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Separability properties of free groups and surface groups

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Abstract

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A subset X of a group G is said to be separable if it is closed in the profinite topology. Separable subgroups are very useful in low-dimensional topology and there has been some interest in separable double cosets. A new method for showing that a double coset is separable is introduced and it is used to obtain a short proof of the result of Gitik and Rips, that in a free group every double coset of finitely generated subgroups is separable. In addition it is shown that this property is shared by Fuchsian groups and the fundamental groups of Seifert fibred 3-manifolds.

1. Introduction

Let G be a group regarded as acting on itself by left and right multiplication. In order to define a topology on G which is equivariant with respect to both of these actions, we only need to specify a base $\mathscr{B}$ for the neighbourhoods of the identity element.

Definition. Let $\mathscr{B}$ be the base consisting of the finite-index normal subgroups of G. The corresponding equivariant topology is the *profinite topology on G.*

Given the profinite topology, G is of course a topological group (i.e., the group operations are continuous) and it is residually finite if and only if it is Hausdorff. (This may be taken as the definition of residual finiteness; as always in a topological group the Hausdorff property is equivalent to the trivial subgroup being closed.) Closed subgroups have proved to be useful in studying the structure of a group, for example Kropholler and Roller have used them to obtain splittings of Poincaré duality groups [2].

Definition. A subset $X \subseteq G$ is *separable* in G if it is closed in the profinite topology on G. G is said to be *subgroup separable* if all of its finitely-generated subgroups are separable, and to be *double coset separable* if for every pair H, K of finitely-generated subgroups of G, and every $g \in G$ the double coset HgK is separable.

In [6] Scott showed that Fuchsian groups and the fundamental groups of Seifert fibred 3-manifolds are subgroup separable, and in [3] Lennox and Wilson showed that every double coset in a polycyclic-by-finite group is separable. In general it is difficult to show that a given subset of a group is separable, though in [4] we gave the following method (the doubling trick) for detecting separable subgroups:

Theorem [4]. *Let H be a subgroup of the group G; if the amalgamated free product $G *_H G$ is residually finite, then H is separable in G.* $\square$

It is not hard to show that if G is residually finite, then the converse to this is also true.

In the main theorem of this note I will give a similar criterion to detect separable double cosets. This is prompted by the paper of Gitik and Rips, [1] and subsequent work by Rips [private communication] where it is shown that free groups are double coset separable. The new criterion given here yields a much shorter proof of their result. We now state the theorem.

Theorem 3.2. *If $H < G$, and $G *_H G$ is subgroup separable, then for any finitely generated subgroup $K < G$ the double coset HK is closed in G.*

The paper is organised as follows. In Section 2 we will examine the definition of coset separability used by Gitik and Rips, and its relationship to the definitions given above. We will also consider the behaviour of double coset separability with respect to subgroups and supergroups. Section 3 contains a proof of Theorem 3.2 and its corollary, that free groups are double coset separable. Section 4 uses this result to show that the fundamental groups of hyperbolic surfaces are double coset separable, and deduces that finitely-generated Fuchsian groups share this property. In Section 5 we apply the foregoing results to the fundamental groups of Seifert fibred 3-manifolds, as well as giving an example to show that the converse to Theorem 3.2 is false. Both of the results in Section 5 depend in some way on the proof by Lennox and Wilson [3] that in a polycyclic-by-finite group every double coset is separable.

2. Coset separability and the profinite topology

In [1] Gitik and Rips make the following definition:

Definition. A group G is said to be *coset separable with respect to the subgroup* H, if for any finitely generated subgroup $K < G$ and any finite subset $X \subseteq G$ there are finite index subgroups $H_0, K_0 < G$ containing H, K respectively such that for all $x, y \in X$, $H_0 x K_0 = H_0 y K_0$ if and only if $HxK = HyK$.

