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Translation Planes of Order q^3 Which Admit $SL(2, q)$

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Kantor [4, No. 7] describes a new class of translation planes of order q^3, $q = 2^{2\omega+1} \equiv 2 \pmod 3$, with kernel $GF(q)$. A plane Π_1 in this class has several interesting properties, but we shall be only interested in the following one possessed by its collineation group: the linear translation complement of Π_1 contains a normal subgroup $G \simeq SL(2, q)$ whose involutions are affine elations. This property of Π_1 is somewhat remarkable: for it is well known that a translation plane of order q^2, on which $SL(2, q)$ acts as a collineation group generated by affine elations, is always desarguesian.

There are further translation planes of order q^3 admitting a collineation group $G \simeq SL(2, q)$ generated by affine elations (due to works of Dye [1] and Kantor [4, 5]): (a) the Dye–Kantor plane Π_2 of order 8^3 having kernel $GF(8)$ (admits $SL(2, 8)$) [1, No. 4] and [4, No. 9]; (b) the Dye–Kantor plane Π_3 of order $4^3 = 8^2$ having kernel $GF(8)$ (admits $SL(2, 4)$) [1, No. 4] and [5, No. 8,.2]. In Π_1 and Π_2 $SL(2, q)$ is completely reducible on the underlying vector space; in Π_3 the action of $SL(2, 4)$ is irreducible. All these planes are of even order.

This article is the report of an attempt to determine all the translation planes of order q^3, with kernel $K \supseteq GF(q)$, admitting a collineation group $G \simeq SL(2, q)$, where the p-elements are affine elations if $q = p^\omega$. (From now on Π will denote such a plane.) We remark ((1.1) and (1.2); see also Schaeffer [8] for q even) that G must be completely reducible. The underlying vector space may be written, as a G-module, in the form

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$V_2 \oplus V_2^\lambda \oplus V_2^\mu$, where V_2 is the canonical representation and λ, μ are automorphisms of $GF(q)$ such that the equations $xx^\lambda = 1$, $xx^\mu = 1$, and $x^\lambda x^\mu = 1$ have no solutions in $GF(q)$ different from ± 1 (Theorem 1.7): in Kantor planes $\lambda = 1$ and μ is the squaring automorphism. Furthermore we determine (Theorem 1.6) the orbits of G on the line at infinity and examine completely the case $\lambda = 1 = \mu$: we prove (No. 2) that Π is desarguesian in such a case. Liebler in [6, No. 3] studies the case $\lambda = 1$ and, for q even, gives some necessary and sufficient conditions for the existence of Π. At the end of the article (No. 3) we introduce a new class of nondesarguesian translation planes for $q = p^{3\omega}$, p an arbitrary prime, which are an example for the case $\lambda \neq 1 \neq \mu \neq \lambda$. The even order planes of this class are not isomorphic to any Kantor plane Π_1.

1. General Case

Let q be a power of the prime p and Π denote a translation plane of order q^3 with kernel containing a field isomorphic to $K = GF(q)$ (see Lüneburg [7] for basic concepts). We may assume that the vectors of $V_6(K)$ (vector space of dimension six over K) are the points of Π and that the cosets of the members of a spread Σ of $V_6(K)$ are the lines. We will write the points of Π as $(x_1, x_2, x_3, y_1, y_2, y_3)$, or (x, y) for short.

Assume Π admits a collineation group $G \simeq SL(2, K)$ such that its elements of order p are affine elations (*shears*): clearly G is contained in the linear translation complement of Π. By Hering [2, Lemma 7], we can choose a basis so that there is a field $K' \simeq K$ of 3×3 matrices such that G is represented by the maps

$$(x, y) \to (x, y) \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, $D \in K'$ and $AD - BC = I$ (the unity matrix). The generator L of the multiplicative group of K' operates reducibly on $V_3(K)$ because, otherwise, $K' \neq \{aI / a \in K\}$ and the ring $K[L]$, the polynomial extension of K by L, would be a field of order q^3 containing two subfields of order q. As $q-1$ is not divisible by p, $V_3(K)$ is a completely reducible L-module by Maschke's theorem: hence we may write the elements of K' in the form

$$A_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^\lambda & 0 \\ 0 & 0 & \alpha^\mu \end{pmatrix}, \tag{1.1}$$

where $\alpha \in K$ and λ, $\mu \in \operatorname{Aut} K$. So we have

(1.2) *There exists a basis of* $V_3(K)$ *by which* G *is represented by the linear maps*

$$g(\alpha, \beta, \gamma, \delta): (x, y) \to (x, y) \begin{pmatrix} A_\alpha & A_\beta \\ A_\gamma & A_\delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in K$ *and* $\alpha\delta - \beta\gamma = 1$ (*here* A_α, A_β, *etc., are used as in* (1.1)).

