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Approximation by $\{f(kx)\}$

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INTRODUCTION

In this paper we study the question of when the linear span of $\{f(kx)\} = E_f$, $k = 0, \pm 1, \pm 2,\ldots \pm n,\ldots$ is dense in certain function spaces B. B will designate any one of the following well-known spaces: $L^p(0, 2\pi)$, $1 \leqslant p < \infty$, the p-integrable functions, $C(0, 2\pi)$, the continuous periodic functions, and l^1, the functions with absolutely convergent Fourier series. The norms used are the usual ones. The results obtained are generalizations of the Weierstrass approximation theorem in B by trigonometric polynomials.

Another problem of interest—When will $\{f(kx)\}$ form a Paley–Wiener basis for L^2?—is of a different nature than the spanning problem in this paper. The methods required to solve the basis problem are different than those used here and are discussed in [3].

We consider functions of the form $f(x) = e^{ix} - \sum_2^n a_k e^{kix}$. This represents no loss of generality as far as the signs of the frequencies are concerned. One replaces e^{kix} by $\cos kx$ ($\sin kx$) and states and proves the theorems for even (odd) functions.

Let $p_1, p_2,\ldots, p_s$ be the primes appearing as divisors of the integers $2,\ldots, n$. We associate with $f(x) = e^{ix} - \sum_{k=2}^n a_k e^{ikx}$ the polynomial

$$P(z_1,\ldots, z_s) = 1 - \sum a_k z_1^{j_1} \cdots z_s^{j_s}, \qquad \text{where} \quad k = p_1^{j_1} \cdots p_s^{j_s}. \tag{1}$$

The density of E_f depends on the location of the zeros of $P(z_1,\ldots, z_s)$. We give necessary and sufficient conditions for the density of E_f in l^1

and L^2 with any number of variables, and all B with one variable. Less complete results for $C(0, 2\pi)$ and $L^p(0, 2\pi)$ with any number of variables are also proven. An effective procedure of approximation in L^2, with any number of variables and L^p with one variable is also given

1. Let $T_m f(x) = f(mx)$, m an integer. T_m is an isometry of each B, and they commute with each other. f can then be written as $f(x) = P(T_{p_1}, \ldots, T_{p_s})\, e^{ix}$. Considering $P(T_{p_1}, \ldots, T_{p_s})$ as a bounded operator from the spaces B to B, it is easy to see that E_f is dense in any B if, and only if, the range $P(T_{p_1}, \ldots, T_{p_s})$ is dense in B.

The problem then becomes one of finding the spectrum $\sigma(T_m)$ of the operator T_m. The spectrum of T_m is independent of m. Because T_m is an isometry, the spectrum is contained in the unit disc, $\{|z| \leqslant 1\}$. We give the determination of the spectrum $\sigma(T_m)$ in the following theorem.

THEOREM 1. (i) *T_m as an operator on l^1 and $C(0, 2\pi)$ has residual spectrum $\sigma_r(T_m)$ equal to the continuous spectrum $\sigma(T_m)$ which equals $\{|\lambda| \leqslant 1\}$, $\lambda \neq 1$. That is, if $\lambda \neq 1$, $|\lambda| \leqslant 1$, then $T_m - \lambda I$ is $1-1$ and range $(T_m - \lambda I)$ is not dense. $\lambda = 1$ is the only eigenvalue and the constants are the only eigenfunctions. For $|\lambda| > 1$, $T_m - \lambda I$ has a bounded inverse.*

(ii) *T_m as an operator on L^p, $1 \leqslant p < \infty$, has residual spectrum $\sigma_r(T_m) = \{|\lambda| < 1\}$, and continuous spectrum, $\sigma(T_m) = \{|\lambda| = 1, \lambda \neq 1\}$. (That is, if $|\lambda| \leqslant 1$, $\lambda \neq 1$, $T_m - \lambda I$ is $1-1$, and if $|\lambda| < 1$, range $(T_m - \lambda I)$ is not dense, but is dense for $|\lambda| = 1$. $\lambda = 1$ is the only eigenvalue and the constants are the only eigenfunctions. For $|\lambda| > 1$, $T_m - \lambda I$ has a bounded inverse.*