Remark. For any $x, y \in G$, $x \in HyK$ if and only if $HxK = HyK$. This enables us to prove the following:

Lemma 2.1. *A group G is coset separable with respect to H if, and only if, for any finitely-generated subgroup $K < G$ and any $y \in G$ the double coset HyK is closed in the profinite topology on G.*

Proof. Suppose that G is coset separable with respect to H, and let K be any finitely-generated subgroup of G; we want to show that every double coset HyK is closed, or equivalently that $G \backslash HyK$ is open. Suppose that $x \not\in HyK$. Since G is coset separable with respect to H, there are finite-index subgroups $H_0, K_0 < G$ containing H, K respectively satisfying $H_0 x K_0 \neq H_0 y K_0$. In particular, by the remark, $H_0 x \cap HyK = \emptyset$. Let $N = \bigcap_{g \in G} H_0^g$ (the *normal core* of H_0 in G); since H_0 has finite index in G the normal subgroup $N \lhd G$ is also finite index, so Nx is an open neighbourhood of x avoiding HyK. It follows that $G \backslash HyK$ is open as required.

Now suppose that for any finitely-generated subgroup $K < G$ all the double cosets HyK are closed. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of G and assume without loss of generality that $Hx_iK = Hx_jK$ if and only if $i = j$. As before, for $i \neq j$, $x_i \not\in Hx_jK$, and since the double coset Hx_jK is closed there is a finite-index normal subgroup $N_{ij} < G$ with $N_{ij} x_i \cap Hx_jK = \emptyset$. $N = \bigcap_{ij} N_{ij}$ is a finite-index normal subgroup such that $Nx_i \cap Hx_jK = \emptyset$ for $i \neq j$. Let $H_0 = HN$, and $K_0 = NK$. Since $Nx_i \cap Hx_jK = \emptyset$, $x_i \not\in NHx_jK = HNx_jNK = H_0 x_j K_0$, so $H_0 x_i K_0 \neq H_0 x_j K_0$. $\square$

Proposition 2.2. *Let G_0, H and K be finitely-generated subgroups of G and put $H_0 = H \cap G_0$, $K_0 = K \cap G_0$. Suppose that $|H : H_0|$ and $|K : K_0|$ are both finite.*

(i) *If G is subgroup separable or $|G : G_0| < \infty$, then the subspace topology induced on G_0 is precisely the profinite topology.*

(ii) *If G is subgroup separable or $|G : G_0| < \infty$, and $H_0 K_0$ is separable in G_0, then HK is separable in G.*

(iii) *If $|G : G_0| < \infty$, then G_0 is double coset separable if and only if G is.*

Proof. (i) If G is subgroup separable, then every finitely-generated subgroup of G is closed in G. Since G_0 is finitely generated, each finite-index subgroup $N < G_0$ is finitely generated, and is therefore closed in the subspace topology. Every left coset of N is therefore closed, and since finite unions of closed sets are closed, $G_0 \backslash N$ is closed. Thus N is open in the subspace topology, and since this is equivariant it is at least as fine as the profinite topology. Now suppose that $X \subset G_0$ is open in the subspace topology and $g \in X$. Then $g^{-1}X$ is an open neighbourhood of 1 in the subspace topology, and there is a finite-index normal subgroup $N \lhd G$ such that $N \cap G_0 \subseteq g^{-1}X$. It follows that X contains the set $g(N \cap G_0)$, and since $N \cap G_0$ is a finite-index normal subgroup of G_0, this is an open neighbourhood of g in the profinite topology on G_0. Since this is true for an arbitrary element $g \in X$, the set X is open in the profinite topology on G_0, so the subspace topology coincides with the profinite topology as required.

Now suppose that $|G : G_0| < \infty$. Then any finite-index normal subgroup of G_0 is of finite index in G, so it is open in G, and hence open in the subspace topology. If on the other hand $X \subseteq G_0$ is open in the subspace topology on G_0, and $g \in G_0$, then as before there is a finite-index normal subgroup $N \lhd G$ such that $N \cap G_0$ lies inside $g^{-1}X$. Since $N \cap G_0$ is a finite-index normal subgroup of G_0, it is open in the profinite topology on G_0, and $g(N \cap G_0)$ is an open neighbourhood of g in the profinite topology of G_0 which lies inside X. So X is open in the profinite topology as required.

