13 December 2001

PHYSICS LETTERS B

ELSEVIER

Physics Letters B 522 (2001) 345–349

www.elsevier.com/locate/npe

Two phases of the noncommutative quantum mechanics

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Received 26 September 2001; accepted 30 September 2001
Editor: P.V. Landshoff

Abstract

We consider quantum mechanics on the noncommutative plane in the presence of magnetic field B. We show, that the model has two essentially different phases separated by the point $B\theta = c\hbar^2/e$, where θ is a parameter of noncommutativity. In this point the system reduces to exactly-solvable one-dimensional system. When $\kappa = 1 - eB\theta/c\hbar^2 < 0$ there is a finite number of states corresponding to the given value of the angular momentum. In another phase, i.e., when $\kappa > 0$ the number of states is infinite. The perturbative spectrum near the critical point $\kappa = 0$ is computed.

1. Introduction

Recently some interest to quantum mechanics on noncommutative space (noncommutative quantum mechanics) arose, inspired by the development of string theory [1]. Beyond the string theory meaning such model models also appear in various systems describing spinning particles. They serve for the study of one-particle sectors of noncommutative field theories arising from string considerations, quantum hall effect, and general phenomenological impacts of the noncommutativity [2–9]. In particular in Refs. [6,9] noncommutative Landau problem on plane, sphere and torus have been considered. A “critical point” was observed in these models when the density of states becomes infinite (see also [10]). From the algebraic point of view it corresponds to the degeneracy of the representation of the Heisenberg algebra [11].

The noncommutativity of coordinates is implemented by the relation,

$$[x^i, x^j] = i\theta^{ij}, \tag{1}$$

where θ^{ij} are c-numbers with the dimensionality $(\text{length})^{-2}$.

In the case when $[p_i, p_j] = 0$, the noncommutative quantum mechanics reduces to the usual one described by Schrödinger equation [12]

$$\mathcal{H}(p, \tilde{x})\Psi(\tilde{x}) = E\Psi(\tilde{x}), \quad \text{where } \tilde{x}^i = x^i - \frac{1}{2}\theta^{ij} p_j. \tag{2}$$

Hence, the difference between noncommutative and ordinary quantum mechanics consists in the choice of polarisation only.

In this Letter we consider a two-dimensional noncommutative quantum mechanical system with arbitrary central potential in the presence of constant mag-

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PII: S0370-2693(01)01304-1

netic field B. It is given by the Hamiltonian,

$$\widehat{\mathcal{H}} = \frac{\mathbf{p}^2}{2\mu} + V\big(|\mathbf{x}|^2\big), \tag{3}$$

where the operators $\mathbf{p}, \mathbf{x}$ obey the commutation relations

$$\big[x^1, x^2\big] = i\theta, \qquad \big[x^i, p_j\big] = i\hbar\delta^i_j,$$
$$[p_1, p_2] = i\frac{e}{c}B, \tag{4}$$

we assume $\theta > 0$.

In the absence of magnetic field, $B = 0$, the leading part of Hamiltonian for large noncommutativity parameter θ is given by the potential term, while the remaining part can be considered as a perturbation [13].

In what follows we show, that in the presence of magnetic field one has a parameter

$$\kappa = 1 - \frac{eB}{c\hbar^2}\theta, \tag{5}$$

which can be made small for arbitrary θ, by choosing a proper value of B. At the critical point,

$$\kappa = 0, \tag{6}$$

the model becomes exactly solvable. This allows to develop the perturbative analysis for the small κ (and arbitrary θ).

Surprisingly, it appears that the global properties of the model are qualitatively different for either κ is positive or negative. Although the perturbative analysis is applicable for both cases, for negative κ we can find some energy levels exactly.

