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Graphs determined by polynomial invariants☆

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Abstract

Many polynomials have been defined associated to graphs, like the characteristic, matchings, chromatic and Tutte polynomials. Besides their intrinsic interest, they encode useful combinatorial information about the given graph. It is natural then to ask to what extent any of these polynomials determines a graph and, in particular, whether one can find graphs that can be uniquely determined by a given polynomial. In this paper we survey known results in this area and, at the same time, we present some new results.

Keywords: Graph invariant; Tutte polynomial; Chromatic polynomial; Matroid

1. Introduction

In the literature we find many polynomials associated to graphs. Besides their intrinsic interest, usually they happen to encode combinatorial information about the given graph. It is natural then to ask to what extent any of these polynomials determines a graph and, in particular, whether we can find graphs, or even better families of graphs, that are uniquely determined by a given polynomial.

Such investigations have been, and are being, carried out for the most common polynomials in graph theory, like the characteristic, the matchings, the chromatic, and the Tutte polynomials. In this paper we provide a survey of this topic, in the hope that it will be useful to get a general view and to foster further research in this area, including research on polynomials that, until now, have received less attention from this perspective.

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The following concepts are basic throughout the paper. Let f be a mapping which associates to any graph G a polynomial $f(G)$ in one or more variables with coefficients in some field, usually the complex numbers. We say that f is a *polynomial invariant* if $f(G)=f(H)$ whenever G and H are isomorphic; in other words, when f is well defined on the class of unlabelled graphs. For the polynomials we discuss in this paper, this basic property is straightforward to establish.

Given a polynomial invariant f, we say that a graph G is *f-unique* if, for any other graph H,

$$f(H) = f(G) \quad \text{implies } H \cong G.$$

In other words, if G is the *only* graph having $f(G)$ as its associated polynomial. If $f(G)=f(H)$ we say that G and H are *f-equivalent*.

In the next sections we recall the definition and basic properties of several polynomial invariants, and review known results about graphs that are unique with respect to these invariants. A special emphasis is put on graphs that are unique with respect to the Tutte polynomial, an area in which the present author has contributed recently, and in which we believe significant progress can be achieved.

Unless otherwise stated, our graphs have no loops or multiple edges. We use the following terminology. If $G=(V,E)$ is a graph with vertex set V and edge set E, then we set $n=|V|$ and $m=|E|$. The girth of G (the length of a shortest cycle) is denoted by g; the minimum degree by δ; and the edge-connectivity (the size of a minimum edge-cut) by λ. A cycle of length three is called a triangle, of length four a square, and so on; K_n and $K_{n_1,n_2,\dots,n_r}$ denote as usual complete and complete r-partite graphs, respectively; $K(n,r)$ denotes the complete r-partite graphs with parts of size n; finally, K_n^- is the graph obtained from K_n by deleting any of its edges. The line graph $L(G)$ of a graph G has as vertices the edges of G, two of them being adjacent in $L(G)$ if they share a vertex. The reader can refer to [6] for terminology and results on graph theory.

2. Polynomial invariants

In this section we review the definition of several polynomial invariants, well known in graph theory. We begin with the most classical of them.

2.1. The characteristic polynomial

The *adjacency matrix* A of a graph G with vertex set $V=\{v_1,\dots,v_n\}$ is the $n\times n$ matrix defined as

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic polynomial* $\phi(G,x)$ is defined as the characteristic polynomial of the adjacency matrix

$$\phi(G,x) = \det(xI - A),$$

where I is the $n \times n$ identity matrix. It may seem that $\phi(G,x)$ depends on the labeling of the vertices; this is not so, since the adjacency matrix for a different labeling is equal to $P^{-1}AP$, where P is a permutation matrix, and similar matrices have the same characteristic polynomial.

The interplay between the algebraic properties of A and $\phi(G,x)$, and the combinatorial properties of G, has given birth to the very active field of Algebraic Graph Theory. Some of the main topics are the study of graphs with strong regularity conditions such as distance regular graphs, automorphisms of graphs, bounds on the diameter, and the spectral analysis of random walks on graphs. A classical and very readable reference is [4]; more recent texts are [16,17,20].

If we write

$$\phi(G,x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

then we have [4, Proposition 7.3]:

Lemma 2.1. *The coefficient a_i is equal to*

$$\sum_{\gamma} (-1)^{\mathrm{comp}(\gamma)}\, 2^{\mathrm{cyc}(\gamma)},$$

where the sum is taken over all subgraphs γ consisting of disjoint edges and cycles, and having i vertices; if γ is such a subgraph, then comp(γ) *is the number of components in it, and* cyc(γ) *is the number of cycles.*

In particular,

$$a_1 = 0, \quad a_2 = -m = -|E|, \quad a_3 = -2t_1(G),$$

where $t_1(G)$ is the number of triangles in G. Thus we see that from the knowledge of $\phi(G,x)$ we can recover the number of vertices, edges and triangles in G. This is a typical example of the information that is employed when characterizing graphs by means of polynomial invariants.

Example. Consider the graph K_4^- with the labeling of vertices shown in Fig. 1. The adjacency matrix is equal to

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\phi(K_4^-, x) = x^4 - 5x^2 - 4x,$$

in agreement with the fact that K_4^- has 4 vertices, 5 edges, and 2 triangles.

