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On solutions of nonlinear time-space fractional Swift–Hohenberg equation: A comparative study

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Homotopy analysis method

Abstract In this paper, a comparison for the solutions of nonlinear Swift–Hohenberg equation with time-space fractional derivatives has been analyzed. The two most promising techniques, fractional variational iteration method (FVIM) and the homotopy analysis method have been chosen for the comparison. The two different definitions of fractional calculus are considered to solve time-fractional derivative separately for the considered approaches. Also, the space fractional derivative is described in the Reisz sense. Analytical and numerical solutions for various combinations of the parameters are obtained. Numerical comparisons have been made for different values of parameters and depicted.

1. Introduction

During the past few decades, the focus of fractional order differential equations [1–5] which has gained mounting interest for some time due to its demonstrated applications in numerous diverse and wide spread, in particular in relation to continuum mechanics, viscoelastic and viscoplastic flow and anomalous diffusion (superdiffusion, non-Gaussian diffusion). The intensive development and constructions of the theory of fractional calculus played an important role for its applications in various fields of sciences such as: electrical circuits, control theory, image processing, viscoelasticity, biology and many other applications. Swift and Hohenberg were first who introduced the mathematical model for the Rayleigh–Benard convective instability of the fluid with thermal fluctuations [6]. The vast field of S–H equation is utilized from hydrodynamics of fluids in physics and engineering problems to the formation of complex patterns [7–10]. The notable applications attract the growing interest of researchers to discover the solutions for time and space fractional equations governed by S–H equations to study the history of the pattern or flow, Akyildiz et al. analyzed the solutions of S–H equations analytically [11], Khan et al. [12] provided the analytical methods for solving the time-fractional Swift–Hohenberg (S–H) equation, Vishal and others examined the approximate analytical solutions of the nonlinear Swift Hohenberg equation with fractional time derivative via homotopy analysis method [13], the fractional variational iteration method (FVIM) with modified Riemann–Liouville derivative has been employed to obtain the approximate solutions of time-fractional Swift–Hohenberg (S–H) equation by Merdan [14], Youshan and Jizhou obtained

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the solutions using a shooting method for the Swift–Hohenberg equation [15].

In the present study, the solutions of nonlinear Swift–Hohenberg equation with time-space fractional derivatives has been found analytically and numerically by employing fractional variational iteration method and the homotopy analysis method. The Reisz definition is used to explain the space factional derivative respectively, as it deals with the derivative of trigonometric and hyperbolic function that cannot be explained by Caputo and Jummarie definitions. While for the time-fractional derivative, Jummarie's definition has been used for FVIM and Caputo's definition for the HAM. Numerical solutions have been analyzed for several combinations of pertaining parameters, i.e., a, L, α and λ. Numerical results have been compared by illustrating graphs and tables.

2. Basics of fractional calculus

There are several definitions of a fractional derivative of order $\alpha > 0$ e.g Riemann–Liouville, Caputo Riesz and Jumarie's fractional derivative. Here, some basic definitions and properties of the fractional calculus theory which can be used in this paper are presented.

Definition 2.1. Caputo's definition of the fractional order derivative is given as

$$D_t^{\Lambda} f(t) = \frac{1}{\Gamma(n-\Lambda)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\Lambda+1-n}} d\xi, \quad n-1 < Re(\Lambda) \leqslant n, \quad n \in N, \quad t > 0, \tag{1}$$

where the parameter Λ is the order of the derivative and is allowed to be real or even complex, a is the initial value of function f. In the present work only real and positive values of Λ are considered. For the Caputo's derivative we have

$$D_t^{\Lambda} C = 0 \text{ (where } C \text{ is a constant)} \tag{2}$$

$$D_t^{\Lambda} t^{\gamma} = \begin{cases} 0; & (\gamma \leqslant \Lambda - 1), \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\Lambda+1)} t^{\gamma-\alpha}; & (\gamma > \Lambda - 1). \end{cases} \tag{3}$$

Definition 2.2. The Caputo time-fractional derivative operator of order $\Lambda > 0$ is defined as

$$D_t^{\Lambda} u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\Lambda)} \int_a^t \frac{1}{(t-\xi)^{\Lambda+1-m}} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi; & m-1 < \Lambda < m, m \in N, \\ \frac{\partial^m u(x,\xi)}{\partial \xi^m}; & \Lambda = m \in N. \end{cases} \tag{4}$$

For establishing our results, we also necessarily introduce following Riemann–Liouville fractional integral operator.

