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Higher-dimensional Scherk's hypersurfaces

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Abstract

In three-dimensional Euclidean space, Scherk second surfaces are singly periodic embedded minimal surfaces with four planar ends. In this paper, we obtain a natural generalization of these minimal surfaces in any higher-dimensional Euclidean space $\mathbb{R}^{n+1}$, for $n \geqslant 3$. More precisely, we show that there exist $(n-1)$-periodic embedded minimal hypersurfaces with four hyperplanar ends. The moduli space of these hypersurfaces forms a one-dimensional fibration over the moduli space of flat tori in $\mathbb{R}^{n-1}$. A partial description of the boundary of this moduli space is also given.

1. Introduction

In three-dimensional Euclidean space Scherk second surfaces come in a one-parameter family $(S_\varepsilon)_{\varepsilon\in(0,\pi/2)}$ which can be described in many different ways. For example it can be described *via* its Weierstrass representation data [1,6]

$$X_\varepsilon(\omega) := \Re \int_{\omega_0}^{\omega} \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right) \mathrm{d}h_\varepsilon,$$

where

$$g(\omega) := \omega \quad \text{and} \quad \mathrm{d}h_\varepsilon := 4\sin\varepsilon\left(\omega^4 + 1 - 2\cos\varepsilon\,\omega^2\right)^{-1}\omega\,\mathrm{d}\omega.$$

Or even more simply as the zero set of the function

$$F_\varepsilon(x_1, x_2, z) := (\cos\varepsilon)^2 \cosh\left(\frac{x_1}{\cos\varepsilon}\right) - (\sin\varepsilon)^2 \cosh\left(\frac{z}{\sin\varepsilon}\right) - \cos x_2. \tag{1}$$

Indeed, it is well known that, the zero set of a function F is a minimal surface if and only if 0 is a regular value of F and

$$\operatorname{div}\left(\frac{\nabla F}{|\nabla F|}\right) = 0,$$

on the zero set of F. Using this, it is straightforward to check that the zero set of F_ε is a minimal surface.

In any of these descriptions, the parameter ε belongs to $(0, \pi/2)$. Observe that we do not consider any dilation, translation or rotation of a minimal surface; in other words we are only interested in the space of surfaces modulo isometries and dilations.

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Now, we would like to point out a few properties of Scherk's second surfaces which will enlighten our construction of their higher-dimensional analogues.

(i) *Periodicity*. Observe that Scherk's second surfaces are singly periodic and, in the above description, their common period has been normalized to be equal to $(0, 2\pi, 0)$. Hence, if we define $T^1 := \mathbb{R}/2\pi\mathbb{Z}$, we can consider S_ε to be a minimal surface embedded in $\mathbb{R} \times T^1 \times \mathbb{R}$.

(ii) *Asymptotic behavior as ε tends to* 0. Another feature which will be very important for us is the study the behavior of Scherk's second surfaces as the parameter ε tends to 0 (a similar analysis can be performed when the parameter ε tends to $\pi/2$). To this aim, we write for all (x_1, x_2) in some fixed compact subset of $\mathbb{R}^2 - (\{0\} \times 2\pi\,\mathbb{Z})$ and for all ε small enough

$$z = \pm \sin\varepsilon \operatorname{acosh}\left((\tan\varepsilon)^{-2}\cosh\left(\frac{x_1}{\cos\varepsilon}\right) - (\sin\varepsilon)^{-2}\cos x_2\right).$$

Using this, we readily see that, away from the set $\{0\} \times 2\pi\mathbb{Z}$, the one parameter family of surfaces S_ε converges to the union of two horizontal planes, as ε tends to 0. In other words, the sequence of surfaces S_ε converges, away from the origin, to two copies of $\mathbb{R} \times T^1 \times \{0\}$ in $\mathbb{R} \times T^1 \times \mathbb{R}$, as the parameter ε tends to 0.

As already mentioned, a similar analysis can be carried out as the parameter ε tends to $\pi/2$ and, this time, we find that the sequence of surfaces S_ε converges, away from the origin, to two copies of $\{0\} \times T^1 \times \mathbb{R}$ in $\mathbb{R} \times T^1 \times \mathbb{R}$.

(iii) *Blow down analysis*. For each fixed $\varepsilon \in (0, \pi/2)$, the surface S_ε has four planar ends which are asymptotic to

$$V_\varepsilon^\pm := \left\{(x_1, x_2, z) \in \mathbb{R} \times T^1 \times \mathbb{R} \mid z = \pm(\tan\varepsilon|x_1| - 2\sin\varepsilon\log\tan\varepsilon)\right\}.$$

More precisely, away from a compact set in $\mathbb{R} \times T^1 \times \mathbb{R}$, the surface S_ε is a normal graph over $V_\varepsilon^\pm$ for some function which is exponentially decaying as x_1 tends to $\pm\infty$. Another way to understand this would be to say that the sequence of surfaces $\lambda\, S_\varepsilon$ converges, as λ tends to 0, to $W_\varepsilon^+ \cup W_\varepsilon^-$, where

$$W_\varepsilon^\pm := \left\{(x_1, x_2, z) \in \mathbb{R} \times T^1 \times \mathbb{R} \mid z = \pm\tan\varepsilon|x_1|\right\}.$$

(iv) *Blow up analysis*. Instead of blowing down the surfaces S_ε as we have done in (iii), we can blow up the surfaces S_ε by considering the sequence of scaled surfaces $\varepsilon^{-1}S_\varepsilon$. As ε tends to 0 this sequence converges on compact to a vertical catenoid. To see this, just define the new set of coordinates

$$(\tilde{x}_1, \tilde{x}_2, \tilde{z}) := \frac{1}{2\sin\varepsilon}(x_1, x_2, z),$$

and, in (1), we expend both $\cos x_2$ and $\cosh(x_1/\cos\varepsilon)$, in terms of powers of ε. We find with little work

$$(\cos\varepsilon)^2\left(1 + 2(\tan\varepsilon)^2\tilde{x}_1^2\right) - (\sin\varepsilon)^2\cosh(2\,\tilde{z}) = 1 - 2(\sin\varepsilon)^2\tilde{x}_2^2 + \mathcal{O}\left(\varepsilon^4\right).$$

Hence,

$$\tilde{x}_1^2 + \tilde{x}_2^2 = \cosh^2\tilde{z} + \mathcal{O}\left(\varepsilon^2\right).$$

Clearly, as ε tends to 0, this converges, uniformly on compact sets, to an implicit parameterization of a vertical catenoid.

To complete this brief description, let us mention that Scherk's second surfaces have recently been used as one of the building blocks of some desingularization procedure, to produce new embedded minimal surfaces in three-dimensional Euclidean space. We refer to the work of M. Traizet [11] and also to the recent work of N. Kapouleas [4,5] for further details.

In order to state our result properly, we need to introduce two ingredients which will be fundamental in our analysis. First observe that, in higher dimensions, there is a natural generalization of the catenoid in Euclidean three-space. This hypersurface, which we will call the unit n-catenoid, is a hypersurface of revolution with two hyperplanar ends. It can be parameterized by

$$\mathbb{R} \times S^{n-1} \ni (s, \theta) \to \left(\varphi(s)\,\theta, \psi(s)\right) \in \mathbb{R}^{n+1},$$

where the function φ is defined by the identity $\varphi^{n-1}(s) = \cosh((n-1)s)$ and where the function ψ is given by

$$\psi(s) := \int_0^s \varphi^{2-n}(t)\,\mathrm{d}t.$$

Using this n-catenoid, S. Fakhi and the author have produced examples of complete immersed minimal hypersurfaces of $\mathbb{R}^{n+1}$ which have $k \geqslant 2$ hyperplanar ends [2]. These hypersurfaces have the topology of a sphere with k punctures and they all have finite total curvature, they generalize the well known k-noids in three-dimensional Euclidean space [3].

Another ingredient in our analysis is the moduli space of flat tori in $\mathbb{R}^m$, for $m \geqslant 1$. We recall a few well known facts about this moduli space and refer to [13] for further details. Any flat torus in $\mathbb{R}^m$ can be identified with $\mathbb{R}^m/A\mathbb{Z}^m$ where $A \in \mathrm{GL}(m,\mathbb{R})$. The volume of the m-dimensional torus $T^m := \mathbb{R}^m/A\mathbb{Z}^m$ is then given by

$$\mathrm{vol}(T^m) = |\det A|.$$

It is a simple exercise to check that two tori $\mathbb{R}^m/A\mathbb{Z}^m$ and $\mathbb{R}^m/B\mathbb{Z}^m$ are isometric if and only if there exist $M \in O(m,\mathbb{R})$ and $N \in \mathrm{GL}(m,\mathbb{Z})$ such that $A = M\,B\,N$. The moduli space of flat tori $\mathcal{T}^m$ is defined to be the space of tori $T^m = \mathbb{R}^m/A\mathbb{Z}^m$ for $A \in \mathrm{GL}(m,\mathbb{R})$, normalized by asking that

$$\mathrm{vol}(T^m) = \mathrm{vol}(S^m),$$

modulo isometries. For later use, it will be convenient to identify any torus $T^m \in \mathcal{T}^m$ with a subset of $\mathbb{R}^m$. To this aim, if

$$T^m = \mathbb{R}^m/A\mathbb{Z}^m,$$

for some $A \in \mathrm{GL}(m,\mathbb{R})$, we identify T^m with the image of $[-\frac{1}{2},\frac{1}{2}]^m$ by A. In particular, we will talk about the origin $0 \in T^m$, simply referring to the origin in $A[-\frac{1}{2},\frac{1}{2}]^m \subset \mathbb{R}^m$. We will also consider, for $\rho > 0$ small enough, $B^n_\rho \subset \mathbb{R}^{n-m} \times T^m$ as the n-dimensional ball of radius ρ in $\mathbb{R}^{n-m} \times A[-\frac{1}{2},\frac{1}{2}]^m$. And so on. Also observe that, granted this identification, T^m is invariant under the action of the following subgroup of $O(m,\mathbb{R})$

$$\mathfrak{D}_m := \{D := \mathrm{diag}(\eta_1,\ldots,\eta_m) \mid \eta_i = \pm 1\}.$$

In this paper, we pursue the quest of higher-dimensional generalizations of classical minimal surfaces which we have initiated in [2]. More precisely, we obtain a natural generalization of Scherk's second surfaces in higher-dimensional Euclidean spaces. Recall that one can view the moduli space of Scherk's surfaces as a one-dimensional fibration over the moduli space of flat tori in $\mathbb{R}$. We will show that, in $\mathbb{R}^{n+1}$, for $n \geqslant 3$, there exists a finite-dimensional family of embedded minimal hypersurfaces satisfying properties which are similar to (i)–(iv). This family, which turn out to be a one-dimensional fibration over the moduli space of flat tori in $\mathbb{R}^{n-1}$, yields a partial description of the moduli space of what might be called "higher-dimensional Scherk's hypersurfaces". More precisely, we obtain a description of the boundary of this moduli space, this boundary turns out to be modeled over the moduli spaces of tori in $\mathbb{R}^m$ for any $1 \leqslant m \leqslant n-1$.

