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Relative Colorings of Graphs

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In this paper, the notion of relative chromatic number $\chi(G, H)$ for a pair of graphs G, H, with H a full subgraph of G, is formulated; namely, $\chi(G, H)$ is the minimum number of *new* colors needed to extend any coloring of H to a coloring of G. It is shown that the four color conjecture (4CC) is equivalent to the conjecture (R4CC) that $\chi(G, H) \leqslant 4$ for any (possibly empty) full subgraph H of a planar graph G and also to the conjecture (CR3CC) that $\chi(G, H) \leqslant 3$ if H is a connected and nonempty full subgraph of planar G. Finally, relative coloring theorems on surfaces other than the plane or sphere are proved.

INTRODUCTION

A pair of graphs (G, H) consists of a graph G and a full (or section) subgraph H of G. We allow the empty graph $\varnothing$ which has no vertices or edges and which is a full subgraph of any graph. An r-coloring c of a graph G for some nonnegative integer r is an assignment of r colors $c_1, \ldots, c_r$ to the vertices of G in such a way that whenever v and w are vertices of G ($v, w \in V(G)$) and v is adjacent to w ($[v, w] \in E(G)$), $c(v) \neq c(w)$. We also assume that each color c_i is actually used. If c is an r-coloring we write $|c| = r$.

As always, $\chi(G)$ denotes the chromatic number of G; i.e., the minimum r for which G has an r-coloring. If (G, H) is a pair of graphs, c is a coloring of G and d a coloring of H, we say that c *extends* d if $c \mid H = d$. We define the *relative chromatic number* $\chi(G, H)$ of the pair (G, H) to be

$$\inf_c \sup_d |c| - |d|,$$

where d is any coloring of H and c is any coloring of G extending d. Thus $\chi(G, H)$ is the minimum number of *new* colors which will be needed to extend any coloring of H to a coloring of G.

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1. Relative Four Color Conjecture

Consider a triple of graphs (G, H, K) where K is full in H and H is full in G. If c is an r-coloring of K, then c can be extended to d, an s-coloring of H, where $s \leqslant r + \chi(H, K)$. Now d can be extended to e, a t-coloring of G, where $t \leqslant s + \chi(G, H)$. Therefore, $t \leqslant r + \chi(G, H) + \chi(H, K)$ and hence we have proved the following lemma.

Lemma 1. *For any triple* (G, H, K), $\chi(G, K) \leqslant \chi(G, H) + \chi(H, K)$.

Corollary 1. *For any pair* (G, H), $\chi(G) \leqslant \chi(G, H) + \chi(H)$.

Lemma 2. *For any pair* (G, H), $\chi(G, H) \leqslant \chi(G - H)$.

Corollary 2. *For any pair* (G, H), $\chi(G) - \chi(H) \leqslant \chi(G, H) \leqslant \chi(G - H)$.

Now suppose that G is planar. Then $G - H$ is planar so by the five color theorem $\chi(G, H) \leqslant 5$ for any H. Conversely, if $\chi(G, H) \leqslant 5$ for all H, then in particular $\chi(G) = \chi(G, \varnothing) \leqslant 5$. This suggests the following.

Conjecture 1 (R4CC). *Let G be planar. Then $\chi(G, H) \leqslant 4$ for all pairs (G, H).*

Of course the preceding argument proves that R4CC is equivalent to 4CC.

Theorem 1. 4CC *holds if and only if* R4CC *holds.*

2. Relative Three Color Conjecture

Suppose now that G is planar and that H is a connected (nonempty) full subgraph of G. One then finds oneself unable to construct an example in which even four new colors are needed to extend some coloring of H to a coloring of G; in fact, three always seem to suffice. Hence, a new conjecture.

Conjecture 2 (CR3CC). *Let G be planar. Then $\chi(G, H) \leqslant 3$ for all pairs (G, H) in which H is connected (and nonempty).*

Theorem 2. 4CC *is equivalent to* CR3CC.

