Endofinite modules and pure semisimple rings

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Abstract

Let $R$ be a right pure semisimple ring, i.e., a ring $R$ such that every right $R$-module is a direct sum of finitely generated modules. It is proved that $R$ is of finite representation type if and only if every finitely presented (indecomposable) right $R$-module is endofinite, if and only if every finitely presented right $R$-module has a left artinian endomorphism ring. As applications, we obtain an alternative proof of the pure semisimplicity conjecture for PI-rings, and new criteria for a right pure semisimple ring to be of finite representation type.

Keywords: Endofinite module; Right pure semisimple ring; Ring of finite representation type

1. Introduction

Let $R$ be an associative ring with identity. $R$ is called right pure semisimple if every right $R$-module is a direct sum of finitely generated right $R$-modules. $R$ is said to be of finite representation type if it is right (and left) artinian and has only finitely many non-isomorphic finitely generated indecomposable right (and left) $R$-modules. It is well-known that a ring $R$ is of finite representation type if and only if it is right and left pure semisimple
The problem of whether right pure semisimple rings are of finite representation type, known as the pure semisimplicity conjecture, still remains open. The reader is referred to [23] and [37] for the history and basic background on the conjecture. So far the conjecture has been verified for several classes of rings satisfying some commutativity condition, namely for artin algebras (Auslander [3]), hereditary PI-rings (Simson [35]), arbitrary PI-rings and rings with Morita self-duality (Herzog [20]). In the general case, the lack of a “good duality” between (finitely generated) right and left \( R \)-modules seems to be the crucial difficulty for solving the problem. Recent work of Simson (see, for example, [36–38]) shows a close connection between potential classes of counter-examples to the conjecture and a generalized Artin’s problem on division ring extensions.

A useful approach in studying right pure semisimple rings is based on an analysis of the existence of almost split morphisms in the category \( \text{fp}(R) \) of finitely presented right \( R \)-modules, and the endofiniteness of certain classes of \( R \)-modules. Following [5], a right \( R \)-module \( M \) is called endofinite if it is of finite length as a left module over its endomorphism ring. It has been observed that rings of finite representation type can be characterized by the property that every right module is endofinite (see [24,27]). On the other hand, if \( R \) is right pure semisimple, then the class of finitely generated right \( R \)-modules has left almost split morphisms if and only if \( R \) is of finite representation type (see [2,3,22]). Note that, for an indecomposable right module over a right pure semisimple ring \( R \), being the source of a left almost split morphism in \( \text{fp}(R) \) implies the endofiniteness [10, Proposition 3.18]. The converse statement, in general, would imply the validity of the pure semisimplicity conjecture (see Remark 4.2). It is thus of interest to examine conditions for an indecomposable endofinite module to be the source of a left almost split morphism. A closely related question is to ask what happens if all or a certain family of finitely presented indecomposable \( R \)-modules are endofinite. Recall that, by a result due to Herzog [19], all finitely presented \( R \)-modules over a right pure semisimple ring \( R \) are endofinite.

One of the points of our paper is to give a complete proof of the result that, for a right pure semisimple ring \( R \), the endofiniteness of all finitely presented indecomposable right \( R \)-modules implies finite representation type. Since finitely presented right modules over an artin algebra are endofinite, this result can be regarded as a generalization of Auslander’s well-known theorem on right pure semisimple artin algebras [3]. Among the applications, we also obtain an alternative proof of the pure semisimplicity conjecture for PI-rings [20].

Our paper is inspired by Gruson’s work [18], where he studies conditions for certain coherent objects in the functor category of an artinian ring to have nonzero socles and from which the above theorem can also be deduced. Published in 1975, Gruson’s paper [18] did not seem to be widely used by experts later on, probably because many of the results and statements in the paper are given either without proofs, or with only sketched proofs. We will develop further Gruson’s ideas for more general contexts, and present several applications to right pure semisimple rings.

Our paper is organized as follows. In Section 2, we present general results on finitely presented injective modules \( M \) over a right locally coherent ring \( S \) with enough idempotents. The category \( \text{Mod}(S) \) will correspond to the functor category of Grothendieck categories or unitary rings, studied in subsequent sections. The main result gives conditions for such a module \( M \) (or more generally, a torsionfree module relative to \( M \)) to have an essential socle. In Section 3, we discuss the existence of left almost split morphisms in
a (pure semisimple) Grothendieck category with all finitely presented objects endofinite. In Section 4, we obtain various new criteria for a right pure semisimple ring to be of finite representation type. Apart from results mentioned above, we show that if $M$ is the (finite) direct sum of all non-isomorphic preprojective right modules over a right pure semisimple ring $R$, and $E = \text{End}(M_R)$, then $R$ is of finite representation type if and only if $E$ is a left artinian ring with left Morita duality. Also, a ring $R$ is of finite representation type if and only if $R$ is right pure semisimple and the Krull–Gabriel dimension of the functor category $D(\text{Mod}(R))$ is at most one, if and only if every pure-projective right $R$-module is endo-artinian.

We refer the reader to [1,39,40] for general properties of rings, modules and categories, and for all undefined notions used in the text.

2. Endofinite modules over rings with enough idempotents

The results in this section develop, in a more general context, the ideas suggested by Gruson [18, Proposition 1 and Corollary].

Recall that $S$ is a ring with enough idempotents if there is a family of pairwise orthogonal idempotents $\{e_\lambda\}_{\lambda \in \Lambda}$ in $S$, so that $S = \bigoplus_{\lambda \in \Lambda} e_\lambda S = \bigoplus_{\lambda \in \Lambda} Se_\lambda$ (see, e.g., Fuller [12]). A right $S$-module will always mean a unitary right $S$-module, and $\text{Mod}(S)$ will denote the category of unitary right $S$-modules. The ring $S$ is right locally coherent if every finitely generated submodule of a finitely presented right $S$-module is finitely presented. Throughout this section, we shall always assume that $S$ is a right locally coherent ring with enough idempotents. Also, $M$ will be a finitely presented injective right $S$-module with a left artinian endomorphism ring $E$. $M$ cogenerates a hereditary torsion theory of the category $\text{Mod}(S)$. Thus a module $X$ is torsion if $\text{Hom}(X, M) = 0$. We shall denote the torsion radical as $t$, and when we speak of torsion modules, torsion free modules, dense submodules, closed (or saturated) submodules, etc., we always refer to this torsion theory.

A right $S$-module $X$ will be said to be finitely $M$-presented if there is an exact sequence

$$M^p \to M^q \to X \to 0$$

where $p$ and $q$ are integers.

Next we fix a module that depends on $M$. Let $J = J(E)$ be the Jacobson radical of $E$, then $\text{Ext}^1_E(J, M)$ will be a right $S$-module. We write $N_M = \text{Ext}^1_E(J, M)$.

Following Faith [11], we call a right $S$-module $L$ finendo if $\text{Hom}(Q, L)$ is finitely generated over the endomorphism ring of $L$, for each finitely generated projective right $S$-module $Q$. In case of unitary rings, a module is finendo if and only if it is finitely generated over its endomorphism ring [11, p. 9]. Recall that a right $S$-module $M$ is endofinite if, for each finitely presented right $S$-module $X$, $\text{Hom}(X, M)$ is of finite length as a module over the endomorphism ring of $M$ (see Crawley-Boevey [5,6]).