2. The model

Let us consider the system (3), (4) in more detail. For this purpose let us split the algebra (4) in two independent subalgebras and pass to the operators π_i and x^i satisfying the following relations

$$\pi_i = p_i - \frac{\hbar\varepsilon_{ij}x^j}{\theta},$$
$$\big[\pi_i, x^j\big] = 0,$$
$$[\pi_1, \pi_2] = -i\frac{\hbar^2}{\theta}\kappa. \tag{7}$$

The $[x, x]$-commutator is given by Eq. (1).

In terms of these operators the Hamiltonian (3) reads

$$\mathcal{H} = \frac{\boldsymbol{\pi}^2}{2\mu} + \hbar\frac{\pi_1 x^2 - \pi_2 x^1}{\mu\theta} + \hbar^2\frac{|\mathbf{x}|^2}{2\mu\theta^2} + V\big(|\mathbf{x}|^2\big). \tag{8}$$

As one can see, there is a "critical point" for

$$\kappa = 0 \Leftrightarrow B = c\hbar^2/e\theta. \tag{9}$$

At this point operators π_i belong to the center of quantum algebra, and consequently, are constant ones. Thus, the system becomes effectively one-dimensional. Also, from the requirement of rotational invariance it follows that

$$\pi_i = 0 \Rightarrow \mathcal{H} = \frac{\hbar^2|\mathbf{x}|^2}{2\mu\theta^2} + V\big(|\mathbf{x}|^2\big). \tag{10}$$

For this Hamiltonian it is easy to find the exact energy spectrum,

$$E_n^{(0)} = \frac{\hbar^2(n+1/2)}{\mu\theta} + V\big(\theta(2n+1)\big), \quad n = 0, 1, \ldots . \tag{11}$$

Consider now the case of nonzero κ. In this case it is convenient to introduce the creation and annihilation operators

$$a^\pm = \frac{x^1 \mp \mathrm{i}x^2}{\sqrt{2\theta}}, \qquad b^\pm = \frac{\sqrt{\theta}}{\hbar}\frac{\pi_1 \mp \mathrm{i}\pi_2}{\sqrt{2|\kappa|}}, \tag{12}$$

with the following nonzero commutators

$$\big[a^-, a^+\big] = 1, \qquad \big[b^-, b^+\big] = -\operatorname{sgn}\kappa. \tag{13}$$

In terms of these operators the Hamiltonian (8) is of the form

$$\begin{aligned}\mathcal{H} = {} & \frac{|\kappa|\hbar^2}{2\mu\theta}\big(b^+b^- + b^-b^+\big) \\ & - \mathrm{i}\frac{\sqrt{|\kappa|}\,\hbar^2}{\mu\theta}\big(b^+a^- - a^+b^-\big) \\ & + \frac{\hbar^2(a^+a^- + a^-a^+)}{2\mu\theta} + V\big(\theta\big(a^+a^- + a^-a^+\big)\big).\end{aligned} \tag{14}$$

The rotational symmetry of the system corresponds to the conserved angular momentum given by the operator,

$$2J = a^+a^- - \operatorname{sgn}\kappa\, b^+b^-, \qquad [\mathcal{H}, J] = 0. \tag{15}$$

As it can be seen, when $\kappa < 0$, the system is naturally formulated in terms of representations of the algebra

$\mathcal{G} = su(2)$. For $\kappa > 0$, one has instead representations of $\mathcal{G} = su(1,1)$. The generators of theses algebras are given by following operators,

$$L_\pm = b^\mp a^\pm, \qquad L_3 = \frac{1}{2}\left(a^+a^- + \operatorname{sgn}\kappa\, b^+b^-\right). \tag{16}$$

It is worthwhile to note, that the angular momentum of the system given by (15), define the Casimir operator of the algebra $\mathcal{G}$

$$J(J + \operatorname{sgn}\kappa) = L_3^2 + \frac{\operatorname{sgn}\kappa}{2}(L_+L_- + L_-L_+). \tag{17}$$

