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Resolutions which split off of the bar construction

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Abstract

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Resolutions which split off of the bar construction are quite common, but explicit formulae expressing these splittings are not often encountered. Given explicit splitting data, perturbations of resolutions can be computed and the perturbed resolutions can be used to make complete effective calculations where previously only partial or indirect results were obtainable.

This paper gives a foundation for the perturbation method in homological algebra by providing a symbolic encoding of binomial coefficient functions which is useful in deriving formulae for an infinite class of resolutions. Formulae for perturbations of those resolutions are then derived. Applications to certain infinite families of groups and monoids are given.

The research for this theory as well as the calculation of closed formulae within the theory was aided by new methods in symbolic computation using the Axiom (formerly called Scratchpad) system.

1. Introduction

1.1. Motivation

Certain formal complexes were associated to groups and formal groups over the ring $\mathbb{Z}/p\mathbb{Z}$ (including the case $p=0$, i.e., the integers) in [16]. These complexes are obtained by *homological perturbation theory* (see [16] and the references cited there) and can be thought of roughly in the following way. If ρ is a formal group law, then $\rho(x, y) = x + y + O(\geq 2)$. One can think of ρ as a 'perturbation' of the affine group $((\mathbb{Z}/p\mathbb{Z})^n, +, 0)$. Given a resolution over the group ring of this affine

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group, one might try to 'deform it' to obtain a 'resolution' over the given formal group. If the formal group law ρ is actually a convergent series, so that one has an ordinary group, this can lead to resolutions suitable for the computation of Ext and Tor over the group ring. In particular, it can lead to computations of the (co)homology of the group. If the group law is actually given by a polynomial function, then in the case $p = 0$, one has a finitely generated, torsion-free nilpotent group, and in the case $p > 0$, one has a finite p-group—and conversely.

For the particular homological perturbation method cited above, it is crucial to have a resolution over the affine group that is embedded in the bar construction (or some other standard resolution) is a special way. When this occurs, the resolution is said to 'split off of' the standard resolution (see Section 2.4). One important aspect of this method is that it produces resolutions that *also* split off of a standard resolution. This makes iteration possible.

Resolutions over group rings of finitely generated abelian groups and which split off of the bar construction are known [4, 6], although the complete data is not often presented. This data is given for a much wider class of objects in (2)–(4) and Lemmas 2.4 and 2.5.

Homological perturbation methods were used in [16] and [17] in the case of linear affine group schemes over the integers and certain convergent power series group laws taking integers to integers. As already mentioned, the method is not restricted to the $p = 0$ case. In fact, it can be applied to groups over rings other than $\mathbb{Z}/p\mathbb{Z}$ as well (with suitable alterations of the 'affine model'). It can also be applied to cases other than group rings or formal groups. A classic example is given by May in [20] for mod-p restricted Lie algebras and Hopf algebras where it is also important to consider resolutions that split off of the bar construction. May's paper as well as the memoirs of Gugenheim and May [12] and the author's previous collaborations with Gugenheim and Stasheff [9–11, 18] provided much inspiration for [16] and this paper.

One of the main goals of this paper is to provide a simple foundation for the calculation and deformation of resolutions that split off of the bar construction.

1.2. General remarks

Let R be a commutative ring with 1, and let A be an augmented algebra over R: $A \xrightarrow{\varepsilon} R$. The bar construction resolution (see Section 4.1) [4–6] of R over A is of the form

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} B(A), s) ,$$

where $B(A) \xrightarrow{s} B(A)$ is an R-module map, and $R \xrightarrow{\sigma} B(A)$ and $B(A) \xrightarrow{\varepsilon} R$ are A-module maps which satisfy $\varepsilon\sigma = 1$, $\partial s + s\partial = 1 - \sigma\varepsilon$. Recall that $B(A) = A \otimes_R \bar{B}(A)$, and

$$\sigma(r) = r[\,]\,,$$

$$\varepsilon(a[a_1 \mid \ldots \mid a_n]) = \begin{cases} 0\,, & \text{if } n > 0\,, \\ \varepsilon(a)\,, & \text{if } n = 0\,, \end{cases}$$

$$s(a[a_1 \mid \ldots \mid a_n]) = [a \mid a_1 \mid \ldots \mid a_n]\,.$$

The relationship between R and $B(A)$ given by ε, σ, and the homotopy s is one of a *strong deformation retraction* of $B(A)$ to R.

Strong deformation retractions are quite common and involve objects other than resolutions, but also involve resolutions in ways other than the one pointed to above. For example, minimal resolutions of a module over an algebra can lead to such retractions. A classic example is given by the Koszul resolution $K = A \otimes E_R[u_1, \ldots, u_n]$ for the ideal $I = (x_1, \ldots, x_n)$ in the polynomial ring $A = R[x_1, \ldots, x_n]$. A is an augmented algebra over R and we may form the *bar construction* $B(A)$ for R over A as above. K is also a resolution of R over A and by the comparison theorem, there is a chain homotopy equivalence $K \to B(A)$. In this case, a map which is the inclusion for a strong deformation retraction $K \rightleftarrows B(A)$ exists. We say that K 'splits off of' the bar resolution (Section 2.4). The small resolutions of Eilenberg and Mac Lane, and of Cartan also give rise to such deformation retractions [4, 6]. The point of this paper is to look more closely at the behavior 'in the large' of such resolutions and to follow up by examining that behavior in some classic examples. All of the general propositions for resolutions that split off of the bar construction given in this paper are applicable to the resolutions for cyclic groups $\mathbb{Z}/p\mathbb{Z}$ found in [4] and [6], but for simplicity in the exposition, for this class, only the case $p = 0$, i.e., the integers will be presented. In fact, well-known resolutions over the free abelian group in n generators and the Koszul resolution will be presented in a new light. These already show some very complicated behavior.