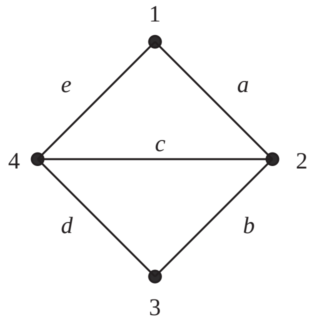

Fig. 1. The graph K_4^-.

2.2. The matchings polynomial

An *r-matching* in a graph is a set of r edges, in which have a vertex in common. Let m_r be the number of r-matchings in G. The *matchings polynomial* $\mu(G,x)$ is defined as

$$\mu(G,x) = \sum_{r\geqslant 0} (-1)^r m_r x^{n-2r},$$

where $n=|V|$ and, by convention, $m_0=1$.

This polynomial has the remarkable property that its zeros are always real, a property first proved by Heilmann and Lieb [22] in the context of statistical mechanics. Another interesting fact is that if G is a tree, then $\phi(G,x)=\mu(G,x)$; this is a direct consequence of Lemma 2.1. Also, many classical families of orthogonal polynomials (like Chebychev, Hermite and Laguerre) appear as matchings polynomials of suitable graphs (see [19] for these and additional properties of matchings polynomials).

It is clear that, in general, we have

$$\mu(G,x) = x^n - mx^{n-2} + \cdots\ .$$

Thus, one can recover from $\mu(G,x)$ the number of vertices and edges. Note that the constant term is the number $\mathrm{pm}(G)$ of perfect matchings in G, which is equal to 0 if n is odd.

Example (Continued). Take again K_4^-; the matchings polynomial is equal to

$$\mu(K_4^-,x) = x^4 - 5x^2 + 2,$$

since K_4^- has 4 vertices, 5 edges, and 2 perfect matchings.

2.3. The chromatic polynomial

A k-coloring of G is a map

$$c : V \to \{1,2,\ldots,k\}$$

such that $c(x) \neq c(y)$ if xy is an edge of G. The chromatic polynomial is defined as

$$P(G,x) = \#\{x\text{-colorings of } G\}.$$

It is not immediate that $P(G,x)$ is indeed a polynomial function; an easy proof is as follows. Let a_i be the number of partitions of V into exactly i stable sets (a set of vertices is stable if there is no edge between any two of them). Then

$$P(G,x) = \sum_{i=1}^{n} a_i x(x-1)\cdots(x-i+1),$$

which is indeed a polynomial in x of degree n (notice that $a_n = 1$).

Graph coloring problems are central in combinatorics and one could say that a substantial part of early graph theory was developed in order to solve the four-color problem. Chromatic polynomials were introduced by Birkhoff [5] precisely in an attempt to solve this problem, since the four-color theorem is equivalent to the fact that $P(G,4) > 0$ for every planar graph G. As we know, the solution of the problem eventually followed a different route, but the interest in chromatic polynomials has increased ever since.

There is an interesting connection between the chromatic and the matchings polynomial. If G is a triangle-free graph and G^c denotes the complement of G, then Farrell and Whitehead [18] proved that $\mu(G,x)$ and $P(G^c,x)$ determine each other.

In general, it can be shown (see [26,39]) that

$$P(G,x) = x^n - mx^{n-1} + \left(\binom{m}{2} - t_1\right)x^{n-2} - \cdots,$$

where t_1 is the number of triangles as before. Moreover, each coefficient can be interpreted in terms of acyclic orientations of G (see [29]). Additional properties of $P(G,x)$ relevant to this paper will be reviewed in the next section.

Example (Continued). To compute the chromatic polynomial of K_4^-, suppose we have x colors available; then the vertices in a triangle can be colored in $x(x-1)(x-2)$ ways, and the remaining vertex in $x-2$ ways since it is adjacent to two vertices colored differently. Hence

$$P(K_4^-,x) = x(x-1)(x-2)^2 = x^4 - 5x^3 + 8x^2 - 4x,$$

in agreement with the previous statement.

2.4. The Tutte polynomial

In order to introduce the Tutte polynomial we need some definitions. Let $G=(V,E)$ be a graph and let $A \subseteq E$ be a subset of edges. The *rank* of A is defined as

$$r(A) = n - c(V,A),$$

where $n = |V|$ and $c(V,A)$ is the number of components of the spanning subgraph (V,A). It is easy to check that $r(A)$ is the maximum number of edges in A containing no cycle.

The term rank suggests an algebraic connection, and this is indeed the case. The incidence matrix $I(G)$ of G is the $n \times m$ binary matrix whose (i, j) entry is 1 if the ith vertex is incident to the jth edge, and 0 otherwise. Then $r(A)$ is the same as the rank over the two element field $\mathbb{F}_2$ of the submatrix formed by the columns corresponding to A. We resume this thread in the last section when discussing matroids.

Now to any $A \subseteq E$ we can associate two numbers, its size and its rank. Define the *rank-size polynomial* as the generating function

$$R(G; x, y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|} = \sum_{i,j} r_{ij} x^i y^j,$$

where r_{ij} is the number of $A \subseteq E$ having rank i and size j.

The *Tutte polynomial*, introduced by Tutte [45], is a variation of the former. It is defined as

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

It should be clear that $R(G; x, y)$ and $T(G; x, y)$ are equivalent in the sense that either of them can be obtained from the other by a simple transformation. Hence, a graph is R-unique if and only if it is T-unique. However, the Tutte polynomial enjoys a number of properties that make it a very remarkable invariant. Most notably the rule

$$T(G; x, y) = T(G-e; x, y) + T(G/e; x, y),$$

where e is any edge of G which is not a loop nor a bridge, and $G-e$ and G/e denote, respectively, the graphs obtained from G by deleting and contracting e. Observe that G/e can have loops and multiple edges.