Definition 2.3. The Riemann–Liouville fractional integral operator of order α is defined as

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi; \alpha > 0, t > 0. \tag{5}$$

For $\alpha \geqslant -1$, $\alpha, \beta \geqslant 0$, $\gamma \geqslant -1$, we have

$$J_t^0 f(t) = f(t), \quad J_t^{\alpha} J_t^{\beta} f(t) = J_t^{\alpha+\beta} f(t), \quad J_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha} \tag{6}$$

$$D_t^{\alpha} J_t^{\alpha} f(t) = f(t), \quad J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, t > 0. \tag{7}$$

Definition 2.4. The Riesz fractional derivative R_x^{λ} is defined as [16,17]

$$R_x^{\lambda} u(x) = -\frac{\left[D_+^{\lambda} u(x) + D_-^{\lambda} u(x)\right]}{2Cos(\lambda\pi/2)}, \quad 0 < \lambda < 2, \quad \lambda \neq 1 \tag{8}$$

where $D_+^{\lambda} u(x)$ and $D_-^{\lambda} u(x)$ are the Weyl fractional derivatives

$$D_{\pm}^{\lambda} u(x) = \begin{cases} \pm \frac{d}{dx} I_{\pm}^{1-\lambda} u(x), & 0 < \lambda < 1, \\ \frac{d^2}{dx^2} I_{\pm}^{2-\lambda} u(x), & 1 < \lambda < 2, \end{cases} \tag{9}$$

$I_{\pm}^{\lambda}$ denote the Weyl fractional integrals of order $\lambda > 0$, and given by

$$I_+^{\lambda} u(x) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{x} (x-\eta)^{\lambda-1} u(\eta) d\eta, \tag{10}$$

$$I_-^{\lambda} u(x) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{x} (x-\eta)^{-\lambda-1} u(\eta) d\eta, \tag{11}$$

and when $\lambda = 0$ the Weyl fractional derivative degenerates into the identity operator

$$D_{\pm}^{0} u(x) = Iu(x) = u(x). \tag{12}$$

For continuity, we get

$$D_{\pm}^{1} u(x) = \pm \frac{d}{dx} u(x), D_{\pm}^{2} u(x) = \frac{d^2}{dx^2} u(x). \tag{13}$$

Obviously, in case $\lambda = 2$, the Riesz fractional derivative takes the form of the second order derivative operator

$$R_x^2 u(x) = \frac{d^2}{dx^2} u(x). \tag{14}$$

For the case $\lambda = 1$ we have

$$R_x^1 \frac{d}{dt} Hu(x) = \frac{d}{dx} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\eta)}{\eta - x} dx. \tag{15}$$

where the Hilbert transform and the integral are understand in the Cauchy principal value sense in order to carry out with iterative steps in series solution.

For $\lambda \in (0,2)$, $\lambda \neq 1$,

$$R_x^{\alpha}(e^{i\lambda x}) = -\lambda^{\alpha} e^{i\lambda x}. \tag{16}$$

$$R_x^{\alpha} \sin(\lambda x) = -\lambda^{\alpha} \sin(\lambda x). \tag{17}$$

$$R_x^{\alpha} \cos(\lambda x) = -\omega^{\alpha} \cos(\lambda x). \tag{18}$$

In addition, we want to give the following some properties of the Jumarie's fractional derivative.

Definition 2.5. The Jumarie's fractional derivative [18–20] is defined as:

$$_0D_{\alpha}^{t} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha} (f(\tau) - f(0)) d\tau, \tag{19}$$

where $t \in [0,1]$, $n-1 \leqslant \alpha < n$, and $n \geqslant 1$.

