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Contingent and Intermediate Tangent Cones in Hyperbolic Differential Inclusions and Necessary Optimality Conditions

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We study some properties of the contingent and intermediate cones to the reachable set and the solution set of hyperbolic differential inclusions. These properties allow us to obtain necessary optimality conditions in terminal problems for the corresponding class of inclusions and dynamic systems with feedback controls.

1. INTRODUCTION

In recent years, set-valued analysis has been widely used in the study of problems of differential inclusions and corresponding dynamic systems: viability problems ([1] and references therein), controllability problems [4, 12, 19], traditional problems of maximum principle [7, 8, 11, 13, 18, 19], and Lyapunov stability problems [21]. The main tools of investigation here are tangent cones and differential calculus of set-valued maps. For example, studying Mayer's problem

$$g(x(T)) \to \inf \tag{1}$$

for the differential inclusion

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = \xi, \tag{2}$$

along a given trajectory $z(t)$, Frankowska [13] introduced the variational inclusion

$$\dot{w}(t) \in A(t)\, w(t), \qquad w(0) = 0, \tag{3}$$

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where $A(t)$ is a closed convex process whose graph is contained in the graph of the derivative of co F (the convexified map) at $(z(t), \dot{z}(t))$ (more precisely, graph $A(t) \subset \text{graph}\, d\,\text{co}\, F(z(t), \dot{z}(t))$ for almost every t). It was proved in [13] that the reachable set $P(T)$ at time T of inclusion (3) is a convex cone and is contained in the intermediate tangent cone $I(S(T); z(T))$ to the reachable set $S(T)$ (at time T) of inclusion (2), at $z(T)$. Therefore, if $z(t)$ is an optimal solution of problem (1), (2) then $\partial g(z(T)/\partial x \in I(S(T); z(T))^+ \subset P(T)^+$ ($^+$ denotes the positive polar) and the maximum principle for $z(t)$ follows from a characterization of the cone $P(T)^+$. Also, in [19] Polovikin and Smirnov proved that when $0 \in F(\xi)$ inclusion (2) is locally controllable around ξ at time T if inclusion (3) is locally controllable around 0 at time T.

In the sequel we will consider hyperbolic differential inclusions of the form

$$\begin{gathered} u_{xy}(x, y) \in F(x, y, Z(u)(x, y)), \\ Z(u)(x, y) := (u(x, y), u_x(x, y), u_y(x, y)) \end{gathered} \tag{4}$$

with the boundary conditions

$$u(0, 0) \in F_0, \qquad u_x(x, 0) \in F_1(x, u(x, 0)), \qquad u_y(0, y) \in F_2(y, u(0, y)). \tag{5}$$

In a previous paper [23], using the set-valued analysis approach we have derived a local controllability condition for these inclusions, in the same spirit as the one for differential inclusions in [12, 19]. On the other hand, using the discrete approach of Pshenichnyi [20], Mahmudov [16] has obtained a Egorov type necessary optimality condition in terminal problems for inclusions (4), (5) with autonomous and convex right hand side. However, this approach does not seem to be suitable for the study of more general classes of optimization problems (optimal time problem, problems with free time, with endpoint constraints, etc.). In the present paper, our purpose is to study the contingent (and intermediate tangent) sets to the reachable and solution sets at given trajectories of inclusions (4), (5), and on the basis of their properties to derive necessary optimality conditions in the maximum principle form in terminal problems for inclusions (4), (5) with nonconvex right hand side and for dynamic systems with feedback controls of corresponding classes (similar problems were studied, e.g., [13, 15] for dynamic systems of ordinary differential equations). In particular, for the dynamic system

$$u_{xy}(x, y) = f(Z(u)(x, y), v(x, y)), \qquad v(x, y) \in V \tag{6}$$

with the boundary conditions

$$\begin{gathered} u(0, 0) \in F_0, \qquad u_x(x, 0) = f_1(u(x, 0), v^1(x)), \qquad v^1(x) \in V^1, \\ u_y(0, y) = f_2(u(0, y), v^2(y)), \qquad v^2(y) \in V^2, \end{gathered} \tag{7}$$

where V, V^1, V^2 are the control ranges, our equations of maximum principle are simpler and more natural than Egorov's equations [9] and do not require twice differentiability of the right hand side as in [9].

The paper consists of five sections. After the Introduction, in Section 2 we study the contingent and intermediate tangent cones to the reachable and solution sets. In Section 3 we study the solution map and its adjoint. In Section 4, we derive necessary optimality conditions in minimization problems with respect to all endpoints for inclusion (4), (5). Finally, Section 5 is devoted to necessary optimality conditions for dynamic systems, including systems with feedback controls.

Throughout the paper Comp R^n (Conv R^n, resp.) will denote the collection of all nonempty compact (convex and compact, resp.) subsets of R^n, K^+ the positive polar of a set K in a Banach space X, i.e.,

$$K^+ = \{h \in X^*: \langle h, a\rangle \geqslant 0 \ \forall a \in K\},$$

$L^1(\tilde{D})$ ($L^\infty(\tilde{D})$, resp.) is the space of integrable (measurable, resp.) functions from $\tilde{D}$ to R^n and Q is the space of absolutely continuous functions from Π to R^n. From [22] a function $u(\cdot, \cdot) \in Q$ if and only if it can be expressed in the form

$$u(x, y) = \int_0^x \int_0^y f(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} + \int_0^x g(\bar{x})\, d\bar{x} + \int_0^y h(\bar{y})\, d\bar{y} + u(0, 0) \qquad \forall (x, y) \in \Pi,$$

where $f(\cdot, \cdot) \in L^1(\Pi)$, $g(\cdot) \in L^1(I_1)$, $h(\cdot) \in L^2(I_2)$. Hence Q becomes a Banach space when endowed with the norm

$$\|u(\cdot, \cdot)\|_Q = \|u(0, 0)\| + \int_0^{a_1} \|u_x(x, 0)\|\, dx + \int_0^{a_2} \|u_y(0, y)\|\, dy + \int_0^{a_1} \int_0^{a_2} \|u_{xy}(x, y)\|\, dy\, dx \qquad \forall u(\cdot, \cdot) \in Q.$$

In inclusion (4), (5),

$$F\colon \Pi \times R^n \times R^n \times R^n \to \text{Comp } R^n, \qquad F_1\colon I_1 \times R^n \to \text{Comp } R^n,$$
$$F_2\colon I_2 \times R^n \to \text{Comp } R^n,$$

are set-valued maps, $F_0 \in \text{Comp } R^n$, $I_1 = [0, a_1]$, $I_2 = [0, a_2]$, $\Pi = I_1 \times I_2$, and $a_1 > 0$, $a_2 > 0$. For any set-valued map G we denote by co G the map

$u \to \text{co}\, G(u)$ (convex hull of $G(u)$) $\forall u \in \text{Dom}\, G$. For the sake of convenience we also use the notations

$$Z(u)(x, y) := (u(x, y), u_x(x, y), u_y(x, y)) \qquad \forall u(\cdot, \cdot) \in Q;$$

$$\Omega := \text{Banach space } R^n \times L^1(I_1) \times L^1(I_2),$$

$$\text{so that } \Omega^* = R^n \times L^\infty(I_1) \times L^\infty(I_2);$$

$$\mathscr{L}u := (u(0, 0), u_x(\cdot, 0), u_y(0, \cdot)) \in \Omega \qquad \forall u(\cdot, \cdot) \in Q,$$

2. Contingent Cones in Hyperbolic Inclusions

We will make the following blanket assumptions on the right hand side of inclusion (4), (5):

(i) The map $F(\cdot, \cdot, u, p, q)$ ($F_1(\cdot, u)$, $F_2(\cdot, u)$, resp.) is measurable on Π (I_1, I_2, resp.) for every fixed (u, p, q) (u, u, resp.) and there is a function $\xi(x, y)$ ($\xi_1(x)$, $\xi_2(y)$, resp.) integrable on Π (I_1, I_2, resp.) such that

$$F(x, y, 0, 0, 0) \subset \xi(x, y)\, O_n \qquad \forall x, y,$$

$$F_1(x, 0) \subset \xi_1(x)\, O_n \qquad \forall x,$$

$$F_2(y, 0) \subset \xi_2(y)\, O_n \qquad \forall y,$$

where O_n denotes the unit ball around O in R^n.

