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Set-theoretic defining equations of the tangential variety of the Segre variety

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ABSTRACT

We prove a set-theoretic version of the Landsberg–Weyman Conjecture on the defining equations of the tangential variety of a Segre product of projective spaces. We introduce and study the concept of exclusive rank. For the proof of this conjecture, we use a connection to the author's previous work and re-express the tangential variety as the variety of principal minors of symmetric matrices that have exclusive rank no more than 1. We discuss applications to semiseparable matrices, tensor rank versus border rank, context-specific independence models and factor analysis models.

1. Introduction

For each i, $1 \leq i \leq n$, let V_i be a complex vector space of dimension $n_i + 1$ and let V_i^* be the dual vector space. The Segre product $Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$ is the variety of indecomposable tensors in $\mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*)$. If V is a complex vector space and $X \subset \mathbb{P}V$ is any variety, the *tangential variety* of X, denoted as $\tau(X)$, is the union of all points on all embedded tangent lines (*i.e.* $\mathbb{P}^1$'s) to X, (see [24] for a comprehensive look at tangential varieties).

Let $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$ denote the irreducible $SL(V_1) \times \cdots \times SL(V_n)$-module associated with the partitions $\pi_1, \ldots, \pi_n$ of d. Readers unfamiliar with this notation may consult [5] or the upcoming [12] for the necessary background on representation theory. Using and cohomological techniques and in particular Weyman's geometric method (see [23]), Landsberg and Weyman identified modules of this form in the ideal of $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ and made the following conjecture.

Conjecture 1.1 (*Conjecture 7.6. [16]*). *$I\left(\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)\right)$ is generated by the quadrics in $S^2(V_1 \otimes \cdots \otimes V_n)$ which have at least four $\bigwedge^2$ factors, the cubics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$.*

The *secant variety* of X, denoted as $\sigma(X)$, is the variety of all embedded secant $\mathbb{P}^1$'s to X, and since every tangent line to X is the limit of secant lines, we have $\tau(X) \subset \sigma(X)$.

If $X = Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$, then $\sigma(Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$ is contained in a subspace variety (or rank variety), namely $Sub_{2,\ldots,2}(V_1^* \otimes \cdots \otimes V_n^*)$, which is all tensors $[T] \in \mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*)$ such that there exist auxiliary subspaces $V_i'^*$ with $\dim(V_i'^*) = 2$ for $1 \leq i \leq n$ and $T \in V_1'^* \otimes \cdots \otimes V_n'^*$.

In [16] Landsberg and Weyman point out that because $\tau(X) \subset \sigma(X) \subset Sub_{2,\ldots,2}(V_1^* \otimes \cdots \otimes V_n^*)$, it is sufficient to answer Conjecture 1.1 in the case where $V_i \simeq \mathbb{C}^2$. We will prove the set-theoretic version of this conjecture in this case.

Our point of departure is to consider the (not immediately obvious) embedding of $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ as a subvariety of the variety of principal minors of symmetric $n \times n$ matrices (denoted as Z_n throughout). We give a precise definition of Z_n and list some of its properties in Section 2.

In the case $n = 3$ the ideal of the tangential variety $\tau(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$ is defined by Cayley's hyperdeterminant of format $2 \times 2 \times 2$, and this is the quartic equation which appears in Conjecture 1.1.

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In [11], Holtz and Sturmfels showed that the ideal of Z_3 is generated by the same polynomial; therefore $\tau(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = Z_3$. In general the two varieties are not equal but one inclusion holds, namely

$$\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right) \subset Z_n$$

for all $n \geq 3$, [18].

Holtz and Sturmfels conjectured that the *hyperdeterminantal module* – the span of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of Cayley's $2\times 2\times 2$ hyperdeterminant – generates the ideal of Z_n. The hyperdeterminantal module is the module of quartic polynomials with three $S_{2,2}$'s and all other factors $S_{4,0}$, *i.e.* the quartics in the Landsberg–Weyman Conjecture. In [19,18] we proved the set-theoretic version of the Holtz–Sturmfels Conjecture:

Theorem 1.2 *([19,18]). Let $Z_n \subset \mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*)$ be the variety of principal minors of symmetric matrices and let HD be the module of quartic polynomials with three $S_{2,2}$'s and all other factors $S_{4,0}$. Then, as sets, $\mathcal{V}(HD) = Z_n$.*

In this paper we study the polynomials in the Landsberg–Weyman Conjecture via their connection to Z_n. Using this connection, we arrive at the following:

Theorem 1.3. *$\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ is cut out set-theoretically by the following modules of equations: the quadrics in $S^2(V_1 \otimes \cdots \otimes V_n)$ which have four $\bigwedge^2$ factors, the cubics that do not occur as a linear form times the quadrics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$.*

Remark 1.4. Notice that we have reduced the number of equations asked for in the Landsberg–Weyman Conjecture, so we are proving something slightly stronger. That is, we consider *fewer* polynomials in the ideal, and show that these suffice for cutting out the variety set-theoretically. In particular, in Section 3.2 we show that one instance of the module of cubics with four $S_{2,1}$ factors and all other factors S^3 occurs as a module of reducible polynomials, and therefore is not a minimal generator of the ideal.

Remark 1.5. Each of these modules of equations has an alternative description, which may be more accessible to those unfamiliar with the representation-theoretic language. The following is a description of the set-theoretic defining ideal as all of the polynomials obtained from just three polynomials via linear span and the natural change of basis by the action of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$:

The quadric equations occur as linear combinations of 2×2 minors of $2 \times 2^{n-1}$ flattenings. Specifically they are the linear span of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of the polynomial F_0 found in Section 3.2.

The cubic equations occur as linear combinations of 3×3 minors of $4 \times 2^{n-2}$ flattenings. Specifically they are the linear span of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of the polynomial F_2 found in Section 3.2.

The quartic equations occur as the linear span of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of Cayley's hyperdeterminant of format $2\times 2\times 2$ (see Example 6.1 below for an explicit expression).

Here is an outline of the rest of the paper and of the proof of Theorem 1.3. We know that all of the polynomials in the conjecture are in the ideal of the tangential variety. This fact appears in [16], deduced using Weyman's geometric technique. The vanishing can also be checked by appealing to the group action and testing a highest weight vector of every module on a generic point of the tangential variety. In light of Remark 1.5, there are three vectors to check, namely the polynomials F_0, F_2 (see Section 3.2) and Cayley's $2 \times 2 \times 2$ hyperdeterminant. For the set-theoretic result, it therefore remains to show that the tangential variety contains the zero set of these polynomials.

By Theorem 1.2 the set of quartics cuts out Z_n so we need to show that the subvariety of Z_n defined by intersecting with the zero set of the quadrics and cubics coincides with the tangential variety.

In Section 3, we explicitly construct the quadrics and cubics in the ideal of the tangential variety and then pull these polynomials back to the space of symmetric matrices. Our intention is that the explicit expressions for polynomials (in coordinates) in this section will be useful for readers unfamiliar with the representation theory notation. We show that the quadric equations are unnecessary for the set-theoretic result. We then consider the subvariety $X \subset S^2\mathbb{C}^n$ defined by this pull-back.

The description of this variety X motivates the introduction of the *exclusive rank* (or *E-rank*) of a matrix. In Section 4 we define E-rank and in Proposition 4.1 we show that E-rank is an invariant of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n \subset GL(2n)$ with a natural action which we describe.

In Section 5 we study the principal minors of E-rank 0 and 1 symmetric matrices. Finally in Proposition 5.2, we show that the image under the principal minor map of the symmetric matrices with E-rank no larger than 1 is exactly the tangential variety. This will show that every point in the zero set of the quadric, cubic and quartic polynomials in Theorem 1.3 has a symmetric E-rank 1 matrix in X mapping to it under the principal minor map. But the image of X under the principal minor map is the tangential variety, the original point must be in the tangential variety and this completes the proof.