Let $f(t)$ denotes a continuous function [18–20] then the solution is defined as:

$$y = \int_0^t f(\tau)(d\tau)^\alpha = \alpha \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad 0<\alpha<1. \tag{20}$$

For example $f(x) = x^\beta$ in Eq. (24) one obtains,

$$\int_0^x \tau^\beta (d\tau)^\alpha = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} t^{\beta+\alpha}, \quad 0<\alpha<1. \tag{21}$$

3. Generalized space-time-fractional Swift Hohenberg equation

The Swift–Hohenberg (S–H) equation is

$$D_t^\alpha u(x,t) - au + (1+\nabla^\lambda)^2 u + u^3 = 0; \quad X \in \Re, \quad t>0. \tag{22}$$

Thus, writing the S–H equation in a more general form, we consider the problem with time-fractional derivative

$$D_t^\alpha u + 2R_x^\lambda u + R_x^{2\lambda} u + (1-a)u + u^3 = 0,$$
$$0<\alpha\leqslant 1, \quad 1<\lambda\leqslant 2, \tag{23}$$

with boundary conditions

$$u=0, u_{xx}=0 \text{ at } x\in(0,l) \quad \text{for all} \quad t>0, \tag{24}$$
$$u(x,0)=h(x), \quad \text{for all} \quad 0<x<l. \tag{25}$$

4. Implementation of the methods

4.1. Solution of the problem by fractional variational iteration method (FVIM)

Fractional variational iteration method was first proposed by Wu and Lee [21,22] and successfully implemented to solve various problems.

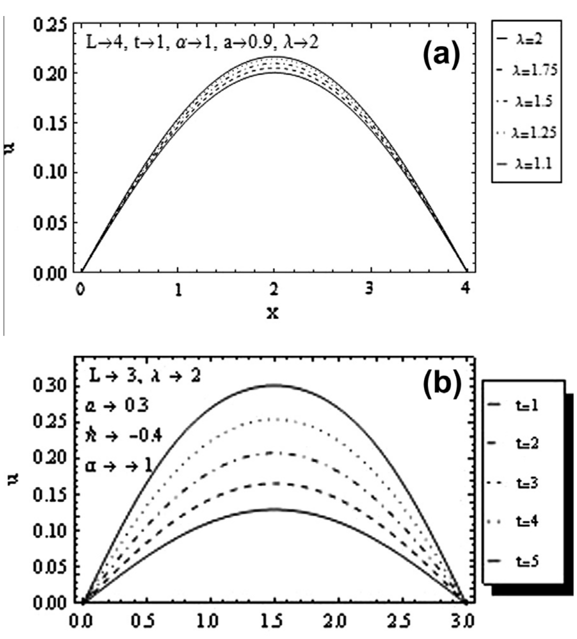

Fig. 1 Profile of $u(x,t)$ vs. x by (a) VIM (b) HAM (for $\hbar = -0.4$).

According to the FVIM, we can build a correct functional formula for Eq. (23) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + I^\alpha\left[\lambda(x,t)\left(D_t^\alpha u_n + 2R_x^\lambda u_n + R_x^{2\lambda}u_n + (1-a)u_n + u_n^3\right)\right], \tag{26}$$

to identify the multiplier, we write (26) in the form

$$u_{n+1}(x,t) = u_n + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda(x,\tau)\begin{pmatrix} D_t^\alpha \tilde{u}_n + 2R_x^\lambda \tilde{u}_n + R_x^{2\lambda}\tilde{u}_n + \\ (1-a)\tilde{u}_n + \tilde{u}_n^3 \end{pmatrix} d\tau. \tag{27}$$

Using Eq. (20), we obtain a new correction functional:

$$u_{n+1}(x,t) = u_n + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(x,\tau)\begin{pmatrix} D_t^\alpha \tilde{u}_n + 2R_x^\lambda \tilde{u}_n + R_x^{2\lambda}\tilde{u}_n + \\ (1-a)\tilde{u}_n + \tilde{u}_n^3 \end{pmatrix} (d\tau)^\alpha. \tag{28}$$