We now obtain a series of lemmas, under the general hypotheses and notations just stated. In the next result, $\text{Hom}(\_, M)$ will be denoted as $(-)^*$. 
Lemma 2.1. Let \( L_S \) be finitely \( M \)-presented by a morphism \( \alpha: M^p \rightarrow M^q \), and let \( \alpha^*: E^q \rightarrow E^p \) have cokernel \( K \). Then \( \tau(L) \cong \text{Ext}_E^1(K, M) \). In particular, \( N_M = \text{Ext}_E^1(\frac{L}{J}, M) \) is a torsion submodule of a finitely presented right \( S \)-module.

Proof. Let \( X = \text{Im}(\alpha) \subseteq M^q \), then we have an epimorphism \( M^p \rightarrow X \rightarrow 0 \) and a short exact sequence

\[
0 \rightarrow X \rightarrow M^q \rightarrow L \rightarrow 0.
\]

By applying the exact functor \( \text{Hom}(\cdot, M): \text{Mod}(S) \rightarrow \text{Mod}(E^{op}) \), we get two short exact sequences in \( \text{Mod}(E^{op}) \)

\[
0 \rightarrow L^* \rightarrow E^q \rightarrow X^* \rightarrow 0,
\]

\[
0 \rightarrow X^* \rightarrow E^p \rightarrow K \rightarrow 0.
\]

Since \( M \) is a left \( E \)-module, we get an exact sequence from the last one

\[
0 \rightarrow K^* \rightarrow M^p \rightarrow X^{**} \rightarrow \text{Ext}_E^1(K, M) \rightarrow 0,
\]

and from the first sequence above we obtain a monomorphism \( X^{**} \rightarrow M^q \). Consequently, there is a commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
M^p & X^{**} & \text{Ext}_E^1(K, M) & 0 \\
\downarrow & \downarrow & \downarrow \\
M^p & X^{**} & L & 0.
\end{array}
\]

We have that \( X^{**} = X_c \), the closure (or saturation) of \( X \) inside \( M^q \) with respect to the torsion theory cogenerated by \( M \), by [16, Proposition 1.1(ii)]. This means that \( \tau(M^q/X) = X^{**}/X \). But this says exactly that \( \tau(L) \cong \text{Ext}_E^1(K, M) \).

Finally, since \( E \) is left artinian, there is a surjective \( E \)-homomorphism \( E^k \rightarrow J \), giving an \( E \)-homomorphism \( \alpha: E^k \rightarrow E \) with cokernel \( \frac{E}{J} \). The corresponding \( S \)-homomorphism \( M \rightarrow M^k \) has a cokernel which we shall denote as \( N \). Then \( N \) is finitely \( M \)-presented and it follows, by the first part of the proof, that \( \tau(N) \cong N_M = \text{Ext}_E^1(\frac{L}{J}, M) \). This proves the last sentence of the lemma. \( \square \)

Recall that, if \( \mathcal{C} \) is any family of objects in an additive category \( \mathcal{A} \), \( \text{add}\mathcal{C} \) denotes the full subcategory of \( \mathcal{A} \) consisting of all objects isomorphic to direct summands of finite direct sums of objects in \( \mathcal{C} \).

Lemma 2.2. If \( U \) is any simple left \( E \)-module, then \( \text{Ext}_E^1(U, M) \) is isomorphic to a direct summand of \( N_M \). Consequently, if \( E \) \( K \) is semisimple of finite length, then \( \text{Ext}_E^1(K, M) \) is isomorphic to an object in the category \( \text{add}(N_M) \).
Proof. It is clear that if $U$ is simple, then $U$ is isomorphic to a direct summand of the semi-simple left $E$-module $E$. Since the functor $\text{Ext}^1_E(-, M)$ is additive, the result is obvious. The second part follows similarly. □

Lemma 2.3. Assume that $N_M$ is finendo as a right $S$-module, and let $P$ be any finitely generated projective right $S$-module. Then, for each integer $n$, there exists a sequence of homomorphisms between finitely generated projective right $S$-modules

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 = P$$

with each homomorphism $w_i : P_{i+1} \rightarrow P_i$ (for $i = 0, \ldots, n - 1$), and each composition $u_i = w_0 \circ w_1 \circ \cdots \circ w_i : P_{i+1} \rightarrow P$, satisfying the following two properties.

1. The cokernel of $u_i$ is a torsion module.
2. If $f : P_i \rightarrow N_M$ is any homomorphism, then $f \circ w_i = 0$.

Proof. By Lemma 2.1, $N_M$ is a torsion submodule of a finitely presented right $S$-module. Therefore, it will be enough to prove the result for any finendo torsion submodule $L$ of a finitely presented right $S$-module $U$.

Let $Q$ be any finitely generated projective right $S$-module. We know that $\text{Hom}(Q, L)$ is finitely generated over the endomorphism ring of $L$. Let us denote by $R_L(Q)$ the reject of $L$ in $Q$. By this finiteness condition, we see that $R_L(Q)$ is the kernel of some homomorphism from $Q$ to $L^s$, for some integer $s$. Now, the image of this homomorphism is a finitely generated submodule of the finitely presented module $U^s$, hence it is finitely presented, implying that $R_L(Q)$ is finitely generated. Moreover, $R_L(Q)$ is a dense submodule of $Q$, that is, the quotient $\frac{Q}{R_L(Q)}$ is torsion.

If we apply this construction to $P = P_0$, we have that $R_L(P)$ is finitely generated, hence there is an epimorphism $P_1 \rightarrow R_L(P)$ for some finitely generated projective right $S$-module $P_1$. This gives a homomorphism $w_0 : P_1 \rightarrow P_0$ such that $\text{Im}(w_0) = R_L(P)$. Setting $u_0 = w_0$, then clearly the two conditions of the lemma are satisfied for $i = 0$, by the construction. We now assume that there exists the stated sequence up to $u_{i-1}$ and we shall make the next step.

Thus, suppose that $u_{i-1} : P_i \rightarrow P_0$ and $w_{i-1} : P_i \rightarrow P_{i-1}$ are defined and with the given properties. We take an epimorphism $P_{i+1} \rightarrow R_L(P_i)$, where $P_{i+1}$ is a finitely generated projective right $S$-module, and we have in this way a homomorphism $w_i : P_{i+1} \rightarrow P_i$, with the image $R_L(P_i)$ that is dense in $P_i$. Let us define also $u_i = u_{i-1} \circ w_i : P_{i+1} \rightarrow P_0$. We have to check the two properties of the lemma for these homomorphisms.

1. The image of $w_i$ is dense in $P_i$, and the image of $u_{i-1}$ is dense in $P_0$, by the induction hypothesis. But a dense submodule of a dense submodule is dense, and this shows condition (1) for $u_i$.
2. Let $f : P_i \rightarrow L$ be any homomorphism. Its kernel contains $R_L(P_i)$ which is the image of $w_i$. Therefore $f \circ w_i = 0$. □
For the next lemma, observe that because $E = \text{End}(M_S)$ is left artinian, its Jacobson radical $J(E)$ is a nilpotent ideal.

**Lemma 2.4.** Suppose that $n$ is the nilpotency index of $J = J(E)$. Given any finitely $M$-presented right $S$-module $L$, there exists a sequence of homomorphisms between right $S$-modules

$$T_n = 0 \to T_{n-1} \to \cdots \to T_1 \to T_0 \cong t(L)$$

such that the cokernel $W_i$ of each homomorphism $\phi_{i+1} : T_{i+1} \to T_i$ embeds in a module of $\text{add}(N_M)$.