The 3-spaces $W(\infty)$, $W(O)$ and $W(I)$ having equations, respectively, $x = 0$, $y = 0$ and $y = x$ are members of Σ because they are axes of shears of G. For any $H \in GL(3, K)$, denote by $W(H)$ the 3-space whose coordinates satisfy the equation $y = xH$ and let $\Lambda = \{M \in GL(3, K)/W(M) \in \Sigma\}$. We have [7, No. 2, Chap. I]:

(1.3) Λ *contains all the matrices* (1.1) *and satisfies the conditions:*

(a) Λ *is sharply transitive (as set of linear maps) on the nonzero vectors of* $V_3(K)$;

(b) *if* $M_1, M_2 \in \Lambda$, $M_1 \neq M_2$, *then* $M_1 - M_2 \in GL(3, K)$;

(c) Π *is desarguesian iff* Λ *is the multiplicative group of a field. The conditions* (a) *and* (b) *are also sufficient in order to define a spread by* Λ.

Denote by Σ_0 the partial spread containing $W(\infty)$, $W(0)$, and all the components $W(A_\alpha)$ of Σ defined by matrices (1.1): Σ_0 is a G-orbit.

Each component of Σ_0 intersects the 4-space V_i satisfying the equations $x_i = y_i = 0$, $i = 1, 2, 3$, in a 2-space: hence Σ_0 "contains" $(q+1)^2$ 1-spaces of V_i. The number of 1-spaces of V_i which are not contained in Σ_0 is so $q^3 + q^2 + q + 1 - (q+1)^2 = q^3 - q$. Therefore

(1.4) *If* $S \in \Sigma - \Sigma_0$, $S \to s_i = S \cap V_i$ *defines a* 1–1 *map of* $\Sigma - \Sigma_0$ *onto the set of* 1-*spaces of* V_i *not lying in components of* Σ_0.

As V_i is G-invariant and $x_i = y_i = 0$, we may omit x_i and y_i in the coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ of a vector of V_i. Let $s = \langle (h, 0, 0, k) \rangle$ (resp. $\langle (0, -k, -h, 0) \rangle$) be a 1-space of V_i: if $hk \neq 0$ no component of Σ_0 contains this 1-space. We have $s^{g(\alpha\beta,\gamma,\delta)} = \langle (h\alpha^{\sigma_1^i}, k\gamma^{\sigma_2^i}, h\beta^{\sigma_1^i}, k\delta^{\sigma_2^i}) \rangle$ (resp. $\langle (-h\gamma^{\sigma_1^i}, -k\alpha^{\sigma_2^i}, -h\delta^{\sigma_1^i}, -k\beta^{\sigma_2^i}) \rangle$), where σ_1^i and σ_2^i take two different values in the set $\{1, \lambda, \mu\}$ (see (1.2)). Therefore

(1.5) *Let* Φ_i *be the multiplicative homomorphism* $x \to x^{\sigma_1^i} x^{\sigma_2^i}$, *then the* 1-*space* $\langle (u, v, w, z) \rangle$ *of* V_i *is contained in the* G-*orbit of* $s = \langle (h, 0, 0, k) \rangle$ (*resp.* $\langle (0, -k, -h, 0) \rangle$), *where* $hk \neq 0$, *iff* $h^{-\sigma_2^i} k^{-\sigma_1^i} (u^{\sigma_2^i} z^{\sigma_1^i} - v^{\sigma_1^i} w^{\sigma_2^i}) \in \operatorname{Im} \Phi_i$. *Moreover* $g(\alpha, \beta, \gamma, \delta) \in G_s$ *iff* $\beta = \gamma = 0$ *and* $\alpha \in \ker \Phi_i$.