(ii) Let $\{h_1, h_2, \ldots, h_m\}$ be a complete set of left coset representatives for H/H_0, and $\{k_1, k_2, \ldots, k_n\}$ be a complete set of right coset representatives for $K_0\backslash K$. Since H_0K_0 is separable in G_0 it is separable in G by (i). Since the profinite topology is equivariant under left and right multiplication in G this makes $h_iH_0K_0k_j$ separable for each i, j. The double coset HK is the (finite) union of these closed sets so is itself closed in G.

(iii) Since the profinite topology is equivariant the double coset, HyK is closed in G if and only if $H^yK = y^{-1}HyK$ is closed, so in order to prove that G is double coset separable it suffices to show that for any finitely generated subgroups H and K in G the double cosets HK are separable. Suppose that G_0 is double coset separable and consider HK where H and K are finitely generated subgroups of G. Since $|G : G_0| < \infty$ the subgroups $H_0 = H \cap G_0$ and $K_0 = K \cap G_0$ are finitely generated, so by hypothesis the double coset H_0K_0 is separable in G_0. We can now apply (ii) to see that HK is closed in G. Conversely suppose that G is double coset separable and let H_0, K_0 be finitely generated subgroups of G_0. The double coset H_0K_0 is separable in G so by (i) it is closed in G_0 as required. □

3. Detecting closed double cosets

Let H be a subgroup of G and form the double $G *_H G$. There is an involution τ defined on the two copies of G which swaps the factors, and by the universal property of amalgamated free products this extends uniquely to an automorphism of $G *_H G$ which we will also denote by τ. The fixed points of τ are precisely the subgroup H. In order to prove Theorem 3.2 we will need the following lemma:

Lemma 3.1. *The function $\partial : G \to G *_H G$ given by $g \mapsto g^{-1}g^{\tau}$ is continuous and for any subgroup $K < G$, $\partial^{-1}(\langle K, K^{\tau}\rangle) = HK$.*

Proof. Since group homomorphisms, group multiplication, and inverse are all continuous, ∂ is a composition of continuous functions.

The subgroup theorem for amalgamated free products tells us that $L = \langle K, K^\tau \rangle$ is actually $K *_H K^\tau$. Suppose that $g \in G \backslash HK$ and $\partial(g) \in L$. Then, as an element of L, $\partial(g)$ can be written as a reduced word $k_1 k_2^\tau \cdots k_{2n-1} k_{2n}^\tau h$, where each $k_i \in K \backslash H$, and $h \in H$. Since each k_i is an element of $G \backslash H$ this word is also reduced in $G *_H G^\tau$. But since $g \in G \backslash H$, $g^{-1} g^\tau$ is another reduced word representing the same element $\partial(g) \in G$. It follows from the uniqueness of reduced form (up to a choice of transversal for H in G) that $g^{-1} \in k_1 H$. Hence $g \in HK$. □

Theorem 3.2. *If $H < G$, and $G *_H G$ is subgroup separable, then for any finitely-generated subgroup $K < G$ the double coset HK is closed in G.*

Proof. Since $G *_H G$ is subgroup separable, and $L = \langle K, K^\tau \rangle$ is a finitely-generated subgroup, it is closed in the profinite topology on $G *_H G$. Lemma 3.1 tells us that HK is the pre-image under ∂ of the closed subgroup L. It follows that HK is itself closed. □

Corollary 3.3. *Free groups are double coset separable.*

Proof. Let H, K be finitely-generated subgroups of a free group G. We want to show that HK is closed in the profinite topology on G. By Burns' strengthening of Hall's theorem [5, Proposition 3.10] free groups are subgroup separable, and furthermore there is a finite-index subgroup $G_0 < G$ such that $G_0 = H * L$ for some subgroup $L < G$. It follows that $G_0 *_H G_0 = L * H * L$ is also free. Now let $K_0 = K \cap G_0$; we can apply Theorem 3.2 to the double coset HK_0 in G_0 to see that HK_0 is closed in G_0. Therefore, by Proposition 2.2(ii), HK is closed in G as required. □

4. Fuchsian groups are double coset separable

A *surface group* G is the fundamental group of a surface F. If F has constant negative curvature, G is said to be *hyperbolic*. By the classification of 2-manifolds G is known to be a torsion free Fuchsian group. F has a double cover which is orientable and if F is not closed, then G is free. In particular, every infinite-index subgroup of G is free.