Another remarkable feature of the Tutte polynomial is that it contains a great amount of information about a graph. For instance, $T(G; 1, 1)$ is the number of spanning trees of G, and $T(G; 2, 0)$ is the number of acyclic orientations. Also, one can obtain the chromatic polynomial from the Tutte polynomial:

$$P(G, x) = (-1)^{r(E)} x^{c(G)} T(G; 1-x, 0),$$

where $c(G)$ is the number of connected components in G. The previous formula is equivalent to a classical result of Whitney [50], namely

$$P(G, t) = \sum_{A \subseteq E} (-1)^{|A|} x^{n-r(A)}.$$

Let us also remark that the Tutte polynomial appears in many different contexts: as the Jones polynomial of a knot, as the partition function of the Potts model in statistical mechanics, as the weight enumerating polynomial in coding theory, and others (see [14,48] for thorough surveys).

Example (Continued). Let us compute $R(K_4^-;x,y)$. For this we consider all possible subsets $A \subseteq E(K_4^-)$ and compute their rank and cardinality. With the edges labeled as in Fig. 1, the results are summarized in the following table:

Rank	Size	A	Number
0	0	$\emptyset$	1
1	1	singletons	5
2	2	doubletons	10
2	3	$\{a,c,e\}$, $\{b,c,d\}$	2
3	3	spanning trees	8
3	4	all 4-subsets	5
3	5	$\{a,b,c,d,e\}$	1

It follows that

$$R(K_4^-;x,y) = x^3y^5 + 5x^3y^4 + 8x^3y^3 + 2x^2y^3 + 10x^2y^2 + 5xy + 1.$$

A simple computation also gives

$$T(K_4^-;x,y) = x^3 + 2x^2 + 2xy + y^2 + x + y.$$

As will be seen in the next section, important structural information about a graph can be obtained from the knowledge of its rank-size generating function (or Tutte polynomial).

2.5. Complexity issues

The characteristic polynomial, being a determinant, can be computed in polynomial time. On the contrary, evaluating the matchings polynomial is a #P-hard problem (see [48] for a good introduction to the class of #P-hard problems, a class that captures the idea of the computational complexity of difficult counting problems). This is because evaluating pm(G), the constant term of $\mu(G,x)$ for graphs of even order, is already #P-hard [47]. For a planar graph G, the evaluation of pm(G) can be done in polynomial time using Pfaffian orientations [25], but the full evaluation of $\mu(G,x)$ is again hard [24].

The chromatic polynomial is hard to evaluate even for planar graphs, since already computing the chromatic number is NP-complete. The complexity of evaluating the Tutte polynomial is most interesting. Jaeger, Vertigan and Welsh proved that evaluating $T(G;x,y)$ is #P-hard at all points (x,y) except those in the hyperbola $(x-1)(y-1)=1$ and a set of eight exceptional points, where it can be done in polynomial time. Among the exceptional points is $(1,1)$; the reason is that $T(G;1,1)$ is the number of spanning trees of G, and this quantity can be computed as a determinant, thanks to the matrix-tree theorem [6]. When G is planar, one has to add the points in the hyperbola $(x-1)(y-1)=2$ to the points where evaluation of $T(G;x,y)$ can be done in polynomial time (see [48] for a full discussion).

3. Uniquely determined graphs

Recall that if f is a polynomial invariant, a graph G is f-unique if it is the only graph having $f(G)$ as associated polynomial. As we mentioned before, much effort has been invested in finding f-unique graphs. In this section we review known results for the four invariants introduced previously.

3.1. Graphs determined by their spectra

The material in this section is taken mostly from [17, Chap. 6; 21].

The *eigenvalues* of a graph are the roots of its characteristic polynomial; they are all real since the adjacency matrix is symmetric. The *spectrum* of a graph is the multiset of eigenvalues, each one counted with the corresponding multiplicity. Instead of a ϕ-unique graph, a common terminology used is a graph determined by its spectrum, or a DS graph. Two graphs are said to be *cospectral* if they have the same spectrum.

A simple example of a DS graph is the complete graph K_n. If $\phi(H,x)=\phi(K_n,x)$, then H as n vertices and $\binom{n}{2}$ edges, so it has to be K_n. More interesting is the case of $K_{n,n}$. If $\phi(H,x)=\phi(K_{n,n},x)$, then H has $2n$ vertices, n^2 edges, and no triangles, since these parameters are determined by $\phi(H,x)$ and must agree with those of $K_{n,n}$. By a well known result of extremal graph theory [6], H must be $K_{n,n}$. In fact, this is a special cases of a more general result (see [17]).

Theorem 3.1. *Regular complete multipartite graphs are determined by their spectra.*

But this is not true for all complete multipartite graphs. For instance, the star $K_{1,4}$ has the same spectrum as the disjoint union of C_4 and an isolated vertex:

$$\phi(K_{1,4},x) = \phi(C_4 \cup K_1,x) = x^5 - 4x^3.$$

A different proof for the case $K_{n,n}$ is based on the following result (for a proof, see for instance [19, Chap. 2]).