**Proof.** By Lemma 2.1 we know that there exists a finitely generated left $E$-module $U$ such that $t(L) \cong \text{Ext}^1_E(U, M)$. For each $i = 0, 1, \ldots, n$, let us set $T_i = \text{Ext}^1_E(U, M)$. Thus $T_0 = 0$ and $T_0 \cong t(L)$. The canonical $E$-homomorphism $U \to T_i$ gives, by applying the contravariant Ext-functor, the $S$-homomorphism $\phi_{i+1} : T_{i+1} \to T_i$. Moreover, the short exact sequence in $\text{Mod}(S)$

$$0 \to \frac{U}{J_{n-i}U} \to \frac{U}{J_{n-i}U} \to \frac{U}{J_{n-i-1}U} \to 0$$

induces the corresponding exact sequence in $\text{Mod}(S)$

$$\text{Ext}^1_E\left(\frac{U}{J_{n-i-1}U}, M\right) \to \text{Ext}^1_E\left(\frac{U}{J_{n-i}U}, M\right) \to \text{Ext}^1_E\left(\frac{J_{n-i-1}U}{J_{n-i}U}, M\right).$$

This means that the cokernel $W_i$ of the homomorphism $\phi_{i+1} : T_{i+1} \to T_i$ embeds in $\text{Ext}^1_E\left(\frac{J_{n-i-1}U}{J_{n-i}U}, M\right)$ which is of the form $\text{Ext}^1_E(K, M)$, where $K$ is a finitely generated semisimple left $E$-module. By Lemma 2.2, it follows that $W_i$ embeds in a module of $\text{add}(N_M)$. $\square$

The next lemma is the key one.

**Lemma 2.5.** Assume that $N_M$ is finendo as a right $S$-module and let $n$ be the nilpotency index of $J = J(E)$. Let $P$ be a finitely generated projective right $S$-module, and let $P_n \to P_{n-1} \to \cdots \to P_0 = P$ be the sequence of homomorphisms between finitely generated projective right $S$-modules as given in Lemma 2.3. Suppose that $L$ is any finitely $M$-presented right $S$-module, and $f : P \to t(L)$ is any homomorphism. Then $f \circ u_{n-1} = 0$.

**Proof.** We have, for the given module $L$, a sequence of homomorphisms between right $S$-modules $\phi_{i+1} : T_{i+1} \to T_i$, with $i = 0, \ldots, n - 1$, as described in Lemma 2.4. Let $f : P \to t(L)$ be any homomorphism. We claim that, for each $i = -1, 0, \ldots, n - 1$, the homomorphism $f \circ u_i : P_{i+1} \to t(L)$ factors through the module $T_{i+1}$. For convenience of notation, let $u_{-1} = 1_{P_0}$, $g_{-1} = f$, $\phi_0 = 1_{T_0}$. We will prove the claim by induction. The case
Assume that $S$ is a finitely presented injective right $S$-module with a left artinian endomorphism ring $E$. Assume that $N_M = \Ext^1_E(\frac{P}{\mathfrak{n}P}, M)$ is finendo as a right $S$-module. Let $L$ be a finitely presented and torsionfree right $S$-module that satisfies the following condition.

Proof. We start by showing the property in the case when $L = P$ is finitely generated projective. We construct the sequence $P_n \to \cdots \to P = P_0$ as in Lemma 2.5, and let $C = \Im(u_{n-1})$. We know that $C$ is dense in $P$. Now, let $X$ be any dense submodule of $P$ such that $\frac{P}{X}$ embeds in some finitely $M$-presented module $K$. Consequently, $\frac{P}{X}$ embeds in $\Im(\frac{P}{\mathfrak{n}P})$. Let $f$ be the canonical epimorphism from $P$ onto $\frac{P}{X}$, then by Lemma 2.5, we have that $f(C) = 0$. Thus $C \subseteq X$.

In the general case, $L$ is a quotient of some finitely generated projective right $S$-module $P$, so that we may assume that $L = \frac{P}{T}$. By the above, there is a dense submodule $H$ of $P$ having the properties of the statement. Then $\frac{H+U}{U}$ is clearly a dense submodule of $\frac{P}{T}$. Now, let $U \subseteq Y \subseteq P$ and $\frac{Y}{T}$ be a dense submodule of $\frac{P}{T}$, so that the quotient $\frac{Y}{T}$ is contained in a finitely $M$-presented module. Then $Y$ is a dense submodule of $P$ and the first part of the proof implies that $Y$ contains $H$. Therefore $\frac{Y}{T}$ contains $\frac{H+U}{U}$, and we are done. □

We now give the central result of this section.

Theorem 2.7. Let $S$ be a right locally coherent ring with enough idempotents, and $M$ a finitely presented injective right $S$-module with a left artinian endomorphism ring $E$. Assume that $N_M = \Ext^1_E(\frac{P}{\mathfrak{n}P}, M)$ is finendo as a right $S$-module. Let $L$ be a finitely presented and torsionfree right $S$-module that satisfies the following condition.
• For every finitely generated submodule $U \subseteq L$, there exists a finitely generated submodule $X \subseteq U$ (with $X \neq 0$ if $U \neq 0$) such that the quotient $\frac{L}{X}$ embeds in a finitely $M$-presented module.

Then $L$ has an essential socle.

Proof. If we apply the condition above for $U = 0$, then we see that $L$ embeds in a finitely $M$-presented module, and since $E$ is left artinian, it follows easily that $\text{Hom}(L,M)$ is of finite length as a left module over the endomorphism ring $E$ of $M$. By applying [1, Corollary 4.2], which also holds for rings with enough idempotents, there is an order-inverting bijection, given by annihilators, between the lattices of all finitely closed submodules of $L$ and of all finitely generated $E$-submodules of $\text{Hom}(L,M)$. But these are all the $E$-submodules of $\text{Hom}(L,M)$. On the other hand, if $U$ is a closed submodule of $L$, then again the fact that $\text{Hom}(L,M)$ is of finite length as a left $E$-module implies that $U$ is also finitely closed. Hence, the above bijection is a bijection between all the closed submodules of $L$ and all the $E$-submodules of $\text{Hom}(L,M)$.

Since every proper submodule of $\text{Hom}(L,M)$ is contained in a maximal submodule, we have that every nonzero closed submodule of $L$ contains a minimal closed submodule. Now, let $X$ be any minimal closed submodule of $L$. This implies that every nonzero submodule of $X$ is a dense submodule. Since $X$ is finitely closed and $L$ is finitely presented, $\frac{L}{X}$ embeds as a finitely generated submodule of a finite direct sum of copies of $M$, and because $S$ is right locally coherent, we have that $X$ is finitely generated. Consequently, by Corollary 2.6, there exists a nonzero submodule $C$ of $X$ which is contained in every nonzero submodule $K$ of $X$ such that $\frac{X}{C}$ embeds in a finitely $M$-presented module. But if $U$ is any nonzero finitely generated submodule of $X$, then our hypothesis implies that there exists a nonzero submodule $K \subseteq U$ such that $C \subseteq K$. This means that $C$ is contained in every finitely generated nonzero submodule of $X$. Therefore $C$ is indeed the smallest of all nonzero submodules of $X$. It follows that $C$ is simple and essential in $X$.

Let $K$ be any nonzero submodule of $L$, and suppose that $K$ does not contain any simple submodule. This implies that $K$ has zero intersection with all the minimal closed submodules of $L$. If $K'$ is the closure of $K$, we know that $K'$ contains some minimal closed submodule of $L$. But $K$ is essential in $K'$, which gives a contradiction. This shows that the socle of $L$ is indeed essential in $L$. \(\square\)

The hypothesis that $N_M = \text{Ext}_E^1(\frac{E}{J(E)}, M)$ is finendo as a right $S$-module seems a bit awkward. Thus we give now some sufficient conditions for this property to hold.