The study of matrices in terms of their E-rank is not new. Of particular interest is a proper subset of matrices of E-rank 1 known as semiseparable matrices, as well as several other related notions [22, Chapter 1]. See for instance [21,25] for an example of recent work on the subject from the point of view of applied linear algebra. In Section 6 we expound on specific applications of the tangential variety: in Section 6.1, quasiseparable and semiseparable matrices, their relation to matrices of E-rank 1, and the relative inverse eigenvalue problem, in Section 6.2 the study of tensor rank (see for instance [15] for recent work on the subject in the case of symmetric tensors), in Section 6.3 the study of context-specific independence models (a topic in algebraic statistics) introduced by Georgi and Schliep in [8] and studied in [3, chap. 2] and the related factor analysis model studied in [4].

2. The variety of principal minors of symmetric matrices

To give a precise definition of Z_n we need some notation. Let $I = (i_1, \dots i_n)$ be a binary multi-index, with $i_k \in \{0, 1\}$ for $k = 1, \dots , n$, and let $|I| = \sum_{k=1}^{n} i_k$. For notational compactness, we will drop the commas and parentheses in the expression of I when there is no danger of confusion.

If A is an $n \times n$ matrix, then let $\Delta^I_J(A)$ denote the minor of A formed by taking the determinant of the submatrix of A with rows indexed by I and columns indexed by J, in the sense that the submatrix of A is formed by only including the kth row (respectively column) of A whenever $i_k = 1$ (respectively $j_k = 1$). When $I = J$, the minor is said to be *principal*, and we will denote it by $\Delta_I = \Delta^I_I$.

For $1 \leq i \leq n$ let $V_i \simeq \mathbb{C}^2$ and consider $V_1 \otimes V_2 \otimes \cdots \otimes V_n \simeq \mathbb{C}^{2^n}$. A choice of basis $\{x_i^0, x_i^1\}$ of V_i for each i determines a basis of $V_1 \otimes \cdots \otimes V_n$. We represent basis elements compactly by setting $X^I := x_1^{i_1} \otimes x_2^{i_2} \otimes \cdots \otimes x_n^{i_n}$. We use this basis to introduce coordinates on $\mathbb{P}\mathbb{C}^{2^n}$; if $P = [C_I X^I] \in \mathbb{P}\mathbb{C}^{2^n}$, the coefficients C_I are the coordinates of the point P.

Let $S^2\mathbb{C}^n$ denote the space of symmetric $n \times n$ matrices. The projective variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by the following rational map:

$$\begin{aligned} \varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) &\dashrightarrow \mathbb{P}\mathbb{C}^{2^n} \\ [A, t] &\longmapsto \left[t^{n-|I|} \Delta_I(A)\, X^I\right]. \end{aligned}$$

The map φ is defined on the open set where $t \neq 0$. Moreover, φ is homogeneous of degree n, so it is well defined on projective space.

3. The pull-back of polynomials in the Landsberg–Weyman Conjecture symmetric matrices via Z_n

3.1. Background and notation

If a variety $X \subset \mathbb{P}(V_{1^*} \otimes \cdots \otimes V_n^*)$ is invariant under the action of $SL(V_1) \times \cdots \times SL(V_n)$ – of which $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ and Z_n are two examples – we say X is a G-variety for $G = SL(V_1) \times \cdots \times SL(V_n)$. The ideal of such a G-variety is a G-submodule of $\bigoplus_d S^d(V_1 \otimes \cdots \otimes V_n)$. Each degree-$d$ piece has an *isotypic decomposition* which Landsberg and Manivel recorded in [13] as follows:

Proposition 3.1 (*Landsberg and Manivel [13], Proposition 4.1*)**.** *Let $V_1, \dots , V_n$ be vector spaces and let $V = V_1 \otimes \cdots \otimes V_n$, and let $G = GL(V_1) \times \cdots \times GL(V_n)$. Then the following decomposition as a direct sum of irreducible G-modules holds:*

$$S^d(V_1 \otimes \cdots \otimes V_n) = \bigoplus_{|\pi_1|=\cdots=|\pi_k|=d} ([\pi_1] \otimes \cdots \otimes [\pi_k])^{\mathfrak{S}_d} \otimes S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$$

*where $[\pi_i]$ are representations of the symmetric group $\mathfrak{S}_d$ indexed by partitions π_i of d, $([\pi_1] \otimes \cdots \otimes [\pi_k])^{\mathfrak{S}_d}$ denotes the space of $\mathfrak{S}_d$-invariants (*i.e. *, instances of the trivial representation) in the tensor product, and $S_{\pi_i}V_i$ are Schur modules.*

For more background on this decomposition formula see [13, Section 4]. For the reader who may be unfamiliar with these concepts, we recall some basic facts. For a more in-depth account, one may consult [5]. The irreducible representations of the symmetric group $\mathfrak{S}_d$ are indexed by partitions π of d, and we write $[\pi]$ for the associated $\mathfrak{S}_d$-module. For a vector space V, the irreducible representations of $SL(V)$ are also indexed by partitions, namely there is a natural action of $\mathfrak{S}_d$ on $V^{\otimes d}$ and if π is a partition of d, we denote by $S_\pi V$ the associated Schur module of $\mathfrak{S}_d$-equivariant linear maps from $[\pi]$ to $V^{\otimes d}$.

Note that the representation theory for $SL(n)$ and $GL(n)$ is the same up to twists by determinants, so we can also use this proposition when $G = SL(V_1) \times \cdots \times SL(V_n)$. When the V_i are all isomorphic to the same V, we also have an $\mathfrak{S}_n$ action. In this case, the irreducible G-modules for $G = SL(V)^{\times n} \ltimes \mathfrak{S}_n$ are direct sums of the modules of the form $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$ where the π_i's occur in every order that produces a non-redundant module. In this case we often drop the superfluous notation for the tensor products and the vector spaces and denote the irreducible $SL(V)^{\times n} \ltimes \mathfrak{S}_n$-modules by $S_{\pi_1} \dots S_{\pi_n}$.

We construct polynomials from Schur modules via Young symmetrizers and Young tableaus. This construction is described in detail in [12] so we do not attempt to repeat its description here, but merely give a brief summary.

The basic idea is that for a partition π of an integer d, each filled Young tableau of shape π provides a recipe for constructing a certain Young symmetrizer, *i.e.* a map $c_\pi : V^{\otimes d} \to V^{\otimes d}$ whose image is isomorphic to $S_\pi V \subset V^{\otimes d}$. The map c_π is defined by skew-symmetrizing over the columns and symmetrizing over the rows of the filled Young tableau of shape π. In particular, one can construct a highest weight vector of $S_\pi V$ as the image under c_π of a simple vector in $V^{\otimes d}$ of the correct weight.

To construct a tensor in a module of degree d polynomials of the from $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$, we must find a clever combination of choices of fillings of the Young tableau of shapes π_i so that the resulting tensor in $S^d(V_1 \otimes \cdots \otimes V_n)$ is nonzero. When $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$ occurs with multiplicity $m > 1$, we must repeat this process until we get m linearly independent vectors to span the highest weight space (which by definition has dimension m).

3.2. Constructing polynomials for the Landsberg–Weyman Conjecture

We first consider the case $n = 4$. After that, we can build the polynomials for the general case from those in the base case.

For these examples, we wrote two simple procedures in Maple. The first, called makeUnsymmetric, takes a list of partitions $\pi_1, \pi_2, \pi_3, \pi_4$ and a list of fillings $T_{\pi_1}, T_{\pi_2}, T_{\pi_3}, T_{\pi_4}$ and constructs a vector of highest weight in the space $S_{\pi_1}V_1 \otimes S_{\pi_2}V_2 \otimes S_{\pi_3}V_3 \otimes S_{\pi_4}V_4 \subset V_1^{\otimes d} \otimes V_2^{\otimes d} \otimes V_3^{\otimes d} \otimes V_4^{\otimes d}$. The second procedure, called unfactor, takes a tensor in $V_1^{\otimes d} \otimes V_2^{\otimes d} \otimes V_3^{\otimes d} \otimes V_4^{\otimes d}$, the degree d, and a list of the dimensions of the vector spaces, and then rearranges the vector spaces and combines the factors so that the resulting tensor is a polynomial in $S^d(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$. When combined, these two procedures produce the highest weight vectors (polynomials) in which we are interested. A copy of the Maple code may be obtained from the author.