Theorem 3.2. *A graph G is bipartite if and only if $\phi(G,x)=(-1)^n\phi(G,-x)$.*

Now, suppose again that $\phi(H,x)=\phi(K_{n,n},x)$; then H has $2n$ vertices, n^2 edges, and is bipartite. This forces H to be $K_{n,n}$.

Recall that the line graph $L(G)$ of a graph G has as vertices the edges of G, two of them being adjacent if they are incident in G. Line graphs have been much studied with respect to their spectral properties. A remarkable result, due to the combined efforts of Chang, Hoffman and Shrikhande, is the following.

Theorem 3.3. (a) *The graphs $L(K_n)$ are determined by their spectrum if $n \neq 8$. For $n=8$, there are three exceptional graphs cospectral but not isomorphic to $L(K_8)$.*

(b) *The graphs $L(K_{n,n})$ are determined by their spectrum if $n \neq 4$. For $n=4$, there is one exceptional graphs cospectral but not isomorphic to $L(K_{4,4})$.*

The graphs in the previous theorem are examples of strongly regular graphs. A graph G is *strongly regular* (s.r.) with parameters (k, λ, μ) if it satisfies the following three conditions:

1. G is k-regular;
2. Any two adjacent vertices in G have exactly λ common neighbors;
3. Any two non-adjacent vertices in G have exactly μ common neighbors.

For instance, $L(K_n)$ is s.r. with parameters $(2n-4, 1, 4)$, and $L(K_{n,n})$ with parameters $(2n-2, n-2, 2)$. The famous Petersen graph is s.r. with parameters $(3, 0, 1)$. This condition is captured by the characteristic polynomial [4].

Theorem 3.4. *A graph G is strongly regular if and only if it has exactly three distinct eigenvalues, the largest of them being simple.*

The proof is based on the fact that the adjacency matrix of a s.r. with parameters (k, λ, μ) satisfies the equation

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where I is the identity matrix and J the all ones matrix. The largest eigenvalue is k, which is simple. The two other eigenvalues α and β, as well as their multiplicities, can be computed from the parameters. It follows that whenever there is a unique s.r. graph with given parameters, it is determined by the spectrum. A classical example is the Petersen graph, which has spectrum $\{3, 1^5, (-2)^4\}$.

Strongly regular graphs are a particular case of *distance regular* graphs. Since the definition is more technical it is not given here (see [4,19,20]), but it is worth mentioning that much effort is being done on finding DS graphs among distance regular graphs [21].

3.2. Matching unique graphs

A graph is *matching unique* if it is unique with respect to the matchings polynomial. Matching unique graphs have not been studied as thoroughly as DS graphs, although several results have been obtained. A very interesting result by Beezer and Farrell is the following [3].

Theorem 3.5. *If G is a d-regular graph and $\mu(H,x) = \mu(G,x)$, then H is also d-regular and has the same girth and number of shortest cycles as G.*

The authors deduce from this that several graphs are matching unique, including cages (regular graphs with minimum number of vertices and given degree and girth) which are unique with given parameters, for instance the Petersen graph. We show here a new application of their result, using the following lemma from [35].

Lemma 3.6. *Among all $(tm - m)$-regular graphs with tm vertices, the complete t-partite graph $K(m,t)$ has the minimum number of triangles.*

Theorem 3.7. *Regular complete multipartite graphs are matching unique.*

Proof. Suppose H has the same matchings polynomial as $K(m,t)$. It follows from Theorem 3.5 that H is $(tm-m)$-regular, has tm vertices, girth three, and the same number of triangles as $K(m,t)$. According to the previous lemma, H is isomorphic to $K(m,t)$. □

3.3. Chromatically unique graphs

A graph that is unique with respect to the chromatic polynomial is known as a *chromatically unique* graph. Instead of P-unique, the standard terminology is χ-unique graphs. There is a prolific literature on χ-unique graphs, a problem introduced by Chao and Whitehead [15]. The reader can find a comprehensive survey in the papers [26,27] by Koh and Teo.

The main tool in proving χ-uniqueness is the following.

Lemma 3.8. *From the chromatic polynomial of G it is possible to deduce the following parameters*:
(1) *the number of vertices and edges*;
(2) *the number of components and*, *if G is connected*, *the number of blocks* (2-*connected components*);
(3) *the number of triangles*;
(4) *the girth g and the number of cycles of length g*;
(5) *the chromatic number.*

Using this lemma, it is an easy matter to establish χ-uniqueness of cycles, complete graphs, and the graphs $K_{n,n}$. More interesting is the case of $K_{n,m}$ for $2 \leqslant m \leqslant n$. The proof in this case relies on an extremal property of $K_{n,m}$ with respect to the number of cycles of length four [44].

The situation with respect to wheels is quite surprising. Let W_n denote the graph obtained from the cycle C_n by adding a new vertex adjacent to every vertex in the cycle. It is known that W_n is χ-unique if n is even. The key fact about even wheels is that they are uniquely 3-colorable. But W_5 and W_7 are not χ-unique, and the chromatic uniqueness of the other odd wheels is not settled yet.

Other families of graphs that have been studied are complete graphs with some edges removed, graphs related to complete multipartite graphs, and several others. In particular, Turán graphs, defined as complete multipartite graphs with nearly equal size parts, are χ-unique; this follows from the previous lemma and Turán's extremal theorem (see [26]).