Proof. $[|\lambda| \leqslant 1]$. Then $T_m f = \lambda f$ implies that $\hat{f}(k) = \lambda \hat{f}(mk)$, where $\hat{f}(k)$ is the k-th Fourier coefficient. So we have $|\hat{f}(k)| \leqslant \hat{f}(m^r k)|$, $k \neq 0$, for all r. By the Riemann–Lebesgue lemma this implies $\hat{f}(k) = 0$. We have $\hat{f}(0) = \lambda \hat{f}(0)$, so if $\lambda \neq 1$, $\hat{f}(0) = 0$ too. This shows that $\lambda = 1$ is the only eigenvalue and the constants are the only eigenfunctions.

$[|\lambda| > 1]$. Then $\sum_{n=0}^{\infty} T_m{}^n / \lambda^n$ is convergent and is a bounded operator for all B, so $T_m - \lambda I$ has a bounded inverse.

$[|\lambda| < 1]$. A function, measure or pseudomeasure $h \in$ range $(T_m - \lambda I)^\perp$, the annihilator of the range of $T_m - \lambda I$, if, and only if,

$$\hat{h}(mk) = \lambda \hat{h}(k) \qquad \text{for all} \quad k. \tag{2}$$

Let $h(x) = \sum_{r=0}^{\infty} \lambda_r e^{im^r x}$. Then $h \in$ range $(T_m - \lambda I)$ in all B because $h \in l^1$. Hence, range $(T_m - \lambda I)$ is not dense in any B.

$[|\lambda| = 1, \lambda \neq 1]$. When $\lambda = 1$, everything we say for B and $|\lambda| = 1$, holds for $B_0 = \{f \in B : f(0) = 0\}$.

We show for any $g \in B$, that the distance $d(g, E_f)$ between g and E_f is given by

$$\lim_r \| S_r(g) \|_B, \qquad \text{where} \quad f(x) = e^{ix} - \lambda e^{imx} = (I - \lambda T_m)\, e^{ix}. \tag{3}$$

Here

$$S_r(g) = \frac{1}{r} \sum_{k=0}^{r-1} \lambda^k g(m^k x). \tag{4}$$

First we have

$$\| g - h \|_B \geqslant \| S_r(g) - S_r(h) \|_B \geqslant \| S_r(g) \|_B - \| S_r(h) \|_B . \tag{5}$$

For $h \in E_f$, $\lim \| S_r(h) \|_B = 0$. To see this, note that h can be approximated by functions of the form $q(x) = (I - \lambda T_m)\, p(x)$. Now we have

$$S_r(q) = p(x) - \lambda^r T_m{}^r p(x)/r \tag{6}$$

and so

$$\| S_r(q) \|_B \leqslant 2/r \| p \|_B .$$

We see $\lim \| S_r(q) \|_B = 0$ and hence the same is true for h. Then

$$d(g, E_f) \geqslant \overline{\lim} \| S_r(g) \|_B . \tag{7}$$

By the Hahn–Banach theorem, $d(g, E) = |\sup L(g)|$, $\| L \| = 1$, $L \in E_f{}^{\perp}$. For $L \in E_f{}^{\perp}$, we have $L(g) = L[S_r(g)]$. So $|L(g)| \leqslant \| S_r(g) \|_B$ and, therefore, $d(g, E_f) \leqslant \underline{\lim} \| S_r(g) \|_B$. This with (7) gives us the desired result. Note this formula is true for $|\lambda| \leqslant 1$. By the Dunford–Schwartz ergodic theorem [2], $\lim S_r(g) = 0$ a.e. and $\lim \| S_r(g) \|_p = 0$, $\| \; \|_p$ being the L^p norm. Hence for $|\lambda| = 1$, range $(T_m - \lambda I)$ is dense in L^p. The continuous and a fortiori the l^1 case is different.