Proposition 2.8. Let $S$ be a right locally coherent ring with enough idempotents. Let $M_S$ be a finitely presented injective endofinite right $S$-module, and $E = \text{End}(M_S)$. Consider the following conditions.

1. Every finitely $M$-presented right $S$-module is endofinite.
2. The ring $E$ has a left Morita duality.
3. $N_M = \text{Ext}_E^1(\frac{E}{J(E)}, M)$ is finendo as a right $S$-module.
4. $M$ has an essential socle.
Then (1) ⇒ (3) and (2) ⇒ (3). Furthermore, if every finitely presented quotient of $M_S$ embeds in a finitely $M$-presented module, then (3) ⇒ (4).

Proof. (1) ⇒ (3) Suppose that (1) holds. By Lemma 2.1, there is a finitely $M$-presented right $S$-module $N$ such that $\tau(N) = N_M$. By hypothesis, $N$ is endofinite. Hence, if $Q$ is finitely generated projective, $\text{Hom}_S(Q, N)$ is of finite length over the endomorphism ring $D$ of $N$. But $\text{Hom}(Q, N_M)$ is clearly a $D$-submodule of $\text{Hom}(Q, N)$, since $\tau(N)$ is invariant under the endomorphisms of $N$. Moreover, every submodule of $\text{Hom}(Q, N_M)$ over the endomorphism ring of $N_M$ is a $D$-submodule, for the same reason. Hence $\text{Hom}(Q, N_M)$ is of finite length over the endomorphism ring of $N_M$. Therefore $N_M$ is finendo as a right $S$-module.

(2) ⇒ (3) Since $M$ is finitely presented and endofinite, $\text{Hom}(M, M)$ is of finite length over $E = \text{End}(M_S)$, hence $E$ is left artinian. Set $J = J(E)$. If (2) holds, there is a finitely generated injective left $E$-module $E_U$, which is an injective cogenerator in $\text{Mod}(E^{op})$. But $\text{Hom}_E(J, U)$ is finitely generated as a left $E$-module whenever $\text{Ext}^1_E(E_J, X)$ is finitely generated as a left $E$-module.

We will compute $\text{Hom}_S(eS, N_M)$, where $e = e_\lambda$ for some $\lambda \in \Lambda$. Starting with the exact sequence of $E$-$S$-bimodules

$$M \to \text{Hom}_E(J, M) \to N_M \to 0$$

and applying the exact functor $\text{Hom}_S(eS, -)$, we see that $\text{Hom}_S(eS, N_M)$ is the cokernel of the morphism of left $E$-modules $Me \to \text{Hom}_E(J, M)e$. Note that $\text{Hom}_E(J, M)e \cong \text{Hom}_E(J, Me)$ again as left $E$-modules. Therefore $\text{Hom}_S(eS, N_M)$ is isomorphic to $\text{Ext}^1_E(E_J, Me)$. But $Me \cong \text{Hom}_S(eS, M)$ is finitely generated over the ring $E$, since $M$ is endofinite. This shows that $\text{Hom}_S(eS, N_M)$ is finitely generated as a left $E$-module. It follows that $\text{Hom}_S(Q, N_M)$ is finitely generated as a left $E$-module for any finitely generated projective right $S$-module $Q$, as $Q$ is a direct summand of a finite direct sum of modules of the type $eS$.

Now, the right $S$-module $N_M$ is a left module both over $E$ and over its own endomorphism ring. The existence of a canonical homomorphism $E \to \text{End}_S(N_M)$ induced by this structure implies that $\text{Hom}(Q, N_M)$ is also finitely generated as a module over the endomorphism ring of $N_M$. This proves that $N_M$ is finendo as a right $S$-module.

Finally, we show that (3) ⇒ (4) under the given hypothesis. Indeed, since every finitely presented quotient of $M_S$ is of the form $M/U$, where $U$ is a finitely generated submodule of $M_S$, it follows immediately from Theorem 2.7 that $M$ has an essential socle. □
3. Grothendieck categories with endofinite finitely presented objects

In this section, \( \mathcal{A} \) will be a locally finitely presented Grothendieck category, with its associated functor category \( \mathcal{D}(\mathcal{A}) \), as defined in [33,34] (cf. [6,9]). \( \text{fp}(\mathcal{A}) \) will denote the subcategory of finitely presented objects of \( \mathcal{A} \). There is a fully faithful additive functor \( T: \mathcal{A} \to \mathcal{D}(\mathcal{A}) \), such that \( T \) carries pure-injective objects of \( \mathcal{A} \) to injective objects of \( \mathcal{D}(\mathcal{A}) \), and carries finitely presented objects of \( \mathcal{A} \) to finitely presented objects of \( \mathcal{D}(\mathcal{A}) \).

Following [33], a category \( \mathcal{A} \) is pure semisimple if every object of \( \mathcal{A} \) is pure-injective, or equivalently, if every object of \( \mathcal{A} \) is a direct sum of finitely presented objects. According to [9], \( \mathcal{A} \) is said to be a category of locally finite representation type if every finitely presented object of \( \mathcal{A} \) is endofinite, and for each finitely presented object \( M \) of \( \mathcal{A} \), there are only finitely many isomorphism classes of finitely presented indecomposable objects \( X \) of \( \mathcal{A} \) such that \( \text{Hom}(M, X) \neq 0 \). Any Grothendieck category of locally finite representation type is locally finite, and pure semisimple (see [9, Theorem 4.2, Proposition 3.3]). In case \( \mathcal{A} \) is the module category over a ring with identity, this gives the usual definition of a ring of finite representation type.

Let \( C = \{ M_i | i \in I \} \) be a family of finitely generated objects of a category \( \mathcal{A} \). For an indecomposable object \( M \) and an object \( N \) in \( \text{add} C \), a morphism \( f: M \to N \) is called a left almost split morphism in \( \text{add} C \) provided \( f \) is not a split monomorphism, and for any object \( K \) in \( \text{add} C \) and a morphism \( g: M \to K \) that is not a split monomorphism, there is a morphism \( h: N \to K \) such that \( g = h \circ f \). If there is a left almost split morphism \( f: M \to N \) in \( \text{add} C \) for each indecomposable object \( M \) in \( \text{add} C \), then we say that the family \( C \) has left almost split morphisms.

We now introduce a new notion. Given two finitely presented objects \( M \) and \( L \) of the category \( \mathcal{A} \), let us say that \( M \) is an almost \( L \)-generator when the following condition holds: For every finitely presented object \( K \), and every morphism \( f: L \to K \) that is not a split monomorphism, there exist a finitely \( M \)-presented object \( X \) and morphisms \( g: L \to X \) and \( h: X \to K \) such that \( f = h \circ g \) and \( g \) is not a split monomorphism. Recall that \( X \) is finitely \( M \)-presented if there is an exact sequence \( M^p \to M^q \to X \to 0 \) where \( p \) and \( q \) are integers. Note that if \( M \) is a generator, then clearly \( M \) is an almost \( L \)-generator for any finitely presented object \( L \). But also if there is a left almost split morphism \( L \to M \), then \( M \) is an almost \( L \)-generator.

We will need the following lemma, proved in [10, Lemma 2.4] (cf. [2,5]).