Consider the $SL(2)^{\times 4} \ltimes \mathfrak{S}_4$-module $\bigwedge^2 \bigwedge^2 \bigwedge^2 \bigwedge^2$. This module is one-dimensional and it occurs with multiplicity 1 in the decomposition of the module of degree 2 homogeneous polynomials. We used our two commands in Maple:

```
T:=makeUnsymmetric([[1,1],[1,1],[1,1],[1,1]], [[1,2],[1,2],[1,2],[1,2]]):
F0:=unfactor(T,2,[2,2,2,2]);
```

to find

$$F_0 = X^{0000}X^{1111} - X^{0001}X^{1110} - X^{0010}X^{1101} + X^{0011}X^{1100} - X^{0100}X^{1011} + X^{0101}X^{1010} + X^{0110}X^{1001} - X^{0111}X^{1000}.$$

Note that this polynomial is a linear combination of 2×2 minors of a 2×8 flattening of a $2 \times 2 \times 2 \times 2$ tensor. Indeed, one can use basic representation theory to check that the module $\bigwedge^2 \bigwedge^2 \bigwedge^2 \bigwedge^2$ occurs in the decomposition of the module of 2×2 minors of any 2×8 flattening of a $2 \times 2 \times 2 \times 2$ tensor, but it does not occur in the module of 2×2 minors of any 4×4 flattening.

We pull back F_0 to $S^2\mathbb{C}^n \oplus \mathbb{C}$ by making the substitution $X^I = t^{n-|I|}\Delta_I(A)$, where the $\Delta_I(A)$ are the principal minors (indexed by I) of a symmetric matrix $A = (a_{i,j})$. We find the polynomial

$$F_0(A) := t^4 \left(a_{1,4}^2 a_{2,3}^2 + a_{1,3}^2 a_{2,4}^2 + a_{1,2}^2 a_{3,4}^2 - a_{1,2}a_{2,3}a_{3,4}a_{1,4} - a_{1,2}a_{2,4}a_{1,3}a_{3,4} - a_{1,3}a_{2,4}a_{2,3}a_{1,4}\right).$$

Notice that $F_0(A)$ is independent of the diagonal entries of A.

Next, consider the module $S_{2,1}S_{2,1}S_{2,1}S_{2,1}$. This module occurs with multiplicity 3 in the decomposition of $S^3(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$. In order to get a basis of the highest weight space, we alter the fillings of the Young tableau in the standard construction of highest weight vectors via Young symmetrizers. There are only two options for standard fillings in each of the four factors: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$. Of the possible 2^4 constructions we must find three which are linearly independent. We found the following three basis vectors of the highest weight space via images of the Young symmetrizers defined by the fillings $T_{\pi_1}, T_{\pi_2}, T_{\pi_3}, T_{\pi_4}$ via the recipes below:

- $T_{\pi_1} = T_{\pi_2} = T_{\pi_3} = T_{\pi_4} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$,

$$F_1 := \left(X^{0000}X^{1111} - X^{0001}X^{1110} - X^{0010}X^{1101} + X^{0011}X^{1100} - X^{0100}X^{1011} + X^{0101}X^{1010} + X^{0110}X^{1001} - X^{0111}X^{1000}\right)2X^{0000},$$

- $T_{\pi_1} = T_{\pi_2} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$, $T_{\pi_3} = T_{\pi_4} = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$,

$$\begin{aligned} F_2 := {} & (2X^{0000}X^{1100} - 2X^{0100}X^{1000})X^{0011} + (-X^{1100}X^{0001} + X^{0101}X^{1000} + X^{0100}X^{1001} - X^{1101}X^{0000})X^{0010} \\ & + (-X^{0010}X^{1100} + X^{1000}X^{0110} + X^{1010}X^{0100} - X^{1110}X^{0000})X^{0001} \\ & + (X^{0011}X^{1100} - X^{0111}X^{1000} - X^{0100}X^{1011} + X^{0000}X^{1111})X^{0000}, \end{aligned}$$

- $T_{\pi_1} = T_{\pi_3} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$, $T_{\pi_2} = T_{\pi_4} = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$,

$$\begin{aligned} F_3 := {} & (2X^{0000}X^{0101} - 2X^{0001}X^{0100})X^{1010} + (-X^{0101}X^{0010} + X^{0100}X^{0011} + X^{0110}X^{0001} - X^{0111}X^{0000})X^{1000} \\ & + (-X^{0101}X^{1000} + X^{0100}X^{1001} + X^{1100}X^{0001} - X^{1101}X^{0000})X^{0010} \\ & + (X^{0101}X^{1010} - X^{0100}X^{1011} - X^{0001}X^{1110} + X^{0000}X^{1111})X^{0000}, \end{aligned}$$

where for example to compute F_2 in Maple, we used the following two commands:

```
T:=makeUnsymmetric([[2,1],[2,1],[2,1],[2,1]],[[1,2,3],[1,2,3],[1,3,2],[1,3,2]]):
F2:=unfactor(T,3,[2,2,2,2]):
```

Notice that $F_1 = 2X^{0000}F_0$. This is an indication of the fact that the copy of $S_{2,1}S_{2,1}S_{2,1}S_{2,1}$ associated with the highest weight vector F_1 is in the ideal generated by $\bigwedge^2 \bigwedge^2 \bigwedge^2 \bigwedge^2$. The other two polynomials occur in the decomposition modules of 3×3 minors of 4×4 flattenings of a $2 \times 2 \times 2 \times 2$ tensor. In particular, $F_2 \in (S_{2,1}V_1 \otimes S_{2,1}V_3) \otimes (S_{2,1}V_2 \otimes S_{2,1}V_4) \subset$

$\bigwedge^3(V_1 \otimes V_3) \otimes \bigwedge^3(V_2 \otimes V_4)$, and $F_3 \in (S_{2,1}V_1 \otimes S_{2,1}V_2) \otimes (S_{2,1}V_3 \otimes S_{2,1}V_4) \subset \bigwedge^3(V_1 \otimes V_2) \otimes \bigwedge^3(V_3 \otimes V_4)$. F_2 and F_3 are readily seen to be the same up to permutation of the indices. So for the construction of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-module of cubics, it suffices to take the linear span of the orbit of one of them. Note also that the difference $F_2 - F_3$ is (up to a scalar multiple) the upper left 3×3 minor of the flattening $(V_1 \otimes V_4) \otimes (V_2 \otimes V_3)$.

3.3. The pull-back of the quadric and cubic polynomials to Z_n

We work on the open set $t \neq 0$, and set $t = 1$. On this set, F_0 and F_1 pull back to the same polynomial. It therefore suffices to just consider the module $\bigwedge^2 \bigwedge^2 \bigwedge^2 \bigwedge^2$ and the 2 copies of the module $S_{2,1}S_{2,1}S_{2,1}S_{2,1}$ corresponding to the span of the orbits of F_2 and F_3.

We find the following polynomials on the entries of the matrix A:

$$F_0(A) = 4(a_{1,2}^2a_{3,4}^2 - a_{1,2}a_{1,3}a_{2,4}a_{3,4} - a_{1,2}a_{1,4}a_{2,3}a_{3,4} + a_{1,3}^2a_{2,4}^2 - a_{1,3}a_{1,4}a_{2,3}a_{2,4} + a_{1,4}^2a_{2,3}^2)$$
$$F_2(A) = 4a_{1,2}^2a_{3,4}^2 - 2a_{1,2}a_{1,3}a_{2,4}a_{3,4} - 2a_{1,2}a_{1,4}a_{2,3}a_{3,4} + a_{1,3}^2a_{2,4}^2 - 2a_{1,3}a_{1,4}a_{2,3}a_{2,4} + a_{1,4}^2a_{2,3}^2$$
$$F_3(A) = a_{1,2}^2a_{3,4}^2 - 2a_{1,2}a_{1,3}a_{2,4}a_{3,4} - 2a_{1,2}a_{1,4}a_{2,3}a_{3,4} + 4a_{1,3}^2a_{2,4}^2 - 2a_{1,3}a_{1,4}a_{2,3}a_{2,4} + a_{1,4}^2a_{2,3}^2.$$

We used Maple for the constructions of F_0, F_1, F_2, and F_3 above. Then we decomposed the ideal generated by $F_0(A), F_2(A), F_3(A)$ in Macaulay2 [9] and found that the radical of this ideal is the single prime ideal

$$\langle a_{1,3}a_{2,4} - a_{1,4}a_{2,3}, \quad a_{1,2}a_{3,4} - a_{1,4}a_{2,3}\rangle. \tag{1}$$

We immediately recognize these equations as special 2×2 minors of the symmetric matrix A. In fact, these minors come from submatrices of A which have no rows or columns in common. We study such minors in Section 4.