(ii) For every fixed x, y the maps $F(x, y, \cdot, \cdot, \cdot)$, $F_1(x, \cdot)$, $F_2(y, \cdot)$ are Lipschitzian with Lipschitz constant $k > 0$.

By solution to inclusion (4), (5) we always mean a solution in the Caratheodory sense, i.e., an absolutely continuous function $u(x, y)$ on Π which has a derivative $u_{xy}(x, y)$ satisfying inclusion (4) almost everywhere on Π and satisfies the boundary conditions (5) almost everywhere on I_1 and I_2.

Now let us recall some definitions from [3, 5, 11–13].

Definition 1. Let X be a Banach space and $K \subset X$. The (Bouligand) contingent cone to K at $a \in \bar{K}$ is the cone

$$T(K; a) = \{v \in X \colon \exists h_i \to 0+,\, v_i \to v \text{ such that } a + h_i v_i \in K\}.$$

The intermediate tangent cone to K at a is the cone

$$I(K; a) = \{v \in X \colon \forall h_i \to 0+,\, \exists v_i \to v \text{ such that } a + h_i v_i \in K\}.$$

DEFINITION 2. Let $G: R^m \to \operatorname{Comp} R^n$ be a set-valued map which is locally Lipschitzian at a and let $b \in G(a)$. The derivative of G at (a, b) is the set-valued map $dG(a, b)$: $R^m \to 2^{R^n}$ such that

$$v \in dG(a, b)(u) \Leftrightarrow (u, v) \in I(\text{graph } G; (a, b))$$

or equivalently,

$$v \in dG(a, b)(u) \Leftrightarrow \lim_{h \to 0+} \operatorname{dist}\left(v, \frac{G(a+hu)-b}{h}\right) = 0.$$

Observe that graph $dG(a, b)$ is a cone in the space $R^m \times R^n$. We refer to [3, 11, 17] for general properties of $T(K; a)$, $I(K; a)$, and $dG(a, b)$.

The Filippov type theorem [10] for inclusion (4), (5) in [23] can be formulated in the following simpler form

THEOREM 1. *Let $\omega(x, y)$ be an absolutely continuous function on Π such that*

$$\operatorname{dist}(\omega_{xy}(x, y), F(x, y, Z(\omega)(x, y))) \leqslant \lambda(x, y) \quad \textit{almost everywhere on } \Pi,$$

$$\operatorname{dist}(\omega_x(x, 0), F_1(x, \omega(x, 0))) \leqslant \lambda_1(x) \quad \textit{almost everywhere on } I_1,$$

$$\operatorname{dist}(\omega_y(0, y), F_2(y, \omega(0, y))) \leqslant \lambda_2(y) \quad \textit{almost everywhere on } I_2,$$

where $\lambda(x, y)$, $\lambda_1(x)$, $\lambda_2(y)$ are integrable functions. Then there exist a constant ℓ depending only on k, a_1, a_2, and a solution $u(x, y)$ to inclusion (4), (5) *satisfying*

$$\|u(x, y) - \omega(x, y)\| \leqslant \ell \left[\int_0^x \int_0^y \lambda(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} + \int_0^x \lambda_1(\bar{x})\, d\bar{x} + \int_0^y \lambda_2(\bar{y})\, d\bar{y} + \operatorname{dist}(\omega(0, 0), F_0) \right],$$

$$\|u_x(x, y) - \omega_x(x, y)\| \leqslant \ell \left[\int_0^x \int_0^y \lambda(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} + \int_0^y \lambda(x, \bar{y})\, d\bar{y} + \int_0^x \lambda_1(\bar{x})\, d\bar{x} + \operatorname{dist}(\omega(0, 0), F_0) \right] + \lambda_1(x) \quad \textit{for a.e. } x \in I_2,$$

$$\|u_y(x, y) - \omega_y(x, y)\| \leqslant \ell \left[\int_0^x \int_0^y \lambda(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} + \int_0^x \lambda(\bar{x}, y)\, d\bar{x} + \int_0^y \lambda_2(\bar{y})\, d\bar{y} + \operatorname{dist}(\omega(0, 0), F_0) \right] + \lambda_2(y) \quad \textit{for a.e. } y \in I_2.$$

Now consider any solution $z(x, y)$ of inclusion (4), (5). For every $(x, y) \in \Pi$ denote

$$F^H(x, y)(u, p, q) = \{v \in R^n: (u, p, q, v) \in I(\text{graph } F(x, y, \cdot, \cdot, \cdot); (Z(z)(x, y), z_{xy}(x, y)))\} \quad \forall (u, p, q) \in R^{3n}, \tag{8}$$

$$F_1^H(x)\, u = \{v \in R^n: (u, v) \in I(\text{graph } F_1(x, \cdot); (z(x, 0), z_x(x, 0)))\} \quad \forall u \in R^n, \tag{9}$$

$$F_2^H(y) = \{v \in R^n: (u, v) \in I(\text{graph } F_2(y, \cdot); (z(0, y), z_y(0, y)))\} \quad \forall u \in R^n, \tag{10}$$

i.e., $F^H(x, y)$ ($F^H(x), F^H(y)$, resp.) is the derivative of the map $F(x, y, \cdot, \cdot, \cdot)$ ($F_1(x, \cdot)$, $F_2(y, \cdot)$, resp.) at $(Z(z)(x, y), z_{xy}(x, y))$ ($(z(x, 0), z_x(x, 0))$, $(z(0, y), z_y(0, y))$, resp.).

Note that the measurability of the map F (F_1, F_2, resp.) in (x, y) (x, y, resp.) implies that of F^H (F_1^H, F_2^H, resp.) [17] and if graph $F(x, y, \cdot, \cdot, \cdot)$ (graph $F_1(x, \cdot)$, graph $F_2(y, \cdot)$, resp.) is locally convex in R^{4n} (R^{2n}, R^{2n}, resp.) then the map F^H (F_1^H, F_2^H, resp.) also satisfies the Lipschitz condition in (u, p, q) (u, u, resp.) with the same Lipschitz constant k [17].