An important distinction between these homological perturbation methods and other methods for calculating resolutions is that they often allow one to calculate resolutions for a class of objects which are parameterized in some way. The perturbation methods allow one to obtain parameterized families of resolutions using the parameters from the given family. This was illustrated in [16] in the case of the 'integral Heisenberg groups' parameterized by q. This class of groups is given by

$$U_3(q) = \left\{ \begin{pmatrix} 1 & x & q^{-1}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}\,.$$

Notice that $U_3(q)$ is a 'deformation' of the free abelian group on three generators, i.e., in the naive sense, $\lim_{q \to 0} U_3(q) = \mathbb{Z}^3$. As a further illustration, parameterized resolutions will be given for a class of non-nilpotent groups in

Section 4. For more information about these sorts of resolutions see [16, 17] and the references cited there.

2. Splitting homotopies and resolutions

Some generalities on algebraic strong deformation retractions are recalled and a general formula for a splitting homotopy (Section 2.2) on the bar construction for a class of resolutions is given in this section.

2.1. Strong deformation retraction

Consider two chain complexes M and N over R. We say that M is a *strong deformation retraction* (SDR) of N if there are chain maps ∇, f, and a chain homotopy ϕ,

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi)\,,$$

such that

$$f\nabla = 1\,, \qquad \nabla f = 1 - (d\phi + \phi d)\,.$$

In addition, it can be assumed that

$$\phi\phi = 0\,, \quad \phi\nabla = 0\,, \quad f\phi = 0\,.$$

Indeed, if these conditions do not hold, they can be obtained by leaving f and ∇ alone and changing ϕ. If the last two do not hold, we can replace ϕ by $\phi'' = D(\phi)\phi D(\phi)$ where $D(\phi) = d\phi + \phi d$. If now the first condition does not hold, all three conditions may be achieved by replacing ϕ'' by $\phi'' = \phi'' d\phi''$. This was pointed out in [18], although the proof was omitted. A special case was given in [7]. A proof, which will contain some notation and observations used elsewhere in this paper will be given here.

By a graded map of degree k we mean an R-module such that $f_n : X_n \to Y_{n+k}$. We use the notation $|f| = k$ for such a map. For convenience, the notation $D(f) = df - (-1)^{|f|}fd$ will be introduced for a map $f : X \to Y$ of degree k of chain complexes X and Y. It is easy to verify that D is a derivation with respect to composition of maps.

Lemma 2.1. *Given a strong deformation retraction*

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi)\,.$$

(1) *If $\phi \neq 0$ or $f\phi \neq 0$ then the vanishing conditions for $\phi\nabla$ and $f\phi$ may be obtained (with no change in f or ∇) by replacing ϕ by $\phi' = D(\phi)\phi D(\phi)$.*

(2) *If $\phi\nabla = 0$ and $f\phi$–0 but $\phi^2 \neq 0$ then the vanishing condition for ϕ^2 may be obtained by replacing ϕ by $\phi' = \phi d\phi$.*

Proof. Let $\pi = \nabla f$. Since $f\nabla = 1$, we have $\pi\pi = \pi$. Also $D(\phi) = 1 - \pi$, and $(1-\pi)(1-\pi) = 1 - \pi$. By hypothesis, $(1-\pi)\nabla = 0$ and $f(1-\pi) = 0$. Consider ϕ' from (1). We have $\phi'\nabla = (1-\pi)\phi(1-\pi)\nabla = 0$ and similarly, $f\phi' = 0$. Also, $D(\phi') = D(\phi)^3$ since $D^2 = 0$ and D is a derivation, but $D(\phi)^3 = (1-\pi)^3 = 1 - \pi$. Now given a ϕ that satisfies the hypotheses of (2), notice that $d\phi\phi d = (1 - \pi - \phi d)(1 - \pi - d\phi)$, but this product is easily seen to vanish. Thus $\phi'\phi' = 0$. □

The conditions obtained in the lemma are often called 'the side conditions'. Also, ∇ is called the 'inclusion', f is called the 'projection', and ϕ is called the 'homotopy' of the given SDR

$$(M \overset{\nabla}{\rightleftarrows} N, \phi)\,.$$

The side conditions will be required in what follows.

2.2. *Resolutions*

Suppose that $\varepsilon : A \to R$ is an augmented algebra. A resolution of R over A is a differential A-module (X, d) which is projective as an A-module and which is 'acyclic on A', i.e., the homology of (X, d) is zero except in degree 0 where it is R. If X is actually a free A-module then (X, d) is called a free resolution. A particularly useful class of free resolutions are those of the form $X = A \otimes \bar{X}$ where $\bar{X}$ is a free R-module. For such free resolutions, let $(\bar{X}, \bar{d}) = (R \otimes_A X, 1_R \otimes_A d)$. We will call the complex $(\bar{X}, \bar{d})$ the 'reduced complex'. We think of $\bar{X} \hookrightarrow X$ via $x \mapsto 1 \otimes x = \bar{x}$.

By the comparison theorem for resolutions, there exists a chain homotopy equivalence $B(A) \to X$. In cases of interest, it often occurs that this chain homotopy equivalence may be completed to an SDR

$$(X \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi)\,.$$

An explicit contracting homotopy on X leads to an implementation of this retraction.

Assume that $X \to R$ is a resolution with an explicit contracting homotopy φ

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} X, \varphi)\,.$$

One has explicit contracting homotopies for the standard 'small resolutions' of Eilenberg and Mac Lane and Cartan mentioned above (see [4, 6]).