Proof. Suppose that $(u, v, w, z) = a(h\alpha^{\sigma_1^i}, k\gamma^{\sigma_2^i}, h\beta^{\sigma_1^i}, k\delta^{\sigma_2^i})$ for some scalar a. Then, by calculation using $\alpha\delta-\beta\gamma=1$ we get that $h^{-\sigma_2^i}k^{-\sigma_1^i}(u^{\sigma_2^i}z^{\sigma_1^i}-v^{\sigma_1^i}w^{\sigma_2^i})=a^{\sigma_1^i}a^{\sigma_2^i}$. Note that in the case $(u, v, w, z)=a(h\alpha^{\sigma_1^i}, k\gamma^{\sigma_2^i}, h\beta^{\sigma_1^i}, k\delta^{\sigma_2^i})$ is a scalar multiple of $(h, 0, 0, k)$ then $\beta=\gamma=0$, $\delta=\alpha^{-1}$ and hence $\alpha\in\operatorname{kern}\Phi_i$. The rest of the proof is left to the reader.

By (1.4) we can determine the length of the G-orbit of a component $S\in\Sigma-\Sigma_0$ by computing the G-orbit of the intersection $s_i=V_i\cap S$: we have $|G_S|=|G_{s_i}|$. If $\operatorname{kern}\Phi_i=\langle 1\rangle$, i.e., $\operatorname{Im}\Phi_i=K^*$ (the multiplicative group of K), then $|SL(2, K)|=q^3-q$ and (1.5) imply G splits Σ into two orbits: Σ_0 and $\Sigma-\Sigma_0$ (q must be even in such a case). Therefore we may assume $\operatorname{Im}\Phi_i\neq K^*$. Let $s_i=\langle(u, 0, 0, 1)\rangle$, $u\in K^*$, and let $s_j=\langle(a_j, b_j, c_j, d_j)\rangle$ for $j\neq i$. If $g(\alpha, \beta, \gamma, \delta)\in G_{s_i}$, from (1.5) it follows $\beta=\gamma=0$ and $\alpha\in\operatorname{kern}\Phi_i$. Assume $\alpha\neq\pm 1$, then $G_{s_j}=G_S=G_{s_i}$ requires either $b_j=c_j=0$ and $a_j\neq 0\neq d_j$ or $a_j=d_j=0$ and $b_j\neq 0\neq c_j$ (the remaining possibilities cannot occur because, otherwise, s_j should be contained in a component of Σ_0). In any case, by (1.5), $\alpha\in\operatorname{kern}\Phi_j$. Thus we have $\alpha\alpha^\lambda=\alpha\alpha^\mu=\alpha^\lambda\alpha^\mu=1$, i.e., $\alpha^2=1$: a contradiction since we assumed $\alpha\neq\pm 1$. Therefore $[K^*: \operatorname{Im}\Phi_i]=2$ (hence q must be odd in this case) and the G-orbit of s_i has length $(q^3-q)/2$. By (1.5)$\langle(u, 0, 0, 1)\rangle$ is in the same G-orbit of $\langle(1, 0, 0, 1)\rangle$ iff $u^{\sigma_2}\in\operatorname{Im}\Phi_i$: so the following theorem holds

THEOREM (1.6). *G splits the components of Σ either into two orbits (q even) or into three orbits (q odd). One of these orbits is Σ_0 and, if q is odd, the remaining orbits have the same length $(q^3-q)/2$.*

In view of (1.4) and (1.5) we have as a direct consequence

THEOREM (1.7). *If G is represented as in* (1.1) *and* (1.2), *then the equations $x^\lambda x^\mu=1$, $xx^\lambda=1$, $xx^\mu=1$ have no solutions in K different from ± 1.*

2. DESARGUESIAN CASE

In this section we will assume $\lambda=1=\mu$ and show that Π is desarguesian in this case, First we prove

(2.1). *If the 3-space $W(H)$ intersects each component $W(A_\alpha)$ of Σ_0 only trivially, the ring $K[H]$ is a field of order q^3. (See the definition preceding* (1.3).)