Next we used Maple to implement a standard procedure using lowering operators to construct a basis of the modules associated with the highest weight vectors F_0, F_2 and F_3. We pulled back these 33 polynomials to the space of symmetric matrices. Then we decomposed this ideal in Macaulay2, and found the same ideal as in (1).

We note that while the polynomials $F_0(A), F_2(A), F_3(A)$ do not depend on the diagonal terms of the matrix A, this does not hold for all of the other basis vectors in the modules. However, the radical of the ideal still does not depend on the diagonal terms of A.

In the general case, we consider the modules $\bigwedge^2 \bigwedge^2 \bigwedge^2 \bigwedge^2 S^2 \dots S^2$ and the modules $S_{2,1}S_{2,1}S_{2,1}S_{2,1}S^3 \dots S^3$ not associated with products of linear forms with the quadrics. These modules have the same highest weight vectors (up to permutation) as those that we considered in the above example, so the pull-back of $S_{2,1}S_{2,1}S_{2,1}S_{2,1}S^3 \dots S^3$ to symmetric matrices must have (at least) all of the 2×2 minors of the matrix A which have no rows or columns in common in our ideal in the general case.

4. Exclusive rank

In this section, motivated by the equations that we found in the previous section, we study the minors whose row and column sets are disjoint. We will say that a minor $\Delta^I_J(A)$ is an *exclusive minor* (or *E-minor*) if $I \cap J = \emptyset$. The Laplace expansion expresses a $(k+2) \times (k+2)$ E-minor as a linear combination of $(k+1) \times (k+1)$ E-minors. Therefore, if all the $(k+1) \times (k+1)$ E-minors vanish, then all the $(k+2) \times (k+2)$ E-minors vanish as well. In light of this, we define the *exclusive rank* (or *E-rank*) of a matrix to be the minimal k such that all the $(k+1) \times (k+1)$ E-minors vanish.

Proposition 4.1. *The E-minors are fixed points under the action of $SL(2)^{\times n}$. In particular, the E-rank is $G \simeq \left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariant.*

Proof. First note that we are considering the action of $SL(2)^{\times n}$ to be the inherited action when considered as a subgroup of $GL(2n)$ which acts on the space of all minors of a generic $n \times n$ matrix by natural change of coordinates. Note also that it suffices to prove that a E-minor is taken to one of the same size under the action of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$.

The $\mathfrak{S}_n$-invariance is clear. So, we need to prove the first statement. We recall the inherited action of $SL(2)^{\times n}$ as a subgroup of $GL(2n)$. Looking at it in this way, we can give a proper definition of the action of $SL(2)^{\times n}$ on the exclusive minors.

Let $V = E \oplus F$ and let $E \simeq F \simeq \mathbb{C}^n$. The Grassmannian $G(n, V)$ can be parameterized by the rational map

$$\begin{aligned} \psi : \mathbb{P}(E^* \otimes F \oplus \mathbb{C}) \dashrightarrow \; & \mathbb{P}\left(\bigwedge^n V\right) = \mathbb{P}\left(\bigoplus_{k=0}^{n}\left(\bigwedge^k E^* \otimes \bigwedge^k F\right)\right) \\ [(A), t] \longmapsto \; & \left[\sum_{|R|=|S|} t^{k-|R|} e^R \otimes f_S(A)\right]. \end{aligned}$$

The map ψ is a variant of the Plücker embedding of the Grassmannian, and it is compatible with the decomposition of $\bigwedge^n V$. In light of this mapping ψ, the Grassmannian $Gr(n, 2n)$ has the interpretation as the variety of (vectors of) minors of $n \times n$ matrices.

For convenience, we will choose a volume form in $\bigwedge^n E$ and identify $\bigwedge^{n-k} E^*$ with $\bigwedge^k E$. Then we will work with the minors as elements of $\bigwedge^{n-k} E \wedge \bigwedge^k F$ – there is no harm in using a wedge between E and F because the vector spaces intersect only at the origin, so we can interchange the tensor symbol with the wedge symbol – and consider $e_R \wedge f_S(A)$ the minor of A found by taking the determinant of the submatrix formed by keeping the rows of A indexed by R^c and the columns of A indexed by S. In this notation, the principal minors of A are $e_R \wedge f_S(A)$ with $R \cap S = \emptyset$ and the E-minors of A are $e_R \wedge f_S(A)$ with $R = S$.

Consider a vector $\sum_{|R|=|S|\geq 1} e_R \wedge f_S(A)$ of all minors of a given $n \times n$ matrix A. Since this vector is in $G(n, 2n)$, we can consider the action of $GL(2n)$ on it, and by the inclusion $SL(2)^{\times n} \subset GL(2n)$ (given below) we can consider the action of $SL(2)^{\times n}$ on $e_R \wedge f_S(A)$.

In [11,18,1], it is shown that the action of $SL(2)^{\times n}$ preserves the variety of principal minors of symmetric matrices. Here we will show that this action fixes the E-minors.

The inclusion that we consider is the following:

$$SL(2)^{\times n} = \left\{ \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \mid D_i - \text{diagonal}, D_1D_3 - D_2D_4 = I_n \right\} \subset \left\{ M \in \begin{pmatrix} E^* \otimes E & F^* \otimes E \\ E^* \otimes F & F^* \otimes F \end{pmatrix} \mid det(M) \neq 0 \right\} = GL(V).$$

Consider the blocked matrix $g = \begin{pmatrix} a_i^i & b_j^j \\ c_k^k & d_l^l \end{pmatrix} \in SL(2)^{\times n}$ with $1 \leq i, j, k, l \leq n$. The individual elements of each $SL(2)$ are the 2×2 matrices constructed from g as $\begin{pmatrix} a_i^i & b_i^i \\ c_i^i & d_i^i \end{pmatrix}$. For simplicity, let all factors of g except the first factor be the identity matrix and consider the action on an exclusive minor

$$\begin{aligned} g.e_R \otimes f_R &= g.\left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{|R|}} \wedge f_{i_1} \wedge \cdots \wedge f_{i_{|R|}}\right) \\ &= \left((a_{i_1}^{i_1} e_{i_1} + c_{i_1}^{i_1} f_{i_1}) \wedge e_{i_2} \wedge \cdots \wedge e_{i_{|R|}} \wedge (b_{i_1}^{i_1} e_{i_1} + d_{i_1}^{i_1} f_{i_1}) \wedge f_{i_1} \wedge \cdots \wedge f_{j_{|S|}}\right). \end{aligned}$$

But if we expand this expression, and use the fact that $e_{i_1} \wedge e_{i_1} = f_{i_1} \wedge f_{i_1} = 0$, we see that the only nonzero term is

$$= (a_{i_1}^{i_1} c_{i_1}^{i_1} - b_{i_1}^{i_1} d_{i_1}^{i_1}) e_R \wedge f_R = e_R \wedge f_R.$$

Therefore the exclusive minors are fixed by $SL(2)^{\times n}$. □

5. Principal minors of symmetric matrices with small exclusive rank

In this section we study the symmetric matrices that have E-rank less than or equal to 1 and their principal minors. The main goal of this section is Proposition 5.2, which is the key to the proof of Theorem 1.3. First, we consider the case of symmetric matrices of E-rank 0. To prove Proposition 5.2 we first consider the principal minors of honest rank 1 symmetric matrices in Proposition 5.4. We show that the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of rank 1 symmetric matrices is $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$. Then we show that the variety of principal minors of symmetric matrices of E-rank 1 is the tangential variety by showing that it is $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariant, irreducible, and has the same dimension as the orbit of principal minors of the honest rank 1 symmetric matrices. The $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariance comes from Lemma 5.5 which is a general statement about how symmetry can be preserved under a projection from a G-variety.