Our main result can be stated as follows:

Theorem 1. *Assume that $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ are fixed. Let $T^m \in \mathcal{T}^m$ be any flat torus of $\mathbb{R}^m$. Then, there exist $\varepsilon_0 > 0$ and $(S_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ a one-parameter family of minimal hypersurfaces of $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$ such that*:

(i) *For all $\varepsilon \in (0,\varepsilon_0)$, the hypersurface S_ε is embedded in $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$ and is invariant under the action of $O(n-m,\mathbb{R}) \otimes \mathfrak{D}_m \otimes \{\pm I_1\} \subset O(n+1,\mathbb{R})$.*

(ii) *As ε tends to 0, the sequence of hypersurfaces $(S_\varepsilon)_\varepsilon$ converges to the union of two copies of $\mathbb{R}^{n-m} \times T^m \times \{0\}$, away from the origin.*

(iii) *For all $\varepsilon \in (0,\varepsilon_0)$, there exist $c_\varepsilon > 0$ and $d_\varepsilon > 0$ such that the hypersurface S_ε has four ends which are asymptotic to*

$$V^\pm_\varepsilon := \{(x_1,x_2,z) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R} \mid z = \pm(c_\varepsilon \zeta_m(x_1) + d_\varepsilon)\},$$

where $\zeta_{n-1}(y) := |y|$, $\zeta_{n-2}(y) := \log|y|$ and $\zeta_m(y) := 0$, when $m \leqslant n-3$. In particular, this means that, up to a translation along the z-axis, the hypersurface S_ε is a normal graph over $V^\pm_\varepsilon$ for some function which is polynomially decaying in $|x_1|$. Furthermore, when $m = n-1$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{1-n} c_\varepsilon = \frac{1}{2}. \tag{2}$$

(iv) *As ε tends to 0, the sequence of rescaled hypersurfaces $(\varepsilon^{-1}S_\varepsilon)_\varepsilon$ converges, uniformly on compact sets, to a vertical unit n-catenoid.*

When $m = n-1$, this result yields minimal hypersurfaces which constitute the natural generalization of Scherk's second surfaces in higher-dimensional Euclidean spaces. More precisely, when $m = n-1$, the above result provides a description of part of $\mathcal{S}_n$, the moduli space of n-dimensional Scherk's hypersurfaces in $\mathbb{R}^{n+1}$, showing that this moduli space is locally a one-dimensional fibration over the moduli space of flat tori in $\mathbb{R}^{n-1}$. Though we have not been able to prove it, we expect this fibration to extend, as it does when $n = 2$, to all $c_\varepsilon \in (0,+\infty)$.

The above result, when $m \leqslant n-2$, yields hypersurfaces which have to be understood as belonging to the boundary of the moduli space $\mathcal{S}_n$, in the same way that any product $\mathbb{R}^{n-m-1} \times T^m$, for $m \leqslant n-2$ corresponds to a point in the compactification of the moduli space of flat tori in $\mathbb{R}^{n-1}$. We expect that the moduli space $\mathcal{S}_n$ can be compactified and that the family of hypersurfaces described in the above result constitutes a collar neighborhood of the boundary of $\overline{\mathcal{S}}_n$. In other words, Theorem 1 provides a local description of $\mathcal{S}_n$, near its boundary.

To conclude, let us briefly describe the strategy of the proof of the result. It should be clear from (ii) to (iv) that, for small ε, Scherk's second surfaces can be understood as a desingularization of two copies of $\mathbb{R} \times T^1 \times \{0\}$ in $\mathbb{R} \times T^1 \times \mathbb{R}$. Keeping this observation in mind, our strategy will be to show that a similar desingularization is possible for two copies of $\mathbb{R}^{n-m} \times T^m \times \{0\}$ in $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$. The proof of this result is very much in the spirit of [2,7] or [8], however, some aspects are simpler in the present paper thanks to the special geometry of our problem.

Our work has been strongly influenced by the recent work of M. Traizet [12] and the work of N. Kapouleas [4,5] in their construction of minimal embedded surfaces in $\mathbb{R}^3$. Indeed, on the one hand, N. Kapouleas has used Scherk's second surfaces to desingularize finitely many catenoids or planes having a common axis of revolution and he has produced embedded minimal surfaces with finitely many ends and very high genus. On the other hand, M. Traizet has used finitely many catenoids to desingularized parallel planes and he has produced minimal surfaces with finitely many ends and genus larger than 2. There is a formal link between these two constructions since, in some vague sense, the surfaces constructed by N. Kapouleas on the one hand and the surfaces constructed by M. Traizet, for a genus large enough, on the other hand, should belong to the same moduli space. It was therefore tempting to try to produce Scherk's second surfaces using some desingularization procedure.

2. Definitions and notations

In this brief section we record some notations and definitions which will be used throughout the paper.

Eigenfunctions of Δ_{T^m}*:* Given $m \geqslant 1$ and $T^m \in \mathcal{T}^m$, we will denote by E_i, $i \in \mathbb{N}$, the eigenfunctions of the Laplacian on T^m with corresponding eigenvalues μ_i, that is $\Delta_{T^m} E_i = -\mu_i E_i$, with $\mu_i \leqslant \mu_{i+1}$. We will assume that these eigenfunctions are counted with multiplicity and are normalized so that

$$\int_{T^m} E_i^2 \, \mathrm{d}x = 1.$$

Though the spectral data of Δ_{T^m} do depend on T^m, we will not write this dependence in the notation.

Functions on T^m *which are invariant under the action of some group:* We will be interested in functions on T^m and eigenfunctions of Δ_{T^m} which have some special symmetry. Namely, the set of functions and eigenfunctions which are invariant under the action of the following subgroup of $O(m, \mathbb{R})$

$$\mathfrak{D}(m) := \bigl\{ D := \mathrm{diag}(\eta_1, \ldots, \eta_m) \bigm| \eta_\ell = \pm 1 \bigr\}.$$

We define $\mathfrak{I}(m) \subset \mathbb{N}$ to be the set of indices i corresponding to eigenfunctions E_i which are invariant under the action of $\mathfrak{D}(m)$, that is

$$\mathfrak{I}(m) := \bigl\{ i \geqslant 0 \bigm| E_i = E_i \circ D, \ \text{for all } D \in \mathfrak{D}(m) \bigr\}. \tag{3}$$

Eigenfunctions of $\Delta_{S^{n-1}}$*:* For all $n \geqslant 2$, we will denote by e_j, $j \in \mathbb{N}$, the eigenfunctions of the Laplacian on S^{n-1} with corresponding eigenvalues λ_j, that is $\Delta_{S^{n-1}} e_j = -\lambda_j e_j$, with $\lambda_j \leqslant \lambda_{j+1}$. We will assume that these eigenfunctions are counted with multiplicity and are normalized so that

$$\int_{S^{n-1}} e_j^2 \, \mathrm{d}\theta = 1.$$

Functions on $\mathbb{R}^n$ *or on* S^n *which are invariant under the action of some group:* Given $1 \leqslant m \leqslant n-1$, we can decompose $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$. We will be interested in functions on $\mathbb{R}^n$ and eigenfunctions of $\Delta_{S^{n-1}}$ which have some special symmetry. Namely, functions which are invariant under the action of the following subgroup of $O(n, \mathbb{R})$

$$\mathfrak{H}(n, m) := O(n-m, \mathbb{R}) \otimes \mathfrak{D}(m).$$

It will be convenient to define $\mathfrak{J}(n, m)$ to be the set of indices $j \in \mathbb{N}$ corresponding to eigenfunctions e_j which are invariant under the action of $\mathfrak{H}(n, m)$, that is

$$\mathfrak{J}(n, m) := \bigl\{ j \geqslant 0 \bigm| e_j = e_j \circ R, \ \text{for all } R \in \mathfrak{H}(n, m) \bigr\}.$$

It will be important to observe that $1, 2, \ldots, n$ do not belong to $\mathfrak{J}(n, m)$ since the eigenfunctions corresponding to the eigenvalues $\lambda_1 = \cdots = \lambda_n$ are not invariant under the action of $-I_n \in \mathfrak{H}(n, m)$.

For all $k \in \mathbb{N}$ and all $\alpha \in (0, 1)$, we define $\mathcal{C}^{k,\alpha}(S^{n-1}, \mathfrak{H}(n, m))$ to be the subset of functions of $\mathcal{C}^{k,\alpha}(S^{n-1})$ whose eigenfunction decomposition only involves indices belonging to $\mathfrak{J}(n, m)$. In other words, $g \in \mathcal{C}^{k,\alpha}(S^{n-1}, \mathfrak{H}(n, m))$ if and only if $g \in \mathcal{C}^{k,\alpha}(S^{n-1})$ and

$$g = \sum_{j \in \mathfrak{J}} g_j e_j.$$

Observe that, by definition, any function of $\mathcal{C}^{k,\alpha}(S^{n-1}, \mathfrak{H}(n, m))$ is orthogonal to $e_1, \ldots, e_n$ in the L^2 sense, on S^{n-1}.