It is easy to see that E_f is dense in $C(0, 2\pi)$ if, and only if, $e^{ix} \in \bar{E}_f$, the closure of E. Now $S_r(e^{ix}) = 1/r \sum_{k=0}^{r-1} \lambda^k e^{im^k x}$ is a gap series. By a theorem of Sidon [7], the sup norm $\| S_r(e^{ix}) \|_\infty$ is equivalent to the l^1 norm $\| S_r(e^{ix}) \|_{1'} = 1/r \sum_{k=0}^{r-1} | \lambda |^k = 1$ $(|\lambda| = 1)$. Hence, $e^{ix} \notin \bar{E}_f$ and so range $(T_m - \lambda I)$ is not dense in E_f. This completes the proof of Theorem 1.

2. *Remarks.* We give two more proofs for the L^p case of the last part of Theorem 1.

An elementary proof can be given by noting that if $h \in E_f^{\perp}$ then $\hat{h}(k) = \lambda \hat{h}(mk)$ ($|\lambda| = 1$), so that $|\hat{h}(k)| \leqslant |\hat{h}(m^r k)|$ for all r and so $\hat{h}(k) = 0$, and, therefore, $h = 0$ a.e.

Another proof can be given by noting that $\| S_r(e^{ix})\|_p$ is dominated by $\| S_r(e^{ix})\|_2$ from a theorem of Paley [7]. But $\| S_r(e^{ix})\|_2 = 1/\sqrt{r}$ which goes to zero, so that $e^{ix} \in E_f$.

3. A question which we do not consider here is when can $g \in L^p$ be written as $g(x) = h(x) - \lambda h(mx)$, for some $h \in L^p$. The case $p = 2$, was discussed by Kac [4] and Rochberg [6].

The continuous case for the last part of Theorem 1 presents another problem of interest. There we showed that when $f(x) = e^{ix} - \lambda e^{imx}$, $|\lambda| = 1$, E_f is not dense in $C(0, 2\pi)$. This implies the existence of nontrivial measures $\mu \in E_f^{\perp}$, such measures satisfying

$$\hat{\mu}(k) = \lambda \hat{\mu}(mk). \tag{8}$$

It is not hard to show such measures are purely singular. It would be of some importance to be able to construct measures satisfying (8). Even the case $\lambda = 1$ is of interest. Besides the Haar and Dirac measures, there is a whole class of measures μ such that $\hat{\mu}(k) = \hat{\mu}(mk)$. These are the Cantor measures μ with Fourier–Stieltjes coefficients

$$\hat{\mu}(k) = \prod_{r=0}^{\infty} \cos \frac{2\pi k}{m^r}, \qquad \text{c.f., [5, 7].}$$

If we have measures μ_j so that $\hat{\mu}_j(k) = \lambda_j \hat{\mu}_j(mk)$, $j = 1, 2$, then $\widehat{\mu_1 * \mu_2}(k) = \lambda_1\lambda_2 \widehat{\mu_1 * \mu_2}(mk)$. So we can build measures satisfying (8) by convolutions. When λ_n is a root of unity ($\lambda_r^r = 1$) we can give a construction of such μ. Let δ_θ be the point mass at θ, and let $\theta_k = 2\pi(m-1)\, m^k/(m^r - 1)$. Then $\mu_r = 1/r \sum_{k=0}^{r-1} \lambda_r^k \delta_{\theta_k}$ is a measure satisfying (8). It would be hoped that by convolving μ's with their λ's roots of unity, we can get measures satisfying (8) for any λ, $|\lambda| = 1$. This does not seem possible because $\{\mu_r\}$ converges w^* to 0, provided infinitely many of the $\lambda_r \neq 1$. This is not hard to prove and we leave this an exercise for the reader.

4. We now prove

THEOREM 2. *Let* $f(x) = e^{ix} - \sum_{k=1}^{n} a_k e^{im^k x}$ *and* $E_f = \operatorname{span}\{f(kx)\}$, $k = 0, \pm 1, \pm 2, \ldots$, *then*

(a) *E_f is dense in l^1 and $C(0, 2\pi)$ if, and only if, the zero set $Z(P)$ of $P(z) = 1 - \sum_{k=1}^{n} a_k z^k$ is contained in $\{z : |z| > 1\}$.*

(b) *E_f is dense in L^p $(1 \leqslant p < \infty)$ if, and only if, $Z(P)$ is contained in $\{z : |z| \geqslant 1\}$.*

Proof. If $Z(P) \leqslant \{|z| > 1\}$, $1/P(z)$ is analytic in a neighborhood of the spectrum T_m. Hence, $P(T_m)^{-1}$ is a bounded operator on any B; c.f., [1]. This suffices for E_f to be dense in B.