Comparison maps $\nabla : X \to B(A)$ and $f : B(A) \to X$ which are A-linear may be defined inductively, using the contracting homotopy φ for X and the standard contracting homotopy s for $B(A)$. One has

$$\nabla(\bar{x}) = s\nabla d\bar{x}\ , \quad \text{for } \bar{x} \in \bar{X}$$

and

$$f(\bar{b}) = \varphi f \partial \bar{b}\ , \quad \text{for } \bar{b} \in \bar{B}(A)\ .$$

A straightforward and useful criterion for when the map ∇ defined above is one–one is given by May in the following theorem:

Theorem 2.2 [20, Section 7]. *Let $X = A \otimes_R \bar{X} \to R$ be a free resolution of R over A. Let $\nabla : X \to B(A)$ be given by $\nabla(\bar{x}) = s\nabla d\bar{x}$, for $\bar{x} \in \bar{X}$. If $d(\bar{X}_n) \cap \bar{X}_{n-1} = 0$, then ∇ is one–one.* □

In general, both composites ∇f and $f\nabla$ are homotopic to the respective identity maps and we may use the given contracting homotopies to explicitly construct these homotopies. Using the mapping cylinder construction, such 'two-sided data' may be factored into the composite of two sets of 'one-sided data'. We shall assume that we are in the one-sided case, i.e., that $f\nabla = 1_X$. Then we have an SDR

$$(X \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi)\ ,$$

where the A-linear chain homotopy $\phi : B(A) \to B(A)$ is given inductively by $\phi(\bar{b}) = s(\nabla f\bar{b} - \bar{b} - \phi\partial\bar{b})$. Note that s vanishes on $\bar{B}(A)$ so this formula reduces to $\phi(\bar{b}) = s(\nabla f\bar{b} - \phi\partial\bar{b})$. These formulae will be summarized in Section 2.4.

2.3. *Splitting homotopies*

Consider an SDR

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi)$$

where M and N are any R-modules. As in Lemma 2.1, let $\pi = \nabla f$. Since π is a projection, there is a direct-sum decomposition

$$N = \operatorname{im}(\pi) \oplus \ker(\pi)\ .$$

In fact, because of the side conditions, it is straightforward to see that the homotopy vanishes on $\mathrm{im}(\pi)$. Furthermore, restricted to $\ker(\pi)$, it is a contracting homotopy. Thus, $\ker(\pi)$ is a totally contractible complex. In addition, $M \cong \mathrm{im}(\pi)$ as differential modules and the original SDR is completely determined by ϕ.

Conversely, given a degree-one map $\phi : M \to M$ which satisfies $\phi\phi = 0$, and $\phi d\phi = \phi$, then by setting $\pi = 1 - (d\phi + \phi d)$ and $M = \mathrm{im}(\pi)$, we obtain an SDR

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi) .$$

These observations were made in [1] where such homotopies ϕ were called 'splitting homotopies'.

In summary, an SDR is completely determined by a splitting homotopy ϕ, i.e., a degree-one endomorphism of a differential module N which satisfies

$$\phi^2 = 0 , \qquad \phi d\phi = \phi .$$

Given an SDR, i.e., a splitting homotopy ϕ with $M = \mathrm{im}(\pi)$, we will often say that M *splits off* of N.

The following lemma can be found in [6] and is used in [2], [7], [18], and [24].

Lemma 2.3. *Given strong deformation retractions*

$$(M_i \underset{f_i}{\overset{\nabla_i}{\rightleftarrows}} N_i, \phi_i) ,$$

for $i = 1,2$, *there are tensor product strong deformation retractions*

$$(M_1 \otimes M_2 \underset{f_1 \otimes f_2}{\overset{\nabla_1 \otimes \nabla_2}{\rightleftarrows}} N_1 \otimes N_2, \phi) ,$$

where either

$$\phi = 1 \otimes \phi_2 + \phi_1 \otimes \pi_2$$

or

$$\phi = \pi_1 \otimes \phi_2 + \phi_1 \otimes 1 .$$

Furthermore, if ϕ_i *satisfy the side conditions for* $i = 1,2$, *then so does* ϕ. □

There is no particular reason to choose one of the homotopies over the other. We will consistently use $\phi = 1 \otimes \phi + \phi \otimes \pi$ and when we consider tensor products of n-resolutions, we will use the tensor product homotopy

$$\phi^{\otimes} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \phi + 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \phi \otimes \pi$$
$$+ \cdots + 1 \otimes \phi \otimes \pi \otimes \cdots \otimes \pi + \phi \otimes \pi \otimes \cdots \otimes \pi \,. \qquad (1)$$

By the lemma, if ϕ satisfies the side conditions, then so does $\phi^{\otimes}$.

2.4. Resolutions which split off of the bar construction

For the remainder of this paper, assume that X is a free A-resolution of R and that an explicit contracting homotopy φ is given, so that we have an SDR

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} X, \varphi) \,.$$

By Lemma 2.1 we may assume the side conditions. In particular, we have $\varphi^2 = 0$. Now assume that

$$\nabla(\bar{x}) = s\nabla d\bar{x} \,, \quad \text{for } \bar{x} \in \bar{X} \,, \qquad (2)$$

$$f(\bar{b}) = \varphi f \partial \bar{b} \,, \quad \text{for } \bar{b} \in \bar{B}(A) \,, \qquad (3)$$

$$\phi(\bar{b}) = s(\nabla f \bar{b} - \phi \partial \bar{b}) \,, \quad \text{for } \bar{b} \in \bar{B}(A) \,, \qquad (4)$$

define maps such that $\nabla f = 1_X$ so that we have an explicit splitting homotopy ϕ on the bar construction,

$$((A \otimes \bar{X}, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi) \,.$$

This situation is not unusual (see for example, [4], [6], [12, Appendix], [14]).