It is obvious that the approximations u_n, $n \geqslant 0$ can be established by determining λ, a general Lagrange's multiplier, which can be identified optimally with the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$ Therefore, we first designate the Lagrange multiplier that will be identified optimally by the use of integration by parts. The straightforward approximations $u_{n+1}(x \cdot t)$, $n \geqslant 0$ of the solution $u(x \cdot t)$, will be willingly obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Accordingly, the exact solution may be acquired by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{29}$$

By taking the initial approximation as:

$$u_0(x,t) = \frac{1}{10} Sin\left(\frac{\pi x}{L}\right) \tag{30}$$

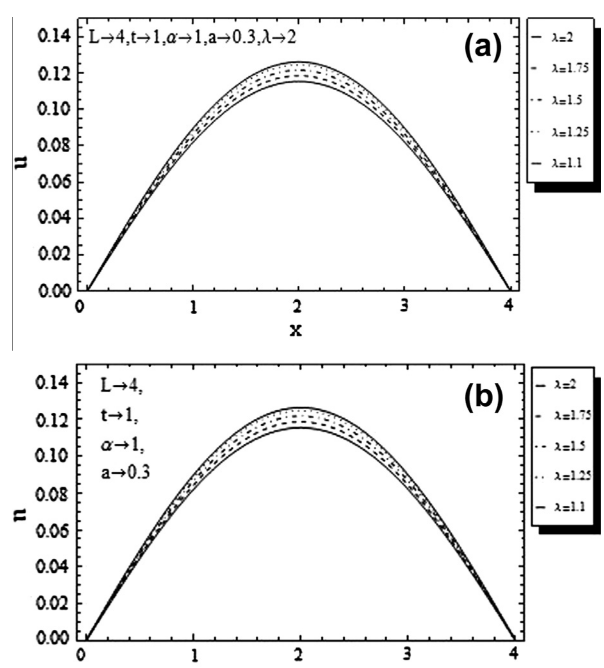

Fig. 2 Profile of $u(x,t)$ vs. x by (a) VIM (b) HAM (for $\hbar = -0.4$).

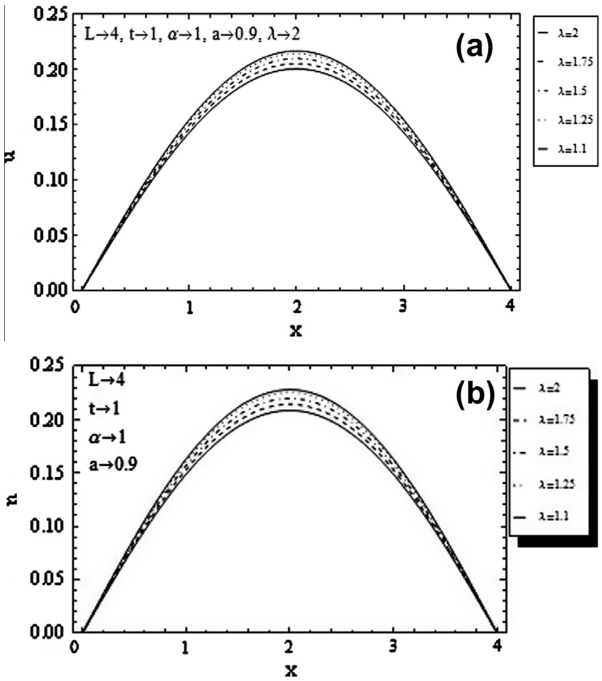

Fig. 3 Profile of $u(x,t)$ vs. x by (a) VIM (b) HAM (for $\hbar = -0.4$).