The general impression one gets through these results, is that the combinatorial information that can be deduced from the chromatic polynomial is often too scarce, and the proofs of χ-uniqueness tend to be rather intricate.

Finally, let us mention some results about families of graphs. It is easy to prove that if T is any tree on n vertices, then $P(T,x)=x(x-1)^{n-1}$. Conversely, suppose G is a graph with $P(G,x)=x(x-1)^{n-1}$. Then from Lemma 3.8 it follows that G is connected and has n vertices and $n-1$ edges; thus it is a tree.

This can be extended to 2-trees, defined as follows. The smallest 2-tree is a triangle. A 2-tree on n vertices, with $n > 3$ is obtained by adding a new vertex adjacent to each end of an edge in a 2-tree on $n-1$ vertices. An example is the graph in Fig. 1. If G is a 2-tree on n vertices, again it is immediate to show that

$$P(G,x) = x(x-1)(x-2)^{n-2}.$$

It turns out that this property characterizes 2-trees [49].

3.4. *T*-unique graphs

A graph that is unique with respect to the Tutte polynomial is called *T-unique*. As we mentioned in the previous section, from the Tutte polynomial one can obtain the chromatic polynomial. It may seem that this implies that a χ-unique graph is also T-unique, but this is true only for 2-connected graphs, and the reason is the following. Suppose G and H are graphs with disjoint vertex sets, and let $G \cdot H$ be a graph obtained by identifying a vertex of G with a vertex of H (an operation known as vertex identification). It is straightforward to show that

$$T(G \cdot H; x, y) = T(G; x, y)T(H; x, y).$$

It follows that a graph which is not 2-connected cannot be T-unique.

Let us mention a converse statement, recently proved in [32]: the irreducible factors of $T(G; x, y)$ in $\mathbb{Z}[x, y]$ correspond precisely to the Tutte polynomials of the 2-connected components of G.

Another operation that is relevant in this context is the following. Suppose G is obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2 as the vertex u of G, and by identifying the vertices v_1 of G_1 and v_2 of G_2 as the vertex v of G. The *twisting* of G about $\{u, v\}$ is the graph G' obtained by identifying, instead, u_1 with v_2 and u_2 with v_1. Then it is easy to see that there exists a bijection from $E(G)$ and $E(G')$ that preserves ranks. If follows that G and G' have the same rank-size generating function, hence the same Tutte polynomial. As a consequence, if a graph G admits a twisting giving rise to a nonisomorphic graph, then G is not T-unique. As an example, we invite the reader to check that the *fan* graph F_6 obtained by joining a vertex to every vertex of a path P_5 on 5 vertices is not T-unique. For more information on the twisting operation, especially the celebrated theorem of Whitney, see [37, Section 5.3].

In view of the previous remarks, in this section we concentrate on 3-connected graphs. Motivated by some conjectures on chromatic uniqueness mentioned in [26], together with Anna de Mier we decided to study T-uniqueness of graphs. Our motivation was that, since the Tutte polynomial contains much more information on a graph than the chromatic polynomial, it should be possible to prove T-uniqueness for graphs conjectured but not known to be χ-unique.

The following result (see [33]) has been our basic tool for proving T-uniqueness.

Lemma 3.9. *Let $G=(V,E)$ be a simple 2-connected graph. Then the following parameters of G are determined by its Tutte polynomial*:

(1) *The number of vertices and the number of edges.*
(2) *The girth g and number of cycles of length g.*
(3) *The edge-connectivity $\lambda(G)$. In particular, a lower bound for the minimum degree $\delta(G)$.*
(4) *The number of cliques of each size. In particular, the clique-number $\omega(G)$.*
(5) *The number of cycles of length three, four and five. For the cycles of length four, it is also possible to know how many of them have exactly one chord.*
(6) *The chromatic polynomial and the chromatic number.*

The first family we studied was that of wheels, and we were able to prove that all wheels are T-unique, independently of the parity. Thus W_5 and W_7 are our first examples of 3-connected T-unique graphs which are not χ-unique. Other families we proved to be T-unique in [33] are the following: prisms (the product of a cycle and K_2), Möbius prisms (even cycles in which every pair of opposite vertices is joined by an edge), squares of cycles (cycles in which every pair of vertices at distance two is joined by an edge), n-cubes, and complete multipartite graphs.

Let us exemplify the general strategy of the proofs in the last case. Complete bipartite graphs were known to be χ-unique and, since they are 2-connected, they are also T-unique. For the general case $r\geqslant 3$, let H be a graph T-equivalent to $K_{p_1,\dots,p_r}$. It follows from the previous lemma that H is 2-connected with $n=\sum p_i$ vertices and $\sum_{1\leqslant i<j\leqslant r} p_i p_j$ edges. We also know that $\chi(H)=r$, that is, H is r-partite. The key point is to show first that H is a *complete* r-partite graph. The argument relies on a double counting of subgraphs in H isomorphic to K_r and to K_{r+1}^-, but is to technical to reproduce it here.

This is not the end, since so far we only know that $H=K_{q_1,\dots,q_r}$ for some positive q_i. By counting the number of complete subgraphs of size from 1 to r, quantities that can be read from the Tutte polynomial, we obtain

$$\begin{aligned}\sum_i p_i &= \sum_i q_i,\\ \sum_{i<j} p_i p_j &= \sum_{i<j} q_i q_j,\\ \cdots &= \cdots\\ p_1\cdots p_r &= q_1\cdots q_r.\end{aligned}$$

The proof finishes by recalling that two sets of numbers with the same elementary symmetric functions must be equal.