Proposition 5.1 *([18]). Consider the open set of tensors* $U_0 = \{[z_I X^I] \in \mathbb{P}(V_1^* \otimes \cdots \otimes V_n^*) \mid z_{[0,\ldots,0]} \neq 0\}$. *Then* $\varphi([A, t]) \in Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*) \cap U_0$ *if and only if* A *is diagonal (has E-rank* 0*).*

Proof. Let $\{x_i^0, x_i^1\}$ be a basis of V_i^* for each i. Let $z = (a^1 x_1^0 + b^1 x_1^1) \otimes \cdots \otimes (a^n x_n^0 + b^n x_n^1)$ be such that $[z] \in Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*) \cap U_0$ and suppose A is a matrix such that $\varphi([A, t]) = [z]$. The following relations on the 0×0 and 1×1 principal minors of $A = (x_{i,j})$ must hold:

$$\begin{aligned} &t^n = (a^1 \ldots a^n) = z_{[0,\ldots,0]} \\ &t^{n-1} x_{i,i} = (a^1 \ldots a^{i-1} b^i a^{i+1} \ldots a^n) = z_{[0,\ldots,0,1,0,\ldots,0]}. \end{aligned}$$

We are assuming that $z_{[0,\ldots,0]} \neq 0$, so this implies that $a^i \neq 0$ for all i and that $t \neq 0$, so we can solve these equations to find $x_{i,i} = \frac{b^i}{a^i} t$. Also, the following relation on 2×2 minors must hold:

$$t^{n-2}(x_{i,i} x_{j,j} - x_{i,j}^2) = (a^1 \ldots a^{i-1} b^i a^{i+1} \ldots a^{j-1} b^j a^{j+1} \ldots a^n),$$

which implies that $x_{i,j} = 0$ for all $i \neq j$. Therefore A must be a diagonal matrix.

For the converse, suppose $A = (x_{i,j})$ is diagonal; we must show that $\varphi([A, 1]) \in Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$. We can further suppose that the $x_{i,i}$ are of the form $x_{i,i} = \frac{b^i}{a^i}t$ for some constants b^i, a^i and t with the a^i assumed to be nonzero and $t^n = a^1 \ldots a^n$. Because A is assumed diagonal, its principal minors are easy to calculate. Let $I(p)$ is a multi-index with 1's in the positions $p_1, \ldots, p_k$ and 0's elsewhere; then

$$t^{n-k}\Delta_{I(p)}(A) = t^{n-k}x_{p_1,p_1} \ldots x_{p_k,p_k} = \left(\frac{b^{p_1}}{a^{p_1}}t\right) \ldots \left(\frac{b^{p_k}}{a^{p_k}}t\right) = \left(\frac{b^{p_1} \ldots b^{p_k}}{a^{p_1} \ldots a^{p_k}}t^n\right) = \left(\frac{b^{p_1} \ldots b^{p_k}}{a^{p_1} \ldots a^{p_k}}a^1 \ldots a^n\right).$$

But the term $(\frac{b^{p_1} \ldots b^{p_k}}{a^{p_1} \ldots a^{p_k}}a^1 \ldots a^n)$ is the $I(p)$ coefficient of the expansion of the tensor $z = (a^1x_1^0 + b^1x_1^1) \otimes \cdots \otimes (a^nx_n^0 + b^nx_n^1)$, so we have $t^{n-k}\Delta_{I(p)}(A) = z_{I(p)}$ and $[z] \in Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*) \cap U_0$ as required. □

Proposition 5.2. *The tangential variety $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ is the image of the symmetric $n \times n$ matrices of E-rank 1 under the principal minor map.*

To prove Proposition 5.2, we will use Proposition 5.4 and Lemma 5.6 below.

Remark 5.3. Though it is not necessary for this paper, it would be interesting to have a similar geometric description of the principal minors of the E-rank k symmetric matrices for all k. This would provide a geometric stratification of Z_n by E-rank and would enhance our understanding of the geometry of Z_n.

Consider the Veronese embedding of $\mathbb{C}^n$ into the $n \times n$ matrices:

$$\begin{aligned} v_2 : \mathbb{C}^n &\longrightarrow S^2\mathbb{C}^n \\ (y_1, y_2, \ldots, y_n) &\longmapsto \begin{pmatrix} y_1^2 & y_2y_1 & \ldots & y_ny_1 \\ y_1y_2 & y_2^2 & \ldots & y_ny_2 \\ \vdots & \vdots & \ddots & \vdots \\ y_1y_n & y_2y_n & \ldots & y_n^2 \end{pmatrix} = \mathbf{y}.^t\mathbf{y}. \end{aligned}$$

This parameterizes the rank 1 complex symmetric $n \times n$ matrices.

Proposition 5.4 ([18]). *The G-orbit of the image (under φ) of the rank 1 symmetric matrices is the tangential variety to the n-factor Segre variety. In particular, $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right) \subset Z_n$.*

Proof. For this proof only, let $Y := \varphi(\mathbb{P}(v_2(\mathbb{C}^n)) \oplus \mathbb{C})$. We want to show that $G.Y = \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$.

Since $\mathbf{y}.^t\mathbf{y}$ is a rank 1 symmetric matrix, all $k \times k$ minors vanish for $k > 1$, and in particular, the $k \times k$ principal minors vanish for $k > 1$. Therefore a generic point in Y has the form

$$P = \left[t\left(x_1^0 \otimes \cdots \otimes x_n^0\right) + \sum_{i=1}^{n} y_i^2\left(x_1^0 \otimes \cdots \otimes x_{i-1}^0 \otimes x_i^1 \otimes x_{i+1}^0 \cdots \otimes x_n^0\right)\right],$$

where $y_i, t \in \mathbb{C}$. Consider a curve

$$\gamma(s) = x_1(s) \otimes \cdots \otimes x_n(s), \quad s \in \mathbb{C}$$

such that $x_i(0) = x_i^0$ and the derivatives $x_i'(0) = x_i^1$. Then it is clear that γ is a curve in $Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$ through $x_1^0 \otimes \cdots \otimes x_n^0$, and that P is on the tangent line to γ at $s = 0$. So $P \in \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$ and therefore

$$Y \subset \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right),$$

and

$$\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n.Y \subset \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$$

by the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariance of $\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$.

In the other direction, suppose we are given an arbitrary point

$$Q = \left[r_0(q_1 \otimes \cdots \otimes q_n) + \sum_i r_i(q_1 \otimes \cdots \otimes q_{i-1} \otimes q_i' \otimes q_{i+1} \otimes \cdots \otimes q_n)\right] \in \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right),$$

where $r_i \in \mathbb{C}$ for $0 \le i \le n$ not all zero, and (without loss of generality) each pair q_i, q_i' is a linearly independent pair such that $\{q_i, q_i'\} = V_i^*$. The form of Q is generic up to the action of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$, so by changing basis for each V_i by an $SL(2)$ action, we can assume $x_i^0 = q_i$ and $x_i^1 = q_i'$ for each i.

$$P = \left[r_0(x_1^0 \otimes \cdots \otimes x_n^0) + \sum_{i=1}^{n} r_i\left(x_1^0 \otimes \cdots \otimes x_{i-1}^0 \otimes x_i^1 \otimes x_{i+1}^0 \cdots \otimes x_n^0\right)\right],$$

which is the image under φ of the point $[\mathbf{y}.{}^t\mathbf{y}, t]$, where t and y_i are chosen such that $t^n = r_0$ and $r_i = y_i^2 t^{n-1}$. This implies that

$$\tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right) \subset \left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n.Y.$$

Therefore $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n.\varphi(v_2(\mathbb{P}^n)) = \tau\left(Seg\left(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*\right)\right)$. □

The following lemma will be used in the proof of Lemma 5.6 below.

Lemma 5.5 ([18]). *Let T be a G-module and let $X \subset \mathbb{P}T$ be a G-variety. Let $H < G$ be a subgroup which splits T — i.e., $T = W \oplus W^c$ as an H-module. Let $\pi : \mathbb{P}(W \oplus W^c) \dashrightarrow \mathbb{P}((W \oplus W^c)/W^c) \simeq \mathbb{P}W$ be the projection map. The map π is obviously H-equivariant, so the image $\pi(X)$ is an H-invariant subvariety of $\mathbb{P}W$.*

This lemma tells us that if we are presented with a variety that is the projection from a G-variety, then we should look for the symmetry group of our variety among subgroups of G.