To inclusion (4), (5) we now associate the inclusion

$$w_{xy}(x, y) \in F^H(x, y)(Z(w)(x, y)) \tag{11}$$

with the boundary conditions

$$\begin{gathered} w(0, 0) \in T(F_0; z(0, 0)), \qquad w_x(x, 0) \in F_1^H(x)\, w(x, 0), \\ w_y(0, y) \in F_2^H(y)\, w(0, y), \end{gathered} \tag{12}$$

or

$$\begin{gathered} w(0, 0) \in I(F_0; z(0, 0)), \qquad w_x(x, 0) \in F_1^H(x)\, w(x, 0), \\ w_y(0, y) \in F_2^H(y)\, w(0, y). \end{gathered} \tag{13}$$

By $S(a_1, a_2)$ denote the reachable set at (a_1, a_2) of inclusion (4), (5), i.e.,

$$S(a_1, a_2) = \{u(a_1, a_2): u(x, y) \text{ is a solution of inclusion (4), (5)}\}$$

and by $P_T(a_1, a_2)$, $P_I(a_1, a_2)$ denote the reachable sets at (a_1, a_2) of inclusions (11), (12) and (11), (13), respectively. Using the Lebesgue dominated convergence theorem and arguing as in the proof of Theorem 2.4 in [13], from Theorem 1 we can easily deduce the following

THEOREM 2. *We have*

$$P_T(a_1, a_2) \subset T(S(a_1, a_2); z(a_1, a_2)),$$
$$P_I(a_1, a_2) \subset I(S(a_1, a_2); z(a_1, a_2)).$$

Suppose now that K is a subset of Ω such that $\mathscr{L}z \in K$ and define the sets in $\Omega \times R^n$,

$$\tilde{P}_I = \{(\mathscr{L}w, w(a_1, a_2)): w(\cdot, \cdot) \in Q, \mathscr{L}w \in I(K; \mathscr{L}z), \\ w_{xy}(x, y) \in F^H(x, y)(Z(w)(x, y)) \text{ a.e. on } \Pi\},$$
$$\tilde{S} = \{(\mathscr{L}u, u(a_1, a_2)): u(\cdot, \cdot) \in Q, \mathscr{L}u \in K, \\ u_{xy}(x, y) \in F(x, y, Z(u)(x, y))) \text{ a.e. on } \Pi\}.$$

THEOREM 3. *We have* $\tilde{P}_I \subseteq I(\tilde{S}; (\mathscr{L}z, z(a_1, a_2)))$.

Proof. By the definition of the intermediate tangent cone it suffices to show that for any $(\mathscr{L}w, w(a_1, a_2)) \in \tilde{P}_I$ and any sequence $h_i \to 0+$ there exists a sequence $(\mathscr{L}z^i, z^i(a_1, a_2)) \in \tilde{S}$ such that we have in $\Omega \times R^n$,

$$\lim_{h_i \to 0+} (\mathscr{L}(z^i - z - h_i w), z^i(a_1, a_2) - z(a_1, a_2) - h_i w(a_1, a_2))/h_i = 0. \tag{14}$$

Tho this end, observe that from the definition of $\tilde{P}_I$, there exists a sequence $(a_i, \alpha_i, \beta_i) \in K$ satisfying

$$\lim_{h_i \to 0+} ((a_i, \alpha_i, \beta_i) - \mathscr{L}(z + h_i w))/h_i = 0, \tag{15}$$

and also, for a.e. $(x, y) \in \Pi$,

$$\lim_{h_i + 0+} \operatorname{dist}(z_{xy}(x, y) + h_i w_{xy}(x, y), F(x, y, Z(z + h_i w)(x, y)))/h_i = 0. \tag{16}$$

On the other hand

$$\operatorname{dist}(z_{xy}(x, y) + h_i w_{xy}(x, y), F(x, y, Z(z + h_i w)(x, y)))/h_i \\ \leqslant \|w_{xy}(x, y)\| + k \, \|Z(w)(x, y)\|,$$

hence, by the Lebesgue Theorem and applying Theorem 1 for $\omega(x, y) = z(x, y) + h_i w(x, y)$, we deduce from (15) and (16) the existence of functions $z^i(x, y)$ with $\mathscr{L}z^i = (a_i, \alpha_i, \beta_i)$, $(\mathscr{L}z^i, z^i(a_1, a_2)) \in \tilde{S}$ satisfying (4). This completes the proof of Theorem 3. ∎

3. Solution Map and Adjoint

Let $\{A(x, y): (x, y) \in \Pi\}$ be a family of closed convex processes from $R^n \times R^n \times R^n$ to R^n satisfying the following assumptions:

(A.1) For all $(u, p, q) \in R^n \times R^n \times R^n$ the map $(x, y) \to A(x, y)(u, p, q)$ is measurable on Π;

(A.2) For all $(x, y) \in \Pi$ the map $(u, p, q) \to A(x, y)(u, p, q)$ is Lipschitzian with Lipschitz constant $k_1 > 0$;

(A.3) graph $A(x, y) \subset$ graph $F^H(x, y, \cdot, \cdot, \cdot)$ for a.e. $(x, y) \in \Pi$.

Consider the solution map $r: \Omega \to 2^{R^n}$ defined by

$$r(a, \alpha, \beta) = \{w(a_1, a_2): w(\cdot, \cdot) \in Q, \mathscr{L}w = (a, \alpha, \beta), \\ w_{xy}(x, y) \in A(x, y)(Z(w)(x, y)) \text{ a.e. on } \Pi\}.$$

It is obvious that r is a convex process. We are interested in obtaining a characterization of the adjoint map r^*.

Recall that, given a set-valued map $G: X \to 2^Y$ from a Banach space X to a Banach space Y, the adjoint map $G^* \to 2^{X^*}$ is defined by the condition $v \in G^*(u) \Leftrightarrow (-v, u) \in (\text{graph } G)^+$ (so when G is a matrix then G^* is the transpose of G). The following result is well known:

Lemma 1 [4, Lemma 1.3]. *Let X, Y be two Hilbert spaces, $\mathscr{R}$ be a continuous linear operator from X to Y, and L be a closed convex cone of Y. Then*

$$\mathscr{R}^{-1}(L)^+ = \mathscr{R}^*(L^+) \qquad \text{if } \operatorname{Im} \mathscr{R} - L = Y.$$

Theorem 4. *The adjoint map $r^*: R^n \to 2^{\Omega^*}$ carries every $b \in R^n$ to the set $r^*(b)$ formed by all (a, α, β) such that*

$$a = \tilde{p}^1(0, 0), \qquad \alpha(x) = \tilde{p}^1(x, 0) - \tilde{p}^2_x(x, 0), \qquad \beta(y) = \tilde{p}^1(0, y) - \tilde{p}^3_y(0, y),$$

where $\tilde{p}^i(\cdot, \cdot) \in Q$, $i = 1, 2, 3$;

$$(\tilde{p}^1_{xy}(x, y), \tilde{p}^2_{xy}(x, y), \tilde{p}^3_{xy}(x, y)) \in A^*(x, y)(\tilde{p}^1(x, y) \\ - \tilde{p}^2_x(x, y) - \tilde{p}^3_y(x, y)) \qquad \text{a.e. on } \Pi;$$

$$\tilde{p}^1(a_1, y) = \tilde{p}^1(x, a_2) = b,$$

$$\tilde{p}^i(a_1, y) = \tilde{p}^i(x, a_2) = 0 \text{ for every } x \in I_1,\ y \in I_2\ (i = 2, 3).$$

Proof. Observe that any linear continuous functional $u^* \in Q^*$ can be expressed in the form

$$\langle u^*, u\rangle = \langle a^*, u(0,0)\rangle + \int_0^{a_1} \langle \alpha^*(x), u_x(x,0)\rangle\,dx + \int_0^{a_2} \langle \beta^*(y), u_y(0,y)\rangle\,dy + \int_0^{a_1}\int_0^{a_2} \langle \bar{\gamma}(x,y), u_{xy}(x,y)\rangle\,dy\,dx \qquad \forall u(\cdot,\cdot)\in Q,$$

$$\text{where } (a^*, \alpha^*, \beta^*)\in \Omega^*,\ \bar{\gamma}(\cdot,\cdot)\in L^\infty(\Pi), \tag{17}$$

