theta_k(x)$ is positive for every x in $\mathbb{R}_+^{r_1+r_2}$, it follows from (3.6) that the coefficients b_m are positive real numbers for all nonnegative integers m.

The identity

$$\frac{z}{(1-z)^2} = \sum_{q=1}^{\infty} q z^q$$

holds for z in the unit disk. It follows from (3.4) and (3.5) that

$$c_k a_j = \sum_{m=0}^{j-1} (j-m)\,b_m \tag{3.7}$$

for every positive integer j. Since b_m are positive numbers for all non-negative integers m, we see that a_j is a positive real number for every positive integer j. ∎

Proof of the Theorem. Since $\xi_k(1)=1$ and $\xi_k(s)=\xi_k(1-s)$, it follows from the product formula (3.1) that

$$\varphi(z) = \prod_{\rho} \frac{1-(1-(1/\rho))z}{1-z}. \tag{3.8}$$

Since $\xi_k(s)$ does not vanish at $s=1$, we can write

$$\varphi'(z)/\varphi(z)=\sum_{n=0}^{\infty}\lambda_{n+1}z^n \tag{3.9}$$

by using the formula (3.8) when $|z|<\varepsilon$ for a sufficiently small positive number ε. Since

$$\sum_{n=1}^{\infty} na_n z^{n-1}=\left(\sum_{i=0}^{\infty} a_i z^i\right)\left(\sum_{j=0}^{\infty}\lambda_{j+1}z^j\right),$$

we have

$$\lambda_n=na_n-\sum_{j=1}^{n-1}\lambda_j a_{n-j} \tag{3.10}$$

for $n=2, 3, ...,$ where $\lambda_1=a_1$ and $a_0=1$.

If the nontrivial zeros of $\zeta_k(s)$ lie on the critical line, it follows from Lemma 3.1 that the numbers λ_n are nonnegative for all positive integers n.

Conversely, assume that the number λ_n is nonnegative for every positive integer n. It follows from (3.10) and Lemma 3.2 that

$$\lambda_n\leqslant na_n$$

for every positive integer n. This inequality together with Lemma 3.2 implies that

$$\sum_{n=1}^{\infty}|\lambda_n z^{n-1}|\leqslant\sum_{n=1}^{\infty} na_n\,|z|^{n-1}=\varphi'(|z|) \tag{3.11}$$

for z in the unit disk. Since $\varphi'(z)$ is analytic in the unit disk, $\varphi'(|z|)$ is finite for z in the unit disk. It follows from (3.9) and (3.11) that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. It is clear that a necessary and sufficient condition for the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ to lie on the critical line is that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. Therefore, the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ lie on the critical line.

This completes the proof of the theorem. ∎

Remark. We know from the proof of Theorem 2 that $\lambda_1=a_1$, which is a positive number by Lemma 3.2. An explicit expression for λ_n is implicit in the recurrence relation (3.10) together with formulas (3.6) and (3.7).

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