**Lemma 3.1.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck category. Let \( \mathcal{D}(\mathcal{A}) \) be the associated functor category of \( \mathcal{A} \), and \( T: \mathcal{A} \to \mathcal{D}(\mathcal{A}) \) be the canonical functor. For a finitely presented object \( M \) of \( \mathcal{A} \) with a local endomorphism ring, \( M \) is the source of a left almost split morphism in \( \text{fp}(\mathcal{A}) \) if and only if \( T(M) \) contains a (finitely presented) simple subobject.

The following result provides a sufficient condition for an indecomposable object of \( \text{fp}(\mathcal{A}) \) to be the source of a left almost split morphism in \( \text{fp}(\mathcal{A}) \).

**Proposition 3.2.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck category, with the associated functor category \( \mathcal{D}(\mathcal{A}) \) and the canonical functor \( T: \mathcal{A} \to \mathcal{D}(\mathcal{A}) \). Let \( M \) be
a finitely presented pure-injective object of \( A \), with a left artinian endomorphism ring \( E = \text{End}_AM \). Assume that \( D(A) \) is equivalent to a module category \( \text{Mod}(S) \), and \( N = \text{Ext}^1_E(\frac{E}{J(E)}, T(M)) \) is finendo as a right \( S \)-module. If an object \( L \) belongs to \( \text{add}(M) \) and \( M \) is an almost \( L \)-generator, then each of the indecomposable summands of \( L \) is the source of a left almost split morphism in \( \text{fp}(A) \).

Proof. First note that \( D(A) \cong \text{Mod}(S) \) is a locally coherent category, where \( S \) is a ring with enough idempotents. Also, \( T(M) \) is a finitely presented injective right \( S \)-module, and \( E \) is isomorphic to the endomorphism ring of \( T(M) \) (see [6]). We will show that \( T(L) \) and \( T(M) \) satisfy the conditions of Theorem 2.7.

We have that \( T(L) \in \text{add}(T(M)) \), hence \( T(L) \) is torsionfree with respect to the torsion theory cogenerated by \( T(M) \). Also, obviously \( T(L) \) is finitely \( T(M) \)-presented. Now let \( U \) be any nonzero finitely generated submodule of \( T(L) \). Then \( \frac{T(L)}{U} \) is finitely presented, hence it embeds in a module of the form \( T(N) \), with \( N \) finitely presented in \( A \) (see, e.g., [9, Corollary 2.7]). This gives a homomorphism \( T(L) \rightarrow T(N) \), with the kernel \( U \). Since the functor \( T \) is full, there is a corresponding morphism \( f : L \rightarrow N \), inducing the above homomorphism. Clearly \( f \) is not a split monomorphism, because otherwise \( T(f) \) would be a split monomorphism, contrary to the fact that it has a nonzero kernel \( U \). By our hypothesis, there exist a finitely presented object \( X \) and morphisms \( g : L \rightarrow X, h : X \rightarrow N \), with \( f = h \circ g \). Moreover, \( g \) is not a split monomorphism and \( X \) is finitely \( M \)-presented.

By applying the functor \( T \), we get a decomposition of the given morphism \( T(L) \rightarrow T(X) \) through the module \( T(X) \). We have that \( T(X) \) is finitely \( T(M) \)-presented, by the right exactness of the functor \( T \). Now, \( T(g) \) is not a split monomorphism, because this would imply that \( g \) is a split monomorphism. As \( T(L) \) is injective, it follows that \( T(g) \) cannot be a monomorphism, and thus it has a nonzero kernel, say \( W \). It is obvious that \( W \subseteq U \). Moreover, the quotient \( \frac{T(L)}{T(L)W} \) embeds in \( T(X) \) which is finitely \( T(M) \)-presented. This shows that the conditions of Theorem 2.7 are indeed fulfilled.

Therefore \( T(L) \) has an essential socle. Since \( E = \text{End}_AM \) is left artinian, \( L \) has a decomposition into indecomposable summands, each with a local endomorphism ring. Combined with Lemma 3.1, it follows that each indecomposable summand of \( L \) is the source of a left almost split morphism in \( \text{fp}(A) \).

Note that examples of Grothendieck categories \( A \) whose functor categories \( D(A) \) are equivalent to module categories include categories of modules over unitary rings, or more generally, categories of modules over rings with enough idempotents (see [9, Theorem 2.9]).

We now focus on categories such that every finitely presented object is endofinite. With this hypothesis, the following corollary gives necessary and sufficient conditions for an indecomposable object of \( \text{fp}(A) \) to be the source of a left almost split morphism in \( \text{fp}(A) \).

**Corollary 3.3.** Let \( A \) be a locally finitely presented Grothendieck category, with the associated functor category \( D(A) \) which is equivalent to a module category \( \text{Mod}(S) \). Assume that every finitely presented object of \( A \) is endofinite. Let \( L \) be a finitely presented indecomposable object of \( A \). Then \( L \) is the source of a left almost split morphism in \( \text{fp}(A) \) if
and only if there exists a finitely presented object $M$ such that $L \in \text{add}(M)$ and $M$ is an almost $L$-generator.

**Proof.** Let $T : \mathcal{A} \to D(\mathcal{A})$ be the canonical embedding functor. The condition that every finitely presented object of $\mathcal{A}$ is endofinite implies that, for any finitely presented object $M$ of $\mathcal{A}$, all finitely $T(M)$-presented right $S$-modules are endofinite. Note that if $M$ is a finitely presented (endofinite) object of $\mathcal{A}$, then $M$ is $\Sigma$-pure-injective in $\mathcal{A}$, and $E = \text{End}_A(M)$ is left artinian. Therefore, the “if” part follows immediately from Proposition 2.8 ($(1) \Rightarrow (3)$), and Proposition 3.2. For the “only if” part, let $L \to X$ be a left almost split morphism in $fp(\mathcal{A})$. Set $M = L \oplus X$. Then it is clear that $M$ is an almost $L$-generator.  

We state now the main result of this section.

**Theorem 3.4.** Let $\mathcal{A}$ be a pure semisimple locally finitely presented Grothendieck category such that its functor category $D(\mathcal{A})$ is equivalent to a module category $\text{Mod}(S)$. Assume that every finitely presented object of $\mathcal{A}$ is endofinite and $\mathcal{A}$ has a finitely presented generator. Then $\mathcal{A}$ is a category of locally finite representation type.

**Proof.** By [10, Proposition 3.15], it is enough to show that every finitely presented indecomposable object $L$ of $\mathcal{A}$ is the source of a left almost split morphism in $fp(\mathcal{A})$. If $H$ is a finitely presented generator of $\mathcal{A}$, then take $M = L \oplus H$ and apply Corollary 3.3.

It would be interesting to know if Theorem 3.4 holds without the finitely presented generator hypothesis. In the case when the given category $\mathcal{A}$ is locally finite, we obtain the following characterization of categories of locally finite representation type. Let us say that a family $\mathcal{C}$ of finitely presented objects of the locally finitely presented Grothendieck category $\mathcal{A}$ is *generably finite* in case there exists a finitely presented object $M$ such that every object of $\mathcal{C}$ is finitely $M$-presented.

**Proposition 3.5.** Let $\mathcal{A}$ be a pure semisimple locally finite Grothendieck category such that its functor category $D(\mathcal{A})$ is equivalent to a module category. Then $\mathcal{A}$ is of locally finite representation type if and only if every finitely presented object of $\mathcal{A}$ is endofinite and for every finitely presented indecomposable object $L$, the family of isomorphism classes of finitely presented indecomposable objects $X$ such that $\text{Hom}(L, X)$ contains a monomorphism is generably finite.