If $P(z)$ has at least one root λ, $|\lambda| = 1$, any measure μ satisfying (8) is in $E_f^{\perp}$. Hence, E_f is not dense in $C(0, 2\pi)$ and l^1.

If $P(z)$ has a root λ with $|\lambda| < 1$, then the l^1 function $g(x) = \sum_{k=0}^{\infty} \lambda^k e^{im^k x}$ is in $E_f^{\perp}$ for all B. So, in this case, E_f is not dense in any B.

Finally, let $\lambda_1, \ldots, \lambda_n$ be the roots of $P(z)$, multiplicities allowed, $(|\lambda_j| \geqslant 1)$. Then $P(T) = \Pi(T_m - \lambda_j I)$. When $B = L^p$, we have from Theorem 1 that range $(T_m - \lambda_j I)$ is dense for each λ_j. Hence, range $P(T_m)$ is dense and this implies E_f is dense in L^p. This completes the proof of Theorem 2.

5. It is easy to see Theorem 2 remains true for $f(x) = e^{ix} - \sum_1^{\infty} a_k e^{im^k x}$ provided $P(z) = 1 - \sum_1^{\infty} a_k z^k$ is analytic in a neighborhood of $\{|z| \leqslant 1\}$.

6. We now consider the case where $f(x) = e^{ix} - \sum_{k=2}^{n} a_k e^{ikx}$. The answer here is not as complete as in Theorem 2.

THEOREM 3. *Suppose $f(x) = e^{ix} - \sum_{k=2}^{n} a_k e^{ikx}$ and $P(z_1, \ldots, z_s) = 1 - \sum a_k z_1^{j_1} \cdots z_s^{j_s}$, $(k = p_1^{j_1} \cdots p_s^{j_s})$. Then*

(a) *if the zero set $Z(P) \leqslant \{\xi = (\xi_1, \ldots, \xi_s) : \max |\xi_j| > 1\}$, E_f is dense in all B;*

(b) *if there is at least one $\xi \in Z(P)$ such that $\max |\xi_j| < 1$, then E_f is not dense in any B.*

Proof. (a) $1/P(z_1 \cdots z_s)$ has a Taylor expansion in

$$|z_1| \leqslant 1 + \epsilon, \ldots, |z_s| \leqslant 1 + \epsilon$$

for some $\epsilon > 0$, and replacing z_i by T_{p_i} gives a series converging in norm to $P(T_{p_1}, \ldots, T_{p_s})^{-1}$.

(b) Let $\xi = (\xi_1 \cdots \xi_s) \in Z(P)$, $|\xi_j| < 1$. Then the l^1 function

$$g(x) = \sum_{0 \leqslant r_j} \cdots \sum \xi_1^{r_1} \cdots \xi_s^{r_s} e^{ip_1^{r_1} \cdots p_s^{r_s} x}$$

is in $E_f^\perp$ for all B. Hence, E_f is not dense. This completes the proof of Theorem 3.

7. We conjecture that if $Z(P) \subseteq \{\xi : \max |\xi_j| \geqslant 1\}$, then E_f is dense in L^p, $1 \leqslant p < \infty$. Further, if there is a $\xi \in Z(P)$ with $|\xi_k| \leqslant 1$ and $|\xi_j| = 1$ for some j, then E is not dense in $C(0, 2\pi)$. We have not been able to prove these assertions. However, it is very easy to show that E_f is not dense in l^1. For let $f_j(x) = e^{ix} - \alpha_j e^{ip_j x}$, $j = 1 \cdots s$, $\alpha_j = 1/\xi_j$, and $E' = \operatorname{span}\{f_1(kx),\ldots, f_s(kx)\}$. Now $E'^\perp \subseteq E_f^\perp$, so if E' is not dense in B then E_f isn't. Let h be the pseudomeasure

$$h \sim \sum_{0 \leqslant r_j < \infty} \xi_1^{r_1} \cdots \xi_s^{r_s} e^{ip_1^{r_1} \cdots p_s^{r_s} x}.$$

Then $h \in (E')^\perp$, so E_f is not dense in l^1.