Notations: Given $1 \leqslant m \leqslant n-1$, we will adopt the following notations:

$$x \quad \text{or} \quad (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^m \sim \mathbb{R}^{n-m},$$

will denote a point in $\mathbb{R}^n$ and

$$(x, z) \in \mathbb{R}^n \times \mathbb{R} \sim \mathbb{R}^{n+1},$$

will denote a point in $\mathbb{R}^{n+1}$. Finally, θ will denote a point in S^{n-1}.

3. Minimal hypersurfaces close to a truncated *n*-catenoid

This section is mainly adapted from [2], we recall some of the technical results of [2] which are needed in this paper and adapt them to our situation.

3.1. *The n-catenoid*

Assume that $n \geqslant 3$ is fixed. We recall some well known fact concerning the unit n-catenoid C_1 which is a minimal hypersurface of revolution in $\mathbb{R}^{n+1}$, further details are available in [2]. By definition, C_1 is the minimal hypersurface of revolution parameterized by

$$X_0 : (s, \theta) \in \mathbb{R} \times S^{n-1} \to \big(\varphi(s)\theta, \psi(s)\big) \in \mathbb{R}^{n+1}, \tag{4}$$

where φ is the unique, smooth, non-constant solution of

$$(\partial_s \varphi)^2 + \varphi^{4-2n} = \varphi^2 \quad \text{with } \varphi(0) = 1,$$

and where the function ψ is the unique solution of

$$\partial_s \psi = \varphi^{2-n} \quad \text{with } \psi(0) = 0.$$

As already mentioned in the introduction, it might be interesting to observe that φ is explicitely given by the identity

$$\varphi^{n-1}(s) = \cosh\big((n-1)s\big).$$

Using this, it is easy to check that the function ψ converges as s tends to $\pm\infty$. We set

$$c_\infty := \lim_{s \to +\infty} \psi.$$

The fact that ψ converges at both $\pm\infty$ implies that the hypersurface C_1 has two hyperplanar ends and is in fact contained between the two asymptotic hyperplanes defined by $z = \pm c_\infty$. In addition, the upper end (respectively lower end) of the unit n-catenoid can be parameterized as a graph over the $z = 0$ hyperplane for some function u (respectively $-u$). It is easy to check that the function u has the following expansion as $r := |x|$ tends to ∞:

$$u = c_\infty - \frac{1}{n-2} r^{2-n} + \mathcal{O}\big(r^{4-3n}\big). \tag{5}$$

3.2. The mean curvature operator

Let us assume that the orientation of C_1 is chosen so that the unit normal vector field is given by

$$N_0 := \frac{1}{\varphi}(\partial_s \psi \theta, -\partial_s \varphi). \tag{6}$$

All surfaces close enough to C_1 can be parameterized (at least locally) as normal graphs over C_1, namely as the image of

$$X_w := X_0 + w\varphi^{\frac{2-n}{2}} N_0,$$

for some small function w. The following technical result is borrowed from [2]. It just states that the mean curvature of the hypersurface parameterized by X_w has some nice expansion in terms of w. Observe that, in order to define X_w, we have used $w\varphi^{(2-n)/2}N_0$ instead of the usual wN_0, there is no loss of generality in doing so and this choice will simplify the notations in the forthcoming result which describes the structure of the nonlinear partial differential equation w has to satisfy in order for the hypersurface parameterized by X_w to be minimal.

Proposition 1 [2]. *The hypersurface parameterized by X_w is minimal if and only if the function w is a solution of the nonlinear elliptic partial differential equation given by*

$$\mathcal{L}w = \varphi^{\frac{2-n}{2}} Q_2\big(\varphi^{-\frac{n}{2}} w\big) + \varphi^{\frac{n}{2}} Q_3\big(\varphi^{-\frac{n}{2}} w\big), \tag{7}$$

where

$$\mathcal{L} := \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4}\varphi^{2-2n},$$

where $\xi \to Q_2(\xi)$ is a nonlinear second-order differential operator which is homogeneous of degree 2 and where $\xi \to Q_3(\xi)$ is a nonlinear second-order differential operator which satisfies

$$Q_3(0) = 0, \qquad D_\xi Q_3(0) = 0 \quad \textit{and} \quad D_\xi^2 Q_3(0) = 0.$$

Furthermore, the coefficients of Q_2 on the one hand and the coefficients in the Taylor expansion of Q_3 with respect to the ξ, computed at any ξ in some fixed neighborhood of 0 in $\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^{n-1})$ on the other hand are bounded functions of s and so are the derivatives of any order of these functions.

The operator $\mathcal{L}$ is clearly equivariant with respect to any action of the form

$$\mathbb{R} \times S^{n-1} \ni (s,\theta) \to (-s, R\theta) \in \mathbb{R} \times S^{n-1},$$

when $R \in \mathfrak{H}(n,m)$. Since in addition the mean curvature is invariant by isometries, we conclude that the nonlinear operator which appears on the right-hand side of (7) also enjoys this equivariance property.

It might be useful to rephrase the properties of the nonlinear operators Q_2 and Q_3 into a slightly weaker form. It follows from the properties of Q_2 and Q_3 that there exist constants $c, c_0 > 0$ such that, for all $s \in \mathbb{R}$ and all $\xi_1, \xi_2 \in \mathcal{C}^{2,\alpha}([s,s+1] \times S^{n-1})$, we have

$$\big\| Q_2(\xi_1) - Q_2(\xi_2)\big\|_{\mathcal{C}^{0,\alpha}} \leqslant c\Big(\sup_{i=1,2} \|\xi_i\|_{\mathcal{C}^{2,\alpha}}\Big)\|\xi_2 - \xi_1\|_{\mathcal{C}^{2,\alpha}}, \tag{8}$$

and, provided $\|\xi_i\|_{\mathcal{C}^{2,\alpha}} \leqslant c_0$, we also have

$$\big\| Q_3(\xi_1) - Q_3(\xi_2)\big\|_{\mathcal{C}^{0,\alpha}} \leqslant c\Big(\sup_{i=1,2} \|\xi_i\|_{\mathcal{C}^{2,\alpha}}\Big)^2\|\xi_2 - \xi_1\|_{\mathcal{C}^{2,\alpha}}, \tag{9}$$

where all norms are computed on $[s,s+1] \times S^{n-1}$. Since Q_2 is homogeneous of degree 2 no assumptions on ξ_i are required in order to get the estimate involving Q_2, however they are required for the estimates involving Q_3.

Let us warn the reader that the operator $\mathcal{L}$ which appears in this result is not the Jacobi operator which is defined to be the linearized mean curvature operator when nearby hypersurfaces are normal graphs over the n-catenoid, that is when they are parameterized by

$$\widetilde{X}_w := X_0 + wN_0.$$

Nevertheless, $\mathcal{L}$ is conjugate to the Jacobi operator.

3.3. Linear analysis

Projecting the operator $\mathcal{L}$ over the eigenspace spanned by e_j, for all j, we are left with the study of the sequence of operators

$$L_j := \partial_s^2 - \lambda_j - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4}\varphi^{2-2n}, \quad j \in \mathbb{N}.$$

The indicial roots of $\mathcal{L}$ at both $+\infty$ or $-\infty$ are given by $\pm\gamma_j$ where

$$\gamma_j := \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}. \tag{10}$$

Let us recall that these indicial roots appear in the study of the asymptotic behavior of the solutions of the homogeneous problem $L_j w = 0$, at $\pm\infty$. More precisely, for each $j \in \mathbb{N}$, one can find $w_j^{\pm}$, two independent solutions of $L_j w = 0$ such that $w_j^+(s) \sim e^{\gamma_j s}$ and $w_j^-(s) \sim e^{-\gamma_j s}$ at $+\infty$. Observe that the functions $s \to w_j^{\pm}(-s)$ are solutions of $L_j w = 0$ such that $w_j^+(s) \sim e^{-\gamma_j s}$ and $w_j^-(s) \sim e^{\gamma_j s}$ at $-\infty$. These indicial roots will play a crucial rôle in the study of the mapping properties of $\mathcal{L}$.

To keep the notations short, we define the second-order elliptic operator

$$\Delta_0 := \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2,$$

which acts on functions defined on $\mathbb{R} \times S^{n-1}$. In particular

$$\mathcal{L} = \Delta_0 + \frac{n(3n-2)}{4}\varphi^{2-2n}.$$

The indicial roots of Δ_0 at both $+\infty$ or $-\infty$ are also given by $\pm\gamma_j$.

It is straightforward to check that Δ_0 satisfies the maximum principle and also that the operator $\mathcal{L}$ does not satisfy the maximum principle because of the presence of the extra potential. Indeed, one can check that the functions

$$\Psi^{0,-} := \partial_s\big(\varphi^{\frac{n-2}{2}}\big), \qquad \Psi^{0,+} := \varphi^{\frac{n-4}{2}}(\varphi\partial_s\psi - \psi\partial_s\varphi), \tag{11}$$

and, for $j = 1, \ldots, n$, the functions

$$\Psi^{j,-} := \varphi^{\frac{n-4}{2}}(\varphi\partial_s\varphi + \psi\partial_s\psi)e_j, \qquad \Psi^{j,+} := \varphi^{-\frac{n}{2}}e_j, \tag{12}$$

are Jacobi fields, i.e. are solutions of the homogeneous problem $\mathcal{L}w = 0$, and that the $\Psi^{j,+}$ are bounded. Nevertheless, the following result, borrowed from [2], asserts that the operator $\mathcal{L}$ still satisfies the maximum principle if it is restricted to the higher eigenspaces of the cross-sectional Laplacian $\Delta_{S^{n-1}}$:

Proposition 2. *Assume that $\delta < (n+2)/2$ is fixed and that w is a solution of*

$$\mathcal{L}w = 0,$$

which is bounded by φ^δ on $(s_1, s_2) \times S^{n-1}$ and which satisfies $w = 0$ on $\{s_i\} \times S^{n-1}$, if any of the s_i is finite. Further assume that, for each $s \in (s_1, s_2)$, the function $w(s, \cdot)$ is orthogonal to $e_0, \ldots, e_n$ in the L^2 sense on S^{n-1}. Then $w \equiv 0$.