Proof. We must consider the fact that π is only a rational map: certainly, $\pi(x) = 0$ if $x \in W^c$, so the map is not defined at all points.

Let U be the open set defined by $U = \{[w_1 + w_2] \mid w_1 \neq 0, w_1 \in W, w_2 \in W^c\}$. Let $U_X = U \cap X$ denote the relatively open set. Let $Y := \overline{\pi(U_X)}$, where the bar denotes Zariski closure. Claim: $H.U_X \subset U_X$. Suppose $h \in H$ and $[w_1 + w_2] \in U$. Then $h.[w_1 + w_2] = [h.w_1 + h.w_2] \in U$ since $0 \neq h.w_1 \in W$ and $h.w_2 \in W^c$. Since X is preserved by G, it is also preserved by any subgroup $H < G$, and therefore we conclude that $H.U_X \subset U_X$.

Let $y \in \pi(U_X)$ and let $h \in H$. By definition, π is surjective onto its image, so let $x \in U_X$ be such that $\pi(x) = y$. Now we use the H-equivariance of π to conclude that $h.y = h.\pi(x) = \pi(h^{-1}.x)$. But by the claim, we know that $h^{-1}.x \in U_X$, so $\pi(h^{-1}.x) \in \pi(U_X)$.

Suppose $y \in \overline{\pi(U_X)}$. Then choose a sequence $y_i \to y \in Y$ such that $\exists x_i \in U_X$ and $\pi(x_i) = y_i$. If $h \in H$ then $h.y_i = h.\pi(x_i) = \pi(h^{-1}.x_i) \in Y$ for all i. If $\{p_i\} \subset Y$ is a convergent sequence such that $p_i \to p$, and f is a polynomial which satisfies $f(p_i) = 0$, then by continuity, $f(p) = 0$ also. So Y must contain all of its limit points, and therefore $h.y_i \to h.y \in Y$, and we conclude that Y is an H-variety. □

Lemma 5.6. *Let X be the variety of $n \times n$ symmetric matrices which have E-rank 1 or less. Then X is an irreducible variety of dimension $2n$, and the image $\varphi(X)$ is an irreducible G-variety for $G \simeq \left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n \subset Sp(2n)$.*

Proof. Claim 1: $\varphi(X)$ is an irreducible $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-variety. The map φ is a rational map, so the fact that $\varphi(X)$ is an irreducible variety will come from the next claim, that X is irreducible. Here we prove the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariance. Our proof is similar to methods used in [18] in the study of the symmetry of Z_n.

Let $\Gamma_n \simeq \mathbb{C}^{\binom{2n}{n}-\binom{2n}{n-2}}$ denote the space of all non-redundant minors of $n \times n$ symmetric matrices. Let $G_\omega(n, 2n) \subset \mathbb{P}\Gamma_n$ denote the Lagrangian Grassmannian embedded by the a variant of the map ψ which we introduced in the proof of Proposition 4.1 that takes a symmetric matrix to a vector of its non-redundant minors. This map and its variants were studied in a more general context by Landsberg and Manivel [14], and the fact that this variant of ψ defines $G_\omega(n, 2n)$ can be found in [14]. $G_\omega(n, 2n)$ is a homogeneous variety and in particular it is invariant under the action of the symplectic group $SP(2n)$.

Let $\pi : G_\omega(n, 2n) \dashrightarrow Z_n$ denote the projection by forgetting the non-principal minors. We will use Lemma 5.5 to prove the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariance of $\varphi(X)$ by checking that the hypotheses are satisfied.

Consider the linear space $L \subset \mathbb{P}\Gamma_n$ defined by setting all $k \times k$ E-minors for $k \geq 2$ equal to 0. Then by definition $\pi(G_\omega(n, 2n) \cap \mathbb{P}L) = \varphi(X)$. Proposition 4.1 implies that L is fixed by the action of $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$.

Γ_n is an $SP(2n)$-module, so by restriction, it is also an $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-module. We further note that L is a vector subspace of Γ_n and an $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-module, so it is an $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-submodule of Γ_n.

One can check that the inclusion of $SL(2)^{\times n}$ as a subgroup of $GL(2n)$ we gave in the proof of Proposition 4.1 actually is an inclusion of $SL(2)^{\times n}$ as a subgroup of $SP(2n) \subset GL(2n)$. So $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$ must act on $G_\omega(n, 2n)$ and leave it invariant, and in particular $G_\omega(n, 2n) \cap \mathbb{P}L$ is $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-invariant. So we have satisfied the hypotheses of Lemma 5.5, with $G = SP(2n)$, $H = \left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$, $T = \Gamma_n$, and $W = L$ (W^c exists because $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$ is reductive). So Lemma 5.5 implies that the image of $G_\omega(n, 2n) \cap L$ under the projection π is an $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-variety.

Claim 2: X is irreducible. We work on the set where $t \neq 0$. Consider the following set of matrices: $Y = \{A \in S^2V \mid A = D + T, D \text{ diagonal}, T \in v_2(\mathbb{P}V)\}$.

The variety Y can be parameterized via

$$\mathbb{P}(\mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C})$$

$$[w_1, \ldots, w_n, y_1, \ldots, y_n, t] \mapsto \left[\begin{pmatrix} w_1^2 & y_1y_2 & \ldots & \ldots & y_1y_n \\ y_1y_2 & w_2^2 & y_2y_3 & \ldots & y_2y_n \\ \vdots & y_2y_3 & w_3^2 & \vdots & \vdots \\ \vdots & \vdots & \ldots & \ddots & y_{n-1}y_n \\ y_1y_n & y_2y_n & \ldots & y_{n-1}y_n & w_n^2 \end{pmatrix}, t^2\right].$$

Then it is clear that all of the 2×2 E-minors vanish on Y, so $Y \subset X$.

For the general case, we need to see that every symmetric matrix having E-rank 1 can be expressed in this form. Work by induction on the size of the matrix. The base case is trivial. Now suppose

$$A = \begin{pmatrix} w_1^2 & y_1y_2 & \dots & \dots & y_1y_n & a_{1,n+1} \\ y_1y_2 & w_2^2 & y_2y_3 & \dots & y_2y_n & a_{2,n+1} \\ \vdots & y_2y_3 & w_3^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & y_{n-1}y_n & a_{n-1,n+1} \\ y_1y_n & y_2y_n & \dots & y_{n-1}y_n & w_n^2 & a_{n,n+1} \\ a_{1,n+1} & a_{2,n+1} & \dots & a_{n-1,n+1} & a_{n,n+1} & a_{n+1,n+1} \end{pmatrix},$$

where we have assumed by induction that the upper left block of A is in the desired form.

The 2×2 E-minors force the vectors $(y_1y_n, y_2y_n, \dots, y_{n-1}y_n)$ and $(a_{1,n+1}, a_{2,n+1}, \dots, a_{n-1,n+1})$ to be proportional, so without loss of generality we may assume that $a_{i,n+1} = y_iy_{n+1}$ for $1 \le i \le n-1$, and y_{n+1} an arbitrary parameter. By comparing to the first column, we find that the vectors $(y_1y_2, \dots, y_1y_n)$ and $(a_{2,n+1}, a_{2,n+1}, \dots, a_{n,n+1})$ must be proportional, and therefore $a_{i,n+1} = y_iy'_{n+1}$ for $2 \le i \le n$, and y'_{n+1} an arbitrary parameter. Combining this information, we must have $a_j = y_jy_{n+1} = y_jy'_{n+1}$ for $2 \le j \le n-1$. If $y_j \ne 0$ for a single j with $2 \le j \le n-1$ then we find that $y_{n+1} = y'_{n+1}$, and in this case, A is in the desired form. Otherwise,

$$A = \begin{pmatrix} w_1^2 & 0 & \dots & \dots & 0 & a_{1,n+1} \\ 0 & w_2^2 & 0 & \dots & 0 & a_{2,n+1} \\ \vdots & 0 & w_3^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & 0 & a_{n-1,n+1} \\ 0 & 0 & \dots & 0 & w_n^2 & a_{n,n+1} \\ a_{1,n+1} & a_{2,n+1} & \dots & a_{n-1,n+1} & a_{n,n+1} & a_{n+1,n+1} \end{pmatrix}.$$

But this is also in the form that we want because (over $\mathbb{C}$) we can set $a_{i,n+1} = y''_iy''_{n+1}$ for $1 \le i \le n$ and $a_{n+1,n+1} = w_{n+1}^2$ for some arbitrary parameters y''_i, y''_{n+1}, and w_{n+1}.