If we can find a measure μ satisfying

$$\hat{\mu}(k) = \alpha_1 \hat{\mu}(P_1 k) = \cdots = \alpha_s \hat{\mu}(P_s k), \qquad |\alpha_j| \geqslant 1, \tag{9}$$

then E_f would not be dense in $C(0, 2\pi)$. When each α_j is an r_j root of unity we can construct measures satisfying (9). We write

$$\mu = \sum\sum_{0 \leqslant k_j \leqslant r_j - 1} \alpha_j^{k_j} \delta(k_1 \cdots k_s),$$

where $\delta(k_1 \cdots k_s)$ is the point mass supported on

$$\theta(k_1 \cdots k_s) = 2\pi \prod_{j=1}^{s} (p_j - 1) p_j^{k_j} \Big/ \prod_{j=1}^{s} (p_j^{r_j} - 1),$$

and this μ satisfies (9). Again it would be of considerable interest to be able to construct measures satisfying (9) but this seems difficult. Similar considerations, as in the case of (8), hold here.

Now when all the α_j are of absolute value 1, the distance of e^{ix} to E' can be calculated by a formula similar to (3). That is $d(e^{ix}, E') = \lim \| S_r(e^{ix})\|_B$, where

$$S_r(e^{ix}) = \frac{1}{r^s} \sum \cdots \sum_{0 \leqslant k_j \leqslant r-1} \alpha_1^{k_1} \cdots \alpha_s^{k_s} e^{ip_1^{k_1} \cdots p_s^{k_s} x}.$$

In this situation, the Sidon theorem doesn't hold. Nevertheless we conjecture that $\| S_r(e^{ix})\| \infty \neq o(1)$.

8. In the case that $Z(P) \subseteq \{\xi : \max |\xi_j| > 1\}$, then for f in B,

$$f = P(T_{p_1}, \ldots, T_{p_s})[P(T_{p_1}, \ldots, T_{p_s})]^{-1} f. \tag{10}$$

We cannot expect this to hold for the case when

$$Z(P) \subseteq \{\xi : \max |\xi_j| \geqslant 1\},$$

but we can expect an approximate formula to hold in L^p. That is, it is reasonable to expect that

$$f = \lim_{r \to 1^-} P(T_{p_1}, \ldots, T_{p_s})[P(rT_{p_1}, \ldots, rT_{p_s})^{-1}]f,$$

convergence taking place in L^p. This of course would complete the density theorem for L^p. We give such a result for all L^p and $P(z)$ a polynomial in one variable; and for L^2 and any number of variables. We first prove some lemmas.

LEMMA 1. *Let $P(z_1, z_2, \ldots, z_m)$ be a polynomial with no zeros in $|z_1| < 1, |z_2| < 1, \ldots, |z_m| < 1$. In the L^2 metric of the torus $|z_1| = |z_2| = \cdots = |z_m| = 1$ the function $P(z_1, z_2, \ldots, z_m)/P(rz_1, rz_2, \ldots, rz_m)$ approaches 1 as $r \to 1^-$.*

Since this fraction does go to 1 a.e. (wherever $P(z_1, \ldots, z_m) \neq 0$) it certainly suffices, by the bounded convergence theorem, to show that it is bounded. Thus, Lemma 1 follows from

LEMMA 2. *Let $P(z_1, z_2, \ldots, z_m)$ be a polynomial of degree D with no zeros in $|z_1| < 1, |z_2| < 1, \ldots, |z_m| < 1$ and let $0 < r < 1$. We have, throughout $|z_1| \leqslant 1, |z_2| \leqslant 1, \ldots, |z_m| \leqslant 1$,*

$$\left| \frac{P(z_1, z_2, \ldots, z_m)}{P(rz_1, rz_2, \ldots, rz_m)} \right| \leqslant 2^{mD}.$$