Denote $P = L^1(\Pi)^4$ and $L = \{(u^1, u^2, u^3, v)\in P\colon v(x,y)\in A(x,y)(u^1(x,y), u^2(x,y), u^3(x,y))$ a.e. on $\Pi\}$. Obviously, L is a closed convex cone in P. Just in the same way as in [11] we have

$$L^+ = \{(-p^1, -p^2, -p^3, q)\in L^\infty(\Pi)\colon (p^1(x,y), p^2(x,y), p^3(x,y)) \in A^*(x,y)(q(x,y)) \text{ a.e. on } \Pi\}. \tag{18}$$

Define now the linear operators

$$D\colon Q\to P, \qquad Du := (u(\cdot,\cdot), u_x(\cdot,\cdot), u_y(\cdot,\cdot), u_{xy}(\cdot,\cdot)),$$

$$\gamma\colon Q\to \Omega\times R^n, \qquad \gamma u := (\mathscr{L}u, u(a_1,a_2)).$$

In view of Theorem 1, from the Lipschitz condition in (u, p, q) of the map $A(x,y)$, it is easily follows that for every $(v^1, v^2, v^3, v^4)\in P$ there exists a solution of the inclusion

$$u_{xy}(x,y)\in A(x,y)(Z(u)(x,y) - (v^1, v^2, v^3)) + v^4(x,y),$$

$$u(x,0) = u(0,y) = 0 \qquad \forall (x,y)\in \Pi.$$

So, for every $(v^1, v^2, v^3, v^4)\in \Pi$ there exists $u\in Q$ such that $Du - (v^1, v^2, v^3, v^4)\in L$, i.e., $\operatorname{Im} D - L = P$, and consequently, by Lemma 1,

$$(D^{-1}L)^+ = D^*(L^+).$$

Because $\gamma(D^{-1}L)\subset \operatorname{graph} r$, this implies that

$$\gamma^*(\operatorname{graph} r)^+ \subset (D^{-1}L)^+ = D^*(L^+).$$

Hence, for any $(b, a, \alpha, \beta)\in \operatorname{graph} r^*$ (i.e., $(-a, -\alpha, -\beta, b)\in (\operatorname{graph} r)^+$) there exists $(-p^1, -p^2, -p^3, q)\in L^+$ satisfying $\gamma^*(-a, -\alpha, -\beta, b) = D^*(-p^1, -p^2, -p^3, q)$, i.e.,

$$\begin{aligned}\langle(-a, -\alpha, -\beta, b), \gamma u\rangle &= -\langle a, u(0,0)\rangle - \int_0^{a_1} \langle\alpha(x), u_x(,0)\rangle\, dx \\ &\quad - \int_0^{a_2} \langle\beta(y), u_y(0, y)\rangle dy + \langle b, u(a_1, a_2)\rangle \\ &= \langle(-p^1, -p^2, -p^3, q), Du\rangle \qquad \forall u(\cdot,\cdot)\in Q.\end{aligned} \tag{19}$$

Therefore, for any $w(x, y)=\int_0^x\int_0^y v(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x}$, such that $w\in \operatorname{Ker}\gamma$, i.e., $\int_0^{a_1}\int_0^{a_2} v(x, y)\, dy\, dx=0$, we have

$$\begin{aligned}0 &= \langle(-a, -\alpha, -\beta, b), \gamma w\rangle = \langle D^*(-p^1, -p^2, -p^3, q), w\rangle \\ &= \langle(-p^1, -p^2, -p^3, q), Dw\rangle = \int_0^{a_1}\int_0^{a_2} (-p^1(x, y)\, w(x, y) \\ &\quad - p^2(x, y)\, w_x(x, y) - p^3(x, y)\, w_y(x, y) + q(x, y)\, w_{xy}(x, y))\, dy\, dx \\ &= \int_0^{a_1}\int_0^{a_2} w_{xy}(x, y)\Big[-\int_x^{a_1}\int_y^{a_2} p^1(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} - \int_y^{a_2} p^2(x, \bar{y})\, d\bar{y} \\ &\quad - \int_x^{a_1} p^3(\bar{x}, y)\, d\bar{x} + q(x, y)\Big]\, dy\, dx \\ &= \int_0^{a_1}\int_0^{a_2} v(x, y)\Big[-\int_x^{a_1}\int_y^{a_2} p^1(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} \\ &\quad - \int_y^{a_2} p^2(x, \bar{y})\, d\bar{y} - \int_x^{a_1} p^3(\bar{x}, y)\, d\bar{x} + q(x, y)\Big]\, dy\, dx.\end{aligned}$$

This implies that

$$\begin{aligned}q(x, y) &= \int_x^{a_1}\int_y^{a_2} p^1(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x} + \int_y^{a_2} p^2(x, \bar{y})\, d\bar{y} \\ &\quad + \int_x^{a_1} p^3(\bar{x}, y)\, d\bar{x} + q_0, \qquad q_0 = \text{const}.\end{aligned} \tag{20}$$

Setting

$$\bar{p}^1(x, y) = \int_x^{a_1}\int_y^{a_2} p^1(\bar{x}, \bar{y})\, d\bar{y}\, d\bar{x},$$

$$\bar{p}^2(x, y) = \int_y^{a_2} p^2(x, \bar{y})\, d\bar{y},$$

$$\bar{p}^3(x, y) = \int_x^{a_1} p^3(\bar{x}, y)\, d\bar{x},$$

we have, by integrating by parts and using Fubini's Theorem

$$\int_0^{a_1}\int_0^{a_2}\frac{\partial}{\partial y}\langle \bar{p}^2(x, y), u_x(x, y)\rangle\, dy\, dx = -\int_0^{a_1}\langle \bar{p}^2(x, 0), u_x(x, 0)\rangle\, dx,$$

$$\int_0^{a_1}\int_0^{a_2}\frac{\partial}{\partial x}\langle \bar{p}^3(x, y), u_y(x, y)\rangle\, dy\, dx = -\int_0^{a_2}\langle \bar{p}^3(0, y), u_y(0, y)\rangle\, dy,$$

$$\begin{aligned}\int_0^{a_1}\int_0^{a_2}&\langle \bar{p}^1(x, y), u_{xy}(x, y)\rangle\, dy\, dx\\ &= -\int_0^{a_1}\langle \bar{p}^1(x, 0), u_x(x, 0)\rangle\, dx - \int_0^{a_2}\langle \bar{p}^1(0, y), u_y(0, y)\rangle\, dy\\ &\quad + \int_0^{a_1}\int_0^{a_2}\langle \bar{p}^1_{xy}(x, y), u(x, y)\rangle\, dy\, dx - \bar{p}^1(0, 0)\, u(0, 0).\end{aligned}$$

Taking account of (20) and the just established relations, the right hand side of (19) becomes

$$\begin{aligned}\langle(-p^1, -p^2, -p^3, q), Du\rangle &\\ = \langle -\bar{p}^1(0, 0) - q_0, u(0, 0)\rangle &+ \int_0^{a_1}\langle -p^2(x, 0) - \bar{p}^1(x, 0)\\ - q_0, u_x(x, 0)\rangle\, dx &+ \int_0^{a_2}\langle -\bar{p}^3(0, y)\\ - \bar{p}^1(0, y) - q_0, u_y(0, y)\rangle\, dy &+ \langle q_0, u(a_1, a_2)\rangle.\end{aligned} \tag{21}$$

Therefore, by setting $\tilde{p}^1(x, y) = \bar{p}^1(x, y) + q_0$, $\tilde{p}^2(x, y) = \int_x^{a_1}\bar{p}^2(x, y)\, d\bar{x}$, $\tilde{p}^3(x, y) = \int_y^{a_2}\bar{p}^3(x, \bar{y})\, d\bar{y}$, substituting (21) into (19), and taking account of (18) we deduce that for any b the set $r^*(b)$ is contained in the set of all (a, α, β) satisfying the relations stated in Theorem 4. Since, on the other hand, for any (a, α, β) satisfying the mentioned relations we obviously have $(-a, -\alpha, -\beta, b) \in (\text{graph } r)^+$, i.e., $(b, a, \alpha, \beta) \in \text{graph } r^*$, the proof of Theorem 4 is complete. ▌

Before closing this section, let us also state the following useful properties of the solution map r and its adjoint r^*. We omit their proof, since it is analogous to the proof of Lemmas 3.5, 3.6, and 4.4 in [13].