In view of the previous result, it is natural to consider the operator $\mathcal{L}$ acting on functions bounded by a constant times a power of the function φ. As in [7,2], we define a family of weighted Hölder spaces by:

Definition 1. For all $\delta \in \mathbb{R}$, the space $\mathcal{C}_\delta^{k,\alpha}(\mathbb{R} \times S^{n-1})$ is defined to be the space of functions $w \in \mathcal{C}_{\text{loc}}^{k,\alpha}(\mathbb{R} \times S^{n-1})$ for which the following norm is finite

$$\|w\|_{\mathcal{C}_\delta^{k,\alpha}} := \sup_{s\in\mathbb{R}}\big(\varphi^{-\delta}|w|_{\mathcal{C}^{k,\alpha}([s,s+1]\times S^{n-1})}\big).$$

Here $|\ |_{\mathcal{C}^{k,\alpha}([s,s+1]\times S^{n-1})}$ denotes the Hölder norm in $[s, s+1] \times S^{n-1}$.

Moreover, for any $S > 0$, the space $\mathcal{C}_\delta^{k,\alpha}([-S, S] \times S^{n-1})$ is defined to be the space of restriction of functions of $\mathcal{C}_\delta^{k,\alpha}(\mathbb{R} \times S^{n-1})$ to $[-S, S] \times S^{n-1}$. This space is naturally endowed with the induced norm.

Though this will not be necessary for the remaining of the analysis, we quote here some well known properties of the operator

$$\mathcal{L}:\mathcal{C}^{2,\alpha}_\delta\big(\mathbb{R}\times S^{n-1}\big)\to\mathcal{C}^{0,\alpha}_\delta\big(\mathbb{R}\times S^{n-1}\big).$$

To keep track of the weighted space over which $\mathcal{L}$ is defined, we will denote the above operator by $\mathcal{L}_\delta$. The most important fact is that the mapping properties of $\mathcal{L}_\delta$ crucially depend on the choice of the weight parameter δ. Indeed, it follows from general arguments that $\mathcal{L}_\delta$ has close range and is even Fredholm if and only if the weight δ is not equal to any of the indicial roots $\pm\gamma_j$, $j\in\mathbb{N}$ (a fact which, given the special structure of our operator, can be easily proven be separation of variables). The fact that the functions given in (12) are Jacobi fields shows that $\mathcal{L}_\delta$ is not injective when $\delta>-n/2$ and it can be proven, with the help of Proposition 2, that $\mathcal{L}_\delta$ is injective if $\delta<-n/2$. This later fact in turn implies that $\mathcal{L}_\delta$ is surjective if $\delta>n/2$ is not equal to any γ_j, $j\geqslant 0$ (this uses the fact that the operator $\mathcal{L}_\delta$ and $\mathcal{L}_{-\delta}$ are, in some sense, dual).

As already mentioned in Section 2, we will only be interested in functions which are invariant under the action of some group. This is the reason why we introduce the:

Definition 2. For all $k\in\mathbb{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m))$ is defined to be the space of functions $w\in\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}\times S^{n-1})$ which satisfy

$$\forall(s,\theta)\in\mathbb{R}\times S^{n-1},\quad w(s,\theta)=w(-s,\theta),$$

and also

$$\forall(s,\theta)\in\mathbb{R}\times S^{n-1},\quad w(s,R\theta)=w(s,\theta),$$

for all $R\in\mathfrak{H}(n,m)$. This space is endowed with the induced norm.

Observe that, any function $w\in\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m))$ can be decomposed as

$$w(s,\theta)=\sum_{j\in\mathfrak{J}}w_j(s)e_j(\theta),$$

where, for all j, all functions $s\to w_j(s)$ are even.

Observe that the Jacobi fields $\Psi^{j,\pm}$, for $j=1,\dots,n$, which are defined in (12), are not invariant with respect to the action of $\{\pm I_1\}\otimes\mathfrak{H}(n,m)$, hence one can show that

$$\mathcal{L}:\mathcal{C}^{2,\alpha}_\delta\big(\mathbb{R}\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m)\big)\to\mathcal{C}^{0,\alpha}_\delta\big(\mathbb{R}\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m)\big),$$

is injective when $\delta<(n-2)/2$ and surjective when $\delta>(2-n)/2$ is not equal to any γ_j, for $j\geqslant 0$. We will not need such a general statement, since we will be working with functions defined on $[-S,S]\times S^{n-1}$.

Among the Jacobi fields defined in (11) and (12),

$$\Psi^{+,0}=\varphi^{\frac{n-4}{2}}(\varphi\partial_s\psi-\psi\partial_s\varphi)$$

is the only one which is invariant with respect to the action of $\{\pm I_1\}\otimes\mathfrak{H}(n,m)$. It is easy to see that this Jacobi field vanishes for finitely many values of s. Hence we can define $s_0>0$ to be the largest zero of the function $\Psi^{+,0}$.

The result we will need reads:

Proposition 3. *Assume that $\delta\in((2-n)/2,(n-2)/2)$ and $\alpha\in(0,1)$ are fixed. There exists some constant $c>0$ and, for all $S>s_0+1$, there exists an operator*

$$\mathcal{G}_S:\mathcal{C}^{0,\alpha}_\delta\big([-S,S]\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m)\big)\to\mathcal{C}^{2,\alpha}_\delta\big([-S,S]\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m)\big),$$

such that, for all $f\in\mathcal{C}^{0,\alpha}_\delta([-S,S]\times S^{n-1},\{\pm I_1\}\otimes\mathfrak{H}(n,m))$, the function $w=\mathcal{G}_S(f)$ is a solution of

$$\mathcal{L}w=f$$

in $(-S,S)\times S^{n-1}$ with $w=0$ on $\{\pm S\}\times S^{n-1}$. Furthermore, $\|w\|_{\mathcal{C}^{2,\alpha}_\delta}\leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\delta}$.

Proof. Our problem being linear, we can assume without loss of generality that

$$\sup_{[-S,S]\times S^{n-1}}\big(\varphi^{-\delta}|f|\big)=1.$$

Observe that, it follows from Proposition 2 that, when restricted to the space of functions w such that, for all s, the function $w(s,\cdot)$ is orthogonal to $e_0,\ldots,e_n$ in the L^2 sense on S^{n-1}, the operator $\mathcal{L}$ is injective over $(-S,S)\times S^{n-1}$. Also, if $s>s_0$ then $\mathcal{L}$ is injective over $(-S,S)\times S^{n-1}$ when restricted to functions which are even and only depend on s. As a consequence, for all $S>s_0$, we are able to solve $\mathcal{L}v=f$, in $(-S,S)\times S^{n-1}$, with $v=0$ on $\{\pm S\}\times S^{n-1}$. In addition, since f is invariant under the action of $\{\pm I_1\}\otimes\mathfrak{H}(n,m)$, so is v.

We claim that there exists some constant $c>0$, independent of $S>s_0+1$ and of f, such that

$$\sup_{[-S,S]\times S^{n-1}}\left(\varphi^{-\delta}|w|\right)\leqslant c.$$

Observe that the result is true when $S>s_0+1$ stays bounded. We argue by contradiction and assume that the result is not true. In this case, there would exist a sequence $S_k>s_0+1$ tending to $+\infty$, a sequence of functions f_k satisfying

$$\sup_{[-S_k,S_k]\times S^{n-1}}\left(\varphi^{-\delta}|f_k|\right)=1,$$

and a sequence v_k of solutions of $\mathcal{L}v_k=f_k$, in $(-S_k,S_k)\times S^{n-1}$, with $v_k=0$ on $\{\pm S_k\}\times S^{n-1}$ such that

$$A_k:=\sup_{[-S_k,S_k]\times S^{n-1}}\left(\varphi^{-\delta}|v_k|\right)\to+\infty.$$

Let us denote by $(s_k,\theta_k)\in[0,S_k)\times S^{n-1}$, a point where the above supremum is achieved, observe that all the functions we consider are even in the s variable, thus we can assume that the above supremum is achieved at some point of $[0,S_k)\times S^{n-1}$. We claim that the sequence S_k-s_k remains bounded away from 0. Indeed, since v_k and $(\partial_s^2+\Delta_{S^{n-1}})v_k$ are both bounded by a constant (independent of k) times $\varphi^\delta(S_k)A_k$ in $[S_k-1,S_k]\times S^{n-1}$ and since $v_k=0$ on $\{S_k\}\times S^{n-1}$, we may apply standard elliptic estimates and conclude that the gradient of v_k is also uniformly bounded by a constant times $\varphi^\delta(S_k)A_k$ in $[S_k-\frac{1}{2},S_k]\times S^{n-1}$. As a consequence the above supremum cannot be achieved at a point which is too close to S_k. Therefore, up to some subsequence, we may also assume that the sequence S_k-s_k converges to $S^*\in(0,+\infty]$. We now distinguish a few cases according to be the behavior of the sequence s_k, which, up to a subsequence, can be assumed to converge in $[0,+\infty]$.

We define the sequence of rescaled functions

$$\tilde{v}_k:=\frac{\varphi^{-\delta}(s_k)}{A_k}v_k(\cdot+s_k,\cdot).$$

Case 1: Assume that the sequence s_k converges to $s_*\in\mathbb{R}$. After the extraction of some subsequences, if this is necessary, we may assume that the sequence $\frac{1}{A_k}v_k$ converges on compact to v some nontrivial solution of

$$\mathcal{L}v=0,$$

in $\mathbb{R}\times S^{n-1}$. Furthermore

$$\sup_{\mathbb{R}\times S^{n-1}}\left(\varphi^{-\delta}|v|\right)=1. \tag{13}$$

Moreover, for each $s\in\mathbb{R}$, the function $v(s,\cdot)$ is orthogonal in the L^2 sense to $e_1,\ldots,e_n$ on S^{n-1}. But, the result of Proposition 2 together with the fact that $\delta\in((2-n)/2,(n-2)/2)$ implies that v only depends on s. Hence, v is a multiple of $\Psi^{0,+}$ and cannot be bounded by a constant times φ^δ unless $v\equiv0$. A contradiction with (13).