Proof. Observe the factorization

$$\frac{P(z_1, z_2, \ldots, z_m)}{P(rz_1, rz_2, \ldots, rz_m)}$$

$$= \frac{P(z_1, z_2, \ldots, z_m)}{P(rz_1, z_2, \ldots, z_m)} \cdot \frac{P(rz_1, z_2, \ldots, z_m)}{P(rz_1, rz_2, z_3, \ldots)} \cdots \frac{P(rz_1, \ldots, rz_{m-1}, z_m)}{P(rz_1, \ldots, rz_m)}.$$

Each factor is a fraction $Q(W_1, W_2, \ldots, W_m)/Q(rW_1, W_2, \ldots, W_m)$, where the W_i are a permutation of variables which are either z_i or $r \cdot z_i$. Thus, $Q(W_1, \ldots, W_m)$ has no zeros in $|W_1| < 1$, $|W_2| < 1, \ldots, |W_m| < 1$, and we show that this implies

$$\left| \frac{Q(W_1, W_2, \ldots, W_m)}{Q(rW_1, W_2, \ldots, W_m)} \right| \leqslant 2^D,$$

which is sufficient for the Lemma. Fix $W_2, W_3, \ldots, W_m$ with $|W_i| < 1$ and view $Q(W_1, W_2, \ldots, W_m)$ as a polynomial in W_1 alone. As such it has all its roots *outside* the open unit disk. Therefore, we can write $Q(W_1, W_2, \ldots, W_m) = c \prod_{j=1}^{d} (W_1 - \alpha_j)$, $c \neq 0$, $d \leqslant D$, $|\alpha_j| \geqslant 1$. Thus,

$$\frac{Q(W_1, W_2, \ldots, W_m)}{Q(rW_1, W_2, \ldots, W_m)} = \prod_{j=1}^{d} \frac{W_1 - \alpha_j}{rW_1 - \alpha_j}.$$

But observe that, for $|W_1| \leqslant 1$,

$$\left| \frac{W_1 - \alpha_j}{rW_1 - \alpha_j} \right| = \left| 1 + \frac{(1-r)W_1}{rW_1 - \alpha_j} \right| \leqslant 1 + \frac{1-r}{|rW_1 - \alpha_j|}$$

$$\leqslant 1 + \frac{1-r}{||\alpha_j| - r|} \leqslant 1 + \frac{1-r}{1-r} = 2.$$

We conclude that

$$\left| \frac{Q(W_1, \ldots, W_m)}{Q(rW_1, \ldots, W_m)} \right| \leqslant 2^d \leqslant 2^D$$

as required.

9. *Remarks.* (a) If $Z(P) \subseteq \{\xi : \max |\xi_j| \geqslant 1\}$, then $P(rT_{p_1}, \ldots, rT_{p_s})$ has a bounded inverse in all B, when $0 < r < 1$.

(b) Under the assumptions in (a), we have

$$\| P(T) P(rT)^{-1} \|_p \leqslant c < \infty \qquad \text{independent of } r, \text{ for } 1 \leqslant p < \infty. \tag{11}$$

Now

$$P(T) P(rT)^{-1} = \prod (T - \alpha_j I)(rT - \alpha_j I)^{-1}, \qquad \text{with } |\alpha_j| \geqslant 1. \tag{12}$$

Further,

$$(T - \alpha_j I)(rT - \alpha_j I)^{-1} = \alpha_j \left[I - \frac{1-r}{r} \sum_{n=1}^{\infty} \left(\frac{r}{\alpha_j} \right)^n T^n \right]. \tag{13}$$

So $\|(T - \alpha_j I)(rT - \alpha_j I)^{-1}\|_B \leqslant 2 |\alpha_j|$ independent of r. (13) combined with (12) gives us the desired result. We prove a similar result for $\| P(T_1 \cdots T_n) P(rT_1 \cdots rT_n)^{-1} \|_2$.

(c) Suppose $R(z_1, \ldots, z_r)$ is analytic in a neighborhood of the polydisc $|z_1| \leqslant 1, \ldots, |z_r| \leqslant 1$. Then if $f(e^{ix}) = R(T_{p_1'}, \ldots, T_{p_r'}) e^{ix}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{ix})|^2 \, dx = \left(\frac{1}{2\pi} \right)^r \int \cdots \int_{0 \leqslant \theta_j \leqslant 2\pi} |R(e^{i\theta_1}, \ldots, e^{i\theta_r})|^2 \, d\theta_1 \cdots d\theta_r. \tag{14}$$

This is just Parseval's equality.