PROPOSITION 1. *The set-valued map r is Lipschitzian on Ω and the map* $\text{cl}\, r$ *defined by* $\text{cl}\, r(a, \alpha, \beta) := \overline{r(a, \alpha, \beta)}$ $\forall (a, \alpha, \beta)$, *is a Lipschitzian closed convex process. Furthermore* $(\text{cl}\, r)^* = r^*$ *is an upper semicontinuous set-valued map carrying bounded sets to bounded sets, and* $\text{Dom}\, r^* = r(0, 0, 0)^+$.

PROPOSITION 2. *For any convex cone E in Ω, r_E denotes the restriction of r to E, i.e.,*

$$r_E(a, \alpha, \beta) = \begin{cases} r(a, \alpha, \beta) & \text{if } (a, \alpha, \beta) \in E, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $r_E^*(b) = r^*(b) - E^*$ for $b \in \operatorname{Dom} r^*$ and $r(E)^+ = r^{*-1}(E^+)$.

PROPOSITION 3. *Let $\{A(x, y): (x, y) \in \Pi\}$ be a family of closed convex processes satisfying conditions* (A.1), (A.2) *stated at the beginning of Section* 3 *and also the condition* graph $A(x, y) \subset$ graph dco $F(x, y, Z(z)(x, y), z_{xy}(x, y))$ *a.e. on Π. Let*

$$\bar{G}(x, y)(u, p, q) = \overline{A(x, y)(u, p, q) + T(\mathrm{co}F(x, y, Z(z)(x, y)); z_{xy}(x, y))}$$
$$\forall (u, p, q).$$

Then $\{\bar{G}(x, y): (x, y) \in \Pi\}$ is a family of closed convex processes satisfying conditions (A.1), (A.2), graph $\bar{G}(x, y) \subset$ graph dco $F(x, y, Z(z)(x, y), z_{xy}(x, y))$ *a.e. on Π, and $A(x, y) \subset \bar{G}(x, y)$ for every x, y. Furthermore, for a.e. $(x, y) \in \Pi$ and $b \in R^n$,*

$$\bar{G}^*(x, y)\, b = \begin{cases} A^*(x, y)\, b & \text{if } b \in (F(x, y, Z(z)(x, y)) - z_{xy}(x, y))^+, \\ \varnothing & \text{otherwise.} \end{cases} \tag{22}$$

4. APPLICATION TO OPTIMIZATION PROBLEM WITH RESPECT TO END POINTS

Consider inclusion (4), (5) with the associated objective functional

$$\phi\left(u(0, 0), \int_0^{a_1} u_x(x, 0)\, dx, \int_0^{a_2} u_y(0, y)\, dy, u(a_1, a_2)\right) \to \inf, \tag{23}$$

where ϕ: $R^n \times R^n \times R^n \times R^n \to R$.

This optimization problem is equivalent to minimizing with respect to end points:

$$\bar{\phi}(u(0, 0), u(a_1, 0), u(0, a_2), u(a_1, a_2)) \to \inf.$$

Let $z(x, y)$ be a solution of inclusion (4), (5) and $\{A_1(x): x \in I_1\}$, $\{A_2(y): y \in I_2\}$ be two families of closed processes from R^n to R^n satisfying the following conditions:

(B.1) For every $u \in R^n$ the maps $x \to A_1(x)\, u$ and $y \to A_2(y)\, u$ are measurable on I_1 and I_2, respectively.

(B.2) For every $x \in I_1$, $y \in I_2$ the maps $u \to A_1(x)\, u$ and $u \to A_2(y)\, u$ are Lipschitzian with Lipschitz constant $k_2 > 0$.

(B.3) graph $A_1(x) \subset$ graph $F_1^H(x)$ a.e. on I_1, graph $A_2(y) \subset$ graph $F_2^H(y)$ a.e. on I_2.

Let K, E be subsets of Ω defined by

$$K = \left\{(a, \alpha, \beta)\colon a \in F_0,\ \alpha(x) \in F_1\left(x, a + \int_0^x \alpha(\bar{x})\, d\bar{x}\right) \text{a.e. on } I_1,\right.$$

$$\left.\beta(y) \in F_2\left(y, a + \int_0^y \beta(\bar{y})\, d\bar{y}\right) \text{a.e. on } I_2\right\},$$

$$E = \left\{(a, \alpha, \beta)\colon a \in I(F_0; z(0, 0)),\right.$$

$$\alpha(x) \in A_1(x)\left(a + \int_0^x \alpha(\bar{x})\, d\bar{x}\right) \text{a.e. on } I_1,$$

$$\left.\beta(y) \in A_2(y)\left(a + \int_0^y \beta(\bar{y})\, d\bar{y}\right) \text{a.e. on } I_2\right\}.$$

LEMMA 2. *We have $E \subset I(K; \mathscr{L}z)$ and the positive polar E^+ of E is the set of all $(a^*, \alpha^*, \beta^*) \in \Omega^*$ for each of which there exist $\tilde{\alpha} \in L^1(I_1)$, $\tilde{\beta} \in L^1(I_2)$ satisfying*

$$a^* - \int_0^{a_1} \tilde{\alpha}(x)\, dx$$

$$\in I(F_0; z(0, 0))^+,\ a^* - \int_0^{a_2} \tilde{\beta}(y)\, dy \in I(F_0; z(0, 0))^+,$$

$$-\tilde{\alpha}(x) \in A_1^*(x)\left(-\int_x^{a_1} \tilde{\alpha}(\bar{x})\, d\bar{x} + \alpha^*(x)\right) \qquad \textit{a.e. on } I_1,$$

$$-\tilde{\beta}(y) \in A_2^*(y)\left(-\int_x^{a_2} \tilde{\beta}(\bar{y})\, d\bar{y} + \beta^*(y)\right) \qquad \textit{a.e. on } I_2. \quad (24)$$

Proof. This is analogous to the proof of Theorem 2 in [17]. ∎

The next theorem gives a necessary optimality condition for a solution $z(x, y)$ to problem (4), (5), (23).