It is clear that in this case the information used from the Tutte polynomial is not at all available from the chromatic polynomial. In fact, we believe that the problem of deciding for which values of the p_i the graph $K_{p_1,\dots,p_r}$ is χ-unique is out of reach with the current techniques.

In some sense, a proof of T-uniqueness for a graph G produces a characterization of G in terms of some numerical invariants. On the other hand, known characterizations of this kind can be used to prove T-uniqueness. Consider the following result by Mulder [2], where the n-cube is the product of n copies of K_2.

Theorem 3.10. *A connected n-regular graph is isomorphic to the n-cube if and only if it has 2^n vertices and every pair of vertices at distance 2 have precisely two common neighbors.*

We show in [33] that the conditions in this theorem can be deduced from the Tutte polynomial by counting the number of cycles of length four. As a consequence, we show that the n-cube is T-unique.

In [31] we studied the product of two cycles $C_p \times C_q$. These graphs are also known as toroidal grids since they have a natural embedding in the torus as a grid of parallels and meridians. This case turned out to be much harder, but it lead us to the study of an interesting family of graphs. The neighborhood of any vertex in $C_p \times C_q$ has the structure of a 3×3 plane square grid. We say that a graph is *locally grid* if it is 4-regular and the neighborhood of any vertex "looks like" ⊞; the precise definition is a bit more technical and is not necessary to reproduce it here. For $p, q \geqslant 6$ we prove the following facts, which together imply that the corresponding toroidal grids are T-unique.

1. Any graph T-equivalent to $C_p \times C_q$ is a locally grid graph.
2. Any locally grid graph is one of three specific families, and each of them has a natural embedding either on the torus or on the Klein bottle.
3. If H is a locally grid graph on pq vertices different from $C_p \times C_q$, then $T(H; x, y) \neq T(C_p \times C_q; x, y)$.

The real difficulty lies in the last statement. For instance, observe that if $pq = p'q'$, then $C_p \times C_q$ and $C_{p'} \times C_{q'}$ have the same number of vertices and edges, and cannot be distinguished just by local properties. In order to prove that they have different Tutte polynomials, we needed to consider global properties. In this case the key point is that, if $p = \min\{p, q, p', q'\}$, then $C_p \times C_q$ and $C'_p \times C'_q$ have different number of cycles of length p. Since we cannot obtain this information directly from the Tutte polynomial (except for small values p), we had to count very carefully subgraphs of size p and rank $p-1$ in both graphs.

In a different direction, in [35] we study Tutte uniqueness of line graphs. Our main result there is the following.

Theorem 3.11. *Let G be a d-regular d-edge-connected graph on n vertices, and assume that either $d > 3$, or $d = 3$ and G is triangle-free. If a graph H is T-equivalent to $L(G)$, then $H = L(G_0)$, where G_0 is a d-regular graph on n vertices.*

The condition of regularity is essential, but the other conditions are technicalities needed in the proof of the theorem. As a consequence, we prove that $L(K_n)$ and the line graphs of regular complete multipartite graphs are T-unique. We also show that the line graph of $K_{m,n}$ is T-unique for every m, n. The reader can compare these results with those mentioned above on graphs cospectral with the line graphs of K_8 and $K_{4,4}$.

4. Related questions

In this section we discuss several topics directly related to the material in the previous sections.

4.1. Other polynomial invariants

Surveying the graph theory literature, one finds an astonishing variety of polynomials that have been defined for graphs. In this section we discuss those which we find more relevant to the context of this paper.

First of all, we should mention the flow polynomial. Given a finite Abelian group A written additively, an *A-flow* on an oriented graph $\vec{G}$ is an assignment of weights from A to the directed edges of $\vec{G}$ so that, at each vertex x, the sum in A of the weights of the edges directed into x equals the sum of the weights of the edges directed out from x (this is known as Kirchhoff's law). The flow is *nowhere zero* if none of the weights is zero. It is an easy fact that, orienting a graph in any way, the number of nowhere zero flows does not depend on the orientation chosen. What is really surprising is that this number depends only on the order $|A|$ and not on the precise structure of the group (see, for instance, [14]).

Let $F(G,k)$ be the number of nowhere zero flows of G on the additive cyclic group $\mathbb{Z}_k$. It can be proved that $F(G,k)$ is a polynomial function of k, called the flow polynomial, and that it is an evaluation of the Tutte polynomial:

$$F(G,k) = (-1)^{|E(G)|-r(G)}T(G;0,1-k).$$

It follows from the basic properties of the Tutte polynomial that if G is a plane graph and G^* its plane dual, then

$$P(G^*,x) = x^{c(G)}F(G,x),$$

where as before $c(G)$ is the number of components of G. Thus, we see that there is a duality between colorings and nowhere zero flows. For example, the four-color theorem is equivalent to saying that every bridgeless planar graph has a nowhere zero 4-flow. A deep conjecture of Tutte is that *any* graph has a nowhere zero 5-flow. To our knowledge, no systematic study has been carried out on graphs determined by their flow polynomials.