It follows that on reduced elements, i.e., on elements of $\bar{X}$, we have the following:

Lemma 2.4. *The A-linear map f is given inductively by*

$$f[b_1] = \varphi(b_1) \,, \qquad f[b_1 \mid \ldots \mid b_n] = \varphi(b_1 f([b_2 \mid \ldots \mid b_n])) \,.$$

Proof. On reduced elements, the inductive formula for f gives

$$\begin{aligned} f[b_1 \mid \ldots \mid b_n] &= \varphi f \partial [b_1 \mid \ldots \mid b_n] \\ &= \varphi(b_1 f[b_2 \mid \ldots \mid b_n]) + \sum \pm \varphi f[b_1 \mid \ldots \mid b_i b_{i+1} \mid b_n] \,. \end{aligned}$$

But if $\bar{x}$ is reduced, then $\varphi f \bar{x} = \varphi \varphi f \partial \bar{x} = 0$. The result follows. □

The following notation [16] is useful. If $b = \sum [b_{i_1} \mid \ldots \mid b_{i_k}] \in \bar{B}(A)$, then for $[x_1 \mid \ldots x_n] \in \bar{B}(A)$, let

$$[x_1 \mid \ldots \mid x_n : b] = \sum [x_1 \mid \ldots \mid x_n \mid b_{i_1} \mid \ldots \mid b_{i_k}] \in \bar{B}(A) .$$

A straightforward induction using the recursive formula for ϕ yields the following:

Lemma 2.5 [16]. *Assuming the hypotheses above for the splitting homotopy* ϕ,

$$\phi[b_1 \mid \ldots \mid b_k] = \sum_{i=0}^{k-1} (-1)^i [b_1 \mid \ldots \mid b_i : s\nabla f[b_{i+1} \mid \ldots \mid b_k]] . \qquad \square$$

It is important to note that Lemmas 2.4 and 2.5 show that f and ϕ are *completely determined* by the contracting homotopy $\varphi : X \to X$ for any resolution X that splits off of the bar construction

$$(X \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi) .$$

The following corollary is also given in [16]. For convenience, its simple proof is indicated here as well.

Lemma 2.6 [16]. *Assuming the hypotheses above, if X is a finite resolution, i.e., X vanishes above degree n for some n, then for all $m > n$*

$$\phi[b_1 \mid \ldots \mid b_m] = (-1)^{m-n} [b_1 \mid \ldots \mid b_{m-n} : \phi[b_{m-n+1} \mid \ldots \mid b_m]] .$$

Thus ϕ is completely determined by $\phi[b_1]$, $\phi[b_1 \mid b_2], \ldots, \phi[b_1 \mid \ldots \mid b_n]$.

Proof. Since $\phi - s\nabla f - s\phi\partial$ on $\bar{B}(A)$ and, necessarily, f vanishes on elements of degree greater than n, we have that for $m > n$, $\phi = -s\phi\partial = -s\phi p$, where $p[b_1 \mid \ldots \mid b_m] = b_1[b_2 \mid \ldots \mid b_m]$. $\square$

There is a wealth of information on finite resolutions in [22].

Since little can be said about the general nature of the differential d in X for an arbitrary resolution that splits off of the bar construction, a general form for the inclusion ∇ cannot be given. However, in many of the classical cases that have already been mentioned, ∇ turns out to be related to the well-known 'shuffle' operation. Two cases that will be examined in detail are the Koszul resolution over a polynomial algebra $P = R[t_1, \ldots, t_n]$ and the analogous resolution over the Laurent polynomials $A = R[t_n^{-1}, \ldots, t_1^{-1}, t_1, \ldots, t_n]$.

3. Resolutions over polynomials and Laurent polynomials

Methods of homological perturbation theory were used in [16] and [17] to calculate explicit resolutions of $\mathbb{Z}$ over the integral group ring A of a finitely generated nilpotent group. The resolutions that were split off of the bar construction as a step in the process will be reviewed along with closed formulae which allow one to reproduce and extend those calculations. The method is illustrated further in Section 4 where a parameterized class of resolutions for some semidirect product groups is calculated.

In addition, we will view the classic Koszul resolution from this perspective. As is pointed to in the cases above, such splittings of the bar construction in general can lead to solutions of other 'transference problems' (in the terminology of [1, 16]) including those involving Lie algebras.

For the remainder of this paper, let $P = R[t_1, \ldots, t_n]$ denote the polynomial algebra and $A = R[t_n^{-1}, \ldots, t_1^{-1}, t_1, \ldots, t_n]$ the Laurent polynomial algebra over R. Of course, A is the group ring of the free abelian group G on n-generators $t_1, \ldots, t_n$ (which we write multiplicatively) and P may be thought of as the monoid ring on the positive cone (with respect to the usual order) in G. If we should need to refer to the number of multiplicative generators n, we will write A_n and P_n.

There are natural augmentations in P and A which are quite distinct. This distinction is at the heart of the difference between Lie algebra cohomology and group cohomology. One striking way in which this manifests itself is the action of the generalized Steenrod algebra in the cohomology of a cocommutative bialgebra. It was shown in [21] that there is such an action for any cocommutative augmented bialgebra and, in fact, an algorithm was given for its action. Both A and P are cocommutative Hopf algebras and it is easy to compute that the generalized Steenrod algebras obtained are different. In fact, they differ in only one relation. For A one has $\mathrm{sq}^0 = 1$, and for P, one has $\mathrm{sq}^0 = 0$. These facts are well known. In the first case, one has the ordinary Steenrod algebra and in the second case, one has the 'Lie algebra version' of the Steenrod algebra [23].

3.1. Exterior algebra SDR

Both A_n and P_n are tensor products of n copies of $A_1 = R[t]$ and $P_1 = R[t^{-1}, t]$ respectively. Thus, we begin with these one-generated algebras. It will be convenient to use the notation $p(t)$ for an element in either A_1 or P_1. When possible, both algebras will be discussed simultaneously. Thus, let Q denote either A_1 or P_1. The augmentation in Q is given by

$$A \xrightarrow{\varepsilon} R \to 0\,, \qquad \varepsilon(p(t)) = p(q)\,,$$

where $q = 0$ if $Q = P_1$ and $q = 1$ if $Q = A_1$. The unit of Q is simply given by $\sigma(r) = r \cdot 1$.

An R-basis of Q is given by the set of all t^i where i ranges over the non-negative integers in the case of P_1 and over all integers in the case of A_1.