THEOREM 5. *If $z(x, y)$ is an optimal solution of problem* (4), (5), (23) *and if the function ϕ is differentiable at* $(z(0,0), \int_0^{a_1} z_x(x,0)\,dx, \int_0^{a_2} z_y(0,y)\,dy, z(a_1,a_2))$ *then there exist functions $\tilde{p}^i \in Q$, $i=1,2,3$, and absolutely continuous functions $\tilde{q}^1(x)$, $\tilde{q}^2(y)$ on I_1 and I_2, respectively, satisfying the relations*

$$(\tilde{p}^1_{xy}(x,y), \tilde{p}^2_{xy}(x,y), \tilde{p}^3_{xy}(x,y)) \in A^*(x,y)(\tilde{p}^1(x,y) - \tilde{p}^1_x(x,y) - \tilde{p}^1_y(x,y)) \quad \text{a.e. on } \Pi, \tag{25}$$

$$\dot{\tilde{q}}^1(x) \in A_1^*(x)(-\tilde{q}^1(x) - \tilde{p}^2_x(x,0) + \tilde{p}^1(x,0)) \quad \text{a.e. on } I_1, \tag{26}$$

$$\dot{\tilde{q}}^2(y) \in A_2^*(y)(-\tilde{q}^2(y) - \tilde{p}^3_y(0,y) + \tilde{p}^1(0,y)) \quad \text{a.e. on } I_2, \tag{27}$$

with the boundary conditions

$$\tilde{p}^1(a_1,y) = \tilde{p}^1(x,a_2) = \overline{\frac{\partial\phi}{\partial u^4}}, \qquad \tilde{p}^i(x,a_2) = \tilde{p}^i(a_1,y) = 0,\ i=2,3\ \forall (x,y) \in \Pi, \tag{28}$$

$$\tilde{p}^1(0,0) + \tilde{q}^1(0,0) + \overline{\frac{\partial\phi}{\partial u^1}} + \overline{\frac{\partial\phi}{\partial u^2}} \in -I(F_0; z(0,0))^+, \tag{29}$$

$$\tilde{p}^1(0,0) + \tilde{q}^1(0,0) + \overline{\frac{\partial\phi}{\partial u^1}} + \overline{\frac{\partial\phi}{\partial u^3}} \in -I(F_0; z(0,0))^+, \tag{30}$$

where

$$\overline{\frac{\partial\phi}{\partial u^i}} := \frac{\partial\phi}{\partial u^i}\left(z(0,0), \int_0^{a_1} z_x(x,0)\,dx, \int_0^{a_2} z_y(0,y)\,dy, z(a_1,a_2)\right), \quad i=1,2,3,4. \tag{31}$$

Proof. Define the set-values maps R, R_K from Ω to R^n,

$$R(a,\alpha,\beta) = \{u(a_1,a_2) : u(\cdot,\cdot) \in Q,\ \mathscr{L}u = (a,\alpha,\beta),\ u_{xy}(x,y) \in F(x,y,Z(u)(x,y)) \text{ a.e. on } \Pi\},$$

$$R_K(a,\alpha,\beta) = \begin{cases} R(a,\alpha,\beta) & \text{if } (a,\alpha,\beta) \in K, \\ \varnothing & \text{otherwise.} \end{cases}$$

From Lemma 2 and Theorem 3 we have

$$I(\text{graph } R_K; (\mathscr{L}z, z(a_1,a_2)))^+ \subset (\text{graph } r_E)^+.$$

Let $\tilde{\phi}: \Omega \times R^n \to R$ be a function defined by

$$\tilde{\phi}(a,\alpha,\beta,b) = \phi\left(a, \int_0^{a_1} \alpha(x)\,dx, \int_0^{a_2} \beta(y)\,dy, b\right).$$

It is easily seen that the Gateaux derivative $\nabla\tilde{\phi}$ of $\tilde{\phi}$ at $(z(0,0), z_x(\cdot,0), z_y(0,\cdot), z(a_1,a_2))$ satisfies

$$\nabla\tilde{\phi} =)\overline{\partial\phi/\partial u^1}, \overline{\partial\phi/\partial u^2}, \overline{\partial\phi/\partial u^3}, \overline{\partial u/\partial u^4}).$$

Since $z(x, y)$ is optimal, the inclusion $\nabla\tilde{\phi} \in I$ (graph R_K; $(z(0,0), z_x(\cdot,0), z_y(0,\cdot), z(a_1,a_2))^+$ holds. Hence $\nabla\tilde{\phi} \in (\text{graph } r_E)^+$ and Proposition 2 implies that

$$(-\overline{\partial\phi/\partial u^1}, -\overline{\partial\phi/\partial u^2}, -\overline{\partial\phi/\partial u^3}) \in r_E^*(\overline{\partial\phi/\partial u^4}) = r^*(\overline{\partial\phi/\partial u^4}) - E^+.$$

Therefore, there exists $(a, \alpha, \beta) \in r^*(\overline{\partial\phi/\partial u^4})$ such that $(a^*, \alpha^*, \beta^*) = (a + \overline{\partial\phi/\partial u^1}, \alpha + \overline{\partial\phi/\partial u^2}, \beta + \overline{\partial\phi/\partial u^3}) \in E^+$. Applying Lemma 2, then setting $\tilde{q}^1(x) = \int_x^{a_1} \tilde{\alpha}(\bar{x})\, d\bar{x} - \overline{\partial\phi/\partial u^2}$, $\tilde{q}^2(y) = \int_y^{a_1} \tilde{\beta}(\bar{y})\, d\bar{y} - \overline{\partial\phi/\partial u^3}$, and applying Theorem 4 we obtain the desired relations (25)–(31). This completes the proof of Theorem 5. ▮

Remark 1. In the case when $F(x, y, u, p, q) \in \text{Conv } R^n$, $F_1(x, u) \in \text{Conv } R^n$, $F_2(y, u) \in \text{Conv } R^n$ for all x, y, u, p, q, we can apply Theorem 5 with $A(x, y)$, $A_1(x)$, $A_2(y)$ replaced by $A'(x, y)$, $A'_1(x)$, $A'_2(y)$ defined as

$$A'(x, y)(u, p, q) = \overline{A(x, y)(u, p, q) + T(F(x, y, Z(z)(x, y)); z_{xy}(x, y))},$$
$$A'_1(x)\, u = \overline{A_1(x)\, u + T(F_1(x, z(x, 0)); z_x(x, 0))},$$
$$A'_2(y)\, u = \overline{A_2(y)\, u + T(F_2(y, z(0, y)); z_y(0, y))}.$$

Then, setting

$$p(x, y) = -\tilde{p}^1(x, y) + \tilde{p}_x^2(x, y) + \tilde{p}_y^3(x, y), \tag{32}$$
$$q_1(x) = \tilde{q}^1(x) + \tilde{p}_x^2(x, 0) - \tilde{p}^1(x, 0), \tag{33}$$
$$q_2(y) = \tilde{q}^2(y) + \tilde{p}_y^3(0, y) - \tilde{p}^1(0, y), \tag{34}$$

and applying Proposition 3 we get, aside from (25)–(30), also the maximum conditions

$$\langle z_{xy}(x, y), p(x, y)\rangle = \max\{\langle v, p(x, y)\rangle : v \in F(x, y, Z(z)(x, y))\} \quad \text{a.e. on } \Pi, \tag{35}$$
$$\langle z_x(x, 0), q_1(x)\rangle = \max\{\langle v^1, q_1(x)\rangle : v^1 \in F_1(x, z(x, 0))\} \quad \text{a.e. on } I_1, \tag{36}$$
$$\langle z_y(0, y), q_2(y)\rangle = \max\{\langle v^2, q_2(y)\rangle : v^2 \in F_2(y, z(0, y))\} \quad \text{a.e. on } I_2. \tag{37}$$

Remark 2. Necessary optimality conditions in optimal time problems and multicriteria problems for inclusion (4), (5) have been given in [24].