Proof. $W(H)\cap W(A_\alpha)=\langle 0\rangle$ requires $H-A_\alpha=H-\alpha I$ nonsingular. Hence H has no eigenvalues in K and the minimal polynomial for H is an irreducible cubic $f(x)$: the ring $K[H]$ is so isomorphic to the field $K[x]/(f(x))$.

Let $W(M) \in \Sigma - \Sigma_0$, then $K[M]$ is a field by (2.1) whence $M' \in K[M]$ for each other component $W(M')$ in the G-orbit of $W(M)$. By (1.3; c) and theorem (1.6) Π is so desarguesian if q is even. Therefore we may assume q odd in the following of this section: Σ splits so into three G-orbits $\Sigma_0, \Sigma_1, \Sigma_2$.

Now we have

(2.2). *There exists a 3-space $W(H)$, intersecting trivially each component of Σ_0, whose nonzero vectors are both in components of Σ, and in components of Σ_2.*

Proof. It is a direct consequence of the following Lemmas (2.3), (2.4), (2.5).

LEMMA (2.3). *There exists a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + x^3$ over K such that $f(x)$ is irreducible over K and $a_2 \neq 0$.*

Proof. If $F \simeq GF(q^3)$ is an extension of K, it is well known that every cubic irreducible over K has three distinct roots in the $q^3 - q$ elements of $F - K$ and every element of $F - K$ satisfies such a cubic. Hence the number of irreducible cubics is $(q^3 - q)/3$. But this is greater than the number of polynomials of the form $a_0 + a_1 x + x^3$.

LEMMA (2.4). *Let $f(x)$ be the irreducible cubic in (2.3); let $t \neq 0$ be an element of K, let $d = (t + a_0) a_2^{-1}$ and let $W(H) = \langle (0, d, -t^{-1}, 0, -a_0, a_2 t^{-1}), (-t, 0, 0, d(a_1 - d), 0, -1), (t^{-1}(a_1 - d), -1, 0, 1, 0, 0) \rangle$. Then the 3-space $W(H)$ has only trivial intersection with each component of Σ_0.*

Proof. Suppose that some linear combination (with coefficients u, v, w) of the basis vectors in $W(H)$ satisfies the equation $y = xm$ for some m in K. Then u, v, w satisfy the equations

$$v(d(a_1 - d) + tm) + w(1 - t^{-1}(a_1 - d) m) = 0,$$

$$-u(a_0 + dm) + wm = 0,$$

$$u(a_2 t^{-1} + mt^{-1}) - v = 0.$$

Using $d = (t + a_0) a_2^{-1}$, the determinant of this system reduces to $m^3 + a_2 m^2 + a_1 m + a_0$ and the determinant is not equal to zero for any m in K because $f(x)$ is irreducible. The case where $W(H)$ contains some point $(0, y)$, $y \neq 0$, is left to the reader. This establishes (2.4).

Remark. In interpreting (1.5) and the preceding arguments for the case $\lambda=\mu=1$, the requirement that σ_1 and σ_2 take two "different" values in the set $\{1, \lambda, \mu\}$ must not be taken to exclude cases such as $\sigma_1^i=1$, $\sigma_2^i=\lambda=1$. In the present context, the 1-space $\langle(u, v, w, z)\rangle$ of V_i is in the G-orbit of $\langle(1,0,0,1)\rangle$ iff $uz-vw$ is a square while $\langle(u, v, w, z)\rangle$ is in a member of Σ_0 iff $uz-vw=0$. Thus the 1-spaces $\langle(u, v, w, z)\rangle$ such that uz-vw is not a square form another G-orbit by (1.4) and Theorem (1.5). We shall say that a 1-space $\langle(u, v, w, z)\rangle$ of V_i is of type $+, 0, -$ depending on whether $uz-vw$ is a square, zero, or a non-square in K. Clearly, two components of Σ contain 1-spaces of V_i of the same type iff they are in the same partial spread Σ_j, $j=0, 1, 2$.