Next, we turn to generalizations of $T(G;x,y)$ to polynomials in many variables. We mention first the U polynomial, defined by Noble and Welsh [36] as follows. Let $G=(V,E)$ be a graph on n vertices, and let $y,x_1,\ldots,x_n$ be a set of commuting indeterminates. Then

$$U(G;\mathbf{x},y) = \sum_{A\subseteq E} x_{n_1}\cdots x_{n_k}(y-1)^{|A|-r(A)},$$

where $n_1,\ldots,n_k$ are the vertex sizes of the connected components of the spanning subgraph (V,A). If we let $x_1=\cdots=x_n=x-1$ in the expression above, we recover the Tutte polynomial $T(G;x,y)$ up to a power of $x-1$. But $U(G;\mathbf{x},y)$ contains information on G that in principle cannot be deduced from $T(G;x,y)$. For instance, it is proved in [36] that from $U(G;\mathbf{x},y)$ one can recover the matchings polynomial $\mu(G,x)$.

The polychromate was defined by Brylawski [13] as a polynomial on graphs, again in as many x variables as vertices. Recently, it has been proved that the polychromate and the U polynomial determine each other [40]. The chromatic symmetric function defined by Stanley [43] also turns out to be equivalent to the U polynomial [36]. Other generalizations of the Tutte polynomial, introduced by Bollobás and Riordan, include Tutte polynomials for colored graphs [8] and Tutte polynomials for graphs embedded in surfaces [9].

Finally, given a 4-regular graph G and a system $W=(\alpha,\beta,\gamma)$ of weights, there is defined a transition polynomial $Q(G;W,x)$ (see [23]). It happens that the Tutte polynomial of a plane graph can be described as the transition polynomial of its medial graph, taking $\gamma=0$. Setting $\alpha=0,\ \beta=1,\ \gamma=-1$ one obtains the Penrose polynomial. This is a very interesting object, first defined by Penrose in a paper with a surprising title [38]. The definition can be extended to all graphs as described in [1].

The former polynomials have not been studied yet with respect to the uniqueness property. In our opinion, the polynomial $U(G;\mathbf{x},y)$ is a good candidate for that, since it contains an enormous amount of information on the graph G.

4.2. Polynomials on matroids

Matroids were introduced by Whitney as an abstraction of the properties of linear dependence in vector spaces [51]. They can be defined using several equivalent systems of axioms, but for our purposes it is convenient to define them in terms of a rank function. A matroid M consists of a finite set E and a function $r:2^E\to\mathbb{N}$ such that for all $A,B\subseteq E$

1. $r(A)\leqslant|A|$;
2. $A\subseteq B\Rightarrow r(A)\leqslant r(B)$;
3. $r(A\cup B)+r(A\cap B)\leqslant r(A)+r(B)$.

These properties are easily checked when E is the set of columns of a matrix over a field and $r(A)$ is the rank of the submatrix defined by the columns in A. Matroids of this kind are called vector matroids.

Given a matroid with ground set E, a set $A\subseteq E$ is independent if $r(A)=|A|$. A basis is a maximal independent set; all bases have the same size, the rank $r(E)$ of the matroid. A circuit is a minimal dependent set. A flat is a set F such that if $r(F\cup a)=r(F)$ then $a\in F$. A matroid is simple if it contains no loops (elements $a\in E$ such that $\{a\}$ is a circuit) or parallel elements (pairs $a,b\in E$ such that $\{a,b\}$ is a circuit). See [37] for a thorough treatment of matroids.

Any graph $G=(V,E)$ gives raise naturally to a matroid with ground set E and rank function the one defined in Section 2. This matroid is denoted by $M(G)$. Matroids of this kind are called graphic. The bases of $M(G)$ are the spanning forests and the circuits correspond to cycles in G. As we mentioned in Section 2, all graphic matroids are vector matroids, but the converse is not at all true. The reader can check that the 2×4 matrix over $\mathbb{R}$ having as rows $(1,0,1,1)$ and $(0,1,1,-1)$ is not graphic (it should come from a graph with four edges, any three of them defining a triangle); this is in fact the smallest non-graphic matroid.

For any matroid M one defines the Tutte polynomial in exactly the same way as for graphs:

$$T(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Given this definition, it is easy to prove that $T(M;1,1)$ is the number of bases of M and that $T(M;2,1)$ is the number of independent sets. Moreover, as for graphs, for a simple matroid M one proves that from $T(M;x,y)$ we can deduce the size of a smallest circuit, the number of such circuits, and many other invariants. Thus we see that $T(M;x,y)$ contains important combinatorial information about M.

As in the case of graphs, we say that a matroid M is T-unique if for any other matroid N with $T(N;x,y)=T(M;x,y)$, necessarily N is isomorphic to M (an isomorphism between matroids is a bijection between their ground sets that preserves ranks).

To prove that a matroid is T-unique is usually much harder that in the case of graphs. The reason is that the class of matroids is much larger than the class of graphs. In fact, in a probabilistic sense that is made precise later, almost no matroid is T-unique. Which matroids are known to be T-unique? Given a finite field $\mathbb{F}_q$, the projective geometry of dimension n is the matroid $PG(n,q)$ having as elements the points of the projective space of dimension n over $\mathbb{F}_q$, and the rank of a set of points is the dimension of the projective subspace they span. They play a central role in matroid theory and have been characterized in several ways.