We will need the following technical result in order to show that hyperbolic surface groups are double coset separable.

Proposition 4.1. *Let G be a hyperbolic surface group, and let H, K be finitely-generated infinite-index subgroups of G. Then there are finite-index subgroups $H_0 < H$ and $K_0 < K$ such that $\langle H_0, K_0 \rangle$ is free.*

Proof. Suppose for a contradiction that for any finite-index subgroups $H_0 < H$, $K_0 < K$, the group $\langle H_0, K_0 \rangle$ has finite index in G.

For any nontrivial, finitely-generated subgroup $L < G$ let $d(L)$ denote the minimum number of elements required to generate L, and $\chi(L)$ denote the Euler characteristic of L. If L has finite index in G, then $\chi(L) = 2 - d(L) < 0$, in particular, $\chi(G) < 0$. If $|G : L| = \infty$, then $\chi(L) = 1 - d(L) \leq 0$.

Since surface groups are subgroup separable, H is closed in G and we can choose a descending chain $\{G_n\}$ of finite-index subgroups of G such that $H < G_n$ and $\bigcap_{n \geq 1} G_n = H$ [6, Theorem 3.1]. Set $K_n = K \cap G_n$. Since $\langle H, K_n \rangle$ is a finite-index subgroup of G_n, $\chi(\langle H, K_n \rangle) = |G_n : \langle H, K_n \rangle| \chi(G_n)$, and since G_n has negative Euler characteristic $\chi(G_n) \geq |G_n : \langle H, K_n \rangle| \chi(G_n)$. It follows that

$$\begin{aligned}
\chi(G_n) &\geq \chi(\langle H, K_n \rangle) = 2 - d(\langle H, K_n \rangle) \\
&\geq 2 - d(H) - d(K_n) \\
&= \chi(H) + \chi(K_n) = \chi(H) + |K : K_n| \chi(K) .
\end{aligned}$$

Since $|G : G_n| \geq |K : K_n|$, and $\chi(K) \leq 0$,

$$|K : K_n| \chi(K) \geq |G : G_n| \chi(K) .$$

Combining these observations we get

$$\chi(G) = \frac{\chi(G_n)}{|G : G_n|} \geq \frac{\chi(H)}{|G : G_n|} + \chi(K) .$$

Since the index $|G : G_n|$ is unbounded we deduce that $\chi(G) \geq \chi(K)$. By symmetry $\chi(G) \geq \chi(H)$, but we may apply exactly the same argument to the groups G_n, K_n and H to deduce that $\chi(G_n) \geq \chi(H)$ for all n and so

$$\chi(G) = \frac{\chi(G_n)}{|G : G_n|} \geq \frac{\chi(H)}{|G : G_n|} .$$

It follows that $\chi(G) \geq 0$, which is a contradiction. □

Corollary 4.2. *Surface groups are double coset separable.*

Proof. Let G be the fundamental group of the surface F; F is said to be *closed* if it is compact with empty boundary. If F is not closed, then G is free so it is double coset separable by Corollary 3.2. Otherwise G has an index-2 subgroup which is the fundamental group of a closed orientable surface, so we may as well assume that G is itself the fundamental group of a closed orientable surface. In this case G is either free abelian of rank 2 or it is a hyperbolic surface group. If G is abelian, then it is double coset separable by the result in [3]. We now consider the case when G is hyperbolic.

Let H, K be finitely-generated subgroups of G; if say H has finite index in G, then HK is a finite union of cosets of H and so is closed, so we may as well assume that both H and K have infinite index in G. By Proposition 4.1 there are finite-index subgroups $H_0 < H$, $K_0 < K$ such that $L = \langle H_0, K_0 \rangle$ is an infinite-index subgroup of G. L is free, so the double coset H_0K_0 is closed in L by Corollary 3.3. But since G is subgroup separable [6, Theorem 3.2] and L is finitely generated, the profinite topology on L is precisely the subgroup topology inherited from G (Proposition 2.2(i)) so, H_0K_0 is closed in G. Now by Proposition 2.2(ii) HK is closed in G. □

Now suppose that G is a finitely-generated Fuchsian group. It is well known that G contains a hyperbolic surface group as a subgroup of finite index. Applying Proposition 2.2(iii) we obtain the following:

Corollary 4.3. *Finitely-generated Fuchsian groups are double coset separable.* □

5. Seifert fibred 3-manifolds

This leads to our final example of double coset separable groups.