Claim 3: X has dimension $2n$. This is clear from the parameterization in the previous claim. The map that we gave is generically finite to one and the source has dimension $2n$. □

Proof of Proposition 5.2. By Proposition 5.4 above,

$$G.\varphi(\mathbb{P}(v_2(\mathbb{C}^n) \oplus \mathbb{C})) = \tau\left(Seg\left(\mathbb{P}V_1^* \times \dots \times \mathbb{P}V_n^*\right)\right),$$

where $v_2(\mathbb{C}^n)$ is the space of rank 1 complex symmetric $n \times n$ matrices. By Lemma 5.6 we have $\varphi(X) = G.\varphi(X)$, and since the condition of having rank 1 is more restrictive than the condition of having E-rank 1, $X \supset \mathbb{P}(v_2(\mathbb{C}^n) \oplus \mathbb{C})$; therefore

$$\varphi(X) = G.\varphi(X) \supset G.\varphi(v_2(\mathbb{P}V) \oplus \mathbb{C}) = \tau(Seg(\mathbb{P}^1 \times \dots \times \mathbb{P}^1)).$$

So we have $\varphi(X) \supset \tau(Seg(\mathbb{P}^1 \times \dots \times \mathbb{P}^1))$, an inclusion of two varieties that are both irreducible and of the same dimension; therefore we must have equality. □

In summary, to prove the set-theoretic version of the Landsberg–Weyman Conjecture, we needed to show that the tangential variety $\tau\left(Seg\left(\mathbb{P}V_1^* \times \dots \times \mathbb{P}V_n^*\right)\right)$ contains the zero set of the polynomials coming from the modules of quadrics with four $\bigwedge^2$ factors, the cubic polynomials not arising from a product of the quadrics and a linear form, with four $S_{2,1}$ factors and the rest S^3, and the quartic polynomials with three $S_{2,2}$ factors and the rest S^4. The quartic polynomials are set-theoretic defining equations of the variety of principal minors of symmetric matrices. We studied the pull-back of the quadric and cubic polynomials to the space of symmetric matrices and found that this pull-back defines the set of E-rank 1 symmetric matrices. Finally, we showed that the image of the set of E-rank 1 symmetric matrices under the principal minor map is precisely the tangential variety. Therefore if z in the zero set of the quadric, cubic and quartic polynomials in our modules, then z has a E-rank 1 symmetric matrix A mapping to it under the principal minor map, thus implying that z is on the tangential variety. This completes the proof of Theorem 1.3.

6. Applications

6.1. Semiseparable matrices

In applied linear algebra a common subject is that of inverse eigenvalue problems. We point out here that Theorem 1.3 answers a special inverse eigenvalue problem. Indeed it is related to (but different than) the eigenvalue problem and its practical aspects considered by [21]. Before we state the eigenvalue problem we recall some definitions and highlight their differences from the E-rank case.

Definition (*[22, Def. 1.1]*). A matrix is called *symmetric semiseparable* if all submatrices taken out of the lower and upper triangular part (both including the diagonal) of the matrix are of rank 1 and the matrix is symmetric.

Definition (*[22, Def. 1.4]*). A matrix is called *symmetric semiseparable plus diagonal* if it can be written as the sum of a symmetric semiseparable and a diagonal matrix.

Notice that the condition of E-rank 1 for 3×3 matrices is no condition, so every 3×3 matrix has E-rank ≤ 1. On the other hand, not every 3×3 symmetric matrix is semiseparable, nor is every 3×3 symmetric matrix semiseparable plus diagonal (cf. [22, Example 1.7]), so the notions are different.

Definition (*[22, Def. 1.5]*). A matrix is called *symmetric quasiseparable* if all the subblocks, taken out of the strictly lower triangular part of the matrix (or the strictly upper triangular part) are of rank 1, and the matrix is symmetric.

Notice here that every 3×3 symmetric matrix is symmetric quasiseparable, so in this case E-rank 1 symmetric and symmetric quasiseparable agree. However consider the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

A is symmetric quasiseparable since the one 2×2 subblock that is entirely above the diagonal, Δ^{12}_{34}, is 0, but does not have E-rank 1 since the E-minor $\Delta^{14}_{23}(A) \neq 0$.

Since the vanishing of the 2×2 E-minors implies that the blocks in the strictly lower (or upper) triangular part of the symmetric matrix have rank 1, "E-rank 1" is strictly stronger than "quasiseparable", which is known to be strictly stronger than "symmetric semiseparable plus diagonal" [22, Chapter 1].

An *inverse eigenvalue problem* is the following; given a set of values $\mathbf{v} \in \mathbb{C}^{2^n}$, does there exist a matrix $A \in \mathbb{C}^{n \times n}$ (with possible extra properties $\mathcal{P}$) with the eigenvalues of all of its principal submatrices given by $\mathbf{v}$? Since this is equivalent to specifying all the coefficients of the characteristic polynomial, it is equivalent to asking that $\mathbf{v}$ be the principal minors of a matrix A (a principal minor assignment problem). If one knows a set $f_1, \ldots, f_s$ of polynomials which cut out the algebraic variety of possible principal minors of matrices A satisfying $\mathcal{P}$, then the answer to the inverse eigenvalue problem is yes if and only if $\mathbf{v}$ is a root of all of the polynomials $f_1, \ldots, f_s$.

This problem has been solved in the case of 4×4 matrices [1,17]. For the general case of $n \times n$ matrices, while the theoretical answer is still unknown, there is an approximate computational test [10] which, assuming a certain non-degeneracy condition, takes an input $\mathbf{v}$ and produces a matrix A with principal minors $\mathbf{v}$ or says that such a matrix does not exist up to some small error.

In the case where $\mathcal{P}$ is the condition that the matrix be skew-symmetric, the answer to the skew-symmetric Pfaffian assignment problem is solved because the related algebraic variety is an isotropic Grassmannian, whose equations are known. This has been studied in a more general context in [14].

To this list of solutions to inverse eigenvalue problems (or equivalently principal minor assignment problems), we add that the case when $\mathcal{P}$ is the condition that the matrix be symmetric is solved in [19,18] — there exists a symmetric matrix A with principal minors $\mathbf{v}$ if and only if $\mathbf{v}$ is a root of all of the equations in the hyperdeterminantal module (see Theorem 1.2 above). Seen from this perspective, Theorem 1.3 answers the inverse eigenvalue problem when $\mathcal{P}$ is the condition that the matrix be symmetric and have E-rank 1. For completeness we reiterate: there exists a symmetric E-rank 1 matrix A with principal minors given by a vector $\mathbf{v} \in \mathbb{C}^{2^n}$ if and only if $\mathbf{v}$ is a root of F_0, F_2 above, the $2 \times 2 \times 2$ hyperdeterminant and all of the polynomials obtained from these by the natural change of coordinates by the action of the group $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$.

Since "E-rank 1" is strictly stronger than "quasiseparable", we point out that Theorem 1.3 gives necessary but not sufficient conditions for the inverse eigenvalue problem when $\mathcal{P}$ is the condition that the matrix be symmetric quasiseparable.

6.2. Tensor rank

The *tensor rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the number r in a minimal expression

$$T = v_1^1 \otimes v_2^1 \otimes \cdots \otimes v_n^1 + \cdots + v_1^r \otimes v_2^r \otimes \cdots \otimes v_n^r,$$

where $v_i^k \in V_i$ for all k. The Segre variety $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ parameterizes the (closed) set of tensors of tensor rank 1. For each k, the secant variety $\sigma_k(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$ is the Zariski closure of the tensors of tensor rank less than or equal to k. The kth secant variety is also known as the set of tensors of *border rank* less than or equal to k.