According to above, the Hilbert space splits in the irreducible representations (irreps) of the algebra $\mathcal{G}$ which are parameterized by the eigenvalues of J. Inside an irrep one can introduce the basis labelled by the eigenvalue of L_3. As a result we have the orthonormal basis in the Hilbert space consisting of states

$$|j,l\rangle = \frac{(a^+)^{j+l}(b^+)^{j-l}}{\sqrt{(j+l)!(j-l)!}}|0,0\rangle, \tag{18}$$

where j and l are eigenvalues of J and L_3, respectively. Let us note that the system of states is equivalent to one of a pair of coupled oscillators. The angular momentum corresponds to the total occupation number

$$2j = n_a - \operatorname{sgn}\kappa\, n_b. \tag{19}$$

One can see that the spectrum has different structure depending on the sign of κ. Indeed, for $\kappa < 0$ (or equivalently, $B > c\hbar^2/e\theta$), the angular momentum $2j$ and the occupation number n_a corresponding to the operator $|\mathbf{x}|^2/2\theta$, take the values

$$n_a = 0, 1, \ldots,$$
$$2j = n_a, n_a + 1, \ldots. \tag{20}$$

For $\kappa > 0$ ($B < c\hbar^2/e\theta$) corresponding to the noncompact case $\mathcal{G} = su(1,1)$, the eigenvalues of the angular momentum $2j$ and of the operator $|\mathbf{x}|^2/2\theta$, respectively, take the values

$$n_a = 0, 1, \ldots,$$
$$2j = -\infty, \ldots, -1, 0, 1, \ldots, n_a. \tag{21}$$

Thus, the character of the spectrum essentially depends on the value of magnetic field: for $B\theta < c\hbar^2/e$ the angular momentum has the upper bound, equal to the eigenvalue of the operator $|\mathbf{x}|^2/2\theta$, while for $B\theta > c\hbar^2/e$ the eigenvalue of $|\mathbf{x}|^2/2\theta$ becomes the lower bound for the angular momentum.

3. The spectrum

In the basis (18) the Hamiltonian (14) splits in the diagonal part given by first and third lines and nondiagonal part given by the second line. Let us consider the diagonal part given by the third line of (14) as the bare Hamiltonian. The remaining part can be considered as a perturbation of the order $|\kappa|^{1/2}$. Then perturbation expansion around the critical point $\kappa = 0$ applies when

$$\sqrt{|\kappa| j} \ll 1 + \mu\theta V(n)/\hbar^2. \tag{22}$$

The energy spectrum of the nonperturbed Hamiltonian is given by the expression (11).

The first order correction to the nth energy level vanishes, while the computation of the second order correction yields the result

$$\begin{aligned} E^{\text{pert}}_{(j,n)} = {} & \frac{\kappa\hbar^2(2j-n)}{\mu\theta}\left(1 + \frac{n+1}{\Omega_{n+1}} - \frac{n}{\Omega_n}\right) \\ & - \frac{|\kappa|\hbar^2(n+1/2)}{\mu\theta\Omega_n} + \frac{\hbar^2(n+1/2)}{\mu\theta} \\ & + V(2\theta n + \theta), \end{aligned} \tag{23}$$

where

$$\Omega_n = \frac{\mu\theta}{\hbar^2}\left(V(2\theta n + \theta) - V(2\theta n - \theta) + \frac{\hbar^2}{\mu\theta}\right).$$

Beyond this, in the compact case ($\mathcal{G} = su(2)$, $\kappa < 0$), one can compute exactly some energy levels in the “lower” (i.e., corresponding to small j) part of the spectrum. In the mentioned case, the half-integer eigenvalues j and l span the range $j = 0, 1/2, 1, \ldots$ and $-j \leqslant l \leqslant j$. This happens due to finite dimensionality of the irreps of $\mathcal{G}$. The Hamiltonian acts invariantly in each irrep because it commutes with the Casimir operator J. Therefore, the problem of diagonalisation of the Hamiltonian in the whole Hilbert space reduces to “smaller” problems of diagonalisation in each finite-dimensional irrep.