Using this, we can get an expression for the norm of $R(T_{p_1'},..., T_{p_r'})$ as an operator on L^2. Let $f(x) = \sum_{k=1}^{m} a_k e^{ikx} = Q(T_{p_1} \cdots T_{p_s}) e^{ix}$, $Q(z_1 \cdots z_s) = \sum a_k z_1^{j_1} \cdots z_s^{j_s}$, $(k = p_1^{j_1} \cdots p_s^{j_s})$. Then $Rf = RQe^{ix}$, RQ is a function of the variables $z_1 ,..., z_s , z_{1'} ,..., z_{r'}$, where repetitions are allowed. Using (14) we find

$$\| Rf \|_2^2 = \left(\frac{1}{2\pi}\right)^{r+s} \int \cdots \int | R(e^{i\theta_1} \cdots e^{i\theta_r})|^2 Q(e^{i\theta_{r+1}} \cdots e^{i\theta_{r+s}})|^2 \, d\theta_1 \cdots d\theta_{r+s} \, . \tag{15}$$

Hence, $\sup \| Rf \|_2 = \max | R(e^{i\theta_1} \cdots e^{i\theta_r})| = \| R \|_2$, where the sup is taken over $f \in L^2$ with norm 1.

Now if $P(z_1 \cdots z_n)$ is a polynomial with $Z(P) \subseteq \{\max | \xi_j | \geqslant 1\}$, then

$$\| P(T_1 \cdots T_n) \, P(rT_1 \cdots rT_n)^{-1} \|_2 = \max \left| \frac{P(e^{i\theta_1} \cdots e^{i\theta_n})}{P(re^{i\theta_1} \cdots re^{i\theta_n})} \right| . \tag{16}$$

Lemma 1 implies (16) is bounded independently of r.

10. We can now state our last two theorems.

THEOREM 4. *Suppose $f \in L^p$, $1 \leqslant p < \infty$ and $P(z)$ is a polynomial with $Z(P) \subseteq \{| z | \geqslant 1\}$. Then*

$$\lim_{r \to 1} \| f - P(T) \, P(rT)^{-1} f \|_p = 0. \qquad (T = T_m). \tag{17}$$

Proof. It is enough to show (17) when $f(x) = e^{ix}$. For if it is true for e^{ix} then it is so for polynomials. From (11) we conclude the truth for all $f \in L^p$. Let $g_r(x) = P(T) \, P(rT)^{-1} \, e^{ix}$. The Fourier series for $g_r(x)$ is of the form $\sum a_k(r) \, e^{im^k x}$, a gap series. So by the Paley Theorem, all the L^p norms of $e^{ix} - g_r(x)$ are dominated by the L^2 norm. From (14) we find that

$$\int_0^{2\pi} | e^{ix} - g_r(x)|^2 \, dx = \int_0^{2\pi} \left| 1 - \frac{P(e^{i\theta})}{P(re^{i\theta})} \right|^2 d\theta.$$

By Lemma 1 we get the desired result.

THEOREM 5. *Suppose $P(z_1 ,..., z_n)$ is a polynomial with*

$$Z(P) \subseteq \{\xi : \max | \xi_j | \geqslant 1\}.$$

Then

$$\lim_{r \to 1} \| f - P(T_1 ,..., T_n) \, P(rT_1 ,..., rT_n)^{-1} f \|_2 = 0 \qquad \text{for all} \quad f \in L^2. \tag{18}$$

Proof. Again it is enough to show (18) true for $f(x) = e^{ix}$. From (14) we have

$$\frac{1}{2\pi}\int_0^{2\pi} | e^{ix} - P(T_1, \ldots, T_n)\, P(rT_1, \ldots, rT_n)^{-1}\, e^{ix}\, |^2\, dx$$

$$= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \left| 1 - \frac{P(e^{i\theta_1}, \ldots, e^{i\theta_n})}{P(re^{i\theta_1}, \ldots, re^{i\theta_n})} \right|^2 d\theta_1 \cdots d\theta_n .$$

Lemma 1 gives the desired result.

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