Note

Balancing Unit Vectors

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Given any family $u_1, \ldots, u_m$ of vectors in Euclidean $n$-space of Euclidean norm at most unity it is shown that at least one of the sums $\pm u_1 + \cdots + \pm u_m$ has norm at most $n^{1/2}$. Probabilistic techniques are used.

**Theorem.** Let $u_1, \ldots, u_m \in \mathbb{R}^n$, all $|u_i| \leq 1$. Then there exist $\epsilon_1, \ldots, \epsilon_m = \pm 1$ so that

$$|\epsilon_1 u_1 + \cdots + \epsilon_m u_m| \leq n^{1/2}.$$ 

This result is not new though we have not found a specific proof in the literature. Our proof makes essential use of the probabilistic method.

**Lemma.** Let $u_1, \ldots, u_n \in \mathbb{R}^n$, all $|u_i| \leq 1$. Let

$$v = \alpha_1 u_1 + \cdots + \alpha_n u_n$$

with $|\alpha_i| \leq 1$ for all $i$. Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$|\epsilon_1 u_1 + \cdots + \epsilon_n u_n - v| \leq n^{1/2}.$$ 

**Proof.** Let $\epsilon_1, \ldots, \epsilon_n$ be independent random variables with distributions

$$\text{Prob}[\epsilon_i = +1] = (1 + \alpha_i)/2 \quad \text{and} \quad \text{Prob}[\epsilon_i = -1] = (1 - \alpha_i)/2$$

so that $\epsilon_i$ has expectation $\alpha_i$ and variance $1 - \alpha_i^2$. Then $\epsilon_1 u_1 + \cdots + \epsilon_n u_n$ has expectation $v$ and the expected value of $|\epsilon_1 u_1 + \cdots + \epsilon_n u_n - v|^2$ resembles a variance. Set $u_j = (a_{i_1}, \ldots, a_{i_n})$ and $v = (b_1, \ldots, b_n)$. For each coordinate $j$
\begin{align*}
E[(\epsilon_1 a_{ij} + \cdots + \epsilon_n a_{nj} - b_j)^2] \\
= \text{Var}(\epsilon_1 a_{ij} + \cdots + \epsilon_n a_{nj}) \\
= \sum_{i=1}^{n} \text{Var}(\epsilon_i a_{ij}) = \sum_{i=1}^{n} (1 - a_i^2) a_{ij}^2.
\end{align*}

Expanding

\[|\epsilon_1 u_1 + \cdots + \epsilon_n u_n - v|^2 = \sum_{j=1}^{n} (\epsilon_1 a_{1j} + \cdots + \epsilon_n a_{nj} - b_j)^2\]

and applying the linearity of expected value

\[E(|\epsilon_1 u_1 + \cdots + \epsilon_n - v|^2) = \sum_{j=1}^{n} \sum_{i=1}^{n} (1 - a_i^2) a_{ij}^2 \]

\[= \sum_{i=1}^{n} (1 - a_i^2) |u_i|^2 \leq n\]

since \(1 - a_i^2 \leq 1\) and \(|u_i| \leq 1\). For some specific \(\epsilon_1, \ldots, \epsilon_n\) the expectation is not exceeded and

\[|\epsilon_1 u_1 + \cdots + \epsilon_n u_n - v| \leq n^{1/2}.
\]

The theorem quickly follows. A linear algebra argument yields \(\alpha_1, \ldots, \alpha_m\) satisfying \(\alpha_1 u_1 + \cdots + \alpha_m u_m = 0\) such that all \(|\alpha_i| \leq 1\) and \(\alpha_i = \pm 1\) for all but at most \(n\) \(i\)'s. Reordering vectors for convenience we have

\[\alpha_1 u_1 + \cdots + \alpha_n u_n + \epsilon_{n+1} u_{n+1} + \cdots + \epsilon_m u_m = 0,\]

where \(\epsilon_i = \pm 1\) and \(|\alpha_i| \leq 1\). The lemma gives \(\epsilon_1, \ldots, \epsilon_n\) so that

\[|\epsilon_1 u_1 + \cdots + \epsilon_m u_m| = |(\alpha_1 u_1 + \cdots + \alpha_n u_n) - (\epsilon_1 u_1 + \cdots + \epsilon_n u_n)| \leq n^{1/2}.
\]