Case 2: Assume that the sequence s_k converges to $+\infty$. After the extraction of some subsequences if this is necessary, we may assume that the sequence $\tilde{v}_k$ converges to v some nontrivial solution of

$$\Delta_0 v=0,$$

in $(-\infty,S^*)\times S^{n-1}$, with boundary condition $v=0$, if S^* is finite. Furthermore

$$\sup_{(-\infty,S^*)\times S^{n-1}}\left(\mathrm{e}^{-\delta s}|v|\right)=1. \tag{14}$$

Independently of the fact that S^* is finite or not. This case is easy to rule out using the eigenfunction decomposition of v

$$v=\sum_{j\in\mathfrak{J}}v_je_j.$$

Indeed, v_j has to be a linear combination of the functions $\mathrm{e}^{\pm\gamma_j s}$ (where γ_j has been defined in (10)) and is bounded by $\mathrm{e}^{\delta s}$. Since we have assumed that $\delta\in((2-n)/2,(n-2)/2)$, it is easy to see that all $v_j\equiv0$, contradicting (14).

We have reached a contradiction in all cases, hence, the proof of the claim is finished. To complete the proof of the proposition, it suffices to apply Schauder's estimates in order to get the relevant estimates for all the derivatives. □

We will also need some properties of the Poisson operator for Δ_0 on $[0,\infty) \times S^{n-1}$. The result we will need is standard and a proof can be found, for example, in [2]:

Lemma 1. *There exists $c > 0$ such that, for all $g \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$, there exists a unique $w \in \mathcal{C}^{2,\alpha}_{(2-n)/2}([0,+\infty) \times S^{n-1})$ solution of*

$$\begin{aligned} &\Delta_0 w = 0 \quad \text{in } (0,+\infty) \times S^{n-1}, \\ &w = g \quad \text{on } \{0\} \times S^{n-1}. \end{aligned} \tag{15}$$

Furthermore, we have $||w||_{\mathcal{C}^{2,\alpha}_{(2-n)/2}} \leqslant c\|g\|_{\mathcal{C}^{2,\alpha}}$ and, for all $s > 0$, the function $w(s,\cdot)$ is invariant with respect to the action of $\mathfrak{H}(n,m)$.

The idea behind the proof of this result is that one can use the eigenfunction decomposition of g to obtain an explicite solution of (15) together with the estimate. In the remaining of the paper, we will denote by $\mathcal{P}(g)$ the solution of (15).

3.4. The nonlinear problem

We fix $\rho \in [0,1]$ and, for all $\varepsilon \in (0,\rho)$, we define $s_\varepsilon > 0$ by the identity

$$\rho = \varepsilon\varphi(s_\varepsilon) > 0.$$

Let us notice that, as ε tends to 0, we have

$$s_\varepsilon \sim -\log\varepsilon.$$

In order to parameterize the unit n-catenoid we use (4) and define the outer unit normal N_0 as in (6). Let us define a smooth function $\xi_\varepsilon : \mathbb{R} \to [-1,1]$ which satisfies $\xi_\varepsilon = -1$ for $s \geqslant s_\varepsilon - 1$, $\xi_\varepsilon = 1$ for $s \leqslant 1 - s_\varepsilon$, $\xi_\varepsilon = -\partial_s \log\varphi$ for $|s| \leqslant s_\varepsilon - 2$ and which interpolates smoothly between those functions when $|s| \in [s_\varepsilon - 2, s_\varepsilon - 1]$. We consider the vector field

$$N_\varepsilon(s,\theta) := \left(\sqrt{1 - \xi_\varepsilon^2(s)}\,\theta, \xi_\varepsilon(s)\right).$$

It turns out that this vector field is a perturbation of the unit normal N_0, and in fact, we have for all $k \geqslant 0$

$$\left|\nabla^k (N_\varepsilon \cdot N_0 - 1)\right| \leqslant c_k \varepsilon^{2n-2},$$

for all $|s| \geqslant s_\varepsilon - 2$.

We look for all minimal hypersurfaces close to the unit n-catenoid which has been rescaled by a factor ε. This means that the hypersurfaces we are looking for can be parameterized by

$$X_w := \varepsilon X_0 + w\varphi^{\frac{2-n}{2}} N_\varepsilon,$$

for $(s,\theta) \in [-s_\varepsilon, s_\varepsilon] \times S^{n-1}$ and for some small function w. It follows from (7) that such an hypersurface is minimal if and only if w satisfies a nonlinear equation of the form

$$\mathcal{L}w = \overline{Q}_\varepsilon(w), \tag{16}$$

where

$$\overline{Q}_\varepsilon(w) := \varepsilon^{2n-2} L_\varepsilon w + \varepsilon\varphi^{\frac{2-n}{2}}\, \overline{Q}_{2,\varepsilon}\left(\varphi^{-\frac{n}{2}}\varepsilon^{-1} w\right) + \varepsilon\varphi^{\frac{n}{2}}\, \overline{Q}_{3,\varepsilon}\left(\varphi^{-\frac{n}{2}}\varepsilon^{-1} w\right).$$

Here $\overline{Q}_{2,\varepsilon}$ and $\overline{Q}_{3,\varepsilon}$ enjoy properties which are similar to those enjoyed by Q_2 and Q_3, namely (8) and (9) still hold uniformly in $\varepsilon \in (0,\rho)$. The linear operator $\varepsilon^{2n-2} L_\varepsilon$ represents the difference between the linearized mean curvature operator for hypersurfaces parameterized using the vector field N_0 and those parameterized using the vector field N_ε. The operator L_ε has coefficients which are supported in $([-s_\varepsilon, 2 - s_\varepsilon] \cup [s_\varepsilon - 2, s_\varepsilon]) \times S^{n-1}$ and which are uniformly bounded in $\mathcal{C}^{0,\alpha}$ topology. The details of the derivation of this formula can be found, for example, in [8] or in [2].

3.4.1. Solutions of (16) which are parameterized by their boundary data

We fix $\delta \in ((2-n)/2, (n-2)/2)$, $\alpha \in (0,1)$ and $\kappa > 0$. Given $h \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$ whose norm satisfies

$$\|h\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa \varepsilon^{n-1},$$

we set

$$g := \varphi^{\frac{n-2}{2}}(s_\varepsilon) h,$$

and we define

$$\widetilde{w}_h := \mathcal{P}_{s_\varepsilon}(g)(s_\varepsilon - \cdot, \cdot) + \mathcal{P}_{s_\varepsilon}(g)(\cdot + s_\varepsilon, \cdot) \in \mathcal{C}^{2,\alpha}\big([-s_\varepsilon, s_\varepsilon] \times S^{n-1}, \{\pm I_1\} \otimes \mathfrak{H}(n,m)\big). \tag{17}$$

We know from Lemma 15 that

$$\|\widetilde{w}_h\|_{\mathcal{C}^{2,\alpha}_{(n-2)/2}} \leqslant c \varepsilon^{\frac{n-2}{2}} \|g\|_{\mathcal{C}^{2,\alpha}} \leqslant c \|h\|_{\mathcal{C}^{2,\alpha}}. \tag{18}$$

Now, if we write $w = \widetilde{w}_h + v$, we wish to find a function $v \in \mathcal{C}^{2,\alpha}_\delta([-s_\varepsilon, s_\varepsilon] \times S^{n-1}, \{\pm I_1\} \otimes \mathfrak{H}(n,m))$ such that

$$\begin{cases} \mathcal{L} v = \overline{Q}_\varepsilon(\widetilde{w}_h + v) - \mathcal{L}\widetilde{w}_h \quad \text{in } (-s_\varepsilon, s_\varepsilon) \times S^{n-1}, \\ v = 0 \quad \text{on } \{\pm s_\varepsilon\} \times S^{n-1}. \end{cases} \tag{19}$$

To obtain a solution of this equation, it is enough to find a fixed point of the mapping

$$\mathcal{N}_\varepsilon(v) := \mathcal{G}_{s_\varepsilon}\big(\overline{Q}_\varepsilon(\widetilde{w}_h + v) - \mathcal{L}\widetilde{w}_h\big),$$

where $\mathcal{G}_{s_\varepsilon}$ is the operator defined in Proposition 7. Using (18) together with Proposition 7 and the properties of $\overline{Q}_\varepsilon$, we can estimate:

$$\big\|\varepsilon^{2n-2} L_\varepsilon \widetilde{w}_h - \mathcal{L}\widetilde{w}_h\big\|_{\mathcal{C}^{0,\alpha}_\delta} \leqslant c\big(1 + \varepsilon^{\frac{3n-2}{2}+\delta}\big)\|h\|_{\mathcal{C}^{2,\alpha}},$$

$$\big\|\varepsilon \varphi^{\frac{2-n}{2}} \overline{Q}_{2,\varepsilon}\big(\varphi^{-\frac{n}{2}} \varepsilon^{-1} \widetilde{w}_h\big)\big\|_{\mathcal{C}^{0,\alpha}_\delta} \leqslant c\big(\varepsilon^{-1} + \varepsilon^{\frac{n}{2}+\delta}\big)\|h\|^2_{\mathcal{C}^{2,\alpha}}$$

and finally, there exists $\varepsilon_0 > 0$ (which depends on κ) such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\big\|\varepsilon \varphi^{\frac{n}{2}} \overline{Q}_{3,\varepsilon}\big(\varphi^{-\frac{n}{2}} \varepsilon^{-1} \widetilde{w}_h\big)\big\|_{\mathcal{C}^{0,\alpha}_\delta} \leqslant c\big(\varepsilon^{-2} + \varepsilon^{\frac{2-n}{2}+\delta}\big)\|h\|^3_{\mathcal{C}^{2,\alpha}}.$$

In the above estimates, the constant $c > 0$ does not depend on ε, nor on κ. Observe that in order to obtain the last estimate, we have implicitly used that fact that $\|h\|_{\mathcal{C}^{2,\alpha}}$ is small enough so that we can apply (9), or rather its counterpart for $\overline{Q}_{3,\varepsilon}$. This explains why the restriction $\varepsilon \in (0, \varepsilon_0)$ is needed in order to obtain the last estimate.