In [12] Bonin and Miller prove that projective geometries over finite fields of dimension at least three are characterized by numerical invariants that can be deduced from the Tutte polynomial, hence they are T-unique. They prove the same result for affine geometries, matroids $M(K_n)$ of complete graphs, and some generalizations known as Dowling geometries. A striking fact is the proof by Kung [28] that projective geometries are characterized by just three numerical invariants, which again can be deduced from the Tutte polynomial. Another interesting result by Bonin and de Mier is that the cycle matroids of wheels are T-unique, not only as graphs, but also as matroids; see [11] for this and related results.

In [34], de Mier and Noy present a survey of T-unique matroids and prove some new results, like the fact that the matroids $M(K_{m,n})$ and the truncations of $M(K_n)$ are T-unique. They also produce the first examples of exponentially large families of T-unique matroids of the same rank and size based on a special kind of planar graphs. If one looks at the proofs of T-uniqueness for matroids, one realizes that the key ingredient is to obtain enough information from $T(M;x,y)$ on the number of circuits and flats of certain sizes and ranks.

4.3. Equivalent graphs

Recall that if f is a polynomial invariant on graphs, G and H are f-equivalent if $f(G)=f(H)$. For any of the polynomials we have discussed, there are examples of nonisomorphic equivalent graphs. For the characteristic polynomial, there is a general technique known as Seidel switching for producing cospectral pairs of graphs. For the Tutte polynomial, the first examples of pairs of T-equivalent 3-, 4-, and 5-connected graphs were found by Tutte [46]. Later Brylawski extended this result for arbitrarily high connectivity [13]; simpler examples are shown in [7].

An interesting question is how many graphs can share the same polynomial invariant. In [7] it is proved that for every large n that is a multiple of 10, there exist $2^{n/10}$ highly connected, nonisomorphic graphs on n vertices, all with the same Tutte polynomial. Notice that they have also the same chromatic polynomial. A similar result is proved for the characteristic polynomial in [42]: for each $n>8$, there exist $2^{n/6}$ nonisomorphic regular graphs on n vertices, all having the same spectrum.

In the case of matroids, Bonin [10] uses inequivalent representations over finite fields to produce large families of nonisomorphic T-equivalent 3-connected matroids.

4.4. *Asymptotics*

Let f be a polynomial invariant, and let $F(n)$ be the proportion of f-unique graphs among all graphs on n vertices. If one could prove that $F(n) \to 1$ as $n \to \infty$ this would indicate that f is a strong invariant in the sense that asymptotically almost all graphs are determined by f. To our knowledge, such an statement has not been proved for any of the polynomials mentioned above. Even if there are many properties that we know hold for almost all graphs, they do not seem strong enough to imply f-uniqueness. However, there are some interesting conjectures; let us mention two of them stated in [7].

Conjecture. (1) *Almost all graphs are χ-unique.* (2) *Almost all graphs are T-unique.*

Of course, (1) implies (2). Given the difficulty in finding pairs of 3-connected T-equivalent graphs, conjecture (2) seems very plausible. The reader can find in [7] an interesting discussion on these conjectures.

The data presented by Haemers [21], based on exhaustive computations for all graphs with up to 10 vertices, show that the proportion of graphs determined by the spectrum decreases from 1.0 when $n=4$ to 0.787 when $n=10$. Since the number of nonisomorphic graphs with n vertices grows very rapidly (for $n=10$ there are 12 005 168), it is not possible to extend these data much further. With respect to the chromatic polynomial, the data in [30] show that the proportion of connected χ-unique graphs with 7 vertices is $100/853=0.117$, and with 8 vertices is $570/11117=0.512$.

We know of very few positive asymptotic results. One of them is due to Schwenk [41], who proved that almost every tree has a cospectral graph, that is, a graph with the same characteristic polynomial. Another result concerns homeomorphs of K_4, that is, graphs obtained from K_4 by repeatedly subdividing edges. Li [26] proved that almost every homeomorph of K_4 is χ-unique.

At a recent meeting, Dominic Welsh asked if we knew of any non-trivial polynomial P for which one could prove that almost no graph is P-unique. We present here a simple example. Given a graph $G=(V,E)$, a set of vertices $U \subseteq V$ is stable if there is no edge joining vertices in U. Let α_i be the number of stable sets of size i, and define the stability polynomial as

$$A(G;x) = \sum_i \alpha_i x^i.$$

We claim that asymptotically almost no graph is determined by $A(G;x)$. Let $g(n)$ be the number of graphs with n vertices up to isomorphism. It is well known that

$$\log g(n) = \frac{1}{2}n^2 + \mathrm{o}(n^2),$$

where logarithms are base 2. We also need the following result, that follows directly from a much stronger result on the clique number of random graphs [6, Section VII.3].

Lemma 4.1. *The maximal size of a stable set in a random graph with n vertices is* $\mathrm{O}(\log n)$.

This implies that for almost all graphs, the coefficients α_i vanish for $i \geqslant c \log n$, where c is a constant. Since each coefficient is an integer between zero and 2^n, it follows that the number of such polynomials is

$$(2^n)^{c \log n} = n^{cn}.$$

Since $\log(n^{cn}) = cn \log n = \mathrm{o}(n^2)$, the claim follows.

We remark that the Tutte polynomial cannot be a strong invariant for matroids in the sense that the proportion of matroids with m elements determined by its Tutte polynomial tends to zero as $m \to \infty$. This is proved using a counting argument in [14, Exercise 6.9].

In conclusion, we wish to stress the interest in proving that asymptotically almost all graphs are unique with respect to some of the polynomial invariants discussed previously.

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