LEMMA (2.5). *The parameter t may be chosen so that $W(H)$ of (2.4) contains nonzero vectors on components of Σ_1 and nonzero vectors on components of Σ_2.*

Proof. $W(H)$ contains the subspaces $\langle(0, d, -t^{-1}, 0, -a_0, a_2t^{-1})\rangle$ and $\langle(t^{-1}(a_1-d), -1, 0, 1, 0, 0)\rangle$. In the condensed notation $\langle(d, -t^{-1}, -a_0, a_2t^{-1})\rangle$ is in V_1 and $(t^{-1}(a_1-d), -1, 1, 0)$ is in V_3. Both are of type $+$. Also $W(H)\cap V_2=\langle(-t, 0, d(a_1-d), -1)\rangle$ which is of type $+$ or $-$ depending on whether t is a square or a non-square in K. Thus (2.5) is trivial.

Let $W(N_i)\in\Sigma_j$, $i=1, 2$, and let M_i be a generator of the multiplicative group of the field $K[N_i]$. If $\langle M_1\rangle=\langle M_2\rangle$, then Π is desarguesian because we have $\Lambda=\langle M_i\rangle$. Suppose $\langle M_2\rangle\neq\langle M_1\rangle$ and let F be a field of 3×3 matrices over K maximal with respect to the condition that F is normalized by $\langle M_1, M_2\rangle$: since F contains $\{aI/a\in K\}$, F is isomorphic either to $GF(q)$ or to $GF(q^3)$. If $F\simeq GF(q^3)$, we may regard $\langle M_1, M_2\rangle$ as a subgroup of $\Gamma L(1, F)$. But each element of order q^3-1 in $\Gamma L(1, F)$ is linear: therefore we have $\langle M_1\rangle=\langle M_2\rangle$, a contradiction. Thus $F\simeq GF(q)$. Now (1.3; a) and Hering [3, Lemma 5.7] imply

$$SL(3, K) \trianglelefteq \langle M_1, M_2\rangle\leqslant GL(3, K). \tag{2.6}$$

Let $W(H)$ be a 3-space satisfying (2.2); by (2.1) $K[H]$ is a field. Let E be a generator of the multiplicative group of $K[H]$ and U the Sylow u-subgroup of $\langle E\rangle$, where u is a q-primitive prime divisor of q^3-1 (see, e.g., [7, pp. 27–28]): U is also a Sylow u-subgroup of $GL(3, K)$. Since U operates irreducibly on $V_3(K)$, Schur's lemma implies the centralizer of U in $GL(3, K)$ is $\langle E\rangle$. U is conjugate in $GL(3, K)$ to the Sylow u-subgroup of $\langle M_i\rangle$ whence $\langle E\rangle$ and $\langle M_i\rangle$ are also conjugate because both centralizers of Sylow u-groups of $GL(3, K)$. Now from (2.6) it follows $\langle E\rangle\leqslant\langle M_1, M_2\rangle$

because the group $\langle M_1, M_2 \rangle$ must contain every element of $GL(3, K)$ having the same determinant of some its element. The linear map

$$(x, y) \to (x, y) \begin{pmatrix} M_i & 0 \\ 0 & M_i \end{pmatrix}$$

fixes each component of Σ_0 and Σ_i, hence leaves $\Sigma_0, \Sigma_1, \Sigma_2$ invariant as sets of vectors. Then the whole $\langle M_1, M_2 \rangle$ behaves in this way. But $V_3(K)$ is an irreducible H-module, hence the group of linear maps of $V_6(K)$ defined by

$$\left\langle \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \right\rangle$$

fixes $W(H)$ and is transitive on its nonzero vectors: according to the choice of $W(H)$, $\langle E \rangle \leqslant \langle M_1, M_2 \rangle$ cannot so happen. Thus $\langle M_1 \rangle = \langle M_2 \rangle$ and Π is desarguesian.

3. A Class of Translation Planes of Order q^3 Admitting $SL(2, q)$.

Let $F = GF(q_0)$, $q_0 = p^\omega$, and K be a cubic extension of F: $K \simeq GF(q)$, where $q = q_0^3$. Also let M be the matrix

$$\begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix},$$

where a, b, $c \in K$ and $t = abc \notin F$. Consider the field of matrices $K' = \{A_\alpha / \alpha \in K\}$ (where A_α is the matrix (1.1)) and set $x^\lambda = x^{q_0}$, $x^\mu = x^{q_0^2}$ for $x \in K$. We have

(3.1). *The matrix* $H(\alpha, \beta, \gamma) = M^2 A_\alpha + M A_\beta + A_\gamma$ *is singular iff* $\alpha = \beta = \gamma = 0$.