Theorem 3.1. *Let Q be either the group ring $R[t^{-1}, t]$ of the free abelian group on one generator t or the polynomial algebra $R[t]$ on one generator t (over an arbitrary commutative ring with one R). Let $E[u]$ denote the exterior algebra on one 1-dimensional generator u.*

Define a Q-linear map $d : Q \otimes_R E[u] \to Q \otimes_R E[u]$ of degree -1 and an R-linear map $\varphi : Q \otimes_R E[u] \to Q \otimes_R E[u]$ of degree $+1$ by

$$d(t^n) = 0\,, \qquad d(u) = t - \varepsilon(t)\,,$$
$$\varphi(t^n) = t^{\{n\}}u\,, \qquad \varphi(t^n u) = 0\,,$$

where, in the polynomial case,

$$t^{\{n\}} = \begin{cases} t^{n-1}\,, & \text{if } n > 0\,, \\ 0\,, & \text{if } n = 0 \end{cases}$$

and, in the group ring case,

$$t^{\{n\}} = \frac{t^n - 1}{t - 1}\,.$$

$(Q \otimes_R E[u], d)$ is a Q-free resolution of R. Extending the augmentation ε and unit σ to $(Q \otimes_R E[u], d)$ in the obvious way ($\sigma(n) = n \otimes 1$, and $\varepsilon(u) = 0$), we have an SDR

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} (Q \otimes_R E[u], d), \varphi)\,.$$

Proof. The identity $d\varphi + \varphi d = 1 - \sigma\varepsilon$ is easily verified. $\square$

The resolutions mentioned in Theorem 3.1 are, of course, well known, as are their tensor products below; however, their contracting homotopies, crucial to many constructions, are less widely distributed. The approach taken here presents both resolution and contracting homotopy as one entity, viz., an SDR. This is a very useful viewpoint. Generally, there are algorithms which transform one or more resolutions into another resolution. This is an important aspect of homological perturbation theory as mentioned in the Introduction. In these cases, the algorithms are designed to produce new SDR's, i.e., the contracting homotopy on the new resolution is also constructed. This makes it possible to iterate.

As already mentioned (see (1)), one can tensor strong deformation retractions to produce a resolution

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} Q_n \otimes_R E[u_1, \ldots, u_n], \varphi)$$

where Q is either the group ring of the free abelian group on n generators $t_1, \ldots, t_n$ or the polynomial ring on n generators $t_1, \ldots, t_n$. In fact, the polynomial case in degree one yields exactly the contracting homotopy of Hochschild [13] for the Koszul resolution. We will present these contracting homotopies in Lemma 3.3 using the notation defined in Theorem 3.1 for $t_i^{\{n\}}$.

The actions of the maps π, φ, and ε for the one-generated case will be recorded in Lemma 3.2. Let Q be either $A = R[t^{-1}, t]$ or $P = R[t]$, as usual.

Lemma 3.2. *φ, ε, and π are R-linear maps such that*

$$\begin{array}{lll} \varphi(1) = 0\,, & \varepsilon(1) = 1\,, & \pi(1) = 1\,, \\ \varphi(t^i) = t^{\{i\}}\,, & \varepsilon(t^i) = \begin{cases} 1\,, & \text{if } Q = A\,, \\ 0\,, & \text{if } A = P,\ i > 0\,, \end{cases} & \pi(t^i) = \varepsilon(t^i)\cdot 1\,, \\ \varphi(t^i u) = 0\,, & \varepsilon(t^i u) = 0\,, & \pi(t^i u) = 0\,. \quad \square \end{array}$$

Note that $\varphi\varphi = 0$ for the contracting homotopy above. In fact, the side conditions hold for

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} Q \otimes E[u], \varphi)\,.$$

The tensor product of these resolutions has the contracting homotopy of (1). It is a sum $\varphi^{\otimes} = \sum_{s=1}^{n} \varphi_s$ where

$$\begin{aligned} \varphi_s &= 1 \otimes \cdots \otimes 1 \otimes \varphi \otimes \pi \otimes \cdots \otimes \pi \\ &= 1^{\otimes s-1} \otimes \varphi \otimes \pi^{\otimes n-s}\,. \end{aligned}$$

From Lemma 3.2 we thus obtain the following:

Lemma 3.3. *There is an explicit contracting homotopy $\varphi^{\otimes}$ such that*

$$(R \underset{\varepsilon}{\overset{\sigma}{\rightleftarrows}} Q_n \otimes_R E[u_1, \ldots, u_n], \varphi^{\otimes})$$

is an SDR satisfying the side-conditions. It is given by $\varphi^{\otimes} = \sum_{s=0}^{n} \varphi_s$ where, on an additive basis element $t_1^{i_1} \ldots t_n^{i_n} u_1^{\xi_1} \ldots u_n^{\xi_n}$ ($\xi_i \in \{0, 1\}$),

$$\begin{aligned} &\varphi_s(t_1^{i_1} \ldots t_n^{i_n} u_1^{\xi_1} \ldots u_n^{\xi_n}) \\ &\quad = (-1)^{\operatorname{sgn}(\xi_1, \ldots, \xi_{s-1})} t_1^{i_1} \ldots t_{s-1}^{i_{s-1}} t_s^{\{i_s\}} \pi_{i_s+1, \ldots, i_n}^{\xi_{s+1}, \ldots, \xi_n} u_1^{\xi_1} \ldots u_{s-1}^{\xi_{s-1}} u_s\,, \end{aligned}$$

$$\pi_{i_k, \ldots, i_l}^{\xi_k, \ldots, \xi_l} = \begin{cases} 0\,, & \text{if } \xi_k + \cdots + \xi_l \neq 0 \\ \pi(t_k^{i_k}) \ldots \pi(t_n^k)\,, & \text{if } \xi_k + \cdots + \xi_l = 0 \end{cases}$$

and sgn *is the function*

$$\operatorname{sgn}(\xi_1, \ldots, \xi_{s-1}) = \begin{cases} 1\,, & \text{if } s = 1\,, \\ \xi_1 + \cdots + \xi_{s-1}\,, & \text{if } s > 1\,. \end{cases} \quad \square$$