The successive iteration can be obtained as:

$$u_1(x,t) = \frac{1}{10}Sin\left(\frac{\pi x}{L}\right) - \frac{t^\alpha Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} + \frac{at^\alpha Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} + \frac{\left(\frac{1}{L}\right)^\lambda \pi^2 t^\alpha Sin\left(\frac{\pi x}{L}\right)}{5\Gamma(1+\alpha)} - \frac{\left(\frac{1}{L}\right)^{2\lambda} \pi^{2\lambda} t^\alpha Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} - \frac{t^\alpha Sin\left(\frac{\pi x}{L}\right)^3}{1000\Gamma(1+\alpha)} \tag{31}$$

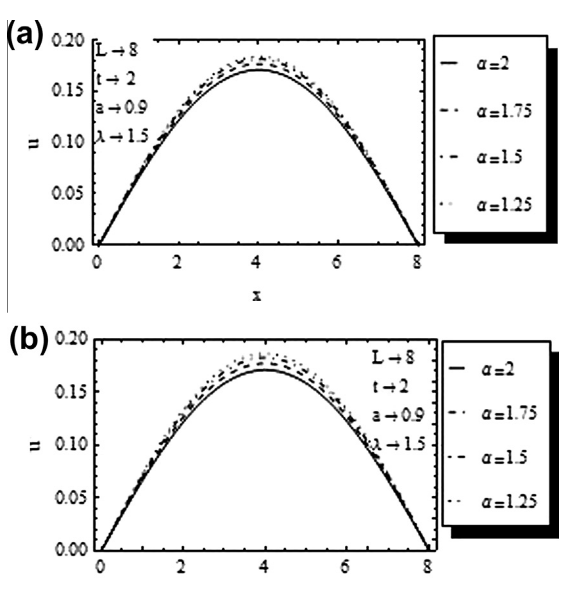

Fig. 4 Profile of $u(x,t)$ vs. x by (a) VIM (b) HAM (for $\hbar = -0.4$).

The solution obtained by next iteration is large enough that only two iterations of FVIM have been done. It can be observed from Eq. (31) that if we replace the space fractional derivative $\lambda = 2$ and simplify it, we can easily accomplish the solution of Merdan [14].

4.2. Solution of the problem by HAM

The homotopy analysis method first introduced by Liao [23,24] efficaciously employed to solve nearly all kinds of integer and fractional order nonlinear differential equations [25,26].

Hence, to solve Eq. (23) by HAM, we choose the initial approximation as:

$$u_0(x,t) = \frac{1}{10}\sin\left(\frac{\pi x}{L}\right), \tag{32}$$

and the linear operator,

$$L[\varphi(x,t;q)] = \frac{\partial^\alpha \varphi(x,t;q)}{\partial t^\alpha}, \tag{33}$$

with the property

$$L[c] = 0, \tag{34}$$

where c is integral constant. Furthermore, for Eq. (23), we define a nonlinear operator as

$$N[\varphi] = D_t^\alpha \varphi(x,t;q) + 2R_x^\lambda \varphi(x,t;q) + R_x^{2\lambda}\varphi(x,t;q) + (1-a)\varphi(x,t;q) + \varphi^3(x,t;q). \tag{35}$$

Now, we construct the zeroth-order deformation equation

$$(1-q)L[\varphi(x,t;q) - u_0(x,t)] = q\hbar N[\varphi(x,t;q)]. \tag{36}$$

Obviously, when $q = 0$ and $q = 1$

$$\varphi(x,t;0) = u_0(x,t) \quad \varphi(x,t;1) = u(x,t) \tag{37}$$

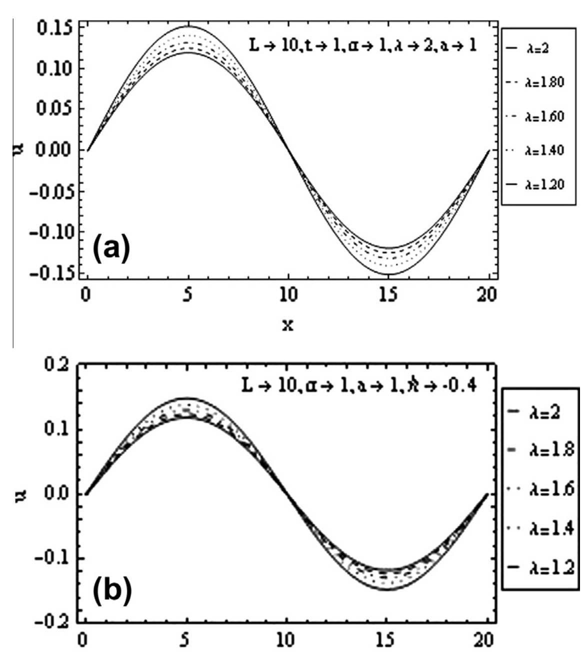

Fig. 5 Profile of $u(x,t)$ vs. x by (a) VIM (b) HAM (for $\hbar = -0.4$).