It is then a simple exercise to show that for any fixed $\kappa > 0$, there exist $c > 0$ and $\varepsilon_0 > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0)$, the nonlinear mapping $\mathcal{N}_\varepsilon$ is a contraction in the ball of radius

$$R(\varepsilon, h) := c\|h\|_{\mathcal{C}^{2,\alpha}},$$

in $\mathcal{C}^{2,\alpha}_\delta([-s_\varepsilon, s_\varepsilon] \times S^{n-1}, \{\pm I_1\} \otimes \mathfrak{H}(n,m))$ into itself, and hence $\mathcal{N}_\varepsilon$ has a unique fixed point v_h in this ball. Therefore, the function $w_h := \tilde{w}_h + v_h$ is a solution of (16) whose boundary data is, up to a constant function, given by h. We can even choose the constant c to be independent of κ, but this will not be useful.

3.4.2. Family of minimal hypersurfaces close to n-catenoid

We summarize the results we have obtained so far and translate them in the geometric framework. Let us fix

$$\delta \in \left(\frac{2-n}{2}, \frac{n-2}{2}\right), \qquad \alpha \in (0,1) \quad \text{and} \quad \kappa > 0.$$

There exists $c > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $h \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$ satisfying

$$\|h\|_{2,\alpha} \leqslant \kappa \varepsilon^{n-1},$$

there exists a minimal hypersurface, which will be denoted by $C_\varepsilon(h) \subset \mathbb{R}^{n+1}$, and which is parameterized by

$$X_h = \varepsilon X_0 + w_h \varphi^{\frac{2-n}{2}} N_\varepsilon \quad \text{in } [-s_\varepsilon, s_\varepsilon] \times S^{n-1},$$

for some function w_h satisfying

$$\|w_h\|_{\mathcal{C}^{2,\alpha}_{(n-2)/2}} \leqslant c\|h\|_{\mathcal{C}^{2,\alpha}}.$$

This hypersurface is symmetric with respect to the hyperplane $z = 0$ and further inherits all the symmetries induces by the symmetries used to define the function spaces in Definition 2, hence, it is invariant with respect to the action of $O(n-m,\mathbb{R}) \otimes \mathfrak{D}_m \otimes \{\pm I_1\} \subset O(n+1,\mathbb{R})$. Furthermore, if we perform the change of variable

$$r := \varepsilon\varphi(s),$$

we see that near its upper boundary, this hypersurface is the graph of the function

$$x \in \overline{B^n_\rho} \setminus B^n_{\rho/2} \to c_\infty \varepsilon - W_h(x) - V_{\varepsilon,h}(x),$$

over the $z = 0$ hyperplane. Here W_h denotes the (unique) harmonic extension of the boundary data h in B^n_ρ and the function $V_{\varepsilon,h}$ satisfies

$$\|V_{\varepsilon,h}\|_{\mathcal{C}^{2,\alpha}} \leqslant c_0 \varepsilon^{n-1}$$

for some constant c_0 which does not depend on κ nor on ε. Here the norms are taken over $\overline{B^n_\rho} - B^n_{\rho/2}$. This last claim, which is a key point of our analysis, follows from (5). Indeed, when $h = 0$, $C_\varepsilon(0)$ is just a rescaled n-catenoid and, using (5) we see that its upper end is the graph of the function

$$x \to c_\infty \varepsilon + \mathcal{O}\big(\varepsilon^{n-1} r^{2-n}\big).$$

We have also used the fact that the solution of (19) we have constructed is equal to $\tilde{w}_h + v_h$ where $\tilde{w}$, defined in (17), is linear in h and where v_h can be estimated by a constant (independent of ε and κ) times $\|h\|_{\mathcal{C}^{2,\alpha}}\varphi^\delta$. Essentially the constant c_0 arises from the term $\mathcal{O}(\varepsilon^{n-1} r^{2-n})$ in the above expansion, the contributions of v_h and the perturbation caused by the change of variable being neglectable when ε is chosen small enough. Indeed, let us denote by $\widetilde{W}_h$ the function defined in $\overline{B^n_\rho} - B^n_{\rho/2}$ by

$$\widetilde{W}_h\big(\varepsilon\varphi(s)\theta\big) = \varphi(s_\varepsilon)^{\frac{n-2}{2}} \varphi(s)^{\frac{2-n}{2}} \tilde{w}_h(s,\theta).$$

One can check that

$$\big\|\widetilde{W}_h - W_h\big\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^{n-2} \|h\|_{\mathcal{C}^{2,\alpha}},$$

where the norm on the left is computed in $\overline{B^n_\rho} - B^n_{\rho/2}$.

Observe that, reducing ε_0 if this is necessary, we can assume that the mapping $h \to V_{\varepsilon,h}$ is continuous and in fact smooth. With little work we also find that

$$\|V_{\varepsilon,h_2} - V_{\varepsilon,h_1}\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^{\frac{n-2}{2}-\delta} \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}}, \tag{20}$$

for some constant $c > 0$ which does not depend on ε. The norm on the left-hand side of this inequality is understood to be the norm on $\overline{B^n_\rho} - B^n_{\rho/2}$. The constant c in (20) can be chosen to be independent of κ but this will be irrelevant for the remaining of the analysis.

4. Minimal hypersurfaces which are graphs over a hyperplane

We are now concerned with both the mean curvature and the linearized mean curvature operator for hypersurfaces which are graphs over the $z = 0$ hyperplane, in $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$.

4.1. *The mean curvature operator for graphs*

We assume that $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ are fixed. Further assume that $T^m \in \mathcal{T}^m$ is fixed. Then, for any function u, defined in $\mathbb{R}^{n-m} \times T^m$, which is at least of class $\mathcal{C}^2$, we can define a hypersurface which is the graph of u

$$\mathbb{R}^{n-m} \times T^m \ni (x_1, x_2) \quad \to \quad \big(x_1, x_2, u(x_1,x_2)\big) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R}.$$

Recall that the mean curvature of this hypersurface, with downward pointing unit normal, is then given by

$$H_u := -\frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right). \tag{21}$$

4.2. Linear analysis

We define the function spaces which are adapted to the analysis of the Laplacian in $\mathbb{R}^{n-m} \times T^m$. Our main concern will be the asymptotic behavior of the functions as $|x_1|$ tends to $+\infty$.

Definition 3. For all $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m)$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n-m} \times T^m)$ for which the following norm is finite:

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu} := |w|_{\mathcal{C}^{k,\alpha}(B_1^{n-m} \times T^m)} + \sup_{r > 1/2} r^{-\nu} \big|w(r\,\cdot)\big|_{\mathcal{C}^{k,\alpha}((B_2^{n-m} - B_1^{n-m}) \times r^{-1} T^m)}.$$

Here $|\ |_{\mathcal{C}^{k,\alpha}(\Omega)}$ denotes the Hölder norm in Ω.

To get a better understanding of these weighted spaces, let us observe that, if $T_m = \mathbb{R}^m / A\mathbb{Z}^m$, we can identify any function defined on $\mathbb{R}^{n-m} \times T^m$ with a function defined on $\mathbb{R}^{n-m} \times \mathbb{R}^m$ which has $\{0\} \otimes A\mathbb{Z}^m$ as its group of periods. In which case functions which belong to $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m)$ are identified with functions defined on $\mathbb{R}^{n-m} \times \mathbb{R}^m$, which are bounded by a constant times $(1 + |x_1|)^\nu$, whose first derivative are bounded by a constant times $(1 + |x_1|)^{\nu-1}$ (if $k \geqslant 1$), and so on.

As in the previous section, we will only work with functions having some special symmetry. To this aim, we introduce the:

Definition 4. For all $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m)$ which are invariant under the action of $\mathfrak{H}(n,m)$.

Observe that, because of the invariance of our function space with respect to the action of $\mathfrak{H}(n,m)$, any function $w \in \mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$ can be decomposed as

$$w(x_1, x_2) = \sum_{i \in \mathfrak{I}} w_i(r_1) E_i(x_2),$$

where $\mathfrak{I}(m) \subset \mathbb{N}$ has been defined in (3) and where

$$r_1 := |x_1|.$$

To begin with let us treat the easy case where $1 \leqslant m \leqslant n-3$. We have the:

Proposition 4. *Assume that $1 \leqslant m \leqslant n-3$. Given $\nu \in (2+m-n, 0)$ and $\alpha \in (0,1)$. There exist some constant $c > 0$ and an operator*

$$G : \mathcal{C}^{0,\alpha}_{\nu-2}\big(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m)\big) \to \mathcal{C}^{2,\alpha}_\nu\big(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m)\big),$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$, the function $w = G(f)$ is a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n-m} \times T^m$. Furthermore, $\|w\|_{\mathcal{C}^{2,\alpha}_\nu} \leqslant c \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}}$.

Proof. The proof of the result is simplified by the fact that

$$\Delta |x_1|^\nu = -\nu(n-m-2-\nu)|x_1|^{\nu-2}.$$

Hence, the function $w(x_1, x_2) := |x_1|^\nu$, which is defined in $(\mathbb{R}^{n-m} - \{0\}) \times T^m$ can be used as a barrier function to prove, for any $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$, the existence of a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n-m} \times T^m$. Furthermore, it also yields the estimate

$$\big|w(x_1, x_2)\big| \leqslant c \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}} |x_1|^\nu,$$

for some constant which does not depend on f. The maximum principle then implies that

$$\big|w(x_1, x_2)\big| \leqslant c \|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}} \big(1 + |x_1|\big)^\nu.$$

Starting from this, Schauder's estimates give

$$\|w\|_{\mathcal{C}^{2,\alpha}_{\nu}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}}.$$

The details are left to the reader. □

When $m = n-2$ or $m = n-1$, the previous result has to be modified since $2+m-n \geqslant 0$ in these two cases. To this aim, we choose χ a cutoff function defined on $\mathbb{R}$ such that $\chi \equiv 1$ for $t \geqslant 2$ and $\chi \equiv 0$ when $t \leqslant 1$. When $m = n-2$, we define the space

$$\mathcal{D}_2 := \operatorname{Span}\bigl\{\chi(r_1)\log r_1\bigr\} \subset \mathcal{C}^{\infty}\bigl(\mathbb{R}^2\bigr),$$

and when $m = n-1$, we set

$$\mathcal{D}_1 := \operatorname{Span}\bigl\{\chi(r_1)r_1\bigr\} \subset \mathcal{C}^{\infty}(\mathbb{R}).$$

This time we have the:

Proposition 5. *Assume that $m = n-2$ or $m = n-1$. Given $\nu \in (-\infty, 0)$ and $\alpha \in (0,1)$. There exist some constant $c > 0$ and an operator*

$$G : \mathcal{C}^{0,\alpha}_{\nu-2}\bigl(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m)\bigr) \to \mathcal{C}^{2,\alpha}_{\nu}\bigl(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m)\bigr) \oplus \mathcal{D}_{n-m},$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$, the function $w = G(f)$ is a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n-m} \times T^m$. Furthermore, $\|w\|_{\mathcal{C}^{2,\alpha}_{\nu} \oplus \mathcal{D}_{n-m}} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\nu-2}}$.