For example, in the group ring case for $n = 4$, let $\bar{t}^{\iota} = t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4}$. Then explicitly,

$$
\begin{aligned}
&\varphi^{\otimes}(\bar{t}^{\iota}) = t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_4 + t_1^{i_1}t_2^{i_2}t_3^{\{i_3\}}u_3 + t_1^{i_1}t_2^{\{i_2\}}u_2 + t_1^{\{i_1\}}u_1\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_1) = -(t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_1u_4 + t_1^{i_1}t_2^{i_2}t_3^{\{i_3\}}u_1u_3 + t_1^{i_1}t_2^{\{i_2\}}u_1u_2)\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_2) = -(t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_2u_4 + t_1^{i_1}t_2^{i_2}t_3^{\{i_3\}}u_2u_3)\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_3) = -t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_3u_4\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_1u_2) = t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_1u_2u_4 + t_1^{i_1}t_2^{i_2}t_3^{\{i_3\}}u_1u_2u_3\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_1u_3) = t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_1u_3u_4\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_2u_3) = t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_2u_3u_4\,,\\
&\varphi^{\otimes}(\bar{t}^{\iota}u_1u_2u_3) = -t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{\{i_4\}}u_1u_2u_3u_4\,.
\end{aligned}
\tag{5}
$$

Using (2)–(4), Lemmas 2.4 and 2.5, Theorem 3.1 and Lemmas 3.2 and 3.3, it is not difficult to see that there is an SDR of the form

$$(Q_n \otimes_R E[u_1, \ldots, u_n] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(Q_n), \phi)\,.$$

Explicit formulae for ∇, f, and ϕ will be given in Lemma 3.5. The one-dimensional group ring case was given in [6]. Using the present notation, ∇, f, and ϕ are Q-linear maps such that

$$
\begin{aligned}
&\nabla(u) = [t]\,,\\
&f(x) = \begin{cases} t^{\{i\}}\,, & \text{if } x = [t^i]\,,\\ 0\,, & \text{if } |x| > 1\,,\end{cases}\\
&\phi[t^{i_1} \mid \ldots \mid t^{i_k}] = (-1)^{k+1}[t^{i_1} \mid \ldots \mid t^{i_{k-1}} \mid t^{\{i_k\}} \mid t]\,.
\end{aligned}
$$

Before giving the n-dimensional case, some new notation will be introduced. Write

$$t^i t^{\{k\}} = t^{i+\{k\}} = t^{\{k\}+i}\,.$$

Notice that if $k = 0$, then $t^i t^{\{k\}} = 0$, and, if $k = 1$, then $t^i t^{\{k\}} = t^i$. It will also be convenient to talk about exponents $e = i + \{k\}$, etc., for the expression $t^e = t^{i+\{k\}}$.

Lemma 3.5. *There is an SDR*

$$(Q_n \otimes_R E[u_1, \ldots, u_n] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(Q_n), \phi)\,.$$

The inclusion ∇ is completely determined by the fact that it is Q-linear, $\nabla(u_i) = [t_i]$, and $\nabla(uv) = \nabla(u) * \nabla(v)$ where '$*$' denotes the well-known shuffle product in the bar *construction* [5, 19]. *In other words, the inductive formula* (2) *produces exactly the classical formula for the inclusion of the exterior algebra in the bar construction given in the cited references.*

The projection f, a Q-linear map, is given by

$$f[b_1 \mid \ldots \mid b_k] = (-1)^{\nu_k} \sum_{1 \le p_1 < \cdots < p_k \le n} f(b_1, \ldots, b_k)^{p_1, \ldots, p_k} u_{p_1} \ldots u_{p_k} ,$$

where

$$f(b_1, \ldots, b_k)^{p_1, \ldots, p_k} = \prod_{s=1}^{p_k} t_s^{e(b_1, \ldots, b_k)_s^{p_1, \ldots, p_k}}$$

and for $s = 1, 2, \ldots, p_k$, *the exponential* $e(b_1, \ldots, b_k)_s^{p_1 \ldots p_k}$ *depends upon which interval of the partition of* $\{1, 2, \ldots, p_k\}$ *determined by* $p_1 < \cdots < p_k$ s *is in. For* $0 \le i \le k-1$, *one has*

$$e(b_1, \ldots, b_k)_s^{p_1, \ldots, p_k} = \begin{cases} \exp(b_1, s) + \cdots + \exp(b_{k-i+1}, s) , \\ \quad p_i < s < p_{i+1} , \\ \exp(b_1, s) + \cdots + \exp(b_{k-i}, s) + \{\exp(b_{k-i+1}, s)\} , \\ \quad s = p_{i+1} ; \end{cases}$$

furthermore, exp *is given by*

$$\exp(t_1^{i_1} \ldots t_n^{i_n}, s) = i_s$$

and the exponent ν_k *is given by* $\nu_k = \binom{k-1}{2}$.

The homotopy ϕ *is given by this and Lemma* 2.5,

$$\begin{aligned} \phi[b_1 \mid \ldots \mid b_k] = s\nabla f[b_1 \mid \ldots \mid b_k] - [b_1 : s\nabla f[b_2 \mid \ldots \mid b_k] \\ + \cdots + (-1)^{k-1}[b_1 \mid \ldots \mid b_{k-1} : s\nabla f([b_k])] . \end{aligned}$$

Proof. The proof is a straightforward application of (2)–(4), Lemmas 2.4 and 2.5, Theorem 3.1 and Lemmas 3.2 and 3.3, using induction. □

As with many inductions, it is useful to investigate the first few cases of Lemma 3.5. Parts of the rank-three and rank-four cases were given in [16] and [17]. Of course, the rank-four case can be worked out completely using (5). It should be emphasized that the Scratchpad computer algebra system was quite useful in investigating general perturbation formulae. In fact, (5) was calculated symbolically within that system. Needless to say, it would have been quite tedious initially to investigate these things 'by hand'. In retrospect, one can see general combinatorial patterns which allow the development of such formulae, but they will not be presented here. It is noted that even when these general formulae are obtained, it is most convenient to use them via computer algebra. The parameterized class of resolutions given in Section 4 below were computed completely symbolically and quite conveniently using Scratchpad. The point of such computations is, as always, to develop a feel for a particular kind of mathematical structure and to reinforce intuition. It is felt that the examples presented here are of sufficient complexity to shed some light on what to expect in other cases, but, it is hoped, are still within the bounds of palatability.