5. Application to Optimal Control of Dynamic Systems

I. Consider problem (6), (7), (23), i.e., that of minimizing the functional (23) over the solutions to system (6), (7).

Let there be given functions $f\colon R^n \times R^n \times R^n \times V \to R^n$, $f_1\colon R^n \times V_1 \to R^n$, $f_2\colon R^n \times V_2 \to R^n$, and the sets $V \in \operatorname{Comp} R^m$, $V_i \in \operatorname{Comp} R^{m_i}$, $i = 1, 2$. The controls $v(x, y)$, $v^1(x)$, $v^2(y)$ are measurable functions such that $v(x, y) \in V$ a.e. on Π, $v^1(x) \in V_1$ a.e. on I_1, $v^2(y) \in V_2$ a.e. on I_2. We also assume that:

(C.1) For every $v \in V$, $v^1 \in V_1$, $v^2 \in V_2$ the functions $f(\cdot, \cdot, \cdot, v)$, $f_1(\cdot, v^1)$, $f_2(\cdot, v^2)$ are Lipschitzian with a Lipschitz constant $k > 0$;

(C.2) For every u, p, q the functions $f(u, p, q, \cdot)$, $f_1(u, \cdot)$, $f_2(u, \cdot)$ are continuous.

Note that by applying Theorem 1 for $\omega(x, y) = 0$ and $F(x, y, u, p, q) = f(u, p, q, v(x, y))$, $F_1(x, u) = f_1(u, v^1(x))$, $F_2(y, u) = f_2(u, v^2(y))$ for every $v(x, y)$, $v^1(x)$, $v^2(y)$, these assumptions imply the following:

(C.2′) There exists a constant M such that $\| f(Z(u)(x, y), v(x, y)) \| \leqslant M$ a.e. on Π for every solution $u(x, y)$ of system (6), (7) corresponding to controls $v(x, y)$, $v^1(x)$, $v^2(y)$.

We now apply the previous results to derive necessary optimality conditions in problem (6), (7), (23). To this end, set

$$F(x, y, u, p, q) = f(u, p, q, V), \qquad F_1(x, u) = f_1(u, V_1), \qquad F_2(x, u) = f_2(u, V_2) \tag{38}$$

and to (4), (5) associate the convexified inclusion

$$u_{xy}(x, y) \in G(Z(u)(x, y)), \tag{39}$$

$$u(0, 0) \in F_0, \qquad u_x(x, 0) \in F_1(u(x, 0)), \qquad u_y(0, y) \in F_2(u(0, y)), \tag{40}$$

where $G(Z(u)(x, y)) = \operatorname{co} f(Z(u)(x, y), V)$.

We need the following result, similar to the Filippov–Wazewski relaxation theorem [2, p. 124].

LEMMA 3 [6]. *Let H be the Banach space of absolutely continuous functions on Π equipped with the norm*

$$\|u\|_H = \max\{\|u(x, y)\| : (x, y) \in \Pi\} + \max\{\{\text{vrai max } \|u_x(x, y)\| : x \in I_1\} : y \in I_2\} + \max\{\{\text{vrai max } \|u_y(x, y)\| : y \in I_2\} : x \in I_1\}.$$

Then in this space every solution of inclusion (39) *with the boundary condition $\mathscr{L}u = (a, \alpha, \beta) \in \Omega$ is a limit of solutions of system* (6) *with the same boundary condition.*

Let $z(x, y)$ be a solution of system (6), (7) corresponding to controls $v(x, y)$, $v^1(x)$, $v^2(y)$. Define

$$C^1(x, y) = \partial f(Z(z)(x, y), v(x, y))/\partial u,\ C^2(x, y) = \partial f(Z(z)(x, y), v(x, y))/\partial p,\ C^3(x, y) = \partial f(Z(z)(x, y), v(x, y))/\partial q,\ C_1(x) = \partial f_1(z(x, 0), v^1(x))/\partial u,\ C_2(y) = \partial f_2(z(0, y), v^2(y))/\partial u. \tag{41}$$

If we set

$$A(x, y) = (C^1(x, y), C^2(x, y), C^3(x, y)),\quad A_1(x) = C_1(x),\quad A_2(y) = C_2(y), \tag{42}$$

then inclusion (25), (26), (27) becomes equations and we can state the next theorem:

THEOREM 6. *Assume that the function ϕ as well as the functions $f(\cdot, \cdot, \cdot, v)$, $f_1(\cdot, v^1)$, $f_2(\cdot, v^2)$ (for any fixed $v \in V$, $v^1 \in V_1$, $v^2 \in V_2$) are differentiable and $f_1(u, V_1) \in \text{Conv } R^n$, $f_2(u, V_2) \in \text{Conv } R^n$ $\forall u \in R^n$. If $z(x, y)$ is an optimal solution of problem* (6), (7), (23) *then there exist a solution $\tilde{p}^i(x, y)$, $i = 1, 2, 3$, $\tilde{q}^1(x)$, $\tilde{q}^2(y)$ of Eqs.* (25)–(27) *with boundary conditions* (28)–(30) *such that the maximum conditions* (35)–(37) *hold with F, F_1, F_2 defined from* (38).

Proof. First, we note that

$$\text{graph } A(x, y) \subset \text{graph } dG(Z(z)(x, y), z_{xy}(x, y)),$$
$$\text{graph } A_1(x) \subset \text{graph } dF_1(z(x, 0), z_x(x, 0)),$$
$$\text{graph } A_2(y) \subset \text{graph } dF_2(z(0, y), z_y(0, y)).$$

From the well-known Castaing–Filippov theorem on implicit functions (see, for example, [26]), the solution sets of inclusion (4), (5) and system (6), (7) coincide. Consequently, instead of system (6), (7) we can consider inclusion (4), (5).

Let the maps $\tilde{R}$, $\tilde{R}_K$ from Ω to R^n be defined by

$$\tilde{R}(a, \alpha, \beta) = \{u(a_1, a_2): u(\cdot, \cdot) \in Q, \mathscr{L}u = (a, \alpha, \beta),$$
$$u_{xy}(x, y) \in G(Z(u)(x, y)) \text{ a.e. on } \Pi\} \qquad \forall (a, \alpha, \beta),$$

$$\tilde{R}_K(a, \alpha, \beta) = \begin{cases} \tilde{R}(a, \alpha, \beta) & \text{for } (a, \alpha, \beta) \in K, \\ \varnothing & \text{otherwise.} \end{cases}$$

From Lemma 3 we obtain

$$I(\text{graph } R_K; (\mathscr{L}z, z(a_1, a_2))) = I(\text{graph } \tilde{R}_K; (\mathscr{L}z, z(a_1, a_2))).$$

Therefore, in view of the optimality of $z(x, y)$, we have, as in the proof of Theorem 5,

$$\nabla\tilde{\phi} \subset I(\text{graph } R_K; (\mathscr{L}z, z(a_1, a_2)))^+ = I(\text{graph } \tilde{R}_K; (\mathscr{L}z, z(a_1, a_2)))^+.$$

The proof can now be completed by applying Theorem 4, Lemma 2, and Proposition 3 to inclusion (39), (40) and the processes $A(x, y)$, $A_1(x)$, $A_2(y)$ and arguing as in the last part of the proof of Theorem 5. ∎

II. We now turn to the optimal control for systems with feedback controls.

Let $f: R^n \times R^n \times R^n \times R^m \to R^n$, $f_i: R^n \times R^{m_i} \to R^n$, $i = 1, 2$, be functions and $\bar{V}: R^n \to \text{Comp } R^m$, $\bar{V}^i: R^n \to \text{Comp } R^{m_i}$, $i = 1, 2$, be set valued maps.

Consider the dynamic system with feedback control

$$u_{xy}(x, y) = f(Z(u)(x, y), v(x, y)), \qquad v(x, y) \in \bar{V}(u(x, y)) \tag{43a}$$

under the boundary conditions with feedback controls

$$\begin{aligned} &u(0, 0) \in F_0 \in \text{Comp } R^n, \\ &u_x(x, 0) = f_1(u(x, 0), v^1(x)), v^1(x) \in \bar{V}^1(u(x, 0)), \\ &u_y(0, y) = f_2(u(0, y), v^2(y)), v^2(y) \in \bar{V}^2(u(0, y)) \end{aligned} \tag{43b}$$

(the controls $v(x, y)$, $v^1(x)$, $v^2(y)$ being measurable functions).