Thus, the eigenvectors of $\mathcal{H}$ can be represented as linear combination of basis elements with the same number j,

$$|j,s\rangle = \sum_{l=-j}^{j} C_l^{(j,s)} |j,l\rangle,$$
$$\mathcal{H}|j,s\rangle = E_{(j,s)}|j,s\rangle, \tag{24}$$

where $C_l^{(j,s)} = \langle j,l|j,s\rangle$, and half-integer number s enumerates energy levels inside irrep.

The second equation in (24) can be rewritten as a set of $2j+1$ linear equations for C_l (we drop the superscripts (j,s)):

$$\begin{aligned}&\big(|\kappa|(j-l)+v(j+l)-\varepsilon\big)C_l\\ &\quad + \mathrm{i}|\kappa|^{1/2}\big(\sqrt{(j-l+1)(j+l)}\,C_{l-1}\\ &\qquad - \sqrt{(j-l)(j+l+1)}\,C_{l+1}\big)=0,\end{aligned} \tag{25}$$

where we introduced shorthand notations for ε and $v(j+l)$ implicitly defined by the equations

$$E = \frac{\hbar^2}{\mu\theta}\left(\varepsilon + \frac{1}{2}\big(1+|\kappa|\big)\right) + V(\theta),$$

and

$$v(j+l) = j+l+\frac{\mu\theta}{\hbar^2}\big(V\big(\theta(j+l+1)\big)-V(\theta)\big).$$

This defines a $(2j+1)$-dimensional eigenvalue problem which can be solved by standard linear algebra methods for not very large j, as well as numerically if j is large. In particular, for $j=0$ and $j=1/2$ the corresponding energy levels are given by,

$$E_{(0,0)} = V(\theta), \tag{26}$$

and

$$\begin{aligned}&E_{(\frac{1}{2},\pm\frac{1}{2})}\\ &\quad = V(3\theta) + \frac{\hbar^2(1+|\kappa|)}{\mu\theta}\\ &\qquad \pm\left[4|\kappa|\left(\frac{\hbar^2}{\mu\theta}\right)^2\right.\\ &\qquad\qquad \left. + \left(\big(1-|\kappa|\big)\frac{\hbar^2}{\mu\theta}\big(V(3\theta)-V(\theta)\big)\right)^2\right]^{1/2},\end{aligned} \tag{27}$$

respectively.

Let us note, however, that the lowest j states do not necessarily correspond to the lowest energy levels. Depending on the form of the potential, the higher j states may have eigenvalues of the Hamiltonian located in the lower part of the spectrum.

Unfortunately, in the case when $\kappa > 0$ we cannot perform the same analysis since in this case the representations of $su(1,1)$ are infinite-dimensional.

4. Example: harmonic oscillator

Consider the particular case of harmonic oscillator,

$$V = \frac{\mu\omega^2|\mathbf{x}|^2}{2}. \tag{28}$$

In this case one can solve the spectrum exactly for any value of κ [6] (see also [14]). Our results agree with mentioned ones. Let us diagonalize the Hamiltonian, performing the appropriate (pseudo)unitary transformation:

$$\begin{pmatrix} a\\ b\end{pmatrix} \to U\cdot\begin{pmatrix} a\\ b\end{pmatrix}, \tag{29}$$

where the matrix U belong to $SU(1,1)$ for $\kappa > 0$ and to $SU(2)$ for $\kappa < 0$. Explicitly,

$$U = \begin{cases}\begin{pmatrix}\cosh\chi\, e^{\mathrm{i}\pi/4} & \sinh\chi\, e^{\mathrm{i}\pi/4}\\ \sinh\chi\, e^{-\mathrm{i}\pi/4} & \cosh\chi\, e^{-\mathrm{i}\pi/4}\end{pmatrix}, & \text{for } \kappa > 0,\\ \begin{pmatrix}\cos\chi\, e^{\mathrm{i}\pi/4} & \sin\chi\, e^{\mathrm{i}\pi/4}\\ -\sin\chi\, e^{-\mathrm{i}\pi/4} & \cos\chi\, e^{-\mathrm{i}\pi/4}\end{pmatrix}, & \text{for } \kappa < 0,\end{cases} \tag{30}$$