4. Application to resolutions over certain monoids

This section relies heavily upon the results in [16] for calculating resolutions by deforming resolutions that split off of the bar construction. The idea will be briefly sketched, but the reader should see [17] and [16] for more details.

4.1. A parameterized class

A very simple class of semi-direct products of the form $G \times_f K$ where $f: K \to \mathrm{Aut}(H)$ is a given group homomorphism will be considered here. In fact, we will look only at the case $K = \mathbb{Z}$ and $H = \mathbb{Z}^2$. A homomorphism f as above is then given simply by an invertible matrix of integers F. Note that F is simply a 2×2 matrix of integers with determinant ± 1. The semi-direct product group G determined by such a matrix

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has underlying set $G = \mathbb{Z}^2 \times \mathbb{Z}$ and has group law given by

$$((x_1, x_2), x_3)((y_1, y_2), y_3) = \left((x_1, x_2) + \left(F^{x_3}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^{\mathrm{t}}, x_3 + y_3\right) \tag{6}$$

where F^{x_3} denotes exponentiation of matrices, and 't' denotes matrix transpose. We could require the condition

$$ab - dc = \pm 1$$

as originally stated, but in any case, the product rule given in (6) gives a *monoid* operation on $\mathbb{Z}^3$, the identity element being $(0, 0, 0)$. For a given F, write the corresponding monoid as G_F.

It is well known that one has a theory of homological algebra over monoids in analogy with the theory for groups. We will consider the class of monoids given in

(6) for arbitrary F and present a small resolution of the integers over the integral monoid ring of G_F.

To begin, note that there is a bar construction for G_F just as in the group ring case (see Section 1.2 and the references cited there). We briefly describe the bar construction here. Let $A = R(G_F)$ be the monoid ring (i.e., the free R-module with basis G_F and multiplication given by extending the multiplication of G_F linearly). The specific augmentation considered on A is analogous to that of the group ring case

$$\varepsilon : A \to \mathbb{Z}\,, \qquad \varepsilon\Big(\sum n_g g\Big) = \sum n_g\,.$$

Recall that different augmentations can give quite different cohomological results (cf. the introduction of Section 3).

The model of $\bar{B}(G_F)$ we use is given as follows. $\bar{B}_0(G_F)$ is the free R-module generated by one element denoted by $[\,]$, $\bar{B}_n(G_F)$ is the free R-module generated by all elements of the form $b = [g_1 \mid \ldots \mid g_n]$ where $g_i \in G_F$ with the convention that the element is zero if one of the g_i is the identity of G_F. The bar construction resolution is given by $B(A) = A \otimes \bar{B}(A)$ where the A-linear differential ∂ is

$$\begin{aligned}\partial[g_1 \mid \ldots \mid g_n] = {} & g_1[g_2 \mid \ldots \mid g_n] \\ & + \sum (-1)^i [g_1 \mid \ldots \mid g_{i-1} \mid g_i g_{i+1} \mid g_{i+2} \mid \ldots \mid g_n] \\ & + (-1)^n [g_1 \mid \ldots \mid g_{n-1}]\,.\end{aligned}$$

Homological perturbation theory may be used to find a small resolution over the monoid ring $R(G_F)$ using the exterior algebra resolution which splits off of the bar construction in Lemma 3.5. As described briefly in Section 1.1, the idea is this. One has an SDR

$$(Q_3 \otimes E[u_1, u_2, u_3] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(Q_3), \phi)\,. \tag{7}$$

Q_3 and $R(G_F)$ have the same underlying R-module structure and because of this, $B(Q_3)$ has the same underlying R-model structure as the bar construction $B(G_F)$. Let B denote this R-module for either case. Thus B supports two different differentials, viz., the bar construction differential ∂^+ for the free abelian group, and the bar construction differential ∂ for the monoid G_F. Let $\wp = \partial - \partial^+$ (i.e., $\wp$ is an *initiator* [1, 9, 18]). In this situation, there is a formal process which, when convergent, gives a new SDR

$$((R(G) \otimes E[u_1, u_2, u_3], \partial') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (B, \partial), \phi')\,.$$

Only the differential for the formal solution will be written here. There are analogous formulae for the inclusion, projection, and homotopy [1, 2, 7, 24].

$$\partial' = f\wp\nabla + \cdots + f(\wp\phi)^n\wp\nabla + \cdots .$$

This formula along with the analogous formulae for the rest of the SDR first occurred in [24] for the study of the normalized chain complex of a simplicial fibration. It was noticed that they could be applied in more general situations in [2] (inspired by conversations with M.G. Barratt [3]). They were developed into what is known now as 'the basic perturbation lemma' in [9–11, 18] and in [7] where some of their formal properties in terms of asymptotic behavior were given as well. A natural explanation of how the formulas arise is given in [1] where they are proven to be essentially unique. Some very concrete applications to resolutions over group rings of nilpotent groups can be found in [17], and, as already mentioned, an application to formal groups occurs in [16] where models for spectral sequences are also developed by their use. Their application to the calculation of resolutions first occurs as a remark in [18], but was expanded in [17]. It was noticed in [16] that the two-stage method mentioned in [18] and [17] could be replaced by one application of the basic perturbation lemma as above.