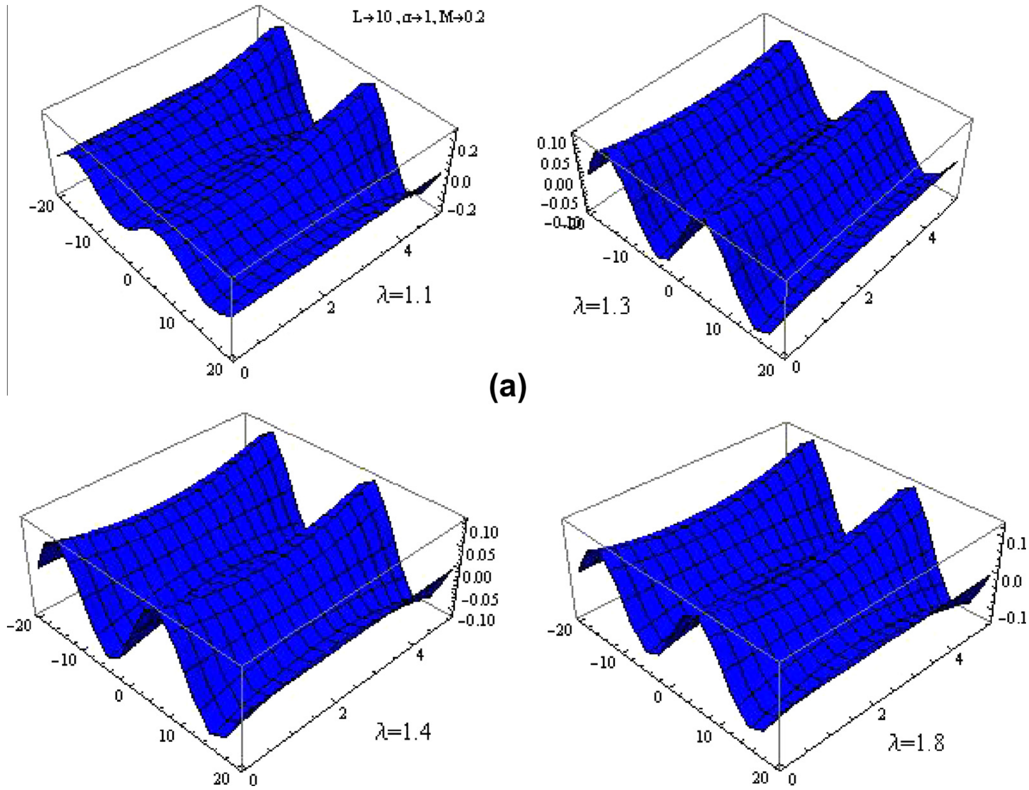

Fig. 6-a The three dimensional surface of $u(x,t)$ by VIM.