The tensor rank and the border rank of a tensor can be quite different. For any variety X, the tangential variety $\tau(X)$ is contained in the secant variety $\sigma_2(X)$. So every tensor in the tangential variety of the Segre variety has border rank no greater than 2. However, the generic point in the tangential variety can have tensor rank n.

In addition, one can also study the case of symmetric tensors, where the Veronese variety $v_d(\mathbb{P}V)$ plays the role of the Segre variety in that it parameterizes the degree d symmetric tensors of rank 1. A generic point of the tangential variety $\tau(v_d(\mathbb{P}V))$ has symmetric border rank 2, but has symmetric tensor rank $d + 1$; see Corollary 4.5 of [15].

Thus the tangential variety is an interesting source of tensors with large rank and small border rank. We point out here that the equations in Theorem 1.3 provide a necessary and sufficient test for a given tensor to be a member of the tangential variety of the Segre product of projective spaces.

6.3. Context-specific independence models

In algebraic statistics one may consider a variety of statistical models. (See [20] for an introduction to this field and for more details.) A standard theme in algebraic statistics is to study the algebraic variety associated with the statistical model of interest. Here we discuss the naive Bayes model, the context-specific independence model, the factor analysis model and their relevant associated varieties.

The *naive Bayes model (NBM)* can be described as follows. Let $X_1, \dots, X_n$ be discrete random variables that have respectively $n_1+1, \dots, n_n+1$ different states, with values taken from vector spaces $V_1, \dots, V_n$. Further assume $X_1, \dots, X_n$ to be pairwise independent of each other but dependent on a hidden or unmeasured random variable Y that has k different states. This model can be depicted as a graphical model by a star shaped tree with one central node and n different leaves.

When $k = 1$ this NBM is just the independence model, and corresponds to the Segre variety $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$. For general k, the Zariski closure of the NBM corresponds to the kth secant variety of the Segre variety [6, Cor. 20]. A general point on this NBM is of the form

$$T = v_1^1 \otimes \cdots \otimes v_n^1 + v_1^2 \otimes \cdots \otimes v_n^2 + \cdots + v_1^k \otimes \cdots \otimes v_n^k,$$

where for each i, $v_i^k \in V_k$ for all k.

A given model may be too complex for certain applications. To overcome this problem, one approach is to add assumptions to the model and thus make the model less complex. *Context-specific independence (CSI) models* take this approach. Indeed, Georgi states "The central idea of the CSI formalism is to increase robustness by making use of regularities in the parameters of a model to reduce and adapt model complexity to the degree of variability observed in the data"; see [7], p.31.

A CSI model restricts the NBM by declaring that some of the parameters are tied together. More specifically, let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be partitions of k and require $v_p^i = v_p^j$ if i and j are in the same block in $\mathcal{A}_p$.

Consider the partitions

$$\begin{aligned} \mathcal{A}_1 &= \{\{1\}, \{2, \dots, n-1, n\}\} \\ \mathcal{A}_2 &= \{\{2\}, \{1, 3, \dots, n-1, n\}\} \\ &\vdots \\ \mathcal{A}_n &= \{\{n\}, \{1, 2, \dots, n-1\}\}. \end{aligned}$$

A general point on this special CSI model is of the form

$$v_1^2 \otimes v_2^1 \otimes \cdots \otimes v_n^1 + v_1^1 \otimes v_2^2 \otimes v_3^1 \otimes \cdots \otimes v_n^1 + \cdots + v_1^1 \otimes \cdots \otimes v_{n-1}^1 \otimes v_n^2,$$

where for each $1 \le i \le n$, $v_i^k \in V_i$ for $k = 1, 2$. This implies that the Zariski closure of this special CSI model is the tangential variety for the Segre variety. In light of this equivalence, we have the following :

Restatement of Theorem 1.3: The CSI model associated with the partitions $\mathcal{A}_i = \{\{i\}, \{1, 2, \dots, i-1, i+1, \dots, n\}\}$, $1 \le i \le n$, is defined set-theoretically by the following equations: the degree 4 polynomials given by the hyperdeterminantal module which is the span of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of the $2 \times 2 \times 2$ hyperdeterminant, and the modules of degree 2 and degree 3 equations given by the span of the $\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n$-orbit of the equations F_0 and F_2 constructed in Section 3.

Here is an example pointed out by Bernd Sturmfels in his course "An Invitation to Algebraic Statistics" during the 2008–09 SAMSI Program on Algebraic Methods in Systems Biology and Statistics.

Example 6.1 (*Sturmfels*)**.** Let random variables X_1, X_2, X_3 respectively take three, two, and two states, and consider the CSI model specified by the partitions $\mathcal{A}_1 = \{\{1, 2\}, \{3\}\}$, $\mathcal{A}_2 = \{\{1, 3\}, \{2\}\}$, and $\mathcal{A}_3 = \{\{2, 3\}, \{1\}\}$. A general point of this special CSI model is of the form

$$v_1^2 \otimes v_2^1 \otimes v_3^1 + v_1^1 \otimes v_2^2 \otimes v_3^1 + v_1^1 \otimes v_2^1 \otimes v_3^2,$$

where $v_l^\epsilon \in V_l$ for $\epsilon = 1, 2$ and $l = 1, 2, 3$. The associated algebraic variety is $\tau(Seg(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1))$. We use coordinates p_{ijk} with $0 \le i \le 2$ and $0 \le j, k \le 1$ to index $3 \times 2 \times 2$ tables — the ambient space for this model. The ideal of this tangential variety is generated by the four 3×3 subdeterminants of the flattening

$$P_{flat} = \begin{pmatrix} p_{000} & p_{001} & p_{010} & p_{011} \\ p_{100} & p_{101} & p_{110} & p_{111} \\ p_{200} & p_{201} & p_{210} & p_{211} \end{pmatrix}$$

and six $2 \times 2 \times 2$ hyperdeterminants, such as

$$\begin{aligned} &p_{000}^2 p_{111}^2 + p_{010}^2 p_{101}^2 + p_{011}^2 p_{100}^2 + p_{001}^2 p_{110}^2 - 2p_{010}p_{011}p_{100}p_{101} - 2p_{001}p_{011}p_{100}p_{110} - 2p_{001}p_{010}p_{101}p_{110} \\ &\quad - 2p_{000}p_{011}p_{100}p_{111} - 2p_{000}p_{010}p_{101}p_{111} - 2p_{000}p_{001}p_{110}p_{111} + 4p_{000}p_{011}p_{101}p_{110} + 4p_{001}p_{010}p_{100}p_{111}. \end{aligned}$$

Note that the reduction of a NBM to this special CSI model is a significant dimensional reduction since for example $\dim(\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))) = 2n$ (see [24]), but Catalisano et al. [2] recently proved that $\sigma_k(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$ for $n \geq 5$ is always of the expected dimension $\min\{kn + n - 1, 2^n - 1\}$.

Implicit in this work is a connection to another statistical model. The *factor analysis (FA) model* in [4] is studied via its parameter space

$$F_{p,m} = \{\Sigma + \Gamma \in \mathbb{R}^{p\times p} \mid \Sigma > 0 \text{ diagonal}, \Gamma \geq 0 \text{ symmetric}, \ rank(\Gamma) \leq m\},$$

where $\Sigma > 0$ and $\Gamma \geq 0$ respectively refer to positive semi-definite and positive definite matrices, and p is the number of observed variables and m is the number of *factors* or hidden variables in the model. Now we can give the following :

Restatement of Proposition 5.2: The image of the parameter space $F_{n,1}$ of the one-factor FA model under the principal minor map is the CSI model associated with the partitions $\mathcal{A}_i = \{\{i\}, \{1, 2, \dots, i-1, i+1, \dots, n\}\}$, $1 \leq i \leq n$.

Here we also reiterate Remark 5.3 in this language, and ask for the precise description of image of the p-factor model under the principal minor map as a CSI model. From our point of view, the invariants for other CSI models might be understood via their pull-back under the principal minor map to the space of symmetric matrices.

We leave to future study the search for the polynomials on 2^n variables which pull back equations on $n \times n$ symmetric matrices such as the $k \times k$ E-minors for $k \geq 3$ and the pentad, septad, etc. equations studied in [4].

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