By using the explicit data from Lemma 3.5 in this case we obtain the following:

Proposition 4.1. *Consider the monoid G_F with underlying set $\mathbb{Z}^3$ and monoid law* (6). *Using the homological perturbation method referred to above, one has a resolution that splits off of the bar construction*

$$((R(G)\otimes E[u_1, u_2, u_3], \partial') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (B, \partial), \phi'),$$

where

$$\begin{aligned}
&\partial' u_i = t_{i-1},\\
&\partial' u_1u_2 = (t_1 - 1)u_1 - (t_2 - 1)u_1,\\
&\partial' u_1u_3 = (t_1^f t_2^h - 1)u_3 + t_1^f t_2^{\{h\}}u_2 + (t_1^{\{f\}} - t_3)u_1,\\
&\partial' u_2u_3 = (t_1^g t_2^k - 1u_3 + (t_1^g t_2^{\{k\}} - t_3)u_2 + t_1^{\{g\}}u_1,\\
&\partial' u_1u_2u_3 = (t_1^f t_2^h - 1)u_2u_3 - (t_1^g t_2^k - 1)u_1u_3\\
&\qquad + (t_1^{\{g\}+f}t_2^{\{h\}} - t_1^{\{f\}+g}t_2^{\{k\}} + t_3)u_1u_2 . \qquad \square
\end{aligned}$$

In order to discuss the reduced complex, i.e., the complex whose homology is the homology of G_F, we need a lemma that is of independent interest because iterates of the construction in Lemma 3.3 occur in general formulae for resolutions obtained by the methods in this paper. The proof is straightforward.

Lemma 4.2. *Let Q be the Laurent polynomial ring in the variable t and let ε be the augmentation* (*Section* 3.1). *As in Lemma* 3.3 *define $t^{\{i\}} = (t^i - 1)/(t - 1)$. Extend this notation for $i \geq 0$ by defining the iterates*

$$t^{\{i\}_2} = \sum_{j=0}^{i-1} t^{\{j\}}, \quad t^{\{i\}_3} = \sum_{j=0}^{i-1} \sum_{k=0}^{j-1} t^{\{k\}}, \quad \ldots;$$

then we have

$$\varepsilon(t^{\{i\}_k}) = \binom{i}{k}. \qquad \square$$

Because of this, the reduced complex whose homology is the homology of the monoid G_F is given by

$$\begin{aligned}
\bar{\partial}' u_i &= 0, \\
\bar{\partial}' u_1 u_2 &= 0, \\
\bar{\partial}' u_1 u_3 &= (f-1)u_1 + hu_2, \\
\bar{\partial}' u_2 u_3 &= gu_1 + (k-1)u_2, \\
\bar{\partial}' u_1 u_2 u_3 &= (1 - \det(F))u_1 u_2,
\end{aligned}$$

where $\det(F) = fk - gh$ is the usual determinant.

Note. Recently the Scratchpad system mentioned in this paper evolved into what is now called the Axiom system.

References

[1] D. Barnes and L.Lambe, A fixed point approach to homological perturbation theory, Proc. Amer. Math. Soc., to appear.

[2] R. Brown, The twisted Eilenberg–Zilber theorem, Celebrazioni Archimedee del Secolo XX, Simposio di Topologia 34–37 (1967).

[3] R. Brown, Personal communication, 1991

[4] H. Cartan, Seminaire Henri Cartan 1954/1955.

[5] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, Princeton, NJ, 1956).

[6] S. Eilenberg and S. Mac Lane, On the groups $H(\pi, n)$ II, Ann. Math. 60 (1954) 49–139.

[7] V.K.A.M. Gugenheim, On the chain complex of a fibration, Ill. J. Math. 3 (1972) 398–414.

[8] V.K.A.M. Gugenheim, On a perturbation theory for the homology of the loop-space, J. Pure Appl. Algebra 25 (1982) 197–205.

[9] V.K.A.M. Gugenheim and L. Lambe, Applications of perturbation theory to differential homological algebra I, Ill. J. Math. 33 (1989) 556–582.

[10] V.K.A.M. Gugenheim, L. Lambe and J. Stasheff, Algebraic aspects of Chen's twisting cochain, Ill. J. Math. 34 (1990) 485–502.

[11] V.K.A.M. Gugenheim, L. Lambe and J. Stasheff, Perturbation theory in differential homological algebra II, Ill. J. Math., to appear.

[12] V.K.A.M. Gugenheim and J.P. May, On the Theory and Application of Differential Torsion Products, Memoirs of the American Mathematical Society, Vol. 142 (1974).

[13] G. Hochschild, Relative homological algebra, Trans. Amer. Math. Soc. 82 (1956) 246–269.

[14] D. Husemoller, J.C. Moore and J. Stasheff, Differential homological algebra and homogeneous spaces, J. Pure Appl. Algebra 5 (1974) 113–185.
[15] J. Koszul, Homologie et cohomologie des algèbres de Lie, Bull. Soc. Math. France 78 (1950) 65–127.
[16] L. Lambe, Homological perturbation theory, Hochschild homology and formal groups, in: Proceedings of the Conference on Deformation Theory and Quantization with Applications to Physics, Amherst, MA, June 1990, Contemporary Mathematics (American Mathematical Society, Providence, RI), to appear.
[17] L. Lambe, Resolutions via homological perturbation, J. Symbolic Comput., to appear.
[18] L. Lambe and J. Stasheff, Applications of perturbation theory to iterated fibrations, Manuscripta Math. 58 (1987) 363–376.
[19] S. Mac Lane, Homology, Grundlehren der Mathematischen Wissenschaften, Band 114 (Springer, New York, 1967).
[20] J.P. May, The cohomology of restricted Lie algebras and of Hopf algebras, J. Algebra 3 (1966) 123–146.
[21] J.P. May, A general algebraic approach to Steenrod operations, Lecture Notes in Mathematics, Vol. 168 (Springer, Berlin, 1970) 153–231.
[22] D. Northcott, Finite Free Resolutions (Cambridge University Press, Cambridge, 1976).
[23] S.B. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. 152 (1970) 39–60.
[24] W. Shih, Homologie des espaces fibrés, Inst. Hautes Études Sci. Publ. Math. 13 (1962) 93–176.