**Proof.** The “only if” part is immediate by the definition of categories of locally finite representation type. To prove the “if” part, by [10, Proposition 3.15], it is enough to show that any indecomposable object $L$ in $fp(\mathcal{A})$ is the source of a left almost split morphism in $fp(\mathcal{A})$. By hypothesis, there exists a finitely presented object $M_0$ such that if there is a monomorphism $L \to X$, with $X$ indecomposable, then $X$ is finitely $M_0$-presented. Also, there is a finite number of isomorphism classes of simple subobjects $S_1, \ldots, S_r$ of $L$. Let $M = M_0 \oplus L \oplus S_1 \oplus \cdots \oplus S_r$. We will show that $M$ is an almost $L$-generator, so that Corollary 3.3 could apply.
Let $f : L \to N$ be a morphism that is not a split monomorphism, with $N$ a finitely presented object. We know that $N$ has an indecomposable decomposition $N = \bigoplus_{i=1}^n X_i$. Let $p_i : N \to X_i$ be the canonical projections, and set $f_i = p_i \circ f : L \to X_i$. If $f_i$ is not a monomorphism, then its kernel contains a simple subobject $S_j$. Then we take $K_i = \frac{L}{S_j}$ and $K_i$ is a finitely $M$-presented object. Moreover, $f_i$ factors through $K_i$ in the form $f_i = h_i \circ g_i$, with $g_i : L \to K_i$ and $h_i : K_i \to X_i$. On the other hand, if $f_i$ is a monomorphism, then we take $K_i = X_i$ and $g_i = f_i$, so that $K_i$ is finitely $M$-presented, and $f_i$ factors through $K_i$, with the first factor being a non-split monomorphism.

This gives a morphism $g : L \to \bigoplus_{i=1}^n K_i$ by adding the morphisms $g_i$. Moreover, we have that $\bigoplus_{i=1}^n K_i$ is finitely $M$-presented, and the morphism $f : L \to N$ factors through $g$. Suppose that $g$ is a split monomorphism, with $k \circ g = 1_L$. Then $k \circ u_i \circ \pi_j \circ g$ is an isomorphism of $L$ for some index $i$, because $\text{End}(L)$ is local, with $u_j$ and $\pi_j$ being the canonical inclusions and projections of the direct sum $\bigoplus_{j=1}^n K_j$. But $\pi_i \circ g = g_i$, which contradicts the fact that $g_i$ is not a split monomorphism. Hence, $g$ is not a split monomorphism, as required. \qed

4. Right pure semisimple rings

Throughout this section, $R$ is an associative ring with identity. We denote by $\text{Mod}(R)$ and $\text{pt}(R)$ the categories of all right $R$-modules, and all finitely presented right $R$-modules, respectively. $R$ is called a Krull–Schmidt ring if every finitely presented right (and left) $R$-module is a finite direct sum of modules with local endomorphism rings. Let $S$ be the functor ring of finitely presented left $R$-modules, or shortly, the left functor ring of $R$ (see, e.g., [12,40]). Then $\text{Mod}(S)$ is equivalent to the functor category $\mathcal{D}(\text{Mod}(R))$ of $\text{Mod}(R)$, so that the results of the preceding section can be applied.

Since finitely presented right modules over an artin algebra are endofinite, the equivalence of (c) and (d) of the following result gives a generalization of Auslander’s theorem [3] that right pure semisimple artin algebras are of finite representation type. As mentioned in the Introduction, this statement can also be obtained from Gruson [18, Corollary], though only a sketched proof was given there. Note that the equivalence of (b) and (d) in the case $R$ is a hereditary ring follows from Simson [35, Corollary 3.2], but with a completely different method. It is well-known that right pure semisimple rings are right artinian (hence finitely generated and finitely presented right $R$-modules are the same). However, as is pointed out in [37], the pure semisimplicity conjecture holds true if and only if every right pure semisimple ring is left artinian.

Theorem 4.1. Let $R$ be a right pure semisimple ring. Then the following conditions are equivalent:

(a) Each finite family of finitely presented indecomposable right $R$-modules has left almost split morphisms.
(b) Each finitely presented right $R$-module has a left artinian endomorphism ring.
(c) Each finitely presented (indecomposable) right $R$-module is endofinite.
(d) $R$ is of finite representation type.
Proof. (a) ⇒ (b) Suppose that (a) holds. Let \( M \) be any finitely presented right \( R \)-module, with \( E = \text{End}(M_R) \). There is an indecomposable decomposition \( M = M_1 \oplus \cdots \oplus M_n \). By (a), the family \( \{M_1, \ldots, M_n\} \) has left almost split morphisms, hence by [7, Lemma 2.3], the Jacobson radical of the left \( E \)-module \( \text{Hom}(M_k, M) \) is finitely generated, for each \( k \). This implies that the Jacobson radical \( J(E) \) of \( E \) is finitely generated as a left \( E \)-module. Note that \( E \) is semiprimary, so it follows easily that \( E \) is left artinian.

(b) ⇒ (c) Suppose that (b) holds, and let \( M \) be any finitely presented right \( R \)-module. Let \( N = R_R \oplus M_R \). Then \( E = \text{End}(N_R) \) is left artinian by hypothesis. Clearly \( N \) is finitely presented as a left \( E \)-module, hence \( N \) is of finite length as a left \( E \)-module, i.e., \( N \) is endofinite. Since the endofiniteness is preserved under taking direct summands [5], this implies that \( M_R \) is endofinite, proving (c).

(c) ⇒ (d) Suppose that each finitely presented indecomposable right \( R \)-module is endofinite. Since the endofiniteness is preserved under finite direct sums [5, Proposition 4.3], each finitely presented right \( R \)-module is endofinite. It follows by Theorem 3.4 that \( R \) is of finite representation type.

(d) ⇒ (a) Suppose that \( R \) is of finite representation type, then it is well-known that every (finitely presented) right \( R \)-module is endofinite (see [24,27]). For any family \( B = \{M_1, \ldots, M_n\} \) of finitely presented indecomposable right \( R \)-modules, let \( M = M_1 \oplus \cdots \oplus M_n \), and \( E = \text{End}(M_R) \). Then \( \text{Hom}(M_k, M) \) is of finite length as a left \( E \)-module, implying that the Jacobson radical of \( \text{Hom}(M_k, M) \) is finitely generated as a left \( E \)-module, for each \( k \). By [7, Lemma 2.3], the family \( B \) has left almost split morphisms. □

Remark 4.2. Let \( R \) be a right pure semisimple ring. If a finitely presented indecomposable right \( R \)-module \( M \) is the source of a left almost split morphism in \( \text{fp}(R) \), then \( M \) is endofinite [10, Proposition 3.18]. It is natural to ask if the converse always holds, i.e., if \( M \) is endofinite, then \( M \) is the source of a left almost split morphism in \( \text{fp}(R) \). Note that if every finitely presented indecomposable endofinite right \( R \)-module is the source of a left almost split morphism in \( \text{fp}(R) \), then \( R \) is of finite representation type (see [10, Proposition 3.17]; cf. [25]). (The referee remarked that this result also follows from Ziegler [41, Theorem 8.6].) Therefore, if the above converse statement holds for all right pure semisimple rings \( R \), it would imply the validity of the pure semisimplicity conjecture, which is believed to fail in general (see Simson’s potential counter-examples, e.g., in [36–38]).

The following result, which is of independent interest, will be useful in the sequel. The special case of modules over unitary rings was observed without a proof by Gruson [18, p. 159].