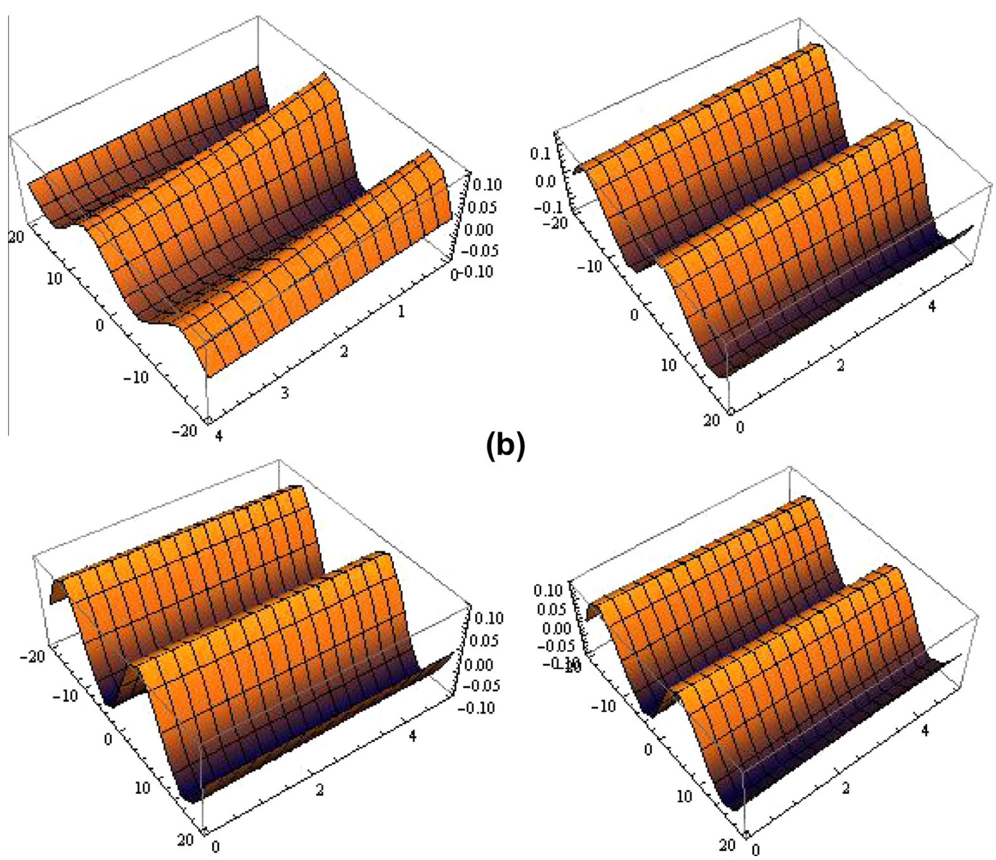

Fig. 6-b The three dimensional surface of $u(x,t)$ by HAM (for $\hbar = -0.4$).

Table 1 Numerical values of function $u(x,t)$ by HAM (in brackets) and by FVIM for different values of α and λ keeping $L = 3$, $t = 1$, $x = 1$, $M = 0.2$ and $\hbar = -0.55$ (for HAM).

λ/α	0.4	0.6	0.8	1	1.2
2.0	(0.107350) 0.107334	(0.106697) 0.106774	(0.105369) 0.105503	(0.103652) 0.103806	(0.101754) 0.101902
1.75	(0.107626) 0.107612	(0.106963) 0.107041	(0.105615) 0.105751	(0.103873) 0.104029	(0.101948) 0.102098
1.50	(0.107863) 0.107849	(0.107190) 0.107270	(0.105825) 0.105963	(0.104062) 0.104219	(0.102113) 0.102266
1.25	(0.108060) 0.108048	(0.107380) 0.107461	(0.106000) 0.106140	(0.104219) 0.104378	(0.102251) 0.102405

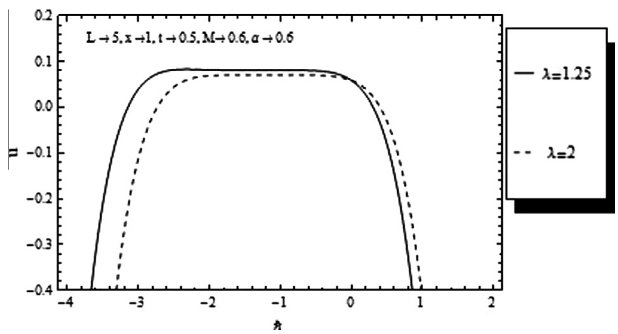

Fig. 7 Plot of $u(x,t)$ vs. ħ.