Proof. We decompose

$$f = f_0 + \sum_{i \in \mathfrak{I}-\{0\}} f_i E_i,$$

and adopt the notation $f = f_0 + f'$. We look for a solution w which will also be decomposed as

$$w = w_0 + \sum_{i \in \mathfrak{I}-\{0\}} w_i E_i,$$

and again we set $w = w_0 + w'$. For notational convenience, $f', v', w', \ldots$ will denote functions whose eigenfunction decomposition only involves indices $i \in \mathfrak{I}(m) - \{0\}$.

Observe that, because of the invariance of our problem with respect to the action of $\mathfrak{H}(n,m)$, the Laplacian in $\mathbb{R}^{n-m} \times T^m$ reduces to the study of the operator

$$L := \partial^2_{r_1} + \frac{n-m-1}{r_1}\partial_{r_1} + \Delta_{T^m},$$

where we have set $r_1 := |x_1|$.

Step 1: We would like to prove the existence of w' and also obtain the relevant estimate. Our problem being linear, we may always assume that

$$\sup_{\mathbb{R}^{n-m} \times T^m} (1+r_1)^{2-\nu}|f'| = 1.$$

Obviously Δ, or L, is injective over any $B^{n-m}_R \times T^m$. As a consequence, for any $R > 1$ we are able to solve $\Delta v' = f'$, in $B^{n-m}_R \times T^m$, with $v' = 0$ on $\partial B^{n-m}_R \times T^m$.

We claim that, there exists a constant $c > 0$, independent of $R > 1$ and of f', such that

$$\sup_{B^{n-m}_R \times T^m} (1+r_1)^{-\nu}|v'| \leqslant c.$$

Observe that the result is certainly true if we assume that R remains bounded. We argue by contradiction and assume that the claim is not true. In this case, there would exist a sequence $R_k > 1$ tending to $+\infty$, a sequence of functions f'_k satisfying

$$\sup_{B^{n-m}_{R_k} \times T^m} (1+r_1)^{2-\nu}\bigl|f'_k\bigr| = 1,$$

and a sequence v'_k of solutions of $Lv'_k = f'_k$, in $B^{n-m}_{R_k} \times T^m$, with $v'_k = 0$ on $\partial B^{n-m}_{R_k} \times T^m$, such that

$$A_k := \sup_{B^{n-m}_{R_k} \times T^m} (1 + r_1)^{-\nu} |v'_k| \to +\infty.$$

Let us denote by $(x_{1,k}, x_{2,k}) \in B^{n-m}_{R_k} \times T^m$, a point where the above supremum is achieved. We now distinguish a few cases according to the behavior of the sequence $r_{1,k} := |x_{1,k}|$ which, up to a subsequence, can always be assumed to converge in $[0, +\infty]$. Observe that, as in the proof of Proposition 7, the sequence $R_k/r_{1,k}$ remains bounded away from 1 and can be assumed to converge in $(1, \infty]$.

We define the sequence of rescaled functions

$$\tilde{v}'_k := \frac{(1 + r_{1,k})^{-\nu}}{A_k} v'_k(r_{1,k}\,\cdot).$$

Case 1: Assume that the sequence $r_{1,i}$ converges to $r_{1,\star} \in [0, \infty)$. After the extraction of some subsequences, if this is necessary, we may assume that the sequence $\frac{1}{A_k} v'_k$ converges to some nontrivial solution of

$$Lv' = 0, \tag{22}$$

in $\mathbb{R}^{n-m} \times T^m$. Furthermore,

$$\sup_{\mathbb{R}^{n-m} \times T^m} (1 + r_1)^{-\nu} |v'| = 1. \tag{23}$$

But, ν being negative, the maximum principle implies that v is identically equal to 0. This clearly contradicts (23).

Case 2: Assume that the sequence $r_{1,k}$ converges to $+\infty$. After the extraction of some subsequences, if this is necessary, we may assume that the sequence $\tilde{v}'_k$ converges to some nontrivial solution of

$$\partial^2_{r_1} v' + \frac{n - m - 1}{r_1} \partial_{r_1} v' + \Delta_{\mathbb{R}^m} v' = 0,$$

in $(\mathbb{R}^{n-m} - \{0\}) \times \mathbb{R}^m$ or in $(B^{n-m}_R - \{0\}) \times \mathbb{R}^m$ in which case we also have $v' = 0$ on $\partial B^{n-m}_R \times \mathbb{R}^m$. In addition, the function v' does not depend on x_2. This last claim follows from the fact that the functions $x_2 \to \tilde{v}'_k(x_1, x_2)$ have a group of period given by $r^{-1}_{1,k} A\,\mathbb{Z}^m$, if $T^m = \mathbb{R}^m / A\,\mathbb{Z}^m$. In addition, $|\nabla_{x_2} \tilde{v}'_k|$ is bounded by a constant only depending on x_1. Passing to the limit, we see that v' does not depend on x_2.

In either case, $x_1 \to v'(x_1)$ solves

$$\partial^2_{r_1} v' + \frac{n - m - 1}{r_1} \partial_{r_1} v' = 0,$$

and satisfies

$$\sup_{(\mathbb{R}^{n-m} - \{0\}) \times \mathbb{R}^m} r_1^{-\nu} |v'| = 1 \quad \text{or} \quad \sup_{(B^{n-m}_R - \{0\}) \times \mathbb{R}^m} r_1^{-\nu} |v'| = 1. \tag{24}$$

It should be clear that $v' \equiv 0$, contradicting (24).

Since we have obtained a contradiction in both cases, this finishes the proof of the claim.

Step 2: We now turn our attention to the existence of w_0 as well as the relevant estimate. Our problem reduces to solve one ordinary differential equation since

$$\partial^2_{r_1} w_0 + \frac{n - m - 1}{r_1} \partial_{r_1} w_0 = f_0.$$

It is easy so check that, when $m = n - 1$ the function w_0 is given by the formula

$$w_0 = r_1 \int_0^{+\infty} f_0 \,\mathrm{d}t + \int_{r_1}^{\infty} \int_{\zeta}^{\infty} f_0 \,\mathrm{d}t \,\mathrm{d}\zeta,$$

while, when $m = n - 2$, the function w_0 is given by

$$w_0 = \log r_1 \int_0^{+\infty} t f_0 \,\mathrm{d}t + \int_{r_1}^{\infty} \zeta^{-1} \int_{\zeta}^{\infty} t f_0 \,\mathrm{d}t \,\mathrm{d}\zeta.$$

Granted the above formula, one can directly check that we can decompose: $w_0 := a_0\chi + \widetilde{w}_0$ when $m = n-1$ and $w_0 := a_0\chi \log r_1 + \widetilde{w}_0$ when $m = n-2$, with

$$|a_0| + \sup_{(0,\infty)} (1+r_1)^{-\nu}|\widetilde{w}_0| \leqslant c \sup_{(0,\infty)} (1+r_1)^{2-\nu}|f_0|.$$

To complete the proof of the proposition, it suffices to sum the two results we have just obtained and apply Schauder's estimates in order to get the relevant estimates for all the derivatives. □

If $\rho > 0$ is fixed small enough, we define the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathfrak{H}(n,m))$ as the space of restrictions of functions of $\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^{n-m} \times T^m, \mathfrak{H}(n,m))$ to $\mathbb{R}^{n-m} \times T^m - B^n_\rho$. This space is naturally endowed with the induced norm.

In order to simplify the notations, we set

$$\mathcal{E}^{2,\alpha}_\nu := \mathcal{C}^{2,\alpha}_\nu\big(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathfrak{H}(n,m)\big)$$

when $1 \leqslant m \leqslant n-3$ and

$$\mathcal{E}^{2,\alpha}_\nu := \mathcal{C}^{2,\alpha}_\nu\big(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathfrak{H}(n,m)\big) \oplus \mathcal{D}_{n-m}$$

when $m = n-2$ or $m = n-1$. We also define

$$\mathcal{F}^{0,\alpha}_{\nu-2} := \mathcal{C}^{0,\alpha}_{\nu-2}\big(\mathbb{R}^{n-m} \times T^m - B^n_\rho, \mathfrak{H}(n,m)\big)$$

when $1 \leqslant m \leqslant n-1$. Using the previous result together with a standard perturbation argument, we obtain the:

Proposition 6. *Assume that $\nu \in (2+m-n, 0)$ when $1 \leqslant m \leqslant n-3$, $\nu \in (-\infty, 0)$ when $m = n-2$ or $m = n-1$, and $\alpha \in (0,1)$ are fixed. There exist $\rho_0 > 0$, $c > 0$ and, for all $\rho \in (0, \rho_0)$, there exists an operator*

$$G_\rho : \mathcal{F}^{0,\alpha}_{\nu-2} \to \mathcal{E}^{2,\alpha}_\nu,$$

such that, for all $f \in \mathcal{F}^{0,\alpha}_{\nu-2}$, the function $w = G_\rho(f)$ is a solution of

$$\Delta w = f,$$

in $\mathbb{R}^{n-m} \times T^m - B^n_\rho$, with $w = 0$ on ∂B^n_ρ. Furthermore, $\|w\|_{\mathcal{E}^{2,\alpha}_\nu} \leqslant c\,\|f\|_{\mathcal{F}^{0,\alpha}_{\nu-2}}$.