Proposition 4.3. Let \( A \) be a pure semisimple locally finitely presented Grothendieck category. Then for every finitely presented object \( M \) of \( A \), the endomorphism ring \( \text{End}_A(M) \) of \( M \) is a right artinian ring with right Morita duality.

Proof. Set \( E = \text{End}_A(M) \), then \( E \) is right artinian (see [8, Corollary 4.2(2)], or [17, Corollary 3]). Let \( S \) be the functor ring of \( A \), with the canonical embedding functor \( G : A \to \text{Mod}(S) \) which identifies objects of \( A \) with flat objects of \( \text{Mod}(S) \) (see, e.g., [6,
Then $G$ carries finitely presented objects of $\mathcal{A}$ to finitely generated projective right $S$-modules. Note that $E$ is also the endomorphism ring of the right $S$-module $G(M)$.

Consider the category $\text{Fl}(S^{\text{op}})$ of all flat left $S$-modules. Then $\text{Fl}(S^{\text{op}})$ is equivalent to the category of unitary left modules over a ring with enough idempotents $T$ such that $T$ is right pure semisimple (see [9, Theorem 2.10]). This means that $S^{\text{op}}$ is the functor ring of the category $\text{Mod}(T^{\text{op}})$ of all left $T$-modules. Hence $\text{Mod}(S)$ is equivalent to the functor category $D(\text{Mod}(T))$ of the category $\text{Mod}(T)$ of all right $T$-modules (see [9, Theorem 2.9]). Thus, $\text{Mod}(S)$ has the property that every finitely presented object embeds in a finitely presented FP-injective object [9, Corollary 2.7]. Note that, since $\mathcal{A}$ is a pure semisimple Grothendieck category, $S$ is a right locally noetherian ring [34]. In particular, FP-injective right $S$-modules are injective, and it follows from the above that injective envelopes of finitely generated right $S$-modules are also finitely generated. By [15, Theorem 3.2], if $P$ is a finitely generated projective right module over a ring $R$ such that each simple quotient of $P$ has a noetherian injective envelope, then injective envelopes of simple right $\text{End}(P_R)$-modules are noetherian. Note that the proof of this result in [15] works also in case $R$ is a ring with enough idempotents. In our situation, since $G(M)$ is a finitely generated projective right $S$-module and simple quotients of $G(M)$ are noetherian, it follows that injective envelopes of simple right $E$-modules are noetherian. As $E$ is right artinian, this shows that $E$ has a right Morita duality.

We will need the following characterization of two-sided artinian PI-rings (i.e., rings satisfying a polynomial identity). The "if" and "only if" parts of the result are due to Schmidmeier [31], and Rosenberg and Zelinski [30], respectively.

**Lemma 4.4.** Let $R$ be a right artinian PI-ring. Then $R$ is left artinian if and only if $R$ has a right Morita duality.

**Proof.** If $R$ has a right Morita duality, indecomposable injective right $R$-modules are finitely generated, hence $R$ is left artinian by [31, Corollary 12]. If $R$ is left and right artinian and PI, then $R$ has a right Morita duality by [30, Theorem 3].

As an application of previous results, we give now an alternative proof of the pure semisimplicity conjecture for PI-rings. The result was established first by Simson [35] for hereditary PI-rings or PI-rings $R$ with $J(R)^2 = 0$. Herzog [20] proved the conjecture for any PI-ring. Other proofs were given later by Krause [25] and Schmidmeier [32].

**Corollary 4.5.** Let $R$ be a right pure semisimple PI-ring. Then $R$ is of finite representation type.

**Proof.** Let $M_R$ be any finitely presented right $R$-module, and $E = \text{End}(M_R)$. By [28], $E$ is again a PI-ring. By Proposition 4.3, $E$ is a right artinian ring with right Morita duality. It follows by Lemma 4.4 that $E$ is left artinian. As this holds for every finitely presented right $R$-module $M_R$, it follows from Theorem 4.1 ((b) $\Rightarrow$ (d)) that $R$ is of finite representation type.
Zimmermann [42, Folgerung 9] showed that if $R$ is an artinian ring, then every non-injective indecomposable projective right $R$-module is the first term of an almost split sequence in $fp(R)$ if and only if $R$ has a left Morita duality. Our next proposition, which is inspired by Gruson [18, Proposition 1], can be regarded as an extension of this result. Recall that, if $M$ is a right $R$-module with $E = \text{End}(M_R)$ and $C$ is a minimal injective cogenerator of $\text{Mod}(E^{op})$, then the left $R$-module $\text{Hom}_E(M, C)$ is called the local dual of $M$, and is denoted by $D(M)$ (see, e.g., [24,32]).

**Proposition 4.6.** Let $R$ be any ring and $M$ a finitely presented endofinite right $R$-module. Consider the following conditions.

(i) Every indecomposable direct summand of $M$ is the source of a left almost split morphism in $fp(R)$.

(ii) $E = \text{End}(M_R)$ has a left Morita duality.

Then we have the implication (i) $\Rightarrow$ (ii). Moreover, if $M$ is an almost $M$-generator in $\text{Mod}(R)$, then (ii) $\Rightarrow$ (i).

**Proof.** To prove (i) $\Rightarrow$ (ii), suppose that (i) holds. Let $S$ be the left functor ring of $R$ and $T : \text{Mod}(R) \to \text{Mod}(S)$ be the canonical embedding functor. Let $C$ be a minimal injective cogenerator of $\text{Mod}(E^{op})$. The ring $E$ is clearly left artinian, and to prove that $E$ has a left Morita duality, we need to show that $EC$ is finitely generated. Let $N = D(M_R)$ be the local dual of $M_R$. By the isomorphism

$$\text{Hom}_E(\text{Hom}_R(M, M), C) \cong M \otimes R \text{Hom}_E(M, C)$$

(see, e.g., [39, Exercise I.33, p. 47]), it follows that $C \cong M \otimes R D(M_R)$ as left $E$-modules. Because $EM$ is of finite length, the claim would follow if we could show that $D(M_R)$ is finitely presented as a left $R$-module. Since the local duality commutes with finite direct sums, provided each summand has a right perfect endomorphism ring (see [32, Theorem 1.6]), it is sufficient to consider an indecomposable direct summand $K$ of $M$. By hypothesis, $K$ is the source of a left almost split morphism in $fp(R)$, thus $T(K)$ is the injective envelope of a simple right $S$-submodule by Lemma 3.1, so $K$ is the source of a left almost split morphism in $\text{Mod}(R)$ [5, Theorem 2.3]. By [26, Lemma 4.6], $K = D(N)$ for some finitely presented indecomposable endofinite left $R$-module $N$. It follows that $D(K) = D(D(N))$. By [42, Lemma 5], we have that $D(D(N)) \cong N$. Thus, $D(K)$, being isomorphic to $N$, is finitely presented as a left $R$-module, as required.