Proof. $H(\alpha, \beta, \gamma)$ has determinant $d(t) = t^2 \alpha^{q_0} \alpha^{q_0^2} + t(\beta \beta^{q_0} \beta^{q_0^2} - \alpha \beta^{q_0^2} \gamma^{q_0} - \alpha^{q_0} \beta \gamma^{q_0^2} - \alpha^{q_0^2} \beta^{q_0} \gamma) + \gamma \gamma^{q_0} \gamma^{q_0^2}$. Note that $d(x)$ is a quadratic in $F[x]$. Hence if $d(t) = 0$, t is in a quadratic extension of F. But K is a cubic extension of F and $t \in K - F$. Hence if $H(\alpha, \beta, \gamma)$ is singular, $\alpha = \beta = \gamma = 0$.

Let $W(M') = W(M)^{g(\alpha, \beta, \gamma, \delta)}$, where $g(\alpha, \beta, \gamma, \delta)$ is defined as in (1.2). Since M normalizes K' (i.e., $A_\eta^M = A_\eta^{q_0}$), $M - M'$ is singular (i.e., $W(M') \cap W(M) \neq \langle 0 \rangle$) iff $H(-\gamma^{q_0}, \delta - \alpha^{q_0}, \beta)$ is singular. Thus by (3.1) $M - M'$ is singular iff $\beta = \gamma = 0$ and $\alpha \alpha^{q_0} = 1$. As $(q_0 + 1, q_0^3 - 1)$ is either 2 or 1 depending on whether q is odd or even, we infer $\alpha = \pm 1$; whence

(3.2). *The orbit of* $W(M)$ *under the group* $G=\{g(\alpha,\beta,\gamma,\delta)/\alpha,\beta,\gamma,\delta\in K,\ \alpha\delta-\delta\gamma=1\}$ *contains either* $(q^3-q)/2$ *3-spaces* (q odd), *or* q^3-q *3-spaces* (q *even*). *Any two of these subspaces in the same orbit intersect trivially.*

Suppose q odd and consider the 3-space $W(MA_\xi)$, where $\xi\neq 0$. Assume that $W(MA_\xi)$ intersects nontrivially some 3-space $W(M')$ of the G-orbit containing $W(M)$, say $W(M')=W(M)^{g(\alpha,\beta,\gamma,\delta)}$. This implies $H(-\xi\gamma^{q_0},\delta-\xi\alpha^{q_0},\beta)$ singular: by (3.1) $\beta=\gamma=0$ and $\xi\alpha\alpha^{q_0}=1$, whence ξ must be a square. Therefore

(3.3.) *If* q *is odd and* ξ *is a nonsquare in* K, $W(MA_\xi)$ *intersects trivially each* 3-*space of the* G-*orbit containing* $W(M)$.

Put $\Sigma_1=\{W(M)^g/g\in G\}$, $\Sigma_2=\{W(MA_\xi)^g/g\in G\}$ and let Σ_0 be as in no. 1; we have

THEOREM (3.4). *If* q *is odd* (*resp. even*) *and* ξ *is a nonsquare in* K, $\Sigma_0\cup\Sigma_1\cup\Sigma_2$ (*resp.* $\Sigma_0\cup\Sigma_1$) *is a spread of* $V_6(K)$.

Proof. Since MA_ξ satisfies the same condition that M (i.e., $t\notin F$), the claim is a direct consequence of (3.1), (3.2), and (3.3).

Remark. The translation plane Π defined by the spread of Theorem (3.4) is not desarguesian because M does not centralize K'. If q is even, Π is not isomorphic to the Kantor plane Π_1 of the same order: for Π should admit a cyclic collineation group C of order $q+1$ fixing every line of Σ_0, [4, Theorem 7.1]. Thus each element of C should be defined by a matrix centralizing K', i.e., a diagonal matrix. But a group of diagonal matrices cannot have order $q+1$.

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