Corollary 5.1. *If M is a Seifert fibred 3-manifold, then $\pi_1(M)$ is double coset separable.*

Proof. Let $G = \pi_1(M)$. If G is finite, then there is nothing to prove. Otherwise, it is an extension of an infinite cyclic subgroup $\langle t \rangle$ by a group Γ which is either finite, polycyclic-by-finite or Fuchsian; in any case the quotient is certainly double coset separable. For any integer n there is a finite-index subgroup $G_n \leq G$ which is a central extension of $\langle t^n \rangle$ by a finite-index subgroup Γ_n in Γ. (To prove this one can use the fact that $\langle t^n \rangle$ is separable in G, which was proved by Scott in [6].)

$$1 \to \langle t^n \rangle \to G_n \xrightarrow{\phi_n} \Gamma_n \to 1 .$$

Let H and K be finitely-generated subgroups of G and denote by H_n, K_n the finitely-generated subgroups $H \cap G_n$ and $K \cap G_n$ respectively. Since $\langle t \rangle$ is normal in G, the subgroup $K_n\langle t^n \rangle$ is finite index in $K\langle t \rangle$.

Claim. The double cosets $HK\langle t^n \rangle$ are all closed in G.

Proof of claim. The double coset $H_nK_n\langle t^n \rangle$ is the pre-image under ϕ_n of the double coset $\phi_n(H_n)\phi_n(K_n)$. Since $\phi_n(H_n)$ and $\phi_n(K_n)$ are finitely generated this double coset is closed in Γ_n; it follows that $H_nK_n\langle t^n \rangle$ is closed in G_n. But then we can apply Proposition 2.2(ii) to see that $H_1K_1\langle t^n \rangle$ is closed in G.

Now let $Y = \bigcap_n H_1K_1\langle t^n \rangle$; Y is closed in G since it is an intersection of closed sets, and we will show that $Y = H_1K_1$; it will follow that H_1K_1 is closed in G so by Proposition 2.2(ii) HK is closed in G as required.

Since $\langle t\rangle$ is central in G_1, $H_1K_1 \cap \langle t\rangle$ is a subgroup of $\langle t\rangle$: if $hk \in \langle t\rangle$, then k commutes with hk and hence with h. So $\langle t\rangle$ contains $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1} \in H_1K_1$; if hk, h_1k_1 are in $\langle t\rangle$ and $x = (hk)(h_1k_1)$, then $x = h(kk_1)h_1$ as $[k_1, h_1] = 1$, and as $h(kk_1)h_1$ is central h_1 commutes with it, and so with $h(kk_1)$. So $x = (hh_1)(kk_1) \in H_1K_1$ as required. If $H_1K_1 \cap \langle t\rangle = \langle t^p\rangle$ for some nonzero integer p, then $H_1K_1 = H_1K_1\langle t^p\rangle$, so $Y = H_1K_1$ as required. We now assume that $H_1K_1 \cap \langle t\rangle = 1$.

Suppose that $hkt^r \in Y$. Thus for each positive integer n there is an integer m and there are elements h_n, k_n such that $hkt^r = h_nk_nt^{mn}$. Using the fact that $\langle t\rangle$ is central in G we can rearrange this to get $t^{r-mn} \in H_1K_1 \cap \langle t\rangle$ so $r - mn = 0$. Hence every positive integer n divides r and $r = 0$ as required. $\square$

We finish with an example to show that the converse to Theorem 3.2 is not true.

Example. Let G denote the $(4, 4, 2)$ triangle group, with presentation

$$\langle a, b, c \mid a^2 = b^2 = c^4 = abc = 1\rangle .$$

G is polycyclic by finite, so by [3] all of its double cosets are closed. On the other hand $G *_{\langle c\rangle} G$ is not subgroup separable as is shown in [4, Theorem 4].

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