Conversely, suppose that (ii) holds, and $M$ is an almost $M$-generator in $\text{Mod}(R)$. By Proposition 2.8 ((2) $\Rightarrow$ (3)), we have that $N_{T(M)} = \text{Ext}^1_{E^{op}}(\frac{K}{T(M)}, T(M))$ is finendo as a right $S$-module. Now an application of Proposition 3.2 yields that each indecomposable direct summand of $M$ is the source of a left almost split morphism in $fp(R)$. \hfill $\Box$

**Remark 4.7.** In the proposition above, the implication (ii) $\Rightarrow$ (i) may not hold in general. Assume that $R$ is a right artinian ring, and $M = eR$ is any simple right $R$-module, where $e$ is a primitive idempotent of $R$. Then $M$ is finitely presented endofinite, and $\text{End}(M_R)$
is a division ring, thus has a left Morita duality. If \( M \) is the source of a left almost split morphism in \( fp(R) \), then the local dual \( D(M) \) of \( M \) is finitely presented as a left \( R \)-module (see, e.g., [10, Proposition 2.5]). Note that \( D(M) \cong \frac{Re}{J(R)e} \), thus \( J(R)e \) is a finitely generated left ideal of \( R \). Hence, if every simple right \( R \)-module is the source of a left almost split morphism in \( fp(R) \), we have that \( J(R) \) is a finitely generated left ideal of \( R \), implying that \( R \) is left artinian. Therefore, if \( R \) is a right artinian ring which is not left artinian, then there exists a simple right \( R \)-module which is not the source of a left almost split morphism in \( fp(R) \) (see also [42, Folgerung 10]).

Let \( R \) be a Krull–Schmidt ring. Following Herzog [20], a finitely presented indecomposable right \( R \)-module \( M \) is called preprojective if there is a finite family \( C \) of non-isomorphic finitely presented indecomposable right \( R \)-modules such that \( M \) is not isomorphic to any module in \( C \), and if \( N \) is a finitely presented right \( R \)-module containing no indecomposable summands isomorphic to some module in \( C \), any epimorphism \( g : N \to M \) splits. Note that if \( R \) is an artin algebra, then the above definition coincides with the concept of preprojective modules, introduced by Auslander and Smalø [4]. If \( R \) is a right pure semisimple ring, then it is known that \( R \) has only finitely many non-isomorphic preprojective right \( R \)-modules (see [20, Corollary 4.3]; cf. [7, Corollary 3.8]). Our next result gives a new criterion for a right pure semisimple ring to be of finite representation type, in terms of the endomorphism ring of the direct sum of all preprojective right \( R \)-modules.

**Theorem 4.8.** Let \( R \) be a right pure semisimple ring. Let \( M \) be the finite direct sum of all non-isomorphic preprojective right \( R \)-modules. Then \( R \) is of finite representation type if and only if \( E = \text{End}(MR) \) is a left artinian ring with left Morita duality.

**Proof.** Suppose first that \( R \) is right pure semisimple, and \( E = \text{End}(MR) \) is a left artinian ring with left Morita duality, where \( M \) is the finite direct sum of all non-isomorphic preprojective right \( R \)-modules. Since each indecomposable projective right \( R \)-module is preprojective, \( R \) belongs to \( \text{add}(M) \) and hence \( M \) is a generator of \( \text{Mod}(R) \). It follows that \( M \) is finitely generated over its endomorphism ring \( E \), which is left artinian by hypothesis, thus \( M \) is endofinite. Now Proposition 4.6 implies that each preprojective right \( R \)-module is the source of a left almost split morphism in \( fp(R) \). By [7, Lemma 3.4], we get that \( R \) is of finite representation type.

Conversely, suppose that \( R \) is a ring of finite representation type. Note that \( M \) is finitely presented and endofinite [24], hence \( E = \text{End}(MR) \) is a left artinian ring. Moreover, it is well-known that left almost split morphisms exist in the category \( fp(R) \) (see, e.g., [2,3], cf. [9, Proposition 4.6]). Now it follows by Proposition 4.6 ((i) \( \Rightarrow \) (ii)) that the ring \( E \) has a left Morita duality.  

The Krull–Gabriel dimension of a Grothendieck category is defined in [14]. Herzog [21, Theorem 3.6] proved that the Krull–Gabriel dimension of the functor category of an artin algebra cannot equal one (cf. Krause [26, Corollary 11.4] for the finite-dimensional algebra case). The next result shows that right pure semisimple rings resemble artin algebras in this aspect. Note that this fact was also stated implicitly without a proof by Gruson [18, p. 159].
We do not know if the result would extend to pure semisimple locally finitely presented Grothendieck categories.

**Proposition 4.9.** Let $R$ be a right pure semisimple ring, with the left functor ring $S$. Suppose that the Krull–Gabriel dimension of the functor category $\text{Mod}(S)$ is at most one. Then $R$ is of finite representation type.

**Proof.** Note that $\text{Mod}(S)$ is equivalent to the functor category $\mathcal{D}(\text{Mod}(R))$ of $\text{Mod}(R)$, with the canonical embedding functor $T: \text{Mod}(R) \to \text{Mod}(S)$. Since $R$ is right pure semisimple, $S$ is right locally noetherian. Let $C_0$ be the hereditary torsion class of all the semiartinian objects of $\text{Mod}(S)$. We know that the Krull–Gabriel dimension of $\text{Mod}(S)$ is at most one if and only if the quotient category $\text{Mod}(S)/C_0$ is locally finite (or zero). Let $p: \text{Mod}(S) \to \text{Mod}(S)/C_0$ be the canonical quotient functor. Suppose now that $M$ is any finitely presented indecomposable right $R$-module. If $M$ is the source of a left almost split morphism in $\text{fp}(R)$, then $M$ is endofinite by [10, Proposition 3.18]. Now assume that $M$ is not the source of a left almost split morphism in $\text{fp}(R)$. Then $T(M)$ does not contain a simple submodule by Lemma 3.1, hence $T(M)$ can be identified as an indecomposable injective object of the quotient category $\text{Mod}(S)/C_0$. Since every injective object of a locally finite category is endofinite (see, e.g., [25, Lemma 3]), it follows that $T(M)$ is endofinite as an object of $\text{Mod}(S)/C_0$. For every finitely generated object $X$ of $\text{Mod}(S)$, we have that $p(X)$ is also finitely generated in $\text{Mod}(S)/C_0$, and $\text{Hom}(X, T(M)) \cong \text{Hom}(p(X), T(M))$ by the adjointness of the localization functor. This implies that $T(M)$ is endofinite as an object of $\text{Mod}(S)$, yielding that $M$ is endofinite as a right $R$-module. Thus every finitely presented indecomposable right $R$-module is endofinite, so it follows by Theorem 4.1 that $R$ is of finite representation type.

Recall that a right $R$-module $M$ is pure-projective if $M$ is a direct summand of a direct sum of finitely presented right $R$-modules, and $M$ is called endo-artinian if it is artinian as a left module over its endomorphism ring. A well-known result, due to Huisgen-Zimmermann and Zimmermann [24] and Prest [27], asserts that a ring $R$ is of finite representation type if (and only if) every right $R$-module is endofinite. We conclude the paper with a generalization of this result as follows.

**Proposition 4.10.** Let $R$ be a ring such that every pure-projective right $R$-module is endo-artinian. Then $R$ is of finite representation type.

**Proof.** Let $M$ be any pure-projective right $R$-module. Since $M$ is endo-artinian, $M$ satisfies the DCC on subgroups of finite definition [5], hence $M$ is $\Sigma$-pure-projective. In particular, every pure-projective right $R$-module has an indecomposable decomposition that complements direct summands, implying that $R$ is right pure semisimple (see, e.g., [12]). Now, let $N$ be any finitely presented right $R$-module, with the endomorphism ring $E$, then $E$ is semiprimary. Consider the finite composition series $N \supseteq J(E)N \supseteq J(E)^2N \supseteq \cdots \supseteq J(E)^mN = 0$, for some integer $m$. Since each quotient $\frac{J(E)^iN}{J(E)^{i+1}N}$ is finitely generated semisimple as a left $E$-module, it follows that $EN$ is of finite length. Therefore, every
finitely presented right $R$-module is endofinite, so $R$ is of finite representation type by Theorem 4.1. □

References