where "angle" χ is given by the following relations,

$$2\chi = \begin{cases}\tanh^{-1}\big(2\sqrt{\kappa}/(\mathcal{E}+\kappa)\big), & \text{for } \kappa > 0,\\ \tan^{-1}\big(2\sqrt{-\kappa}/(\mathcal{E}+\kappa)\big), & \text{for } \kappa < 0.\end{cases} \tag{31}$$

We have used here the notation $\mathcal{E} = 1 + (\mu\omega\theta/\hbar)^2$.

The diagonalized Hamiltonian reads,

$$\begin{aligned}\mathcal{H}_{\text{osc}} = {}&\frac{1}{2}\hbar\omega_+\big(b^+b^- + b^-b^+\big)\\ &+ \frac{1}{2}\hbar\omega_-\big(a^+a^- + a^-a^+\big),\end{aligned} \tag{32}$$

where

$$\begin{aligned}&\frac{2\mu\theta\omega_\pm}{\hbar}\\ &\quad = \begin{cases}\pm(\mathcal{E}-\kappa) + \sqrt{(\mathcal{E}+\kappa)^2 - 4\kappa}, & \text{for } \kappa > 0,\\ (\mathcal{E}-\kappa) \pm \sqrt{(\mathcal{E}+\kappa)^2 - 4\kappa}, & \text{for } \kappa < 0.\end{cases}\end{aligned} \tag{33}$$

Hence, the spectrum is of the form

$$E^{\mathrm{osc}}_{n_1,n_2} = \hbar\omega_+\left(n_1 + \frac{1}{2}\right) + \hbar\omega_-\left(n_2 + \frac{1}{2}\right). \qquad (34)$$

Let us recall that the transformation (30) belongs to the symmetry group of the rotational momentum J. Therefore in new variables its eigenvalues are given by the following equation,

$$2j = n_1 - \operatorname{sgn}\kappa\, n_2, \qquad (35)$$

where $n_1, n_2 = 0, 1, 2, \ldots$.

In particular, the vacuum energy corresponding to different signs of κ looks as follows,

$$h_0 = \frac{\hbar^2}{2\mu\theta}\sqrt{(\mathcal{E}-\kappa)^2 + 4\kappa(\mathcal{E}-1)}, \quad \text{for } \kappa > 0,$$

and

$$h_0 = \frac{\hbar^2}{2\mu\theta}(\mathcal{E}-\kappa), \quad \text{for } \kappa < 0.$$

5. Concluding remarks

In this Letter we considered a two-dimensional central symmetric noncommutative mechanical model in the presence of magnetic field. We have shown that in the case when magnetic field is smaller than some critical value the spectrum of the model is organized according to representations of algebra $su(1,1)$ while for the magnetic field beyond this value it “reorganizes” according to representations of $su(2)$. This algebras are symmetry algebras of the rotational momentum operator in these two cases. These cases are physically different. In particular in the first one the possible values of rotational momentum span both positive and negative half-integer numbers while in the second case only positive orbital numbers are allowed. This may lead to the conclusion that in the presence of a strong magnetic field properly oriented with respect to inverse noncommutativity parameter θ^{-1} the spinning properties of the noncommutative particle are gravely affected.

As an example we considered the particular case of the harmonic oscillator. Our results appear to be in agreement with the ones previously known in the literature.

Acknowledgements

We thank Ph. Pouliot for criticism leading, hopefully, to the improving of the manuscript and useful discussions on the oscillator example. A.N. thanks INFN for the financial support and hospitality during his stay in Frascati, where this work was started. A.N. and C.S. were partially supported under the INTAS project 00-00262. C.S. was also supported by RFBR grants: #99-01-00190 and young scientists support grant, Scientific School support grant # 00-15-96046.

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