We first conjectured the preceding theorem in a preprint of this (see also [4]). It was then proved by Roy Levow [3] and independently by Frank Bernhart. Bernhart's proof has the virtue that if H is k-colored for $k \geqslant 3$, then the existence of an r-coloring of G, which extends the given coloring of H and with $r \leqslant k + 3$, does not depend on the validity of 4CC. Moreover, his proof may well go through when $k = 2$. Of course, $k = 1$ is a different matter! The proof given below is different from that of Levow and also from the original version of Bernhart's proof. Later, Bernhart [1] independently found the same proof as we give here.

Proof. One half of the theorem is trivial. For suppose that G is any planar graph and let H be a single vertex. By Corollary 1,

$$\chi(G) \leqslant \chi(G, H) + \chi(H) \leqslant 3 + 1 = 4.$$

Conversely, suppose that G is any planar graph and that H is a connected full subgraph. Let c be an r-coloring of H with colors $c_1, \ldots, c_r$. We must extend c to a coloring d of G using at most 3 new colors.

Since H is connected, we may find a spanning tree T in H, i.e., $T \subset H$, T is a tree, and $V(T) = V(H)$. Now shrink T to a single point x and let $\bar{G}$ be the corresponding graph. Specifically, $V(\bar{G}) = V(G) - V(H) \cup \{x\}$. Two vertices other than x are adjacent in $\bar{G}$ if and only if they were adjacent in G. A vertex v is adjacent to x if and only if v was adjacent in G to some vertex w in $V(H)$. Finally, we delete all loops and parallel edges so that $\bar{G}$ *is* a graph. Note that $\bar{G}$ is still planar since we have collapsed a contractible subgraph.

By the 4CC, $\chi(\bar{G}) \leqslant 4$ so we can 4-color $\bar{G}$ by a coloring e. Moreover, we can assume that the color $e(x)$ which e assigns to x is one of the original r colors $c_1, \ldots, c_r$, say c_1, while the other three colors are all new.

We now define d as follows:

$$d(v) = \begin{cases} c(v) & \text{if } v \in V(H); \\ e(v) & \text{if } v \in V(G) - V(H). \end{cases}$$

Obviously, the only thing which needs checking is that if $v \in V(G) - V(H)$, $w \in V(H)$, and $[v, w] \in E(G)$, then $d(v) \neq d(w)$. Suppose $d(v) = d(w)$. Then since $d(v) = e(v)$, we must have $d(v) = e(v) = c_1$. But $[v, w] \in E(G)$ means v is adjacent to x in $\bar{G}$ and hence $e(v) \neq e(x) = c_1$.

3. Relative Coloring Theorems

Since any graph G in the torus can be 7-colored, the same argument as in the proof of Theorem 2 shows that $\chi(G, H) \leqslant 6$ when H is connected. More generally, we have the following result.

THEOREM 3. *Let S be any surface and suppose that for all G embeddable in S, $\chi(G) \leqslant k$. Then $\chi(G, H) \leqslant k - 1$ for G embeddable in S and H connected.*

The only detail of proof which may need some elucidation is that it still is valid to shrink T to a point. One proceeds as follows. First note that if G is embeddable in S, then G is in fact piecewise linearly embeddable in S with respect to some triangulation J of S. Subdividing appropriately, some subdivision G' of G is embedded as a subcomplex of a subdivision J' of J.

Let J'' denote the second barycentric subdivision of J' and let N be the regular neighborhood of T in J''; that is, $N = \bigcup \{\sigma \mid \sigma$ is a (closed) 2-simplex in J'' and $\sigma \cap T \neq \varnothing\}$. Then N is a ball, so pinching N to a point does not change the homeomorphism type of S. Moreover, $\bar{G} = G/T$ is embedded in $S/N \simeq S$. For all the details and justifications in the above argument, see Hudson [2].

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