Define the maps F, F_1, F_2 as

$$\begin{aligned} &F(x, y, u, p, q) = f(u, p, q, \bar{V}(u)), \\ &F_1(x, u) = f_1(u, \bar{V}^1(u)), \qquad F_2(y, u) = f_2(u, \bar{V}^2(u)). \end{aligned} \tag{44}$$

LEMMA 4. *Assume that for any fixed v, v^1, v^2 the functions $f(\cdot, \cdot, \cdot, v)$, $f_1(\cdot, v^1)$, $f_2(\cdot, v^2)$ are Lipschitzian and that for any fixed u, p, q the functions $f(u, p, q, \cdot)$, $f_1(u, \cdot)$, $f_2(u, \cdot)$ are continuous. If the set-valued maps $\bar{V}$, $\bar{V}^1$, $\bar{V}^2$*

are Lipschitzian, the solution sets of inclusion (4), (5) *and system* (43a), (43b) *coincide.*

Proof. Obviously, any solution of system (43a), (43b) is a solution of inclusion (4), (5). To prove the converse, let $u(x, y)$ be any solution of inclusion (4), (5) and let $g: \Pi \times R^n \to R$ be the function defined by

$$g(x, y, v) = \|u_{xy}(x, y) - f(Z(u)(x, y), v)\| \qquad \forall (x, y, v).$$

Since the function g is measurable in (x, y) and continuous in v, by Himmelberg's Theorem [14, Theorem 6.4] the map $\hat{V}$ defined by

$$\hat{V}(x, y) = \{v \in R^m: g(x, y, v) = 0\} \qquad \forall (x, y) \in \Pi$$

is measurable on Π. Hence, the map $(x, y) \to \bar{V}(x, y) \cap \hat{V}(x, y)$ is also measurable and by Theorem 4.1 of [25] (or Theorem 1.7.7 of [26]) there exists a measurable selector $v(x, y) \in \bar{V}(x, y) \cap \hat{V}(x, y)$ satisfying (43a) obviously. Similarly, there exist measurable selectors $v^1(x) \in \bar{V}^1(u(x, 0))$, $v^2(y) \in \bar{V}^2(u(0, y))$ satisfying (43b). This means that $u(x, y)$ is also a solution of system (43a), (43b). ▮

Now let $z(x, y)$ be any solution of system (43a), (43b) corresponding to feedback controls $\bar{v}(x, y)$, $\bar{v}^1(x)$, $\bar{v}^2(y)$, and let $\{B(x, y): (x, y) \in \Pi\}$, $\{B_1(x): x \in I_1\}$, $\{B_2(y): y \in I\}$ be families of closed convex processes from R^n to R^n satisfying the conditions

(C.3) For every $u \in R^n$, the maps $(x, y) \to B(x, y)\, u$, $x \to B_1(x)\, u$, $y \to B_2(y)\, u$ are measurable.

(C.4) For every $(x, y) \in \Pi$, the maps $u \to B(x, y)\, u$, $u \to B_1(x)\, u$, $u \to B_2(y)\, u$ are Lipschitzian with Lipschitz constant $k > 0$.

(C.5) graph $B(x, y) \subset I(\text{graph } \bar{V}; (z(x, y), \bar{v}(x, y)))$ a.e. on Π, graph $B_1(x) \subset I(\text{graph } \bar{V}^1; (z(x, 0), \bar{v}^1(x)))$ a.e. on I_1, graph $B_2(y) \subset I(\text{graph } \bar{V}^2; (z(0, y), \bar{v}^2(y)))$ a.e. on I_2.

Let $C^i(x, y)$, $i = 1, 2, 3$, $C_1(x)$, $C_2(y)$ be defined by (41). Set

$$C^4(x, y) = \partial f(Z(z)(x, y), \bar{v}(x, y))/\partial v,$$
$$C_3(x) = \partial f_1(z(x, 0), \bar{v}^1(x))/\partial v^1,$$
$$C_4(y) = \partial f_2(z(0, y), \bar{v}^2(y))/\partial v^2,$$
$$\tilde{A}(x, y) = (C^{1*}(x, y) + B^*(x, y)\, C^{4*}(x, y), C^{2*}(x, y), C^{3*}(x, y)),$$
$$\tilde{A}_1(x) = C_1^*(x) + B_1^*(x)\, C_3^*(x), \qquad \tilde{A}_2(y) = C_2^*(y) + B_2^*(y)\, C_4^*(y),$$

The following result expresses the maximum principle for problem (43a), (43b), (23).

THEOREM 7. *In addition to the conditions in Lemma* 4, *assume that functions* f, f_1, f_2, ϕ *are differentiable and* $f(u, p, q, \bar{V}(u)) \in \operatorname{Conv} R^n$, $f_1(u, \bar{V}^1(u)) \in \operatorname{Conv} R^n$, $f_2(u, \bar{V}^2(u)) \in \operatorname{Conv} R^n$ *for every* u, p, q. *If* $z(x, y)$ *is an optimal solution of problem* (43a), (43b), (23) *corresponding to feedback controls* $\bar{v}(x, y)$, $\bar{v}^1(x)$, $\bar{v}^2(y)$ *then there exists solution* $\tilde{p}^i(x, y)$, $i = 1, 2, 3$, $\tilde{q}^1(x)$, $\tilde{q}^2(y)$ *of the inclusion*

$$(\tilde{p}^1_{xy}(x, y), \tilde{p}^2_{xy}(x, y), \tilde{p}^3_{xy}(x, y)) \in \tilde{A}(x, y)(\tilde{p}^1(x, y) - \tilde{p}^2_x(x, y) - \tilde{p}^1_y(x, y)),$$

$$\dot{\tilde{q}}^1(x)\tilde{A}_1(x)(-\tilde{q}^1(x) - \tilde{p}^2_x(x, 0) + \tilde{p}^1(x, 0)),$$

$$\dot{\tilde{q}}(y) \in \tilde{A}_2(y)(-\tilde{q}^2(y) - \tilde{p}^3_y(0, y) + \tilde{p}^1(0, y))$$

with boundary conditions (28)–(30) *such that the maximum conditions* (35)–(37) *hold for* F, F_1, F_2 *defined from* (44).

Proof. Define

$$\bar{A}(x, y) = (C^1(x, y) + C^4(x, y)\, B(x, y), C^2(x, y), C^3(x, y)),$$

$$\bar{A}_1(x) = C_1(x) + C_3(x)\, B_1(x), \bar{A}_2(y) = C_2(y) + C_4(y)\, B_2(y),$$

$$A(x, y)(u, p, q) = \overline{\bar{A}(x, y)(u, p, q) + T(F(x, y, Z(z)(x, y)); z_{xy}(x, y))},$$

$$A_1(x)\, u = \overline{\bar{A}_1(x)\, u + T(F_1(x, z(x, 0)); z_x(x, 0))},$$

$$A_2(y)\, u = \overline{\bar{A}_2(y)\, u + T(F_2(y, z(0, y)); z_y(0, y))},$$

in the same way as in [13, Theorem 5.6] it is easily seen that

$$A^*(x, y) \subset \tilde{A}(x, y), \qquad A_1^*(x) \subset \tilde{A}_1(x), \qquad A_2^*(y) \subset \tilde{A}_2(y).$$

Theorem 7 is now a direct consequence of this relation, Proposition 3, and Theorem 5. ∎

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