4.3. The nonlinear problem

Using (21), one can check that the hypersurface parameterized by

$$\mathbb{R}^{n-m} \times T^m - B_\rho \ni (x_1, x_2) \to \big(x_1, x_2, u(x_1, x_2)\big) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R},$$

has mean curvature 0 if and only if the function u is a solution of

$$\Delta u - Q(u) = 0, \tag{25}$$

where we have set

$$Q(u) := -\frac{1}{1+|\nabla u|^2}\nabla^2 u(\nabla u, \nabla u).$$

4.3.1. Solutions of (25) which are parameterized by their boundary data

Let us assume that

$$\nu \in (2+m-n, 0) \quad \text{when } 1 \leqslant m \leqslant n-3, \quad \text{or} \quad \nu \in (-2, 0) \quad \text{when } m = n-2, \quad \text{or} \quad \nu \in (-\infty, 0) \quad \text{when } m = n-1,$$

is fixed. The new restriction on ν when $m = n-2$ is needed to ensure that the nonlinear operator

$$u \to \Delta u - Q(u),$$

maps $\mathcal{E}^{2,\alpha}_\nu$ into $\mathcal{F}^{0,\alpha}_{\nu-2}$. Thanks to the result of Proposition 6, it is possible to apply the implicit function theorem to solve (25) with w on ∂B^n_ρ equal to some given function $h \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$ which satisfies $\|h\|_{2,\alpha} \leqslant c_0$ for some fixed constant $c_0 > 0$. The solution of (25) provided by the implicit function theorem will be denoted by w_h. By construction, the graph of w_h is a minimal hypersurface whose boundary is parameterized by the boundary data h.

4.3.2. Family of minimal hypersurfaces which are close to $\mathbb{R}^{n-m} \times T^m$

Let us summarize what we have proved. We fix ν according to the above choice, $\alpha \in (0,1)$ and $\kappa > 0$. There exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$, for all $h \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$ satisfying

$$\|h\|_{2,\alpha} \leqslant \kappa\, \varepsilon^{n-1},$$

we have been able to find a minimal hypersurface, which is a graph over $\mathbb{R}^{n-m} \times T^m - B^n_\rho$. This hypersurface, once translated by $\varepsilon\, c_\infty$ along the z-axis, will be denoted by $M_\varepsilon(h)$.

Moreover, there exists a constant c_h such that $M_\varepsilon(h)$ is asymptotic to

$$\left\{(x_1, x_2, z) \in \mathbb{R}^{n-m} \times T^m \times \mathbb{R} \;\middle|\; z = c_h \zeta_m(x_1) + c_\infty \varepsilon\right\},$$

where $\zeta_{n-1}(y) := |y|$, $\zeta_{n-2}(y) := \log|y|$ and $\zeta_m(y) := 0$, when $1 \leqslant m \leqslant n-3$.

Also observe that the hypersurface $M_\varepsilon(h)$ inherits all the symmetries induces by the symmetries used to define the function spaces in Definition 4, hence it is invariant with respect to the action of $O(n-m, \mathbb{R}) \otimes \mathfrak{D}_m \otimes \{\pm I_1\} \subset O(n+1, \mathbb{R})$.

Furthermore, near its boundary this hypersurface can be parameterized as the graph of

$$\overline{B^n_{2\rho}} - B^n_\rho \ni x \to c_\infty \varepsilon - \widehat{W}_h(x) - \widehat{V}_h(x),$$

where $\widehat{W}_h$ is the unique (bounded) harmonic extension of the boundary data h in $\mathbb{R}^{n-m} \times T^m - B^n_\rho$ which belongs to $\mathcal{E}^{2,\alpha}_\nu$. Here the function $\widehat{V}_h$ satisfies

$$\left\|\widehat{V}_h\right\|_{\mathcal{C}^{2,\alpha}} \leqslant c_0 \varepsilon^{2n-2},$$

for some constant c_0 which depends on κ but does not depend on ε. The norm is taken over $\overline{B^n_{2\rho}} - B^n_\rho$.

Reducing ε_0 if this is necessary, we can assume that the mapping $h \to \widehat{V}_h$ is continuous and in fact smooth. It follows from standard properties of the solutions obtained through the application of the implicit function theorem that

$$\left\|\widehat{V}_{h_2} - \widehat{V}_{h_1}\right\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^{n-1} \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}}, \tag{26}$$

for some constant $c > 0$ which does not depend on ε, but depends on κ. Here the norms are understood on $\overline{B^n_{2\rho}} - B^n_\rho$.

5. The gluing procedure

We fix $\kappa > 0$ large enough and apply the results of the previous sections. There exists $\varepsilon_0 > 0$ and for all $g, h, \in \mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n,m))$ satisfying $\|g\|_{2,\alpha} \leqslant \kappa\varepsilon^{n-1}$ and $\|h\|_{2,\alpha} \leqslant \kappa\varepsilon^{n-1}$, we define the hypersurface $M_\varepsilon(g)$ and the hypersurface $C_\varepsilon(h)$. Our aim will be now to find g and h in such a way that

$$\left(M_\varepsilon(g) \cup C_\varepsilon(h)\right) \cap \mathbb{R}^{n-m} \times T^m \times (0, +\infty),$$

is a $\mathcal{C}^1$ hypersurface. Then applying a reflection with respect to the hyperplane $z = 0$, we will obtain a complete $\mathcal{C}^1$ hypersurface in $\mathbb{R}^{n-m} \times T^m \times \mathbb{R}$. Finally, it will remain to apply standard regularity theory to show that this hypersurface is in fact $\mathcal{C}^\infty$.

By construction, the two hypersurfaces $M_\varepsilon(g)$ and $C_\varepsilon(h)$ are graphs over the $z = 0$ hyperplane near their common boundary. That is, $M_\varepsilon(h)$ is the graph of the function

$$x \in \overline{B^n_{2\rho}} \setminus B^n_\rho \to c_\infty \varepsilon - \widehat{W}_g(x) - \widehat{V}_g(x),$$

and $C_\varepsilon(g)$ is the graph of the function

$$x \in \overline{B^n_\rho} \setminus B^n_{\rho/2} \to c_\infty \varepsilon - W_h(x) - V_{\varepsilon,h}(x).$$

Hence, to produce a $\mathcal{C}^1$ hypersurface, it remains to solve the equations

$$\widehat{W}_g + \widehat{V}_g = W_h + V_{\varepsilon,h}, \qquad \partial_r \widehat{W}_g + \partial_r \widehat{V}_{\varepsilon,g} = \partial_r W_h + \partial_r V_{\varepsilon,h}, \tag{27}$$

where all functions are evaluated on ∂B^n_ρ. The first identity is obtained by asking that the Dirichlet data of the two graphs coincide on ∂B^n_ρ and already ensures that the hypersurface is $\mathcal{C}^0$, while the second is obtained by asking that the Neumann data of the two graphs coincide on ∂B^n_ρ and ensures that the hypersurface is $\mathcal{C}^1$.

Let us recall that the mapping

$$\mathcal{U} : h \in \mathcal{C}^{2,\alpha}\left(S^{n-1}, \mathfrak{H}(n,m)\right) \to \rho\partial_r\left(W_h - \widehat{W}_h\right)(\rho\,\cdot) \in \mathcal{C}^{1,\alpha}\left(S^{n-1}, \mathfrak{H}(n,m)\right)$$

is an isomorphism. Indeed, this mapping is a linear first order elliptic pseudo-differential operator and, in order to check that it is an isomorphism, it is enough to prove that it is injective. Now if we assume that $\mathcal{U}(h) = 0$ then the function w defined by $w := \widehat{W}_h$ in $\mathbb{R}^{n-m} \times T^m - B^n_\rho$ and $w := W_h$ in B^n_ρ is a global solution of $\Delta w = 0$ in $\mathbb{R}^{n-m} \times T^m$, and furthermore, w belongs to $\mathcal{E}^{2,\alpha}_\nu$. It is easy to check that necessarily $w \equiv 0$ and, as a consequence, $h \equiv 0$. This proves the injectivity of $\mathcal{U}$.

Using the above claim, it is easy to see that (27) reduces to a fixed point problem

$$(g, h) = \mathbf{C}_\varepsilon(g, h),$$

in $\mathcal{E} := (\mathcal{C}^{2,\alpha}(S^{n-1}, \mathfrak{H}(n, m)))^2$. However, (20) and (26) imply that $\mathbf{C}_\varepsilon : \mathcal{E} \to \mathcal{E}$ is a contraction mapping defined in the ball of radius $\kappa\varepsilon^{n-1}$ of $\mathcal{E}$ into itself, provided ε is chosen small enough. Hence, we have obtained a fixed point of the mapping $\mathbf{C}_\varepsilon$, in this ball. This completes the proof of the existence of the hypersurfaces S_ε which are described in the Theorem 1. Most of the properties states in Theorem 1 follow readily from the construction itself except the derivation of (2).

Proof of (2). This follows from the application of the balancing formula for minimal hypersurfaces. In the case where $m = n - 1$ we know from the construction itself that the hypersurface S_ε is, away from the origin, the graph of the function

$$(x_1, x_2) \to c_\varepsilon |x_1| + c_\infty \varepsilon + \mathcal{O}\big(\varepsilon^{n-1} |x_1|^\nu\big),$$

for some $\nu < 0$. Observe that necessarily $c_\varepsilon \geqslant 0$, otherwise we easily get a contradiction by the maximum principle. Moreover, near 0 the hypersurface is a graph over the rescaled n-catenoid. It remains to identify the constant c_ε. In order to do so, we apply the balancing formula of [10] (Theorem 7.2) between the hyperplane $z = 0$ and $z = z_0$ for z_0 tending to $+\infty$. This yields

$$2\mathrm{Vol}\big(T^{n-1}\big) c_\varepsilon \sim \varepsilon^{n-1} \mathrm{Vol}\big(S^{n-1}\big).$$

And (2) follows at once from our normalization of the volume on an $(n-1)$-dimensional torus. □

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