Expanding $\varphi(x,t;q)$ in Taylor series with respect to q, one can find

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \quad u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{38}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter ħ are properly chosen, the above series is convergent at $q = 1$, the series form becomes:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{39}$$

The mth order deformation equation is

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}(x,t)), \tag{40}$$

where

$$R_m(\vec{u}_{m-1}(x,t)) = \left(\begin{array}{c} D_t^{\alpha} u_{m-1} + 2R_x^{\lambda} u_{m-1} + R_x^{2\lambda} u_{m-1} + (1-a)u_{m-1} + \\ \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} u_j u_{i-j} \right) u_{m-1-i} \end{array} \right) \tag{41}$$

The values of $u_m(x,t)$ for $m = 1, 2, 3, \ldots$ can be obtained from Eq. (40).

The first component of the function $u(x,t)$ by HAM is found as

$$u_1 = \frac{\hbar t^{\alpha} Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} - \frac{\hbar a t^{\alpha} Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} - \frac{\hbar \left(\frac{1}{L}\right)^{\lambda} \pi^2 t^{\alpha} Sin\left(\frac{\pi x}{L}\right)}{5\Gamma(1+\alpha)} + \frac{\hbar \left(\frac{1}{L}\right)^{2\lambda} \pi^{2\lambda} t^{\alpha} Sin\left(\frac{\pi x}{L}\right)}{10\Gamma(1+\alpha)} + \frac{\hbar t^{\alpha} Sin\left(\frac{\pi x}{L}\right)^3}{1000\Gamma(1+\alpha)} \tag{42}$$

As specified by Liao, homotopy analysis method provides the auxiliary parameter ħ to choose the convergence region of the obtained solution. The form of series is obtained through sixth order of approximation.

It is examined by Liao that HPM is the special case of HAM and if the value of parameter ħ is replaced by $\hbar = -1$, it represents the same solution as obtained through HPM. Here, for the considered problem if we put $\hbar = -1$ and $\lambda = 2$ in Eq. (42), the solution obtained by Khan et al. [12] can be recovered.

5. Results and conclusion

In this paper, the nonlinear time-space fractional Swift–Hohenberg equation has been solved and compared employing the HAM with time-fractional Caputo derivative and FVIM with time-fractional Jumarie derivative, whereas the Reisz definition has been used for space fractional derivative in both approaches.

Following results have been obtained which provide the insight into the function, they are compared through Figs. 1–6 (a- by FVIM and b- by HAM) for various values of related parameters.

- Table 1 provides the insight into the influence of space fractional derivative λ and time-fractional derivative α on the function $u(x,t)$. It can be observed that the function $u(x,t)$ is the increasing function of λ and decreasing function of α.
- In Fig. 1(a and b), the profile gives the variation in the function $u(x,t)$ on space co-ordinate x with respect to time t. The graph predicts that, with the increase in time the function $u(x,t)$ is also increasing.
- The effect of space fractional derivative λ on the function is displayed in Figs. 2 and 3(a and b) (2- for the constant parameter $a = 0.3$, 3- for the constant parameter $a = 0.9$). It shows the slight increase in the function $u(x,t)$ with the decrease in λ.
- The time histories of the function $u(x,t)$ are displayed in Fig. 4, and it gives the previous path of the function at different stages of time.
- Fig. 5 is plotted with the large length of the domain for the change in the space fractional derivative which shows two periods of the function.
- The three dimensional surfaces have been displayed for various values of space fractional derivative in Fig. 6.

- Fig. 7 provides the convergence range of the solution obtained through HAM by 6th order of approximation.

It can be seen from the obtained results that both the methods are efficient in solving time-space fractional Swift–Hohenberg equation. With the use of Reisz derivative for space function, the generalized analytic solution can be obtained by employing both the techniques. The comparison made, proves the existence and uniqueness of solution. Also, both the methods are in good agreement with the previous results.

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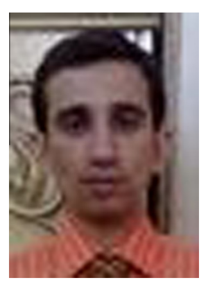

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